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The double nature of didactic models in conceptualizing the evolution of number systems: A mathematical model and a learning tool

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This paper is a small part of an on-going theoretical study on didactic models as a form of didactic transformations of mathematical notions, concepts and ideas, i.e. of adjusting mathematics for teaching. In what we propose here we argue that regarding this adjustment as mathematical modeling should be inherent to the mathematics teaching: they may enhance the concept development as the on-going result of students' learning; foster embedding the "big ideas" approach to mathematics learning, and lead to more self-consistently evolving mathematical knowledge. The "big idea" is that numbers are to be studied in the context of number structures, i.e. together with operations defined on them and properties of these operations, and that a familiar number system may serve a model for studying a new one. We illustrate the didactical model's approach at the initial stage of learning fractions.

Keywords: Didactic models, acquiring notion of number systems, evolution of mathematical knowledge, arithmetic at elementary school.

Introduction

This paper is a small part of an on-going theoretical study on didactic models as a form of didactic transformations of mathematical notions, concepts and ideas, i.e. of adjusting mathematics for teaching in a way that would preserve to a maximal possible extent its structure and spirit. "We want the students to be exposed as early as possible to the idea that beyond the nuts and bolts of mathematics, there are unifying undercurrents that connect disparate pieces" (Wu, 2009). The theoretical framework for the research is inspired by ideas such as Wu's idea cited above, and sprouts from the works by Freudenthal (in particular, Freudenthal, 1975), Kirsch (2000) and from the discussion on applied-mathematic nature of didactic transformations (Borovik, 2012). These and other sources reflect the need for the merged input of deep mathematical, psychological and didactic considerations in constructing the mathematics feasible and meaningful for students of various ages and levels of mathematics learning.

Any form of teaching mathematics involves adjusting it for the students. In what we propose here we argue that regarding this adjustment as mathematical modeling should be inherent to the mathematics teaching. The philosophy behind mathematic models is applying a user-attainable mathematical apparatus to study an unknown subject or phenomenon. No mathematical model fully represents the subject being studied. One should always be aware of limitations of a model being used along with its purpose and benefits, and also of what Freudenthal (1975) presents as the dual character of mathematical models: "*Models of something* are after-images of a piece of given reality; *models for something* are pre-images for a piece of to be created reality" (p. 6).

An educated usage of models is supposed to shed light on phenomena and subjects being studied and seems to be indispensable as a tool at any level of mathematic studies. This implies that the mathematical models usage should become an integral part of the teaching/learning procedure in

mathematics lessons. In particular, concerning the *models for something* – the “to be created reality” in school mathematics is first and foremost the new mathematical knowledge hopefully to become in due time reality for the pupils. Thus, when the unknown subject to be studied belongs to mathematics, the model illustrating it serves didactic purposes; in this case, we are referring to didactic models (DMs) (see Figure 1):

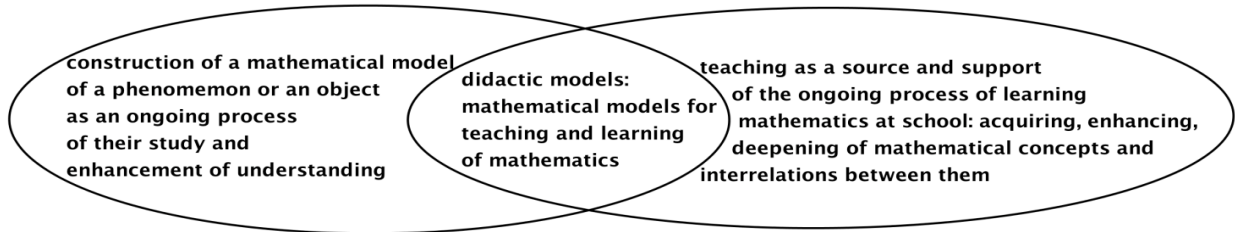


Figure 1. Didactic models as mathematical models and as a learning tool

Examples of well-known and widely used DMs are the Dienes model demonstrating the principles of the decimal system, and the rectangle-area model used to impart some properties of multiplication and division. In this paper, we would like to look closer at didactic models as mathematical models of the *to be created reality*, in order to appreciate their educational value provided they are used knowingly and systematically. We suggest that mindfully and systematically applied to teaching, they may enhance the concept development as the on-going result of students’ learning; foster embedding the “big ideas” approach to mathematics learning, and lead to more self-consistently evolving mathematical knowledge.

A more-or less usual applied-mathematics scheme for the mathematical model usage is (Figure 2):

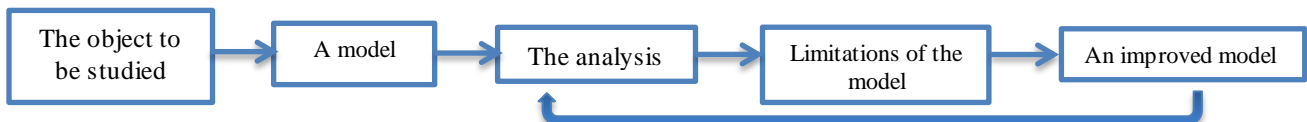


Figure 2. Applied- mathematics scheme for the mathematical model

An initial model is the result of a simplification rendering the phenomenon being studied mathematically feasible, solvable, analyzable. The mathematical model is applied to obtain results, which are supposed to reflect at least to a certain extent the “real thing” – the phenomenon or object being studied. The analysis of the results of the model application usually indicates situations at which the model fails to reflect adequately and fully the “real thing”, and should therefore be improved to better reflect it. The improved model leads to more understandings concerning the object, provided it is mathematically feasible for the user. This looks like a never-ending story, and in mathematics it usually is.

Applying this concept of mathematical model *as is* in the didactic context, i.e., as a didactic model, does not differ conceptually from application of mathematical models to any other field. It is just that the “real thing” being studied is of mathematical nature. In what follows, we illustrate the approach, emphasizing the need for the expertise in using it, which means knowledgeably following the main steps represented in the scheme above, accounting for the students’ level, so that the model being used is feasible to them; otherwise it cannot serve a basis for the further learning.

Didactic models

Didactic models are the result of didactic transformation, being a form of applied mathematical activity aimed at teaching: „We have to accept that, in mathematics, didactic transformation is indeed a form of mathematical practice. Moreover, it is in a sense applied research since it is aimed at a specific application of mathematics teaching.” (Borovik, 2012, p.99). Didactically transforming a mathematical concept is no trivial matter, since it is supposed to cater to both the mathematical and the didactic aspects of the concept: to simplify without distorting the mathematical concept and to present it to pupils in an accessible form.

Shternberg & Yerushalmy consider didactic models to be a means for learning mathematics on the basis of mathematics already familiar to the students and rigorous mapping of learned operations onto the formal mathematical operations (2004). In line with this, we argue that a properly and consistently applied DM approach is a way to enhance the concept-development aspect of mathematical learning.

Prior to presenting the examples illustrating this assertion, we will sum up the main principles on which we propose to base the DMs approach in school. First of all, following Kirsch who claims that activating the existing knowledge is the way to attain accessibility of the new knowledge; DMs should be based on the existing mathematical knowledge, skills and understandings of a student (Kirsch, 2000). Second, the mathematical idea in the basis of the notion should not be distorted as a result of simplifications leading to a didactic model. The definition, operations and properties of a mathematical object should be lucid to those who construct a DM for its learning and to those who use it in its teaching (teachers, textbook writers, curriculum designers etc.)¹ Kirsch (2000) asserts that simplification is a way of making mathematics accessible, but explicitly refers to the “dividing line between legitimate simplification and falsification that does not get past critical pupils” (p. 267). “Not getting past critical pupils” does not ensure that less critical and mathematically aware students do not acquire the falsification as a true image of the mathematical concept. Third, properly used DM approach is a link between the student’s existing mathematical knowledge, the knowledge being currently acquired and the future study of the subject, exactly as the mathematical model is the main tool of the on-going upgrading of the mathematics-based understanding of a phenomenon or an object. Thus, properly used DM approach is a tool for inherently mathematical way of studying it.

In addition, no DM is unique in presentation of a mathematical object (as any mathematical model is not the unique mathematical presentation of any object, for that matter). No contradiction should exist between various DMs; they are supposed to complete each other in the representation of the mathematical object. A student may be exposed only to some DMs representing the concept, appropriate to the didactic circumstances (such as the stage of acquaintance with the subject; level of mathematical development of the students; aims of the specific lesson etc.); the properties represented by a DM must be coherent with mathematics, even if it is not explicitly presented to a student.

The efficient usage of DMs involves two equally important components of the DM-based approach: regarding DM as a mathematical model and as a learning tool. As a mathematical model, the proposed approach enables gradual building-up of an appropriate and consistent concept using the mathematical phenomena, objects and skills familiar to a student. As a learning tool, it provides a

¹ This is obviously the matter of Specialized Mathematics Knowledge for Teaching (see Hill et al., 2004).

precious experience of utilizing models in the process of acquiring a new piece of mathematical knowledge, which necessitates critical and mindful insight into the existing knowledge being used. In what follows, we consider two possible appearances of DMs at school. The outline of the first one appearance is presented in Examples 1-4. The reference to the second one we found appropriate to include as a part of the Discussion.

The beginning of fractions

To illustrate what we consider to be a consistent and educated mode of DMs usage in elementary school, we will apply it to the initial stage of fractions learning. This is an example that we believe to be especially valuable at the elementary school level, when the young pupils do not yet have experience in the process of developing a mathematical concept, while they gradually accumulate some mathematical knowledge. Beliefs, skills and concepts they have acquired are supposed to serve them for the further study. The properly planned and applied model usage for learning may be one of the most important experiences in learning mathematics (Van Den Heuvel-Panhuizen, 2003).

Much too often the term “fraction” is used as a synonym to “a number smaller than 1”, which is the more problematic since in the very beginning of fractions learning the pupils really meet mostly fractions smaller than 1. Moreover, dominating approaches to the beginning of the fractions teaching are based on the “part-of-the-whole” concept and on geometric-visual representations (Hurst & Hurrel, 2014). Important and intuitively supportive as they are, they are detached from the only arithmetic and the only number system the students have come to know to a certain extent at this stage, which is the system of natural numbers. Hurst & Hurrel (ibid.) suggest that it might be plausible to present fractions already at the early stages of learning in a way that will not inhibit, but rather support the future acquiring of the fraction concept without having to significantly change it. Their approach is that of “big ideas”, which we interpret as constructing coherent DMs consistent with the future evolution of fractions into (final and infinite) decimal fractions, notion of ratio, algebraic fractions, the slope of a line and the derivative, and other advanced mathematical appearances of fractions. We suggest that the big idea behind the notion of fraction is the division operation (Mamede & Vasconcelos, 2016). In mathematics, a fraction is either the division operation itself or its result (quotient) (not necessarily a number). If the numerator and the denominator are both natural numbers, the fraction represents a rational number. Fraction is also an operator acting on other mathematical objects, and this is also directly related to its being the division operation. Hence, “the big idea” we propose as the mathematical background, is fraction as division: operation or result. Needless to mention that the idea itself is not intended for elementary school pupils, but the teachers should be cognizant of it.

We illustrate the DMs approach at the initial stage of learning fractions, the model being the arithmetic of natural numbers. We are fully aware of the risk of inhibition effect of this approach. Davis (1989) includes whole number schemes among inhibitors on the way to the rational numbers. Nevertheless, we assert that there is no other mathematical knowledge to build upon for the simple reason that the natural number arithmetic is more or less everything the pupils know before their first encounter with fractions, but for their possible acquaintance with $\frac{1}{2}$ (also justly included by Davis among inhibitors), some primary geometric intuition and some idea of a number line.

Following Shternberg & Yerushalmy (2004), we provide here examples of “mapping” ideas familiar to pupils from the natural numbers arithmetic onto the new mathematical object - fractions, applying the usual scheme for a mathematical model use presented above.

We use it in the first example to impart a meaning of fractions needed for the understanding of addition of fractions; in the second example - to impart conventions of fractions presentations; in the third example – to adjust to fractions a handy geometric model used for integers. In all three examples we refer briefly both to advantages and to limitations of the chosen model, and propose an improved model. Last but not the least is the fourth example of a meaning of natural numbers inapplicable to fractions.

Example 1: Addition of fractions.

The model: a natural number as a cardinal number of a finite set of objects. In a fraction whose numerator and denominator are natural numbers, the numerator serves as a *cardinal number* of a set of equal parts - unit fractions, into which the whole is divided. The denominator indicates the number of parts and their magnitude. Different unit fractions are different objects and cannot be added, unless they are united into one set, just as apples and pears are to be united into the set of fruit to be counted together. For unit fractions, this means representing them with a common denominator. *Limitations:* applicable only to rational numbers. Any other fraction, for example, $\frac{\sqrt{2}}{1+a}$, has to be understood otherwise, namely, as the division operation $\sqrt{2}:(1+a)$ written in another form.

Example 2: Conventions concerning representation of fractions.

The model: the decimal representation of natural numbers. The decimal representation is an equally important appearance of two ideas: of a *representation* of numbers *per se*, and of *conventions* in mathematics. As a decimal representation, it is the *model* applied almost as is to decimal fractions, when the pupils are prepared to deal with them. As an example of a *representation convention*, it may pave the way to the understanding that in mathematics there may be different forms of presenting commonly used objects; these forms should be familiar to everybody; this is the part of the mathematical language. $\frac{7}{5}$ is just another form of writing 7:5, meaning either the operation or the number resulting from it. *Limitations:* the final decimal representation is inapplicable for some numbers; it has to undergo adjustments to infinite (periodic or non-periodic) decimal representations, and provide meanings for their truncations of various kinds.

Example 3: The area model

To adjust the useful *area model* from a rectangle whose sides' length are integers to the rectangle whose sides are rational numbers, it suffices to count “unit rectangles” whose sides are unit fractions corresponding to the factors' denominators, instead of unit squares. *Limitations:* the area model “as is” is hardly applicable, for example, to infinite decimal fractions², to fractions with irrational nominators or denominators, and would demand serious amendment to apply it to negative rational

² Some ideas as presented e.g. in Nelsen (1993, pp.118-122) are based on this type of visual reasoning linking the area notion to numerical reasoning and convergence ideas.

numbers. Nevertheless, speaking of irrational numbers - the segments division in an arbitrary ratio is defined for incommensurable segments as well, for example, by Thales similarity theorems in geometry, on the basis of segments measurement directly related to the number line. Having recognized that the segments ratios is attainable for irrational lengths as well, one can happily keep using the rectangle model for distribution properties of multiplication and division provided it is transfigured so that a subdivision neither into unit squares nor into small “unit” rectangles is needed anymore to apply it. Moreover, the basic fact that the whole segment of length a may be represented as the sum of the two parts, for example, $\frac{a}{(1+\sqrt{2})}$ and $\frac{a\sqrt{2}}{(1+\sqrt{2})}$, is consistent with a similar idea for rational ratio, which again is beneficial for the further goal of regarding the system of real numbers as a whole.

Example 4: Addition of natural numbers as continued counting.

Consider the addition of natural numbers as continued counting: $m+n$ as n times the addition of 1 to m , or m times the addition of 1 to n . Here the *limitations* of the model render it inapplicable as a model for fractions. Obviously, these examples are not meant to be used simultaneously and immediately and not necessarily explicitly in the beginning of acquaintance with the notion of fraction. We do assert though that the ideas represented in these examples must be intertwined in appropriate detail in the course of primary school arithmetic as a general approach to mathematics teaching and learning (DM being a learning tool) and as a groundwork to further encounter with irrational numbers (DM being a mathematical model).

Discussion

The examples above include instances of appearance of new features when the object evolves from an existing one, of transforming the existing feature to adjust to the evolving object, and instances when some features disappear in the new object. Systematically focusing on such occurrences as a teaching norm may foster the concept development as an integral part of learning, provided the notion being taught is regarded as a concept to be permanently developed as a result of teaching and not merely as a topic in a curriculum. One important observation should be made here: should this approach be adopted for fractions or for real numbers, it has to be kept in mind already in the natural numbers teaching. More generally, it will hardly be useful if applied sporadically instead of being a systematic mindful approach. The more so in view of constraints of educational systems: in Israel, for example, and in many other countries, the primary schools are separated from the secondary and the mode of mathematics teaching at different levels is not always coordinated. This transition between the levels is therefore intrinsically discontinuous. We believe that systematic adoption in the primary school of DMs may help to cope with this discontinuity. We regard this to be an issue worth theoretical and empiric study.

Speaking of the encounter with the real number system, we refer here again to the double-sided role of DMs. *DMs as a mathematical model*: similarly to the initial encounter with fractions which is based on natural numbers as a model, in the case of irrational and in general, real numbers, the initial models to build upon are those originating from the system of rational numbers more or less familiar to the students. *DMs as a learning tool*: should the students have acquired appropriate mathematical concepts and learning skills prior to the encounter with real numbers system, these will determine their ability to take-in this new, rather advanced concept, and the extent to which they may take it in. They should have experienced testing the properties of new numbers and operations on them vs. the

familiar ones and the appearance of a new system that includes the previous one not only as a set of numbers, but also as a number system.

Examples of the challenges anticipating the students in their encounter with irrational and in general with real numbers in which the DM approach seems to be promising and worth a close empiric study, are appearance of the root operation and sign; operations on roots (arithmetic and later - algebraic), on expressions of like $a + \sqrt{b}$, and rules of these operations; decimal representations, in particular, decimal approximations of irrational numbers and the necessity to decide when, whether and how to approximate; inclusion of rational and irrational numbers in the same number system, etc. One of the key problems with the notion of irrational numbers is based first and foremost on *impossibility* of writing an irrational number as a fraction of two integers. Thus, their mere existence seems to claim for a new model because of the impossibility of using the previous one. On the other hand, *any* number a may be written as a fraction at least in a trivial way as $\frac{a}{1}$ meaning nothing more than $a:1$. A representation of an irrational number by a fraction means just that at least one of the two parts of the fraction: its nominator or denominator or both, are not rational³. This does not prevent one from using operations on these fractions the way they were used on fractions as rational numbers; sometimes this representation calls for formulation of new rules. For example, to avoid fractions with irrational denominator, the students are sometimes taught to expand them following the rule familiar from rational fractions and based on division properties, for example, $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. On the other hand, an equality like $\frac{\sqrt{3}}{\sqrt{5}} = \sqrt{\frac{3}{5}}$ represents actually a new rule, to be both understood and adopted into the set of mathematical skills. Thus, the notion of fraction as division operation and its properties retains its usefulness. Comprehension of this can be the result of recurrent examination of the notion of fraction and operations defined on it and by it for “new” numbers, based on the DM approach.

Not less important, we suggest that a process of learning that systematically involves DMs is intrinsic to mathematics. No DM adequately represents “the real thing”, in our case eventually the system of real numbers. Various facets of the same complicated mathematical object awaiting the students in their forthcoming studies based, at least to some extent, on the analytic abilities acquired with the help of DMs, is a didactic challenge not less that it was a mathematical challenge, and mathematical and didactic tools should be combined in their teaching and learning. We suggest that the didactic models should be seriously regarded as a tool for this type of learning and closely studied in various theoretical and empiric aspects.

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³ This seems to be a vicious cycle but it is not: we are referring not to a definition but to a possibility of using fractions to write a division operation.

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