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# Models' use and adaptation aiming at conceptual understanding of decimal numbers

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*In this paper, we report part of a study carried out within a design research methodology. An initial conjecture was made that included the importance of the hundred square model to facilitate the discussion about decimal number system features and connections among and within different rational numbers' representations. We present how this model was used and why it led to changes into different models, 10x100 grid, and decimat, during the teaching experiment. Finally, we reflect on how these changes inform the initial conjecture.*

*Keywords: Models, decimals, elementary education, design research.*

## Introduction

Currently, in the Portuguese official curriculum (Ministério da Educação, 2013), rational numbers are first approached in Grade 2 (7/8 years old students), in its fraction representation and measure meaning. This meaning has a central role and the use of line segments in the number line is recommended. Decimal numbers<sup>1</sup> are first introduced in Grade 3 and operations with both representations are highly valued. Given the students' age, we believe that these guidelines are too focused on procedures, and, instead, a learning path aiming at conceptual understanding should be privileged.

We will focus part of a broader study in which we follow a design research methodology. In this paper, we discuss how the decimal number system features can be addressed and discussed in the hundred square model, and why this representation was changed into other representations, during the teaching experiment. In order to better frame this paper, we first present an overview of the characteristics of design research methodology, because the use of this approach allowed the constant analysis and adaptation of the model explored during the teaching experiment. We then organize the paper according to the different stages of design research: in the preparation phase section, we present how the literature informed the design principles and initial conjecture; in the experimentation phase, episodes regarding the use and adaptations of the hundred square model will be presented, and in the third phase a deeper analysis of the episodes will be made as well as its impact on the initial conjecture.

## Design research methodology

Design research is a methodology that has gaining ground in Mathematics Education research. It can allow the construction, or “engineering”, as Cobb, Jackson, and Dunlap (2016) describe it, of

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<sup>1</sup> Term used in this paper to identify positive rational numbers written accordingly to the decimal system notation, using the decimal comma or point.

instructional means to promote the learning of a particular topic, while constantly studying the development of that learning, considering all elements of the instructional means, not only the designed tasks but also the context in which they are carried on. Therefore, in design research theoretical and pragmatic components are highly dependent on each other.

There are five crosscutting features of design research that, together, distinguish it from other methodologies: (i) the purpose is to develop theories about both the learning process and the means designed to support it; (ii) it has a highly interventionist nature, since it can be a powerful methodology to design an approach to promote the development of a particular content or form of practice, in a real classroom; (iii) it has two interrelated components, a conjecture is made regarding students' learning (prospective component), that is constantly confronted to the actual learning (reflective component); which can lead to changes in the initial conjecture that is tested again, giving design research its (iv) iterative design; and finally there is an attempt to (v) develop humble theories that address the learning of a particular topic (Cobb et al., 2016).

One of the main characteristics of this methodology is its cyclic nature. Each cycle develops in three phases: (i) teaching experiment preparation and design; (ii) teaching experiment; and (iii) retrospective analysis that can lead to revisions and a new cycle (Cobb et al., 2016). The conjecture can be refined during the teaching experiment, resulting in micro cycles, or in between experiments, in macro cycles, or both types of refinements can happen (Prediger, Gravemeijer, & Confrey, 2015). The present study builds on micro cycles.

### **Study's rationale (*preparation and design phase*)**

#### **Decimal numbers in the research field**

Studies related to decimal numbers reveal important evidence regarding difficulties that arise when dealing and operating with this number representation (e.g., Steinle & Stacey, 2003), whole numbers' knowledge influence (e.g., Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989) or about knowledge of the decimal number system, specific of this representation (e.g., Baturu, 2000). Studies present evidence that those difficulties and whole number interference can remain in adulthood (e.g., Vamvakoussi, Van Dooren, & Verschaffel, 2012), which reveals the demanding conceptual understanding needed regarding this representation of rational numbers.

According to Post, Cramer, Behr, Lesh, and Harel (1993), the development of rational number understanding is related to (i) flexibility with translations between rational number representations; (ii) flexibility with transformations within a representation; and (iii) progressive independence from concrete embodiments of rational numbers. In a broader perspective, representations can be a tool to or reflect students' development of mathematical ideas of a particular concept. Therefore, representations that can become models assume an important role, establishing a link between knowledge connected to reality and the mathematical knowledge to be developed (Van den Heuvel-Panhuizen, 2003). Initially, models named as models of (Gravemeijer, 1999), are closely connected with the task context and cannot be used in other situations. This type of model should evolve to a model for (Gravemeijer, 1999), that can be applied in various situations, focusing mathematical connections and not the situation described in the task. In order for this development to happen, the model should have certain characteristics that make it suitable to be used in diverse situations. As

Van den Heuvel-Panhuizen (2003) highlights, to support learning, the model should allow students not only to progress into the mathematical ideas but also to go back to the reality context if needed.

Research show evidence of decimal numbers' learning fostered by the appropriation of different models. We will now focus two of the models considered in the teaching experiment: the hundred square and the decimat. The hundred square is a model often used when teaching decimal numbers, until hundredths, that allows connections between the iconic representation of the grid and symbolic representations, not only decimals but also fractions and percentages. The grid also facilitates students to understand the meaning of each digit and, consequently, its decimal number system notation. It is also an important model to compare and order decimal numbers and, later on, can be used to add and subtract decimal numbers (Cramer, Monson, Wyberg, Leavitt, & Whitney, 2009). Decimat is a similar model, however, by being rectangular and divided into two rows and five columns, allows a clear visualization of each part again divided by 10, emphasizing the multiplicative structure of the decimal place value in the decimal number system (Roche, 2010).

### **Structuring the cycle**

Based on literature review, the following six design principles were elaborated to guide the conjecture and instructional means: (1) use of tasks which context appeals to the use of rational numbers in its decimal representation; (2) promote movements among decimal numbers and other rational number's representations highlighting their relations; (3) promote the use of representations that support their transformations into models to think about rational numbers in its decimal representation; (4) encourage the use of prior knowledge; (5) promote the discussion of whole number interferences and common misconceptions; and (6) establish a learning environment where students are encouraged and feel confident to share and discuss their own mathematical ideas.

Supported by these principles, an initial conjecture was made: A teaching experiment comprised by different types of sequenced tasks, explorations and exercises, focusing decimal numbers in measure and part-whole meanings and the use of number line and hundred square model, considering students' whole number and informal knowledges and evoking the need for the use of decimal numbers, as well as its connections with other rational numbers representations, in a learning environment where students have an active role and small group work and whole-class discussions are privileged, will promote a meaningful understanding of decimal numbers.

A set of tasks was planned (some new and others adapted from existing materials) and students' understanding was anticipated. Tasks were open to adjustments or to be completely revised depending on the understanding students revealed along the way. The teaching experiment was intended to be carried out in Grade 3, from February to June 2014, however, the last tasks were conducted at Grade 4. The teaching experiment was, generally, carried out once per week, in one 90 minutes lesson, involving a total of 16 weeks over the two school years.

The participants were 25 students and their teacher. A diagnostic study was made with the same students, in Grade 2, that provided information about students' ideas of different rational numbers' representations, which supported the design of the initial tasks and also help to gather information concerning the teacher's role. Consequently, the classroom teacher asked for a detailed plan for each task. The plan was made by the researcher (first author) and discussed previously with the

teacher, and included suggestions to support teacher inquiry, possible students' answers and solutions and potential students' difficulties.

In the data presented in the next section, students will be referred to with fictitious names. The tasks were solved in small group work or in pairs, and whole-class discussions were privileged. Records of all the students' written work, along with participant observation by the researcher supported by audio/video recordings and field notes, constituted the main data sources. Meetings between the researcher and the teacher, prior and after each lesson, were also audio-recorded.

One of the expected products of the broader study is a set of indicators of decimal number understanding that can be helpful both for teachers and researchers. We intend that these indicators address two different levels: (i) what is specific of this rational number representation, and, (ii) the intertwinement between this representation and other rational number representations. In this paper, we outlined some indicators of students' understanding of decimal numbers to be supported by the appropriation of the models here presented. As an ongoing research, these indicators are preliminary and open to revision. Regarding the first level, we consider identifying the partitioning and grouping by powers of ten to create units of tenths, hundredths, and thousandths, and reveal an understanding of the decimal numeration properties (positional value, multiplication and addition properties, in addition to base ten property). In relation to the second level, we consider recognition of a decimal number in different representations; identifying the unit, and establish equivalences between numbers represented as decimals, percentages, and fractions.

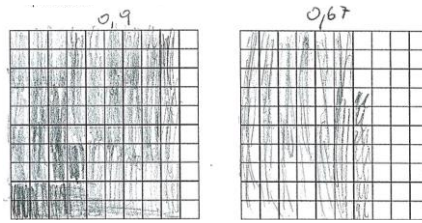
### **Classroom episodes (*teaching experiment phase*)**

We present three illustrative episodes of the use of three models throughout the teaching experiment, focusing on part-whole meaning. The examples presented concern the use of each model by students (representations as models of). The first two occurred in Grade 3 and the third in Grade 4. We focus our analysis on the indicators of decimal number understanding as mentioned above.

The hundred square was presented to students as a towel, divided into tenths and hundredths. After some exploration of this model, a task was presented to promote the discussion of common misconceptions, such as the comparison of decimal numbers based on its number digits. The hundred square model was showed to help students explain their answers. One of the questions was "Do you think 0,67 is bigger than 0,9?". In whole-class discussion, Jorge revealed how he used the hundred square model to compare both numbers:

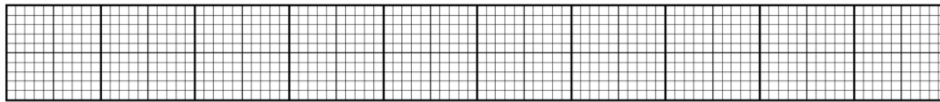
Jorge: Initially I thought that 0,67 was bigger than 0,9 because at first sight 67 seems bigger than 9. . . but then I realized that I could think in a different way. So, if we think that each column has ten-hundredths, we would have to paint six of these columns, without the seven (in 0,67) it would be only sixty. And the other one (0,9) would be 90, it was bigger, nine columns are 90 hundredths, so it was bigger than painting 67 hundredths.

Due to the appropriation of the hundred square, Jorge could visualize and compare the quantities represented by both numbers (Figure 1). The hundred square has shown to have great potential, as its use helped Jorge to overcome the initial, and expected, interference of whole number knowledge.



**Figure 1: Jorge's work record at a task with the hundred square model**

When preparing the teaching experiment, it was anticipated that students could visualize each small square in the model (one hundredth) divided into ten equal parts, each representing ten thousandths. However, it was important that students, in fact, saw the thousandths, instead of inferring that from this model. Given its shape, the hundred square doesn't allow further divisions into thousandths, in the same manner, thus another model was thought. Later, in the classroom, this model started to be called as "thousandths bar". With a rectangular shape, a bar represents the unit that is divided into ten large squares, representing tenths, and each one is then divided into ten columns, the hundredths, that are again divided into ten equal parts, the thousandths (Figure 2).



**Figure 2: Thousandths bar model**

At first, students were encouraged to find out how many "small squares" were in the whole bar. Many looked into one "big square" divided into quarters and calculated  $4 \times 25$ , which was 100, and then multiplied 100 by 10, reaching 1000. Initially, students thought that each "big square" was like the hundred square, representing a hundred hundredths, or one. This was probably due to the fact that each tenth in this model was similar to the hundred model. A unit change was implied: before, one big square represented one unit, now a similar but smaller square represents one tenth. Nonetheless, the model allowed students to relate tenths, hundredths, and thousandths.

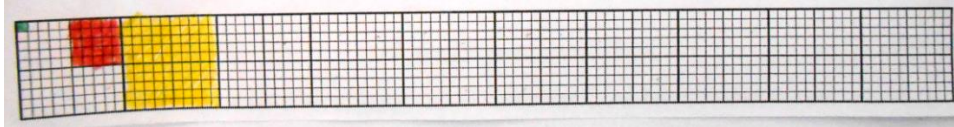
An example of these connections made by students is shown in Figure 3. Artur's answer relates to a question where students were asked to paint in the thousandths bar 0,001 with green, 0,01 in red and 0,1 in yellow. After, they were asked about what connections they could find among these parts.

*As relações que encontramos: verde x 100 = amarelo  
 verde x 10 = vermelho vermelho x 10 = amarelo  
 vermelho : 10 = verde amarelo : 10 = vermelho  
 amarelo : 100 = verde*

"The relations we found were:  
 green x 100 = yellow, green x 10 = red,  
 red x 10 = yellow, red : 10 = green,  
 yellow : 10 = red, yellow : 100 = green"

**Figure 3: Artur's work record at a task with the thousandths bar model, with translation**

Artur, like other colleagues, could clearly state the partitioning and grouping by powers of 10 that create the units of tenths, hundredths, and thousandths. It is important to refer that it was the first time that these relations were clearly stated by the students. We believe that visualizing each painted unit in the same model promoted the establishment of these connections. On the other side, and even though some students correctly identified the tenth and the thousandth in the bar, they had painted one hundredth as one-quarter of one-tenth. Students said that 0,1 was the biggest square in the bar, then 0,01 was the “middle” square and 0,001 was the smallest square (Figure 4).



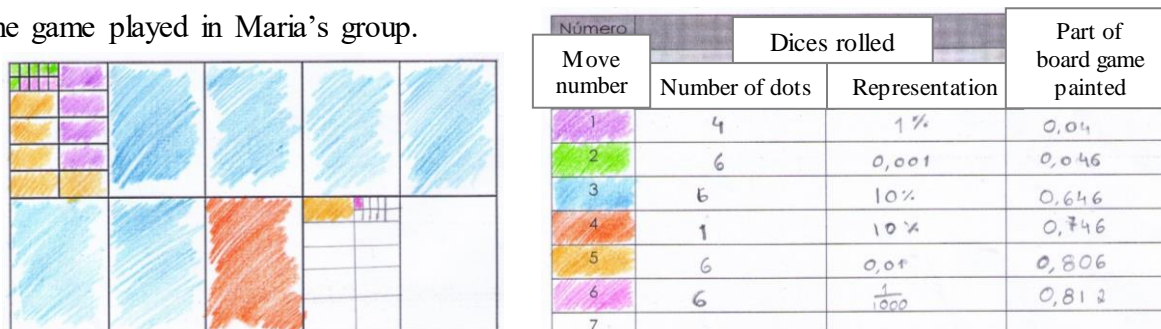
**Figure 4: Mafalda's work record at a task with the thousandths bar model**

We weren't expecting this response. It can be linked to two factors: due to the strategy used to count the total of “little squares” in the bar, or probably due to the bar layout that misleads students to think about the different decimal units in terms of squares. If a big square is 0,1 and a small square is 0,001, the middle square will be, incorrectly, 0,01.

Thus, together with the classroom teacher, we felt the need to adapt the model again. We needed a model that, like the hundred square, clearly allowed to see the connections between different units, and like the thousandths bar, allowed the extension to the thousandths and students' inference of further partitions by powers of 10, to develop the idea of density. After scratching a model with such features, followed by searching for a similar model in the research field, we came across the decimat model, as described by Roche (2010). Therefore, we included some tasks adapted from this author's work in the teaching experiment.

The model was presented with one tenth divided into hundredths, and one of which divided again into thousandths. When students first saw it, they called it “towel”, relating it to the hundred square. They immediately recognized the model shown tenths, one of which divided into hundredths and thousandths. It was said that the model could be further divided if they wanted or needed.

One of those tasks was a game adapted from the one proposed by Roche (2010). In groups of about four students, two dices were given: one regular dice with dots and other with different symbolic representations, specifically 0,01; 0,001;  $\frac{1}{1000}$ ,  $\frac{1}{10}$ , 1%, and 10%. The students had to roll both dices and multiply the numbers represented in them. Then, they had to color that part in the decimat and say which part of the decimat had already been painted, altogether. Figure 5 shows the record of the game played in Maria's group.



**Figure 5: Maria's group record of game plays using the decimat model, with translation**

Besides the flexibility in the movement within symbolic representations (decimal, percentage, and fraction) and the operations with decimal numbers as multiplication and addition, this example illustrates the potential of this model. Only one tenth is further divided, however, when needed, the students easily did the divisions on another tenth, revealing an at-ease use of the model.

### **Looking back and adjusting (*retrospective analysis phase*)**

We addressed the use of part-whole models to promote students' understanding of rational number in its decimal representation. Both the hundred square and the thousandths bar models can foster connections between the unit partitioning and grouping by powers of 10, and the decimal number system. However, the hundred square only extends to hundredths and the features of the thousandths bar can hinder the idea of partitioning the unit by powers of 10.

We want to highlight that students had already worked with the hundred square and thousandths bar when the decimat was introduced, which influenced its successful use. We also need to refer that we weren't seeking for a single and perfect model. In fact, students should explore different models. In the present study, the students continued to use all models. Nevertheless, a model should promote the visualization of specific mathematical connections to support student's learning (Van den Heuvel-Panhuizen, 2003), thus, the model's features should allow its evolving alongside the development of students' understanding. The adjustment of the initial model, done to highlight the decimal number system properties, such as the partitioning and grouping of units and place value, led us to the revision of our initial conjecture, in which we will now emphasize the decimat as an important part-whole model.

However, the use of a model by itself is not enough for students to establish mathematical connections, so the connections intended by the use of models should be focused (Prediger, 2013). The results help us to understand that the decimat can be the first model approached, initially divided into tenths, then fully divided into hundredths, in the same manner that the hundred square was also first approached, and, finally, divided into thousandths. We believe that such an approach can promote the understanding of partitioning by powers of ten connected with decimal place value. Besides that, it will allow to order and compare different representations, promote the development of a benchmark number system, unit conceptualization and support decimal numbers' operations. All these connections are strong foundations for the development of decimal number understanding.

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