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Seeking symmetry in distributive property: Children developing structure sense in arithmetic

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Structure sense can be mobilized by pupils to compare and to transform arithmetical expressions, however sometimes it can lead to mathematical inconsistency that pupils might be not aware of. This paper provides evidence of this type of phenomenon. Through the analysis of an interview with a third grader, it is shown that the development of structure sense can result in transformations as $a \times b + a \times c \rightarrow (a+a) \times (b+c)$. It is concluded that a development of structure sense requires a dialectical control between the syntactic and semantic interpretations of symbolic sentences.

Keywords: Structure sense, distributive property, arithmetic, syntactical transformations.

Learning distributivity

Distributive property appears to be less accessible to young students if compared to other multiplication's properties (Larsson, 2016). This phenomenon could depend on the fact that it is not a property of one operation but it states a relation *between* two operations. Lo and colleagues (2008) found that many prospective primary teachers show difficulties in applying the distributive property: a frequent erroneous transformation is $18 \times 26 = 10 \times 20 + 8 \times 6$ in which tens are multiplied just by tens and units are multiplied just by units (*ibidem*).

According to Carpenter et al. (2005) “an implicit understanding of the distributive property can provide students a framework for learning multiplication number facts by relating unknown facts to known facts” (Carpenter et al., 2005, p.55). For this reason they sustain that it is important to foster the use of fundamental properties of operations to transform mathematical expressions rather than simply calculating. An awareness of the structure of arithmetical expressions appears as fundamental to recognize the equivalence of two arithmetical sentences without carrying out calculations. Mason et al. (2009) use the expression ‘*structural thinking*’ to refer to such awareness.

We wonder which difficulties might students face when they are introduced to structural thinking, and specifically when they elaborate expressions through operations’ properties – distributive property in particular.

Structure sense in arithmetic

Apparently, before Mason et al. (2009) introduced the construct of structural thinking, different words have been used to express similar ideas. Linchevski and Livneh (1999) found that many of the typical difficulties faced by students while interpreting algebraic expressions can be found also in the arithmetical context. In particular, they notice students’ difficulties in determining the order in which additions and subtractions have to be performed both in the arithmetical and in the algebraic context. These authors conclude that “difficulties revealed in children’s understanding of structural properties of the algebraic system originate in their understanding of the number system” (Linchevski & Livneh, 1999, p.192).

Undoubtedly, students must be exposed to the structure of algebraic expressions. However, it must be done in a way that enables them to develop *structure sense*. This means that they will be able to use equivalent structures of an expression flexibly and creatively. (*ibidem*, p. 191)

Similarly, Caspi and Sfard remarked how “structures of algebraic formulas are not unlike those of arithmetic expressions” (Caspi & Sfard, 2012, p. 64) and they interpreted such similarity as based on the fact that school algebra can be conceived as a gradual formalization of meta-arithmetic (*ibidem*); thus the development of an effective algebraic calculation competence has been referred to as the development of structure sense in arithmetic.

Hoch and Dreyfus (2004) characterize structure sense in the context of high school algebra. They define it in terms of a collection of abilities. According to the cited literature, these abilities can be related to similar abilities in arithmetic. So we propose to modify Hoch and Dreyfus’ (*ibidem*) definition to adapt it to the context of primary arithmetic. Thus, structure sense in arithmetic can be described as a set of competences:

1. recognising an arithmetical expression or sentence as an entity, for instance comparing two arithmetical sentences without calculating partial results.
2. Recognising an arithmetical expression or sentence as a previously met structure, for example noticing that $3 \times 4 + 5$ is less than $10 + 3 \times 4 + 5$ because the second one is a sum which includes the first one.
3. Recognising sub-expressions in which an arithmetical expression can be divided, as in the case of a student who can describe $5 \times 7 + 8 \times 7$ as composed by two multiplications.
4. Recognising mutual connections between sub-expressions, that means being able to identify which are the operations connecting the terms of an arithmetical sentence, even when such terms are not just single numbers but shorter expressions.
5. Recognising which manipulations are possible to perform. For instance, on an arithmetical sentence like $7 + 8 \times 7 + 3 + 4$, many transformations could be done ($9 \times 7 + 7$ or $14 + 8 \times 7$) but there are also transformations that are not executable (as $15 \times 7 + 3 + 4$).
6. Recognising which manipulations are useful to perform. According to the aim of transformations (comparing or calculating), some manipulations can be more useful. In the case mentioned above, the usage of associative property of sum ($3 + 4 = 7$) and distributive property ($7 + 8 \times 7 + 7 = 10 \times 7$) allows to notice that $7 + 8 \times 7 + 3 + 4$ is equivalent to 10×7 .

This definition is coherent with and specifies those given by Mason et al. (2009) and Linchevski and Livneh (1999). Some of the competences listed above can be activated by pupils to compare and to transform arithmetical expressions: thus, the presence of these abilities can witness the emergence of structure sense, however sometimes this same abilities can lead to mathematical inconsistency that pupils might be not aware of.

This paper aims to provide evidence of this type of phenomenon, that can be considered as a an indication of an incomplete development of structure sense due to a lack of control on the numerical interpretation of a specific structure.

Data collection and analysis

The results that we are going to present are part of broader research study (Maffia & Mariotti, 2016) aimed at investigating the teaching/learning of multiplication properties in the primary school. The empirical design included long-term teaching experiments involving, among others, a group of second graders. The results presented in this paper concern data coming from this specific group. Grade 2 was chosen to promote structural thinking in the case of multiplication since the very first introduction of this operation, that usually takes place at this school level in Italy. Among others, following Linchevski and Livneh's suggestion to "promote the search for decomposition and recomposition of expressions" (1999, p. 191), we designed and implemented activities aimed at introducing the pupils to the distributive property as a transformation of numerical expressions (Maffia & Mariotti, 2015). The rectangular model of multiplication was introduced: activities of cutting and pasting rectangles with the same height (or width) were proposed to explore the relationship between different arithmetical expressions, eventually generalized and symbolically expressed in the distributive property. Examples of such cutting and pasting are given in Figure 1.

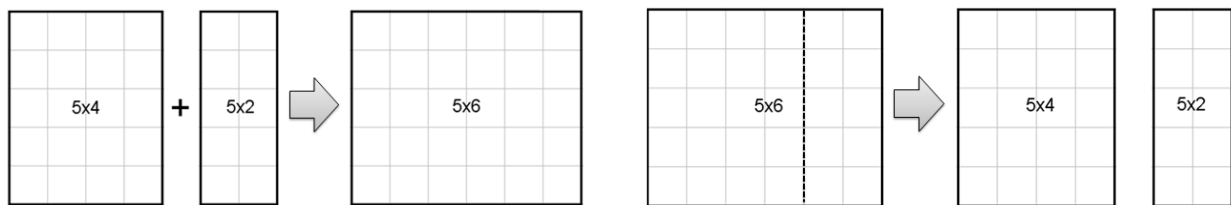


Figure 1: Composition and decomposition of rectangles

Starting from the activities with paper rectangles, the teacher realizes a mediation process to guide students till the usage of conventional arithmetical symbols to represent the relation between multiplication and sum according to the distributive property (*ibidem*).

In this paper, we show data from semi-structured interviews conducted one year after the end of the teaching experiment. Thus, at the moment of the interviews the children are third graders (aged 8-9).

The interviewer shows an image of two children who are writing the equalities $4 \times 7 = 7 \times 4$ and $6 \times 8 + 6 \times 3 = 6 \times 11$ on a blackboard (Figure 2) and he asks if what these children are writing can be considered correct. During the teaching experiment, children were asked to produce compositions and decompositions of multiplications, using paper rectangles and then writing them with arithmetical symbols. This is the first time that they have to validate or refute an already written equation.

After the equalities shown in Figure 2, three other numerical sentences are shown and the interviewee is asked to comment about their correctness. These sentences are $5 \times 6 = 5 \times 2 + 5 \times 4$; $5 \times 6 = 5 \times 3 + 5 \times 4$; $5 \times 4 + 5 \times 3 = 5 \times 2 + 5 \times 5$. The structure of the first one is similar to the one shown in the image, but the position of expressions is inverted in respect to the equal symbol. The second one is like the first one, except for one number (so it is wrong), and the last one has a different structure but it relates two expressions with the same structure – a sum of products – and specifically, the structure of one of the members of the other equalities. So, the different sentences are designed to allow the child to compare or contrast the structures in the different equalities and, eventually, to apply arithmetical properties. During the interview, paper and pen are provided.

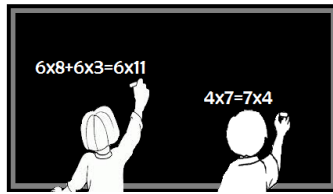


Figure 2: Image showed to students at the beginning of the interview

The interviews have been videotaped and then fully transcribed. Students' transcribed utterances were analysed seeking for evidences of structure sense, through identifying instances of the characterizing abilities. In the following section we discuss some examples, showing specific aspects emerging from this analysis. In the analysis, the six competences of the list are indicated through the corresponding number in the list that is indicated between square brackets.

Seeking symmetry in distributive property

We begin with some excerpts, starting from the end of an interview: Francis comments about the equality $5 \times 4 + 5 \times 3 = 5 \times 2 + 5 \times 5$.

- 53 Interviewer: Now I will show you a very long one. What do you think about this one [he shows the equality $5 \times 4 + 5 \times 3 = 5 \times 2 + 5 \times 5$]?
- 54 Francis: [he writes the equality on his paper and then he answers quickly] It's right!
- 55 Interviewer: Did you already do it?
- 56 Francis: Yes.
- 57 Interviewer: Tell me how. I am not as fast as you are.
- 58 Francis: Wait. I'll write it. 5×4 , is 20. [he writes 20 under the first multiplication. Then he writes the results of the other multiplications; second line in Figure 3] If I would put the 3 and I put 2 [he circles the 3 and he writes a 2 above it] and here I put a 5 [he circles the 4 and he writes a 5 above it] it would be the same operation.

Figure 3: Francis' inscriptions for the last equality

In his explanation (line 58) Francis recognises the possibility of decreasing one of the factors of the second multiplication and increasing one of the factors of the first one, still maintaining the same result (he says "it would be the same operation"). We can recognize an occurrence of the first component of structure sense because Francis is jointly and consistently acting on each part of the arithmetical sentence to maintain its value: he is recognising that the transformation of one multiplication affects the other one, thus he is considering the arithmetical expression on the left side as a unique entity [1].

The expression $5 \times 4 + 5 \times 3$ is transformed in $5 \times 5 + 5 \times 2$ to show the equivalence with $5 \times 2 + 5 \times 5$; so Francis recognises a useful transformation for his purpose [6]. However, in the obtained expression $5 \times 5 + 5 \times 2$, the order of the two multiplications is inverted in respect to $5 \times 2 + 5 \times 5$. Stating that the two

expressions are equivalent, Francis is considering the expression as a sum of two multiplications [4] and so – according to addition’s commutative property – the order of the addends 5×2 and 5×5 can be inverted [5]. The child is also recognising that the expression is composed of two multiplications [3]; this interpretation is strengthened by the written operations in the second line of Figure 3.

So far, we have instances of five of the competences that characterize the structure sense; we can say that Francis is showing some evidence of structure sense. As a matter of fact, Francis’ explanation not only shows his awareness of structure regularities, but it is completely consistent in terms of the mathematical meaning of the expressions.

However, this has not always been the case. At the very beginning, when the image (Figure 2) was firstly showed, he recognized the equality $6 \times 8 + 6 \times 3 = 6 \times 11$ as incorrect and stated that the equivalence would have been true if 6×11 was replaced with 12×12 . Here is his explanation:

- 11 Interviewer: Wait. Tell me how did you get twelve and twelve.
 12 Francis: Six times eight plus six times three [he writes it] I would do six plus six [he draws circles around the 6s, as shown in Figure 4a] that makes twelve.
 13 Interviewer: I understand. So you get the first twelve.
 14 Francis: And eight plus three [he circles 8 and 3, Figure 4a] that makes twelve.
 15 Interviewer: I don’t agree. How much is eight plus three?
 16 Francis: Eight plus three... eleven [he corrects the second 12 writing a 1 over the 2].
 17 Interviewer: Eleven. Ok.
 18 Francis: So it wouldn’t be twelve times twelve but twelve times eleven.

(a) $6 \times 8 + 6 \times 3$ with circles around the 6s and lines connecting them to a new 12×12 expression above. (b) $6 \times 11 = 71x$ (crossed out) and 66 . (c) $48 + 18 = 66$.

Figure 4: Francis’ inscriptions for the first equality

Francis seems to recognize the expression $6 \times 8 + 6 \times 3$ as relating two parts [3], two multiplications connected by an addition [4], and he elaborates this structure according to a *syntactic rule* clearly respecting some kind of “structure sense”, but unfortunately it is inconsistent from the mathematical point of view. The transformations he operates (Figure 4a) are strictly at the syntactical level: he is transforming the expression as if the addition would operate in the same way on both the first and second factors of the two multiplications.

The interviewer asks Francis to check the correctness of his conjecture. Francis proposes to calculate the operations’ results.

- 27 Interviewer: How can we get the result of this thing?
 28 Francis: We calculate forty-eight plus six times three that is... eighteen. Forty-eight plus eighteen is... and six times eleven is... [he performs the written calculation in Figure 4b]. Forty-eight plus eighteen... is... [he performs the written calculation in Figure 4c] sixty-six. So it’s right!

- 29 Interviewer: Is it? So, what was wrong here? [he points Francis inscription in Figure 4a] In your initial check. Because you said that it wasn't right.
- 30 Francis: I thought we had to do 12×11 .
- 31 Interviewer: And is 6×11 enough?
- 32 Francis: [...] Yes, because we have to calculate the results of the two multiplications, to calculate the result of the third one and see if the first two ones equal that... their result.

In line 28, Francis is able to divide the expression into its parts: he recognizes that it is composed of two multiplications [3], then he recognizes that he has to sum the two products, so he is recognizing the connection between the two parts [4]. This interpretation is made explicit again in line 32. Francis is showing two of the competences that characterize structure sense: number 3 and 4 in the list. This time, though using his structure sense, the pupil is interpreting the equivalence between the two expressions in a different way. Previously he considered the expressions $6 \times 8 + 6 \times 3$ and 12×11 to be equivalent because one could be transformed into the other according to a syntactical manipulation. In the following, he recognizes two expressions to be equivalent when they give the same result (lines 28 and 32). We consider the first case an occurrence of a *syntactical interpretation* of the equivalence between numerical expressions, the second one as an occurrence of a *semantic interpretation*. Though not yet well harmonized, both types of interpretations seem to be available to Francis, at the same time, the semantic interpretation seems to maintain its primacy.

When the other two equalities are shown, Francis resorts again to the semantic interpretation. He calculates the results of the expressions on the two sides of the equal sign and then he checks if the results equal each other:

- 43 Interviewer: What if I show you this one? [he shows $5 \times 6 = 5 \times 3 + 5 \times 4$]
- 44 Francis: Thirty [he writes 30]. Fifteen, [he writes 15] twenty [he writes 20 next to 15 and then he puts a + sign between the last two numbers. Then he writes =35 obtaining the inscription shown in Figure 5a]. It doesn't work.
- 45 Interviewer: Can we modify it to make it correct? [Francis doesn't answer] If I would keep this as it is [he points the right side of the equality] what should I write on this side? [he points the left side of the equality]
- 46 Francis: Ehm... [he puts the pen on the sheet of paper]
- 47 Interviewer: Let's write it on the paper [Francis writes the equality] Ok. Let's say that I want this [he points the right side of the equality in Francis' inscription] as it is, but I would change the other to make it correct.
- 48 Francis: We should change the 6 [he circles it] into a 7 [he writes 7 above the 6, Figure 5b]

In this excerpt the interviewer tries to push Francis to go back to a syntactical interpretation. However, though Francis responds in a mathematically consistent way, it is impossible to determine if the proposed modification depends on a syntactical transformation ($3+4=7$) or on a comparison of the expressions' results. His behaviour in lines 27-32 and 53-58 suggests that both the interpretations are plausible.

(a) $15 + 20 = 35$

(b) $5 \times 7 = 5 \times 3 + 5 \times 4$

Figure 5: Francis' inscriptions for the third equality

Discussion and conclusion

As discussed in the introduction of this paper, the development of what we have called “structure sense” can be considered a main objective of the teaching and learning of algebra.

Starting from adapting the definition given by Hoch and Dreyfus (2004) to the case of arithmetic expressions, we set up a list of competences characterizing structure sense and we used it to evaluate students' behaviours as evidences of the presence of structure sense. The aim of this paper is not to discuss about the effectiveness of the classroom intervention; indeed, it presents a recurrent phenomenon that was possible to identify in the development of the structure sense: it is characterized by an unstable relationship between the syntactic and the semantic level in treating numerical expressions. The case of Francis can be considered a paradigmatic example.

The pupil shows all the competences we used to characterize structure sense but in order to check the correctness of an equality he adopts a syntactical manipulation of operations that is not mathematically consistent: an expression as $a \times b + a \times c$ is transformed into $(a+a) \times (b+c)$. We interpret this behaviour as coherent with a structural sense, but also as a case of corrective action aimed at overcoming what can be seen as a structural flaw, a *seeking for symmetry* in the distributive property. The perceived lack of symmetry could be twofold. On the one hand there is no symmetry in the role of the terms: the common factor in the multiplications plays a different role than the others. On the other hand the structure of the equality $a \times b + a \times c = a \times (b+c)$ is asymmetrical because there is a sum of multiplications on one side of the equal sign and just one multiplication on the other side. This interpretative hypothesis is reinforced by the fact that the student does not show difficulties in treating an equality like $5 \times 4 + 5 \times 3 = 5 \times 2 + 5 \times 5$, which has a symmetrical structure. This urgent demand of symmetry may be based also on the experience with other properties, such as the commutative and associative properties, and can be considered as a particular source of difficulty in dealing with distributive property.

The wrong transformation $(a+c) \times (b+d) \rightarrow a \times b + c \times d$ is well known in the context of school algebra and it is found also in the arithmetic context in equalities like $18 \times 26 = 10 \times 20 + 8 \times 6$ (Larsson, 2015; Lo et al., 2008). In this paper we have evidence of the application of the *opposite* transformation $a \times b + c \times d \rightarrow (a+c) \times (b+d)$ in the arithmetic context. As far as we know, this particular transformation has not been documented in literature before. It has to be stressed that this transformation is shown by four students out of nineteen pupils who were involved in our research. So, we have a too small sample to state anything about its spreading.

In any case, the emergence of this kind of erroneous transformation appears relevant from the didactic point of view: if we expect teachers to promote structural thinking they have to know the potential difficulties that students could meet. Literature shows that this is not always the case (Lo et al., 2008). One clear suggestion emerging from our study is that an approach privileging pure syntactical

transformations seems risky, whilst educating pupils on the danger of losing the semantic interpretation of an expression can help them to reach mathematical consistency.

Fostering structural thinking requires the development of semantic control assuring that any syntactic transformation has a consistent arithmetical interpretation. Further investigation is needed in order to fully describe how such a semantic control can be efficiently developed.

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