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Number line as a conceptual and didactical resource for teaching fractions using applets

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Results of secondary school students' performance who participated in a pilot study of a general research project whose purpose is to contribute to building better mental objects for fractions are described. A Local Theoretical Model for fractions is used as a theoretical and methodological framework to design and develop a seven stages teaching sequence based on the use of applets created with GeoGebra and the number line as a conceptual and didactical resource. In this paper details of the design, development, and results of the first two stages are given. Results show children's preferences to represent fractions on the unit segment, that is, they think of proper fractions. A majority of the participants paid more attention to the graphical aspects of the applet.

Keywords: Fractions, local theoretical models, applets for teaching, number line, improper fractions.

Introduction

The teaching and learning of fractions and rational numbers have been studied during several decades by researchers such as Freudenthal (1983), Behr, Lesh, Post and Silver (1983), Kieren (1988) and Figueras (1988). In the last decade, fractions have been considered as one of the most complex concepts studied in basic education (see for example López-Bustamante (2009) in Mexico, Contreras (2012) in Spain and Petit, Laird & Marsden (2010) in the United States of America).

The construction of better mental objects of fractions during elementary school is considered important because that concept is widely accepted as an integral part of mathematics curriculum. Moreover, Siegler, Duncan, Davis-Kean, et al (2012) characterized the knowledge of fractions and division as unique predictors of students' mathematics performance from elementary to high school.

On the other hand, the technological tools as a resource for teaching mathematics have been incorporated into the curriculum because there is evidence that those are cognitive resources (e.g., Kieran & Yerushalmy, 2004). In this sense, an attempt is made to design a teaching model that includes these tools to promote the building up of students' better mental objects of fractions.

Research objectives

The principle aim of the general study is to construct a Local Theoretical Model for fractions in order to enrich the actual teaching model (in Figueras' sense, 1988, pp. 21- 22) for Mexican elementary school. To achieve this aim a seven stage teaching sequence based on the use of applets was designed. The focus of this paper is the design and results of the first two stages of a pilot study. The purposes of those stages are: 1) to characterize the type of fractions students keep in mind to represent them on the number line displayed on the screen, 2) to make an exploration inquiry about students' ideas

related to density and order of fractions, and 3) to identify if the students relate numerical aspects of fractions with their graphical representation.

Theoretical framework and related literature

The idea of Local Theoretical Model (LTM) developed by Filloy (see Filloy, Rojano & Puig, 2008) is used as a theoretical and methodological framework. From the theoretical point of view, the LTM serves to focus on the object of study through four interrelated components: (1) formal competence, (2) teaching models, (3) models for cognitive processes, and (4) models of communication. The construction of these components allows having an interpretative framework to identify the different aspects of fractions, which according to Freudenthal (1983) appear as fracturer, comparer, measurer, fraction operator and numbers.

The building up of the teaching models component enabled the detection in Mexican and Spanish curricula, that the study of properties of order, density, the equivalence of fractions, and proper and improper fractions were considered in the last years of primary school. Bright, Behr, Post, and Wachsmuth (1988) and Saxe, Taylor, McIntosh, and Gearhart (2005) support the idea that working with the number line gives students an approach to the notion of above concepts. However, in some sixth-grade' textbooks used in Spain (e.g., González et al, 2015) continuous and discrete models are more often than not used for teaching fractions, and activities that include the number line are lacking. These are some reasons for using the number line in this research as a didactical resource.

A revisiting of specialized literature was carried out to construct the cognitive processes' models component. Some of the papers analysed report students' difficulties faced when they learn with the number line. Michel and Horne (2008) mentioned three principal misconceptions: (1) *Instrumental part-whole knowledge* -difficulties with unit-forming, that is, children consider any line segment as a unit-. (2) *Counting lines, not spaces* -to represent a fraction on the number line some students count the lines or points (considering the zero point) instead of counting intervals-. (3) *Decimalising the count* -to represent any fraction, some children always divide the unit segment in ten parts-.

Finally, the communication component is formed by the observation of communication processes between student-applet interactions. The building up of the four components grounds the design of the teaching sequence and its trial. The latter is detailed in the following sections.

Methodology and methods

From the methodological point of view the LTM serves to organize the research project in two main parts: (1) the building up of the four components of an initial LTM for fractions as a reference framework of the general research project and (2) an experimentation with students.

Three phases comprise the experimental part. The first one relates to the design of a pre-test, a post-test and a seven-stage teaching sequence. The second phase is the application of the teaching sequence and is structured as shown in Figure 1. Each stage is composed of two parts, one is a GeoGebra applet and the other a series of questions posed with the purpose that students show the ideas they bring into play about fractions when interacting with the applets. Applets and questions are set up on a Webpage that is associated with a database to record students' responses and interactions with the applets. The third phase corresponds to data analysis and characterization of students' performance.

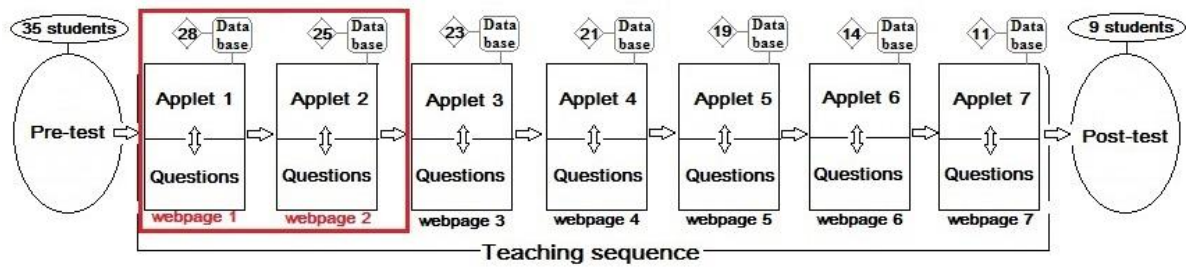


Figure 1: Structure of the teaching sequence design and its trial

The first two stages of the teaching sequence were done in one 45-minute session. Student individual interactions with applets were collected in a non-invasive manner. Answers given were stored in a database and collected with computers provided by researchers. At the end of the experimental phase, stored information was joined and organized to proceed with its analysis. The applets are used as a resource to teach fractions and to collect data.

Setting and participants

The pilot study was carried out with 45 students from 12 to 14 years old in a secondary school located in a troubled urban area of Valencia in Spain. According to their mathematics teacher, participants have a large history of difficulties in mathematics. The students have serious problems of truancy, for this reason, not all of them completed the trial of the teaching sequence. Due to this fact, only data from students that completed sequential steps were considered, that is, 28 students made the first stage and 25 completed the second stage and so on (see Figure 1). The students worked alone during the teaching sequence trial. In this study, the teacher applied the pre-test and post-test.

Applets' design and results

The applets were constructed in a learning environment for fostering the development of conceptual understanding of fractions, taking into account the didactical functions of technology in mathematics education adapted by Drijvers (2013, p. 3). As aforementioned, each stage of the teaching sequence has an applet with an exploration/interaction component (Figure 2 and 4) and a list of questions for students to reflect on what they observe during the interaction. To respond, students can turn to applets and observe the animation or representations of fractions.

To characterize the answers given by students, schemes that enable a codification have been constructed (Figure 3 and 5). Answers given by pupils were grouped in different types determined by the form in which questions are posed. Type i are answers to questions that are general statements with diverse interpretations. Type ii collects answers that can be classified solely as correct or incorrect. Type iii groups answers to questions where a justification is required and Type iv are answers to questions that requires information students must write on the applets' windows.

Applet design for the first stage. The applet's tasks for the first stage were developed considering two parallel lines of action. One directed towards the student's familiarity with the interactive environment. The other leads the student to represent different aspects of fractions on the number line and to introduce them to a proper use of the fractions' mathematical sign system (Figure 2). Three indications appear on the screen in Figure 2. The first one -'Move the sliders and watch what happens on the number line'-, has the purpose to focus students on the effects of the numerator and

denominator sliders that appear at the upper left corner of the screen and to relate those to the graphical and symbolic representations of fractions also shown on the screen.

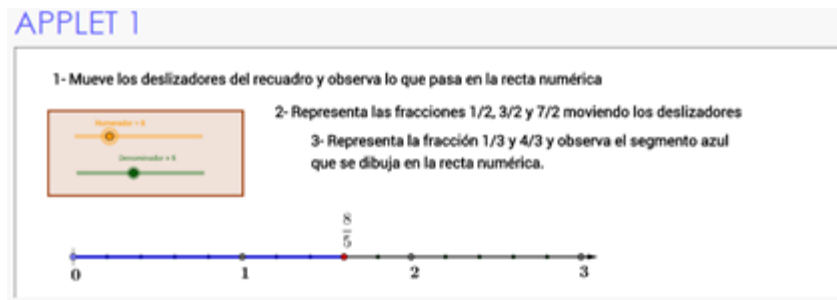


Figure 2: Screenshot of the applet for the first stage

With the second indication –“Represent fractions $1/2$, $3/2$ and $7/2$ by moving the sliders”-, students are asked to represent the first two fractions to see them in the line segment on the screen. The main idea for asking students to represent $7/2$ is to promote reflection regarding characteristics of fractions that can be visualized on the screen and of those that cannot.

The third indication -‘Represent the fractions $1/3$ and $4/3$ and observe the blue segment that is drawn on the number line’- is provided in order to identify the point representing the fraction or the fraction as number, but also to focus students’ attention on the magnitude representing the fraction, that is the length of the segment that represents the fraction. Thus, fraction as a measurer emerges, taking into account the part-whole relationship.

In addition to the above information, seven questions (Figure 3) are posed to make students write their ideas about the observations made during exploration/interaction period. Students can read questions and explore the applet as many times as necessary to answer them.

Data analysis and results of the first stage. For applet 1 there are only questions of types i and ii. The codification of students’ answers is done using the scheme shown in Figure 3.

Questions (1, 2, 4, 5 y 6)	Type i	
Q1. Represent the fraction $1/4$. What happens to the fraction if you move the numerator slider and the denominator slider is fixed? Q2. Represent the fraction $7/8$. What happens with the fraction if you move the denominator slider and the numerator slider is fixed? Q4. What will happen if the number 25 is shown in the denominator slider? Q5. What happens when the numerator and the denominator are equal? Q6. Why the fraction $7/2$ cannot be seen on the screen?	Students answers are referred to: <ol style="list-style-type: none"> 1. Numerical aspects 2. Graphical aspects 3. Numerical and graphical aspects. 4. Could not be interpreted 	The interpretation is considered: <ol style="list-style-type: none"> a. Complete b. Incomplete c. Incorrect or ambiguous
Questions (3 y 7)	Type ii	
Q3. Represent the fraction $6/7$. In how many parts does the line segment that starts at 0 and ends at 1 is divided? How many of those parts are coloured? Q7. Write two fractions that cannot be represented on the line segment shown on the screen.	The purpose of these questions is that the students observe the graphical representation (line segment). The answers can be: <ol style="list-style-type: none"> a. Complete b. Incomplete c. Incorrect or ambiguous 	

Figure 3: Scheme to characterize the answers to questions posed in the first stage

One of the most common mistakes to respond Q3 was that students focused on counting the lines or points considering the zero point instead of counting spaces. 22 students identified at least one fraction greater than three to respond Q7. In the answers to questions type i (for example, Q1, Q2,

Q4, Q5 and Q6), pupils orient their attention on what happens in the line segment that appears on the screen (Table 1). Of the 140 responses (28 students x 5 questions), 63 (45%) were classified in this group (code 2, in grey). Eight of these answers are considered complete (code 2a), since a consistent explanation is offered, 25 incomplete (code 2b) and 30 incorrect or ambiguous (code 2c). Two answers related to these results are included to exemplify the way the coding is done.

(1) The answer given by student S1 to question Q1 is: "the bar is moving to the right". This response is coded as 2b (see Table 1) because the focus is posed on what happens on the line segment. Specifically, it refers to the movement that occurs in the blue segment ("the bar" named by the student) representing the fraction when the slider moves to the right, that is when the value of the fraction is increased. The response is not considered complete because the student centres his attention on the movement to the right and presumably does not move the slider to the left. (2) The answer given by S3 to question Q1 is: "there are more points between numbers (denominators), in particular, four points between numbers". In this case, the answer also reveals a focus on the number line; the student observed the partition of the line segment, but the interpretation is incorrect or ambiguous because the student did not explore using the sliders (code 2c).

The focus on numerical aspects of the fraction (code 1, in green) was observed in 38 of the 140 responses to the questions of type i (27.15%); two answers were classified as complete (1a), 14 as incomplete (1b) and 22 as incorrect or ambiguous (1c). The answer given by student S17 to question Q6 is: "because the denominator is 2, and this is smaller than 7, so the numerator is bigger, so it is not possible". The student observed the values of the numerator and denominator, makes a comparison between the numerical values and justifies his answer; it was coded as 1c (see Table 1).

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	
Q1	1b	2c	2c	3b	2b	3b	2c	1c	1c	1c	2c	2b	2b	2b	3b	2c	1c	2b	1b	2c	4c	3b	2c	1c	2b	2b	2c	4c	
Q2	1b	1c	2c	2b	2b	3b	3b	1c	1c	1c	2c	2b	2c	1c	3b	1b	3a	2c	3b	2b	1b	3b	2c	2c	3a	2b	2c	2c	
Q4	2b	2c	2c	2b	2b	2c	2c	4c	4c	4c	4c	2c	4c	4c	2b	2b	1b	2b	2a	1c	1c	4c	4c	2c	1c	4c	4c	4c	
Q5	2a	3a	2a	2c	4c	2b	1b	1c	1c	1b	4c	1a	1b	4c	2a	2a	1b	2a	3a	1b	4c	1b	1a	2b	2b	2a	4c	2a	
Q6	3b	1c	2c	1c	1b	1c	3b	4c	2c	1c	2c	1c	1c	2b	2b	2b	1c	3b	3a	4c	2c	4c	2c	2c	4c	2b	2c	1b	
Q3	c	a	b	a	b	a	c	b	b	b	a	b	b	b	a	c	c	c	a	c	b	b	c	c	c	b	a	a	
Q7	b	c	a	a	c	c	c	b	b	a	b	b	a	b	b	b	a	a	a	b	a	b	b	b	b	c	c	b	b

Table 1: Characterizations of students' answers in the first stage

Only in 17 of the 140 (12.14%) answers to the questions of type i (in purple) the focus on numerical and graphical aspects is made evident. Five of them were classified as complete (3a) and twelve as incomplete (3b). The answer given by student S19 to Q6 is: "because in the line, one can only represent numbers between 0 and 3, and $7/2$ is greater than 3". The student observed the structure of the line segment; his focus is posed on the graphical representation but also refers to the fraction as a number. For this reason, the answer was coded as 3a.

Applet design for the second stage. A new form of symbolic representation is introduced in this applet (see Figure 4). In this case, students are asked to write five fractions in a pop-up window when they click the start button (INICIO in Spanish). This button is associated to a JavaScript subroutine that offers feedback and stores the student actions.

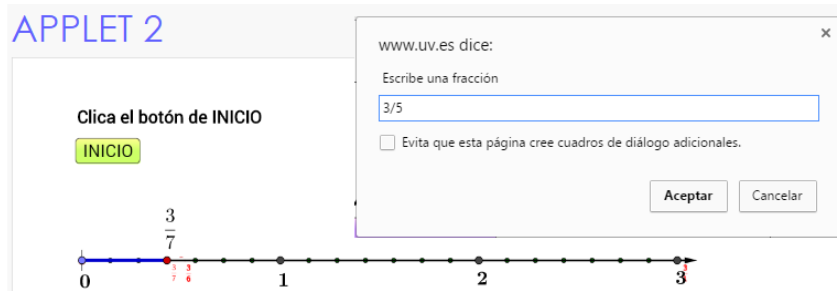


Figure 4: Screenshot of the applet for the second stage

Fractions are limited to those with denominator between 2 and 10 and numerator between 0 and 40. If a pupil writes a fraction that does not meet these conditions, a pop-up window appears with a message advising to take into account the characteristics of the numbers involved. When introducing a fraction greater than three, another alert window appears to indicate that the fraction cannot be seen on the number line. Fractions that are written by the user leave a trace in the form of red dots and the value of the fraction on the number line (see Figure 4). The visible trace on the screen helps the students answer questions posed in the Webpage in which the applet is embedded.

Data analysis and results of the second stage. In the applet that corresponds to the second stage, there are questions of type i, ii and iv. The codification is done using the scheme in Figure 5.

Questions (1, 3 and 4)	Type i
Q1. The 5 fractions you wrote appear on the number line on the screen? Why?	Students answers are referred to: <ul style="list-style-type: none"> 1. Numerical aspects 2. Graphical aspects 3. Numerical and graphical aspects. 4. Could not be interpreted
Q3. Of the 5 fractions written, which one is the smallest? Why?	
Q4. Of the 5 fractions written, which one is the greatest? Why?	
Q2. Write the fractions you wrote from greatest to smallest.	
Questions (2, 5 and 6)	Type ii
Q5. How many fractions could you write between 0 and 1?	The answers can be: <ul style="list-style-type: none"> a. Correct; b. Incorrect; c. Incomplete a. Finite b. Infinite, many, a lot of... c. No answer (blank)
Q6. How many fractions could you write between 1 and 4?	
Write 5 fractions	
The student must write five fractions	Type iv
	Written fractions can be classified as: <ul style="list-style-type: none"> 1. Proper; 2. Improper; 3. Unit

Figure 5: Scheme to characterize the questions of the second stage

Only 12 of 25 students were able to answer correctly Q2 (code 1). This result has an effect on the answers to questions Q3 and Q4, as shown in Table 2. The students who order fractions correctly chose correctly the greater or smallest fraction for questions Q3 and Q4 respectively. To justify the order of those fractions S1 considered the length of the blue segment that represents the fraction on the number line (2a). S2 considered the position of the point representing the fraction on the number line (2a), i.e. graphical aspects of fractions. Students S5 and S18 considered the characteristics of the numerator and denominator of the fraction, that is, numerical aspects of fractions (code 1a).

The justification for most students who do not respond correctly to questions Q3 and Q4 is based on comparing the numerators and denominators of the fractions. Two of these cases are the following: (1) Student S8 choose 8/3 as the smaller fraction, "because the denominator is the smallest." The

comparison made with fractions $4/10$, $3/9$ and $12/5$. He also chose $4/10$ as the greater fraction "because the denominator is the largest." (2) Student S7 chose $2/3$ as the smaller fraction between $5/4$, $3/7$, $3/4$, "Because that [fraction] has the smaller numbers."

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15	S16	S17	S18	S19	S20	S21	S22	S23	S24	S25
Q1	2a	1a	2b	1b	4a	4b	2b	4b	4b	4b	4a	4a	4b	4b	4a	2a	4b	2b	1a	1b	4b	4a	4b	2a	4b
Q2	1	1	0	0	1	1	0	0	0	1	1	1	1	0	0	0	0	1	1	0	1	0	0	0	1
Q3	2a	2a	4b	1b	1a	4a	1b	1b	2b	4b	1a	1a	2a	4b	4b	2b	2b	1a	1a	1b	1a	4b	2b	2b	2a
Q4	2a	1a	4b	1b	1a	4a	4b	1b	2b	4b	4a	2b	4a	4b	4b	4b	2b	1a	2a	1b	1a	4b	2b	2b	2a
Q5	b	a	a	a	a	b	a	a	a	a	a	a	a	a	a	a	a	a	b	a	c	a	c	a	b
Q6	b	a	a	a	a	b	a	c	a	c	a	a	b	a	a	a	a	a	b	a	c	a	c	a	b
P	3	5	3	3	1	2	3	3	1	3	0	1	2	3	1	3	3	0	2	3	1	3	1	4	2
Im	0	0	0	2	1	3	1	2	2	1	4	2	2	1	4	1	1	2	1	0	3	0	3	1	1
U	0	0	0	0	2	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	5

Table 2: Characterizations of students' answers in the second stage

Of the 50 answers (25 students x 2 questions) related to the density of fractions (questions Q5 and Q6), 35 were classified with code a, because the answer refers to a finite number of fractions, 6 were blank (code c), and only in 9 answers the density property was mentioned in some sense (code b), because some of the responses were "infinite fractions" or "as much as one wants". Finally, Table 2 shows that students wrote more often proper (56) than improper fractions (38).

Conclusions

Results of the first two stages described before allow to highlight the fact that few students were able to relate the numerical and graphical representation of fractions. However, students who are able to relate these two representations can give complete and correct answers. To justify questions related to fractions' order, some students relied solely on the numerical aspect, for example, comparing the numerator and the denominator, which led to incorrect or ambiguous answers. Whereas answers in which graphical aspects are used, for example, the position of the point on the number line or the length of the blue segment, led to correct answers. Although most students focused on the latter aspect, they encountered difficulties in representing a fraction on the number line. The most common mistakes are instrumental part-whole knowledge and counting lines, not spaces, also reported by Michel and Horne (2008).

On the other hand, the idea of density that students had seems to be strongly related to the number of fractions they represented during their interactions with the applet, these results are a warning to continue investigating these ideas in later stages and reflect the influence of applets. The students wrote more proper fractions even though the number of proper fractions with small denominators as 2 or 3 are few compared with the number of improper fractions. These results can be related to fact that teaching models favouring the recognition of proper fractions are widely used.

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