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Middle school students' difficulties in proportional reasoning

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The purpose of this study was to investigate 6th, 7th and 8th grade students' common difficulties regarding rate and ratio problems. The data was collected from 149 sixth, seventh, and eighth grade students enrolled in public middle schools in Balıkesir in response to three typical proportionality questions. Data analysis revealed that the confusion of unit rate identification and algorithmically based mistakes were identified as major difficulties in solving the missing-value proportion and comparison problems. To address the potential difficulties students have regarding rate and ratio problems, students should be exposed to different types of proportional problems.

Keywords: Proportional reasoning, unit rate identification, algorithmically based mistakes.

Introduction

Proportional reasoning is considered to be a keystone of students' mathematical development and is required to access more advanced high school mathematics like algebra, geometry, probability, and statistics (Lesh, Post, & Behr, 1988). Lesh and his colleagues identify the proportional reasoning involving multiple comparisons between quantities (Lesh et al., 1988).

Previous studies have shown that children can manipulate part-whole relationships within sets of data, but they tend not to employ a systematic approach or to use categories (Inhelder & Piaget, 2013). Although children's mathematical development contains preproportional reasoning knowledge, which has always been seen as the hallmark of the formal operations stage (Inhelder & Piaget, 2013), this knowledge is insufficient for an understanding of proportional reasoning and the solution of ratio and proportion problems (Lamon, 1993).

In order to promote a conceptual understanding of ratio as the first step of proportional reasoning, it is necessary to create comparisons, beginning with additive and moving on to multiplicative comparisons. Additive comparisons compare two quantities to decide how much greater or less one quantity is than another by finding their difference, which is not the same thing as ratio. On the other hand, multiplicative comparisons provide a ratio by comparing two quantities to decide how many times larger one quantity is than another. Therefore, providing some key activities carefully designed to link these two concepts will be useful to help students to construct the aspect of ratio (Kaput, 1985).

Difficulties in proportional reasoning

Teachers have focused on teaching how to arrive at correct answers by applying rules instead of explaining the reasons behind those rules (Hart, 1988). While most of the students will be able to do computation properly, they are not encouraged to build links between concepts. Furthermore, some of the methods used by students lead to problems due to incorrect solution strategies or incorrect use of correct strategies (Hart, 1988).

Regarding proportional difficulties, algorithmically based mistakes are usually the result of lack of attention during the learning process or weak conceptual understanding (Hart, 1988). According to

researchers, problems may also arise from the attempt to find a fast way to compute ratio problems. In other words, students may be rushing to solve the problem quickly without thinking about the relationship between the given quantities and tend to simply copy the procedures identically. In addition, even though they may be able to construct a correct algorithm for cross multiplication, some students may not correctly explain the reasoning behind the algorithm (Lobato, Ellis, & Zbiek, 2010). Routine problems may lead students to assume that they can mimic solution procedures, but when they come across different problem types or non-standard language, they may struggle. Literature review showed that a number of studies have dealt with the reasons for mathematical difficulties in the field of ratio (Ellis, 2013; Hart, 1988; Lamon, 1993; Misailidou & Williams, 2003; Sarwadi & Shahrill, 2014). However, few studies have examined common difficulties among 6th, 7th and 8th grades students connected with unit measures approach for the different types of proportionality problems.

Statement of the problem

This study was conducted to investigate 6th, 7th and 8th grade students' difficulties regarding rate and ratio problems. To put it another way, study aimed to answer the research question: "What are the common difficulties encountered by 6th, 7th and 8th grade students' while the missing-value proportion and comparison problems involving rate and ratio concepts?"

Method

Sample

The sample of this study, which included 66 males and 83 females, consisted of students from sixth to eighth grade at a public middle school in Balıkesir, Turkey. The school addresses a wide variety of neighborhoods and income levels ranging from low income to upper middle class.

Measuring tool

Three proportional word problems in real world contexts were used to investigate students' proportional reasoning difficulties by grade levels (see Table 1). When we reviewed the literature on proportional reasoning, these three questions were cited as the most widely known problems. The first and the second questions were chosen from Lamon's (1993, 1999) studies. The last question was an adapted version of an orange-juice task identified by Noelting's (1980). In terms of problem types in the domain of ratio, the first and third questions were comparative, and the second question was missing value problem. While comparison problems provide four values and the aim is to specify the order relation between the ratios, in a missing-value problem three of four values are given and the last value is asked (Karplus et. al, 1983b; Lamon, 2012). These adapted questions were applied to sixth, seventh and eighth grade students, and the students were given one class hour to complete the written test. The questions are given in Table 1 below.

Ratio Achievement Test

Please solve the problems by using appropriate strategies

1. Ayşe bought four bananas paid 3.6 liras from Market A. Berna bought three bananas paid 3.3 liras from Market B. Where would you buy your bananas to make profit?
2. Derya, Ahmet and Kaan bought three helium-filled balloons and paid 1.5 liras for all three. They decided to go back to the store and buy enough balloons for everyone in the class, How much did they pay for 24 balloons?
3. Zeynep and Sinan tested three juice mixes. Which juice will have the stronger lemon flavor?

Mix A

2 cups lemon concentrate
3 cups cold water

Mix B

1 cup lemon concentrate
4 cups cold water

Mix C

4 cups lemon concentrate
8 cups cold water

Table 1: Questions about proportional reasoning

Data analysis

The solution strategies were analyzed along the strategies as unit rate, scale factor, ratio tables, and cross multiplication (e.g., Bart, 1994; Hart, 1988; Lesh, Post, & Behr, 1988). Then, incorrect solutions were separated from correct solutions, and then qualitative analysis was conducted to capture the difficulties behind the incorrect answers. The students' solutions were categorized with regards to mistake (error) strategies that have been stated in the literature: misusing a correct strategy (e.g., Hart, 1988; Karplus et al, 1983), using additive strategy (e.g., Hart, 1988; Inhelder & Piaget, 2013) and faulty application of a correct results that deviates from the unit rate (e.g., Tourniaire & Pulos, 1985). These mistake strategies were used to characterize students' difficulties while solving the given proportional problems (see Table 1) involving rate and ratio concepts. Table 2 showed the types of difficulties.

| Types of difficulties | Explanation | Example |
|---|---|--|
| Confusion of the unit rate identification | Using an arbitrary unit value (it is the guessing method without adjustment/ the unit value is the number of objects the problem starts with. | The student assumes that each banana is 3.6 liras in Market A (Q1) |
| Algorithmically based mistakes | Having computational mistakes | $\frac{4 \text{ cups}}{8 \text{ cups}} = 2$ Mixture C (Q3) |

Table 2: Classification of Student' difficulties on Ratio Achievement Test

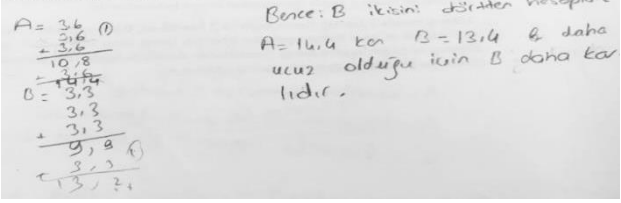
Findings

Students' incorrect solutions in solving rate problems were analyzed through category building (see Table 2) to reveal their difficulties and grouped under two headings: "confusion of the unit rate identification" and "algorithmically based mistakes". These two difficulties were identified as the major problems in three typical proportionality questions. Data analysis revealed that of 149 students, 45 (27%) for the Q1, 25 (15%) for the Q2 and 94 (56%) students for the Q3 gave either no answer or an incorrect solution, as recorded in Table 3.

| | 6 Grade (69) | | | 7 Grade (39) | | | 8 Grade (41) | | |
|---|-----------------|--------|-----------------|--------------|--------|----------------|--------------|---------|----------------|
| | Q1 | Q2 | Q3 | Q1 | Q2 | Q3 | Q1 | Q2 | Q3 |
| Incorrect Solutions | 13(19%) | 4(6%) | 30(44%) | 8(16%) | 3(8%) | 17(44%) | 8(20%) | 2(5%) | 23(56%) |
| Confusion of the unit rate identification | <u>10 (77%)</u> | 2(50%) | <u>30(100%)</u> | 5(63%) | 1(33%) | <u>12(71%)</u> | 4(50%) | 2(100%) | <u>20(87%)</u> |
| Algorithmically based mistakes | 3 (23%) | 2(50%) | 0(0%) | 3(37%) | 2(67%) | 5(19%) | 4(50%) | 0(0%) | 3(13%) |
| No answer | 7(7%) | 8(12%) | 9(13%) | 5(13%) | 2(5%) | 8(16%) | 4(10%) | 6(15%) | 7(17%) |

Table 3: Number and percentages of students' incorrect solutions**Confusion of the unit rate identification**

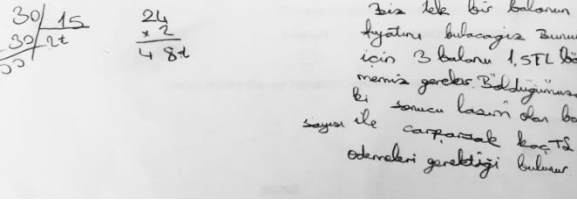
Unitizing is identified as a cognitive process that occurs after identifying the unit. This process allows subjective preference by composing two quantities to create a new unit called composed unit (Lobato et al., 2010). Students employed a number of different measurement units. However, some types of questions require the use of a standard measurement unit; the use of any other unit than this one was not allowed (Lamon, 1999). The main difficulty in answering these problems was the identification of the unit/rate. To answer the Q1 (see Table1), it was necessary to measure the amount of stuff using the concept of unit. Students can also find a different number of measures with regards to their measuring unit. When solutions to the first problem were analyzed, confusion about the unit rate could be plainly seen, indicating that the student was not able to conceptualize the unit of measurement.



(a)

I think the answer is B. We can compute [corresponding two prices] for 4 [bananas]. They [corresponding prices] are Market A=14.4 and Market B= 13.4. Therefore Market B is more economical.

(c)



(b)

We need to find price for one balloon. Therefore we should divide 3 balloons by 1.5 liras. And then we should multiply the result by the number of balloons that they need to get how much money is required.

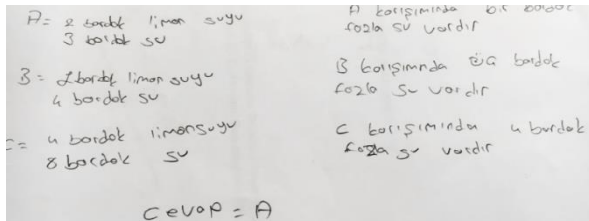
(d)

Figure 1: Student's original (a) and translated (c) solution on ignoring the number of objects, Student's original (b) and translated (d) solution on finding the amount of balloons for one lira as original versions

As can be clearly seen from Figure 1-a, although the unit was defined explicitly (single banana cost) the student thought that a banana cost 3.6 liras in Market A and 3.3 liras in Market B. In other words, the student ignored the number of objects. Then two prices 14.4 and 13.2 were compared and this solution did not contribute to the correct answer. 11 (16%) of 69 sixth grade students, 6 (15%) of 39 seventh grade students and 7 (17%) of 42 eight grade students stated that the best place to buy a banana was Market B because they spend 13.2 for four bananas and they get profit.

Another misunderstanding regarding the unit rate given in the question can be seen in Figure 1-b. This student could not decide the exact and correct numbers to find the unit rate in Q2. Therefore,

instead of finding the price for one balloon, as can be seen here, he calculated the amount of balloons for one lira by dividing 3 by 1.5. Then this value was multiplied by 24 to find the price of whole balloons. Indeed, it makes sense to divide 3 by 1.5 to find the amount of balloons per unit lira. This student seems to follow the directions provided in terms of finding single units. Even though in her written explanation on the right hand side of Figure 2 she stressed finding the value of one balloon, she noted that three would be divided by 1.5.



(a)

A=2 glasses of lemon concentrate
3 glass of water
B=1 glass of lemon concentrate
4 glasses of water
C=4 glasses of lemon concentrate
8 glasses of water

Mixture A has one more glass of water,
Mixture B has three more glasses of water
Mixture C has 4 more glasses of water

Answer: A

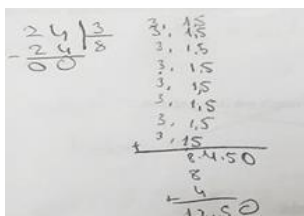
(b)

Figure 2: Student's original (a) and translated (b) solution on comparing the amount of lemon juice with the amount of water

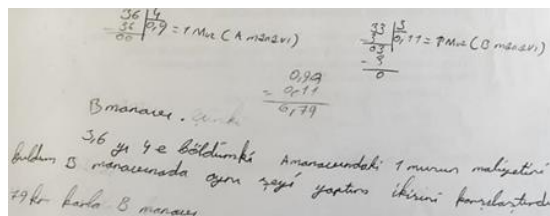
The Q3 provides more evidence for students' difficulties regarding typical ratio comparison unit rate problems. Findings revealed that 56% of 149 students think about the unit rate with given quantities in a reasonable way. For instance, as can be seen in Figure 2 without thinking about fractional relativity, the student compares the amount of lemon juice used (2 glasses, 1 glass and 4 glasses) with the amount of water to find the stronger tasting lemonade. Even though this student came to the correct solution that Mix A has a stronger taste than the others, this solution contains misconceptions. He matched the glass of water with the lemon concentrate, and decided that there is one extra cup of water in Mix A. However, if this student was asked to sort the concentration of juice for each mix, the answer would likely be wrong, because, according to his solution, the order would be Mixture A and then Mixture B and then Mixture C. Thus, the less tasty mixture would have been Mix C.

Algorithmically based mistakes

The other difficulty which emerged from the data was algorithmically-based mistakes and emerged in the computation process. Ashlock (2001) identifies these errors as 'buggy' algorithms that involve more than one incorrect step in the procedure and do not attain the desired purpose. When the details of this difficulty were examined, it was revealed that 8% of the students made basic fact errors and conducted incorrect operations while dividing the decimals.



(a)



(b)

I divided [corresponding to price] 3.6 by [corresponding to amount] 4 to get the price of one banana for Market A. And I did the same computation for Market B. I made a comparison [between Market A and Market B]. Market B is more economical with 79 cent (kuruş).

(c)

Figure 3: Students' original (a) solution on an algorithmically based mistake, Student's original (b) and (c) translated solutions on a difficulty with division

Figure 3-a illustrates these algorithmically based difficulties for Q2. First, by dividing 24 by 3, this student got 8 groups. As shown above, this student attempted to add 1.5 eight times. She wrote 3 and left a space and then wrote 1.5. It can be seen in the calculation that the student first multiplied the whole part by 8 and then the decimal part by 8. However, she did not recognize the multiplicative structure even in a familiar computation and she started with an incorrect computation. Then it seems she apparently lost track and then made algorithmic error by adding .50 to 12. Another example for algorithmically based mistakes can be seen in Figure 3-b. This student divided 3.3 by 3 and found 0.11 as a cost of one banana. However, the correct answer was 1.1 and this mistake led the student to make the wrong comparison between the profits of two markets. According to this solution 0.9 is bigger than 0.11, so Market B is the best place to buy a banana.

Discussion and conclusion

The aim of this study was to investigate the common difficulties faced by 6th, 7th and 8th grade students while solving typical proportionality problems involving rate and ratio concepts. The first common difficulty resulted from confusion of unit rate identification. The second major finding that algorithmically based mistakes were another common difficulty in computation process. These findings show that current difficulties are consistent with the Turkish context beginning from 6th grades (Kaplan, Isleyen, & Ozturk, 2011). Unitizing is a different process from determining the unit, because different systems of units are based on different choices of base units. The most obvious finding to emerge from this study is that students mostly preferred the unit rate method as a solution strategy. The reason might be related to their tendency to retreat to more familiar strategies in their solutions.

The cognitive process emerges after making a decision about the unit and misunderstanding arises when students think about the unit, from their computations especially while explaining unitizing (Lamon, 1999). The results of this study showed that sixth, seven and eight graders had problems about conceptualization of the unit of measurement. More specifically, the lemonade juice problem showed that students from all grades used unit rate strategy as a part of faulty application of a correct result by comparing the numerical differences additively rather than the multiplicatively. As Noelting (1980) states, they focused on the basis of the number of glasses of orange juice instead of proportional relations between given quantities. Especially sixth graders who offered a rich repertoire of unit rate mistakes tended to apply unit rate for the lemonade problem, and their incorrect results seems to be deviated from unit rate method (see Table 3). Based on this point, lemonade problem juice experiments might be beneficial for students in terms of experiencing their own strategies in real contexts instead of just explaining them verbally. Findings suggest that they might be performing this operation without realizing the difference between additive and multiplicative reasoning (Lamon, 2012; Lobato et al., 2010). This situation can make a noteworthy contribution in terms of providing some indications of the complexity of these mixture problems not only for sixth but also for seventh and eighth graders. This suggests the need for more in-depth investigation into student thought-processes when making these specific mistakes (Ashlock, 2001; Son, 2013). These results are consistent with other studies regarding emergent difficulties within Turkish context and suggest that proportional reasoning should involve more than just applying rules, and that there is a need for more information about what students perceive the unit to be (Sarwadi & Shahrill, 2014). Teachers should be more aware of student conceptions about unit rate while teaching proportional reasoning. This

corroborates with Lamon (1993) and Singh (2000) who state that teachers must encourage the spend time to connect composed units with multiplicative comparisons by setting different types of problems for students as much as possible to enable them to build flexible and complex unit structures develop thinking strategies.

Findings of the present study showed that algorithmically based mistakes are commonly seen in the student computation process. Son (2013) identifies three categories of error in the responses received in her study: concept-based errors, procedure-based errors and diagnosis errors. As Ashlock (2001) states, one of erroneous steps lead to emerge these kind of mistakes and then as a result the intended purpose is not systemically accomplished. On the other hand, the lack of clinic interviews with students in this study suggests caution in identifying the exact causes of these fundamental mistakes. In another words, it is difficult to decide whether students' algorithmically based mistakes stem from limited conceptualization of the problems or whether there was a mistake in their algorithmic procedures. It is possible that some of the difficulties revealed in this study may result from concentration on algorithmic computation. This suggests the need for further investigation into the exact reasons for algorithmically-based mistakes through clinical interviews.

The findings of the present study and previous research suggest several implications. As Lamon (1999) states, textbooks don't provide the flexibility of using unit rates, and under these limited conditions students will not be able to compose units. Turkish mathematics textbooks do not provide sufficient examples of these types of problems which might promote the development of an understanding of unit rate. Besides, exposing students to a variety of proportionality problems can help them to develop multiplicative reasoning skills, and to promote flexibility in unitizing.

Even though the scope of this study includes only determining the potential difficulties students have regarding rate and ratio problems, we might make some implications and suggestions to overcome those difficulties. Future research might continue to investigate students' verbal reasoning process and provide more detailed insight into the reasons for incorrect solutions. Such exploration might yield more informative insights into the reasons for student identification of unit-rate confusion and algorithmically-based errors, and provide valuable implication for the development of students' multiplicative reasoning ability.

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