



# The index of $G$ -transversally elliptic families in KK-theory

Alexandre Baldare

## ► To cite this version:

Alexandre Baldare. The index of  $G$ -transversally elliptic families in KK-theory. Journal of Noncommutative Geometry, 2020, 14 (3), pp.1129-1169. 10.4171/JNCG/389 . hal-01873318

**HAL Id: hal-01873318**

**<https://hal.science/hal-01873318>**

Submitted on 13 Sep 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The index of $G$ -transversally elliptic families in KK-theory

Alexandre Baldare<sup>\*1</sup>

<sup>1</sup>IMAG, Univ Montpellier, CNRS, Montpellier, France

May 7, 2018

## Abstract

We define and study the index map for families of  $G$ -transversally elliptic operators and introduce the multiplicity for a given irreducible representation as a virtual bundle over the base of the fibration. We then prove the usual axiomatic properties for the index map extending the Atiyah-Singer results [1]. Finally, we compute the cup product of our index class against the K-homology class of an elliptic operator on the base. Our approach is based on the functorial properties of Kasparov's cup product, and relies on some constructions due to Connes-Skandalis and to Hilsum-Skandalis.

**Keywords:** Transversally elliptic operators, KK-theory, K-homology, compact group actions, crossed product algebras, fibration.

**MSC2010 classification:** 19K35, 19K56, 19L47.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>The index class of a family of <math>G</math>-transversally elliptic operators</b>	<b>4</b>
2.1	Some preliminary results . . . . .	4
2.2	The index class . . . . .	6
2.3	Unbounded version of the index class . . . . .	10
2.4	K-theory Multiplicity . . . . .	13
<b>3</b>	<b>The properties of the index class</b>	<b>15</b>
3.1	The index map . . . . .	15
3.2	The case of free actions . . . . .	16
3.3	Multiplicative property . . . . .	18
3.4	Excision property . . . . .	21
3.5	The naturality and induction theorems . . . . .	23

---

<sup>\*</sup>alexandre.baldare@umontpellier.fr

4	Cup product of the index class with an elliptic operator on the base	27
A	Some results on families of pseudodifferential operators	29

# 1 Introduction

The study of transversely elliptic operators was initiated in the seminal work of Atiyah-Singer, see [1]. For such operators, the equivariant Fredholm index is not well defined but they proved that the multiplicity of any irreducible representation still makes sense and that one can still introduce an important index invariant which is now an invariant slowly increasing distribution on the given compact Lie group. They also proved many axiomatic properties for their distributional index which allowed them to reduce its computation to the case of actions of tori on euclidean spaces, and to totally compute it in many special cases including locally free actions and general actions of the circle. Although important progress have been achieved during the last decades, the problem of computing the Atiyah-Singer distributional index for transversely elliptic operators, or equivalently the multiplicities of the irreducible representations, is still open. The goal of the present paper is to extend the Atiyah-Singer results to the context of families of transversely elliptic operators, by using the powerfull properties of Kasparov's bivariant K-theory and especially the associativity of the cup-product.

Let  $G$  be a compact Lie group and let  $M$  be a compact  $G$ -manifold. A  $G$ -transversally elliptic pseudodifferential operator on  $M$  is a pseudodifferential operator whose symbol becomes invertible when restricted to the (non-zero) covectors which are transverse to the orbits of the  $G$ -action. For this class of operator, Atiyah and Singer define an index class  $\text{Ind}(P)$  living in the space  $C^{-\infty}(G)^G$  of  $Ad$ -invariant distributions on  $G$ , so

$$\text{Ind}(P) \in C^{-\infty}(G)^G.$$

More precisely, for a  $G$ -transversally elliptic operator  $P$ , the evaluation of the index class against a smooth function  $\varphi \in C^\infty(G)$  is defined by the formula (see [1])

$$\text{Ind}(P)(\varphi) = \int_G \varphi(g) \left( \text{Tr}_{|\ker(P)}(g) - \text{Tr}_{|\text{coker}(P)}(g) \right) dg.$$

This actually induces the Atiyah-Singer index map

$$\text{Ind} : K_G(T_G^*M) \longrightarrow C^\infty(G)^G$$

whose properties, already investigated in [2], lead Atiyah-Singer to reduce its computation to the case of a Euclidean space equipped with a torus action [2].

In [18], Julg associated with any  $G$ -transversally elliptic operator  $P$  a K-homology class  $[P] \in \text{KK}(C(M) \rtimes G, \mathbb{C})$  as well as its restricted class in  $\text{KK}(C^*G, \mathbb{C})$ , an index class closely related with the Atiyah-Singer distribution through the Connes-Chern character. In [8], Berline and Vergne proved an index theorem for  $G$ -transversely elliptic operators, in the world of equivariant cohomology. More recently, Kasparov extended all known K-theory constructions for transversely

elliptic operators to the larger class of proper actions on complete manifolds. In this context, although the index class is not defined, the Julg class  $[P]$  still makes sense and plays the role of the index class, see [21]. It is worth pointing out that Kasparov even proved many index theorems as equalities in his bivariant K-theory.

We investigate here the extension of the classical Atiyah-Singer results to the case of families and we restrict ourselves to the case of actions of compact Lie groups on compact fibrations. Exactly as in the case of families of elliptic operators [3], we point out that the base of the fibration can be assumed to be a compact space, and the smoothness condition is only used when we compute the cup product with an elliptic operator on  $B$ . So, let  $p : M \rightarrow B$  be a given compact locally trivial  $G$ -fibration and denote by  $T^V M = \ker p_*$  the vertical tangent bundle. Using a  $G$ -invariant metric on  $M$  we identify as usual the cotangent bundle  $T^*M$  with the tangent bundle  $TM$ . Assume that  $G$  acts trivially on the base  $B$  then a family of pseudodifferential operators  $A = (A_b)_{b \in B}$  in the sense of [3], is  $G$ -transversely elliptic if its restrictions to the fibers are  $G$ -transversally elliptic pseudodifferential operators. Said differently, we assume that the principal symbol  $\sigma(A)$  of the family  $A$  is invertible in restriction to the non-zeros covectors in  $T_G^V M := T^V M \cap T_G^* M$ . Following [18] and [21], we associate with such family an index class, now living in the bivariant Kasparov group  $\text{KK}(C^*G, C(B))$ .

**Definition 1.1.** Given a  $G$ -invariant family  $P_0 : C^{\infty,0}(M, E^+) \rightarrow C^{\infty,0}(M, E^-)$  of  $G$ -transversally elliptic operators, the index class  $\text{Ind}_G^{\text{M|B}}(P_0)$  of  $P_0$  is defined as the KK-class:

$$\text{Ind}_G^{\text{M|B}}(P_0) := [\mathcal{E}, \pi, P] \in \text{KK}_G(C^*G, C(B)),$$

where  $\mathcal{E}$  is the Hilbert  $C(B)$ -module associated with the hermitian  $\mathbb{Z}_2$ -graded bundle  $E$ ,  $\pi$  is the representation of  $C^*G$  on  $\mathcal{E}$  induced by the  $G$ -action on  $E$ , and  $P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ . We denote by  $\text{Ind}^{\text{M|B}}(P_0)$  the image of this class in  $\text{KK}(C^*G, C(B))$ .

The index class then only depends on the homotopy class of the principal symbol then induces the index map

$$\text{Ind}^{\text{M|B}} : K(T_G^V M) \longrightarrow \text{KK}(C^*G, C(B)).$$

Finally, the notion of K-theory multiplicity can be introduced as a virtual bundle over the base  $B$ , generalizing the standard multiplicity for  $G$ -transversally elliptic operators [1].

Following [1], this paper is mostly devoted to the investigation of the basic properties of our index map, i.e. free actions, multiplicativity, excision and also induction and behaviour with respect to topological shriek maps. These properties allow us to reduce the computation of the index map to the case of a trivial fibration  $B \times V \rightarrow B$ , where  $V$  is a Euclidean space which is equipped with a torus action, giving the full generalization of the Atiyah-Singer work. The compatibility theorem for instance can be states as follows (see section 3.5 for the details):

**Theorem 1.2.** [1] *Let  $j : M \hookrightarrow M'$  be a  $G$ -embedding over  $B$  with  $M$  compact. Then the following*

diagram is commutative:

$$\begin{array}{ccc}
K_G(T_G^V M) & \xrightarrow{j!} & K_G(T_G^V M') \\
\text{Ind}^{M|B} \downarrow & & \downarrow \text{Ind}^{M'|B} \\
KK(C^*G, C(B)) & = & KK(C^*G, C(B)).
\end{array}$$

Given an elliptic operator  $Q$  on the base  $B$ , and using results from [14], we compute the Kasparov product of our index class with the K-homology class of  $Q$ . In order to achieve this computation, we exploit the convenient description of the index class through the morphism that it induces from the representation ring  $R(G)$  to the topological K-theory  $K(B)$ . This latter cup-product computation is crucial in defining a distributional index à la Atiyah for families of  $G$ -transversally elliptic operators, and to prove a Berline-Vergne formula for families. These cohomological results are postponed to the forthcoming paper [5].

This article follows the following plan. The second section is devoted to the definition of the index class for families of  $G$ -transversally elliptic operators and to the introduction of the K-multiplicity. In the third section, we define the index mapping and investigate its basic properties. More precisely, we give the exact computations for free  $G$ -actions, we prove the expected multiplicativity theorem as well as the excision theorem which are now equalities in the appropriate KK-groups. Induction for closed subgroups and especially for the maximal torus is investigated right after, with the compatibility with shriek maps. In the fourth and fifth sections, we further study the index class as a morphism between  $R(G)$  and  $K(B)$  and we compute the Kasparov product with an elliptic operator on the base, preparing the equivariant cohomological computations of our Berline-Vergne theorem.

**Acknowledgements** This work is part of my PhD thesis under the supervision of M.-T. Benameur. I would like to thank my advisor for very helpful discussions, comments and corrections. I also thank W. Liu, P.-E. Paradan and V. Zenobi for several during the preparation of this work. I am also indebted to M. Hilsum and M. Puschnigg for reading the PhD version of this work.

## 2 The index class of a family of $G$ -transversally elliptic operators

In this section we define the index class of a family of  $G$ -transversally elliptic operators and we introduce the K-multiplicity of a unitary irreducible representation in the index class, a virtual vector bundle over the base  $B$ .

### 2.1 Some preliminary results

This first paragraph is devoted to a brief review of some standard results. For most of the classical properties of Hilbert  $C^*$ -modules and regular operators between them that we use here, we refer the reader to [22] and [30]. The constructions given below extend the standard ones, see for instance

[11], [13] and [16]. Our hermitian scalar products will always be linear in the second variable and anti-linear in the first. Let  $G$  be a compact group and denote by  $C^*G$  the  $C^*$ -algebra associated with  $G$ . Recall that any irreducible unitary representation of  $G$  naturally identifies with a finitely generated projective module on  $C^*G$  [17].

Let  $p : M \rightarrow B$  be a  $G$ -equivariant fibration of compact manifolds, with typical fiber  $F$ , a compact manifold. We denote by  $M_b = p^{-1}(b)$  the fiber over  $b \in B$ , by  $T^V M = \ker p_*$  the vertical subbundle of  $TM$ , and by  $T^V M^*$  its dual bundle. We choose a  $G$ -invariant riemannian metric on  $M$  and hence will identify  $T^V M^*$  with a subbundle of  $T^*M$  when needed.

Let  $\pi : E = E^+ \oplus E^- \rightarrow M$  be a  $\mathbb{Z}_2$ -graded vector bundle on  $M$  which is assumed to be  $G$ -equivariant with a fixed  $G$ -invariant hermitian structure. Denote by  $\mathcal{P}^m(M, E^+, E^-)$  the space of (classical) continuous families of pseudodifferential operators on  $M$  as defined in [3]. A family  $P \in \mathcal{P}^m(M, E^+, E^-)$  is  $G$ -equivariant if  $g \cdot P = g \circ P \circ g^{-1} = P$  for any  $g \in G$ . As usual,  $C^{\infty,0}(M, E)$  will be the space of continuous fiberwise smooth sections of  $E$  over  $M$ , see [3].

We fix from now on a  $G$ -invariant continuous family of Borel measures  $(\mu_b)$  which are of Lebesgue class, constructed using a partition of unity of  $B$ . So, for any  $f \in C(M)$ , the map  $b \rightarrow \int_{M_b} f(m) d\mu_b(m)$  is continuous, and each measure  $\mu_b$  is fully supported in the fiber  $M_b$ . Since  $E$  is equipped with a hermitian structure, the  $C(B)$ -modules  $C^{\infty,0}(M, E^\pm)$  of continuous fiberwise smooth sections over  $M$ , are naturally equipped with the structure of pre-Hilbert right  $G$ -equivariant  $C(B)$ -modules with the inner product given by:

$$\langle s, s' \rangle(b) = \int_{M_b} \langle s(m), s'(m) \rangle_{E_m^\pm} d\mu_b(m), \quad \text{for } s, s' \in C^{\infty,0}(M, E^\pm).$$

We denote by  $\mathcal{E}^\pm$  the completion of  $C^{\infty,0}(M, E^\pm)$  with respect to this Hilbert structure. So,  $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$  is our  $G$ -equivariant  $\mathbb{Z}_2$ -graded Hilbert module on  $C(B)$ .

**Proposition 2.1.** *We define an involutive representation  $\pi$  of  $C^*G$  in the Hilbert  $G$ -module  $\mathcal{E}$  by setting:*

$$(\pi(\varphi)s)(m) := \int_G \varphi(g)(g \cdot s)(m) dg, \quad \forall s \in \mathcal{E}, \varphi \in L^1(G).$$

*Moreover, if we endow  $C^*G$  with the conjugation  $G$ -action, then the representation  $\pi$  is  $G$ -equivariant.*

*Proof.* Let  $s, s' \in \mathcal{E}$  and  $\varphi \in L^1(G)$ . By using the  $G$ -invariance of the metric on  $E$  together with a Cauchy-Schwartz argument, we deduce that

$$\| \langle s', \pi(\varphi)s \rangle \|_{C(B)} \leq \int_G |\varphi(g)| \|s\| \|s'\| dg \leq \|\varphi\|_{L^1(G)} \|s\|_{\mathcal{E}} \|s'\|_{\mathcal{E}}.$$

Recall that the action of  $G$  on  $C(B)$  is trivial here. We then easily see that  $\pi(\varphi)$  is an adjointable operator. Again by the  $G$ -invariance of the hermitian metric on  $E$ , and using the  $G$ -invariance of the family of measures  $(\mu_b)_{b \in B}$ , we also get that  $\pi(\varphi)^* = \pi(\varphi^*)$ . Finally, the relation  $\pi(\varphi \star \psi) = \pi(\varphi) \circ \pi(\psi)$  is verified as follows:

$$\begin{aligned} \pi(\varphi \star \psi)s &= \int_G (\varphi \star \psi)(g) g \cdot s dg \\ &= \int_G \int_G \varphi(h) \psi(h^{-1}g) g \cdot s dg dh. \end{aligned}$$

So setting  $h^{-1}g = k$ , we get  $\pi(\varphi \star \psi) = \pi(\varphi) \circ \pi(\psi)$ .

On the other hand, we also have  $\pi(g \cdot \varphi)s = \int_G \varphi(g^{-1}hg)h \cdot s dh$  and setting  $k = g^{-1}hg$  we get

$$\pi(g \cdot \varphi)s = \int_G \varphi(k)gkg^{-1} \cdot s dk = (g \cdot \pi(\varphi))s.$$

Moreover, we have:

$$(g \cdot \varphi g \cdot \psi)(h) = \int_G (g \cdot \varphi)(k)(g \cdot \psi)(k^{-1}h)dk = \int_G \varphi(g^{-1}kg)\psi(g^{-1}k^{-1}hg)dk.$$

Finally, setting  $g^{-1}hg = t$ , we obtain:

$$(g \cdot \varphi g \cdot \psi)(h) = \int_G \varphi(t)\psi(t^{-1}g^{-1}hg)dt = (g \cdot (\varphi \star \psi))(h).$$

By density of the range of  $L^1(G)$  in  $C^*G$ , we get the result.  $\square$

**Remark 2.2.** The Hilbert  $C(B)$ -module  $\mathcal{E}$  is associated with the continuous field of Hilbert spaces  $(\prod_{b \in B} L^2(M_b, E_b), \Delta)$ , where  $\Delta$  is the space of continuous sections of  $E$ , see [10].

Since  $C^{\infty,0}(M, E^+)^\perp = \{0\}$ , any pseudodifferential family  $Q$  as before is densely defined and admits a formal adjoint with respect to the inner product of  $\mathcal{E}$  which will be denoted  $Q^*$ . The domain of  $Q^*$  is

$$\text{Dom}(Q^*) = \{s \in \mathcal{E}^- / \exists s' \in \mathcal{E}^+ \text{ such that } \langle Qs'', s \rangle = \langle s'', s' \rangle, s'' \in \text{Dom}(Q)\}.$$

Recall that  $Q^*$  is defined by its graph  $G(Q^*)$  given by

$$G(Q^*) = \{(s', s) \in \mathcal{E}^- \times \mathcal{E}^+, \forall s'' \in C^{\infty,0}(M, E^+), \langle s, s'' \rangle = \langle s', Qs'' \rangle\}.$$

If  $Q$  has pseudodifferential order  $m$  and principal symbol  $q$ , then  $Q^*$  is also a continuous family of pseudodifferential operators of order  $m$  with principal symbol  $q_b^*(m, \xi)$ . If  $Q$  is  $G$ -equivariant, then so is  $Q^*$ .

## 2.2 The index class

Assume now that  $G$  is a compact *Lie* group with Lie algebra  $\mathfrak{g}$ , and that the action of  $G$  on the base  $B$  is trivial. We start by extending some results from [21, Section 6] and use the notations from there. For  $m \in M$ , we denote by  $f_m : G \rightarrow M$  the map given by  $f_m(g) = g \cdot m$ , by  $f'_m : \mathfrak{g} \rightarrow T_m M$  its tangent map at the neutral element of  $G$  and by  $f_m^* : T_m^* M \rightarrow \mathfrak{g}^*$  the dual map. So, any  $X \in \mathfrak{g}$  defines the vector field  $X^*$  given by  $X_m^* := f'_m(X)$  which, under our assumptions, is a vertical vector field, i.e. it belongs to  $T^V M$ . Notice also that  $g \cdot f'_m(X) = f'_{g \cdot m}(\text{Ad}(g)X)$ , for any  $g \in G$ ,  $m \in M$  and  $X \in \mathfrak{g}$  [21].

Let  $\mathfrak{g}_M := M \times \mathfrak{g}$  be the  $G$ -equivariant trivial bundle of Lie algebras on  $M$ , associated to  $\mathfrak{g}$  for the action  $g \cdot (m, v) = (g \cdot m, \text{Ad}(g)v)$ . We endow  $\mathfrak{g}_M$  with a  $G$ -invariant metric and we will

denote by  $\|\cdot\|_m$  the associated family of Euclidean norms. The map  $f' : \mathfrak{g}_M \rightarrow TM$  defined by  $f'(m, v) = f'_m(v)$  is a  $G$ -equivariant vector bundle morphism. Up to normalization, we can always assume that  $\forall v \in \mathfrak{g}, \|f'_m(v)\| \leq \|v\|_m$ . Here  $\|f'_m(v)\|$  is the norm given by the riemannian metric at  $m$ . We thus assume from now on that  $\|f'_m\| \leq 1, \forall m \in M$ . These metrics on  $\mathfrak{g}_M$  and  $TM$  are also used to identify  $\mathfrak{g}_M$  with  $\mathfrak{g}_M^*$  and  $TM$  with  $T^*M$ . Then we can define the map  $\phi : T^*M \rightarrow T^*M$  by setting  $\phi_m = f'_m f_m^*$ . Again according to the Kasparov's notations [21], we introduce the quadratic form  $q = (q_m)_{m \in M}$  on the fibers of  $T^*M$  by setting:

$$q_m(\xi) = |\langle f'_m f_m^*(\xi), \xi \rangle| = \|f_m^*(\xi)\|_m^2, \quad \forall (m, \xi) \in T^*M.$$

We have  $q_m(\xi) \leq \|\xi\|^2, \forall (m, \xi) \in T^*M$ . If  $\xi \in T_m^*M$ , then it is easy to see that  $\xi$  is orthogonal to the  $G$ -orbit of  $m$  if and only if  $q_m(\xi) = 0$ .

**Definition 2.3.** Denote by  $T_G M := \{(x, \xi) \in TM / q_m(\xi) = 0\}$  the closed subspace of  $TM$ . We define

$$T_G^V M := T_G M \cap T^V M.$$

Following [21] which extends the standard definitions, see [1] and [7], we state two definitions of families of  $G$ -transversally elliptic operators, the naive definition and the technical definition.

**Definition 2.4** (naive definition). We will say that a  $G$ -equivariant selfadjoint family  $A$  of pseudodifferential operators acting on a vector bundle  $E = E^+ \oplus E^-$  of order 0, odd for the graduation of  $E$  is a family of  $G$ -transversally elliptic operator or that  $A$  is a  $G$ -transversally elliptic family if

$$\sup_{m \in M} \|\sigma_A(m, \xi)^2 - \text{id}_{E_m}\| \rightarrow 0$$

when  $(m, \xi) \rightarrow \infty$  in  $T_G^V M$ .

**Remark 2.5.** By a classical argument, the symbol of a  $G$ -transversally elliptic family as in Definition 2.4 represents a class in the Kasparov group  $\text{KK}_G(\mathbb{C}, C_0(T_G^V M))$  which is simply given by the Kasparov cycle  $(C_0(\pi^* E), \sigma_A)$ , where  $C_0(\pi^* E)$  is the space of continuous sections of  $\pi^* E \rightarrow T_G^V M$  vanishing at infinity. Hence, it defines a class in the topological  $G$ -equivariant K-theory group  $\text{K}_G(T_G^V M)$ .

**Definition 2.6** (technical definition). We will say that a  $G$ -equivariant selfadjoint family  $A$  of pseudodifferential operators of order 0, is a family of  $G$ -transversally elliptic operators or that  $A$  is a  $G$ -transversally elliptic family if its principal symbol  $\sigma_A$  satisfies the following condition:  $\forall \varepsilon > 0, \exists c > 0$  such that for any  $(m, \xi) \in T^V M$ , we have

$$\|\sigma_A^2(m, \xi) - \text{id}\|_{(m, \xi)} \leq c(1 + q_m(\xi))(1 + \|\xi\|^2)^{-1} + \varepsilon$$

where  $\|\cdot\|_{(m, \xi)}$  means the operator norm on  $E_m$ .

The following proposition is stated in [21] in the case of a single operator:



**Proposition 2.7.** *Let  $T$  be a selfadjoint family of pseudodifferential operators of order 0 on  $M$ , with principal symbol  $\sigma_T$ . Denote by  $F$  a selfadjoint family of pseudodifferential operators with principal symbol given by  $\sigma_F(m, \xi) = (1 + q_m(\xi))(1 + |\xi|^2)^{-1}$ . Suppose that  $\forall \varepsilon > 0, \exists c > 0$  such that  $\forall (m, \xi) \in T^V M$*

$$\|\sigma_T(m, \xi)\| \leq c \sigma_F(m, \xi) + \varepsilon.$$

*Then  $\forall \varepsilon > 0$ , there exist  $c_1, c_2 > 0$  and two selfadjoint families of integral pseudodifferential operators with continuous (vertical) kernel such that*

$$-(c_1 F + \varepsilon + R_1) \leq T \leq c_2 F + \varepsilon + R_2.$$

*Proof.* We adapt the classical proof to our setting, see also Appendix A. For  $\varepsilon' > \varepsilon$ , we have:

$$\lim_{\substack{\xi \rightarrow \infty \\ \xi \in T^V M^*}} \operatorname{Re} \left( c \sigma_F(m, \xi) + \varepsilon' - \sigma_T(m, \xi) \right) > 0, \quad (1)$$

since  $|\operatorname{Re}(\sigma_T(m, \xi))| \leq \|\sigma_T(m, \xi)\|_{E_m} \operatorname{id}$ . Indeed, we have

$$0 < c \sigma_F(m, \xi) + \varepsilon' - \operatorname{Re}(\sigma_T(m, \xi)) = \operatorname{Re}(c \sigma_F(m, \xi) + \varepsilon' - \sigma_T(m, \xi)).$$

By Proposition A.2, we can find families of operators  $S_2$  and  $R_2$  such that

$$S_2^* S_2 - (cF + \varepsilon' - T) = R_2 \text{ and hence } 0 \leq -T + cF + \varepsilon' + R_2.$$

Replacing  $T$  by  $-T$ , the inequality (1) remain true. Hence, we can also find families of operators  $S_1$  and  $R_1$  such that  $0 \leq T + cF + \varepsilon' + R_1$ . When  $\varepsilon' \rightarrow \varepsilon$ , we get the result.  $\square$

Let now  $P_0 : C^{\infty,0}(M, E^+) \rightarrow C^{\infty,0}(M, E^-)$  be a  $G$ -invariant family of  $G$ -transversally elliptic pseudodifferential operators of order 0. We will denote by  $P : C^{\infty,0}(M, E) \rightarrow C^{\infty,0}(M, E)$ , the pseudodifferential family  $\begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ .

**Remark 2.8.** Since  $P$  is  $G$ -invariant, for any  $\varphi \in C^*G$ , we have  $[\pi(\varphi), P] = 0$ .

We fix, for later use, a  $G$ -equivariant continuous family  $\Delta_G$  of second order differential operators (along the fibers) with the principal symbol  $q$ , see also [1]. If  $X \in \mathfrak{g}$  and  $s \in C^{\infty,0}(M, E)$ , we denote by  $\mathcal{L}(X)(s)$  the Lie derivative associated with the corresponding action, i.e.

$$\mathcal{L}(X)(s) := \frac{d}{dt} \Big|_{t=0} (e^{tX} \cdot s).$$

In the same way, using the action of  $G$  on itself by left translations, we denote for any given basis  $\{V_k\}$  of  $\mathfrak{g}$  with dual basis  $\{v_k\}$  and any  $\varphi \in C^\infty(G)$ , by  $\frac{\partial \varphi}{\partial V_k}$  the derivative along the one-parameter subgroup of  $G$  corresponding to the vector  $V_k$ . We then define the family of differential operators

$$d_G = (d_{G,b})_{b \in B} : C^{\infty,0}(M, E) \rightarrow C^{\infty,0}(M, E \otimes \mathfrak{g}_M^*),$$

given by  $d_G(s) = \sum_k \mathcal{L}(V_k)s \otimes v_k$ .

Notice that the representation  $\pi$  can be extended by setting

$$\pi(d\varphi)s = \sum_k \int_G \frac{\partial \varphi}{\partial V_k}(h) h \cdot s \otimes v_k dh.$$

**Proposition 2.9.** [21] For  $\varphi \in C^\infty(G)$  and  $s \in C^{\infty,0}(M, E)$ , we have  $d_G(\pi(\varphi)s) = \pi(d\varphi)s$ .

*Proof.* We have:

$$\begin{aligned} d_G(\pi(\varphi)s) &= \sum_k \mathcal{L}(V_k)\pi(\varphi)s \otimes v_k \\ &= \sum_k \frac{d}{dt}\bigg|_{t=0} e^{tV_k}\pi(\varphi)s \otimes v_k \\ &= \sum_k \frac{d}{dt}\bigg|_{t=0} \int_G \varphi(g)(e^{tV_k}g) \cdot s \otimes v_k dg. \end{aligned}$$

Substituting  $e^{tV_k}g = h$ , we get:

$$\begin{aligned} d_G(\pi(\varphi)s) &= \sum_k \frac{d}{dt}\bigg|_{t=0} \int_G \varphi(e^{-tV_k}h)h \cdot s \otimes v_k dh \\ &= \sum_k \int_G \frac{\partial \varphi}{\partial V_k}(h)h \cdot s \otimes v_k dh \\ &= \pi(d\varphi)s. \end{aligned}$$

□

We are now in a position to state the main result of this section:

**Theorem 2.10.** [21] The triple  $(\mathcal{E}, \pi, P)$  is an even  $G$ -equivariant Kasparov cycle for the  $C^*$ -algebras  $C^*G$  and  $C(B)$ . It thus defines a class in  $\text{KK}_G(C^*G, C(B))$ .

*Proof.* The  $\mathbb{Z}_2$ -grading is clear. By Proposition 2.1, the representation  $\pi$  is  $G$ -equivariant and by Proposition A.4, the operator  $P$  is  $C(B)$ -linear and belongs to  $\mathcal{L}_{C(B)}(\mathcal{E})$ . By Remark 2.8, we know that  $[\pi(\varphi), P] = 0$ . Moreover,  $P$  is selfadjoint, and it thus remain to check that  $(\text{id} - P^2) \circ \pi(\varphi) \in \text{k}_{C(B)}(\mathcal{E})$ . Denote by  $\sigma_P$  the principal symbol of  $P$ . As  $P$  is a family of  $G$ -transversally elliptic operators, the principal symbol of  $\text{id} - P^2$  which coincides with  $\text{id} - \sigma_P^2$ , satisfies the assumption of Proposition 2.7. Therefore  $\forall \varepsilon > 0$ , there exist  $c_1, c_2 > 0$  and selfadjoint integrals operators with continuous kernels  $R_1$  and  $R_2$  such that

$$-(c_1F + \varepsilon + R_1) \leq 1 - P^2 \leq c_2F + \varepsilon + R_2,$$

where  $F$  is a continuous family of operators with symbol  $\sigma_F(m, \xi) = (1 + q_m(\xi))(1 + |\xi|^2)^{-1}$ . We may take for  $\Delta_G$  the operator  $d_G^*d_G$ . Indeed, the symbol of  $d_G$  at  $(m, \xi) \in T^VM$  is given by  $\sum_k i\langle \xi, f'_m(V_k) \rangle \text{ext}(v_k) = \sum_k i\langle f_m'^*(\xi), (V_k) \rangle \text{ext}(v_k) = i \text{ext}(f_m'^*(\xi))$ . So the symbol of  $d_G^*$  is given by  $-i \text{int}(f_m'^*(\xi))$ . Then we get that the symbol of  $d_G^*d_G$  is given by  $\langle f_m'^*(\xi), f_m'^*(\xi) \rangle = q_m(\xi)$  modulo symbols of order 1. Denote also by  $\Delta$  a vertical Laplace operator associated with the fixed metric, that is a family of second order differential operators with principal symbol  $\|\xi\|^2$  for  $(m, \xi) \in T^VM^*$ . Modulo families of operators of negative order, the family of operators  $F$  coincides with the operator  $d_G^*(1 + \Delta)^{-1}d_G$ .

By Proposition 2.9, we know that  $d_G \circ \pi(\varphi) = \pi(d\varphi)$  is a bounded operator on  $\mathcal{E}$ . Moreover,  $d_G^*(1 + \Delta)^{-1}$  has negative order, so by Corollary A.6, it is a compact operator of the Hilbert module  $\mathcal{E}$ . It follows that  $d_G^*(\text{id} + \Delta)^{-1}d_G\pi(\varphi)$  is compact as well. In order to show that the operator  $(\text{id} - P^2) \circ \pi(\varphi)$  is compact, we first notice that for  $\psi = \varphi^* \star \varphi$ , we have:

$$-\pi(\varphi)^* \circ (c_1 F + \varepsilon + R_1) \circ \pi(\varphi) \leq \pi(\varphi)^* \circ (\text{id} - P^2) \circ \pi(\varphi) \leq \pi(\varphi)^* \circ (c_2 F + \varepsilon + R_2) \circ \pi(\varphi),$$

i.e.

$$-(c_1 F + \varepsilon + R_1) \circ \pi(\psi) \leq (\text{id} - P^2) \circ \pi(\psi) \leq (c_2 F + \varepsilon + R_2) \circ \pi(\psi),$$

since all the operators are  $G$ -invariant. Therefore, projecting in the Calking algebra and putting  $\varepsilon \rightarrow 0$ , we deduce that  $(\text{id} - P^2) \circ \pi(\psi)$  is compact for any non-negative  $\psi \in C^*G$ . Now, using continuous functional calculus, we may write any  $\varphi \in C^*G$  as a linear combination of non-negative elements and conclude.  $\square$

In the following definition and as before,  $P = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ .

**Definition 2.11.** The index class  $\text{Ind}_G^{\text{M|B}}(P_0)$  of a  $G$ -invariant family  $P_0$  of  $G$ -transversally elliptic operators is defined as:

$$\text{Ind}_G^{\text{M|B}}(P_0) := [\mathcal{E}, \pi, P] \in \text{KK}_G(C^*G, C(B)).$$

We also denote by  $\text{Ind}^{\text{M|B}}(P_0)$  its image in  $\text{KK}(C^*G, C(B))$ .

**Remark 2.12.** If the family  $P_0$  is invariant with respect to an extra action of a compact group  $H$ , whose action commutes with the action of  $G$  of  $G$ , then the previous construction yields, for any  $G \times H$ -equivariant family a  $G$ -transversely elliptic operators to an  $H$ -equivariant index class

$$\text{Ind}_{G \times H}^{\text{M|B}}(P_0) \in \text{KK}_{G \times H}(C^*G, C(B)).$$

We then denote accordingly by  $\text{Ind}_H^{\text{M|B}}(P_0)$  its image in  $\text{KK}_H(C^*G, C(B))$ .

### 2.3 Unbounded version of the index class

Since the geometric operators play an important part in the index theory of  $G$ -transversely elliptic operators, we now introduce the definition of the index class for operators of order 1, using the unbounded version of Kasparov's theory, see for instance [4]. The unbounded version simplifies the computation of some Kasparov products in the next sections.

**Definition 2.13** (Unbounded Kasparov module [4]). Let  $A$  and  $B$  be  $C^*$ -algebras. An  $(A, B)$ -unbounded Kasparov cycle  $(E, \phi, D)$  is a triple where  $E$  is a Hilbert  $B$ -module,  $\phi : A \rightarrow \mathcal{L}(E)$  is a graded  $\star$ -homomorphism and  $(D, \text{dom}(D))$  is an unbounded regular selfadjoint operator such that:

1.  $(1 + D^2)^{-1}\phi(a) \in k(E)$ ,  $\forall a \in A$ ,

2. The subspace of  $A$  composed of the elements  $a \in A$  such that  $\phi(a)(\text{dom}(D)) \subset \text{dom}(D)$  and  $[D, \phi(a)] = D\phi(a) - \phi(a)D$  is densely defined and extends to an adjointable operator on  $E$ , is dense in  $A$ .

When  $E$  is  $\mathbb{Z}_2$ -graded with  $D$  odd and  $\pi(a)$  even for any  $a$ , we say that the Kasparov cycle is even. Otherwise, it is odd.

In [4], appropriate equivalence relations are introduced on such (even/odd) unbounded Kasparov cycles, which allowed to recover the groups  $\text{KK}^*(A, B)$ . When the compact group  $G$  acts on all the above data, one recovers similarly  $\text{KK}_G^*(A, B)$  by using the equivariant version of the Baaj-Julg unbounded cycles of the previous definition. We now recall some useful properties of unbounded operators on Hilbert modules over our commutative  $C^*$ -algebra, our main reference is [12]. Let  $X$  be a locally compact space and let  $\mathcal{F}$  be a Hilbert  $C_0(X)$ -module. We denote by  $\mathcal{F}_x$  the fiber of  $\mathcal{F}$  at  $x \in X$  and denote for any  $\xi \in \mathcal{F}$  by  $\xi_x \in \mathcal{F}$  its value at  $x$ . If  $\mathcal{G} \subset \mathcal{F}$  is a subset of  $\mathcal{F}$ , we denote by  $\mathcal{G}_x = \{\xi_x, \xi \in \mathcal{G}\}$ . The following lemma is proved in [12].

**Lemma 2.14.** [12] *Let  $\mathcal{G}$  be a  $C_0(X)$ -submodule of  $\mathcal{F}$ , then*

1. *For any  $x \in X$ ,  $\overline{\mathcal{G}_x} = (\overline{\mathcal{G}})_x$ .*
2.  *$\mathcal{G}$  is dense in  $\mathcal{F}$  if and only if  $\mathcal{F}_x$  is dense in  $\mathcal{G}_x$ ,  $\forall x \in X$ .*

Let, for any  $x \in X$ ,  $T_x$  be a given unbounded operator on  $\mathcal{F}_x$  with domain  $\text{dom}(T_x) \subset \mathcal{F}_x$ . We then define an unbounded operator  $T$  on  $\mathcal{F}$  by setting

$$\text{dom}(T) = \{\xi \in \mathcal{F} / \xi_x \in \text{dom}(T_x), \forall x \in X \text{ and } T\xi := (T_x \xi_x)_{x \in X} \in \mathcal{F}\}.$$

We shall also use the following proposition.

**Proposition 2.15.** [12] *Assume that for any  $x \in X$ , the operator  $T_x$  is selfadjoint, and that  $T$  is densely defined. Then  $T$  is selfadjoint and the following assertions are equivalent:*

1.  *$T$  is regular;*
2. *for any  $x \in X$ ,  $(\text{dom}(T))_x$  is an essential domain for  $T_x$ ;*
3. *for all  $x \in X$ , we have  $(\text{dom}(T))_x = \text{dom}(T_x)$ .*

Let us get back now to our situation with the Hilbert  $C(B)$ -module  $\mathcal{E}$  associated with the bundle  $E \rightarrow M$  and with the previous data.

**Definition 2.16.** Let  $P_0 : C^{\infty,0}(M, E^+) \rightarrow C^{\infty,0}(M, E^-)$  be a family of  $G$ -invariant pseudodifferential operators of order 1. We say that  $P_0$  is a family of  $G$ -transversally elliptic operators if its principal symbol  $\sigma(P_0) : \pi^* E^+ \rightarrow \pi^* E^-$  is invertible over  $T_G^V M \setminus M$ .

Let  $P_0 : C^{\infty,0}(M, E^+) \rightarrow C^{\infty,0}(M, E^-)$  be a  $G$ -invariant family of  $G$ -transversally elliptic pseudodifferential operators of order 1. Denote by  $\hat{P}$  the operator  $\begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ , where  $P_0^*$  is the family  $\{(P_0^*)_b\}_b$  corresponding to the formal adjoints along the fibers.

**Proposition 2.17.** *Let  $P_0 : C^{\infty,0}(M, E^+) \rightarrow C^{\infty,0}(M, E^-)$  be a  $G$ -invariant family of  $G$ -transversally elliptic pseudodifferential operators of order 1, and let  $\hat{P} = \begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$  as above. Then with the domain  $\text{dom}(\hat{P}) := \{\xi \in \mathcal{E}, \xi_t \in \text{dom}(\hat{P}_t) \text{ and } \hat{P}\xi \in \mathcal{E}\}$ , the unbounded operator  $\hat{P} : \mathcal{E} \rightarrow \mathcal{E}$ , is essentially selfadjoint. So if we denote by  $P$  its closure then  $P$  is regular and selfadjoint.*

*Proof.* Clearly  $\hat{P}$  is then densely defined in  $\mathcal{E}$  because  $\text{dom}(\hat{P})$  contains  $C^{\infty,0}(M, E)$ . For any  $b \in B$ , the operator  $\hat{P}_b$  with domain  $C^\infty(M_b, E_b)$  is essentially selfadjoint, see [11]. As  $\text{dom}(P) = \{\xi \in \mathcal{E}, \xi_b \in \text{dom}(\overline{P}_b) \text{ et } \overline{P}\xi \in \mathcal{E}\}$ , we get by Proposition 2.15 that  $P$  is selfadjoint and it remains to check that  $P$  is regular which again by Proposition 2.15 reduces to proving that for any  $b \in B$ ,  $(\text{dom}(P))_b$  is an essential domain for  $P_b$ . We have  $C^\infty(M_b, E_b) \subset (\text{dom}(P))_b$ , so  $P$  is regular.  $\square$

We are now in position to state our result. Recall that  $G$  acts on  $C^*G$  by conjugation, that  $\mathcal{E}$  is the  $G$ -equivariant Hilbert  $C(B)$ -module associated with the continuous field of Hilbert spaces  $\{L^2(M_b, E_b)_{b \in B}\}$ , and that  $\pi : C^*G \rightarrow \mathcal{L}_{C(B)}(\mathcal{E})$  is the representation defined in Proposition 2.1.

**Theorem 2.18.** *Let  $P_0 : C^{\infty,0}(M, E^+) \rightarrow C^{\infty,0}(M, E^-)$  be a  $G$ -invariant family of  $G$ -transversally elliptic pseudodifferential operators of order 1, and let  $P$  be the associated regular self-adjoint operator defined above out of  $\begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ . Then the triple  $(\mathcal{E}, \pi, P)$  is an even  $(C^*G, C(B))$ -unbounded Kasparov cycle, which defines a class in  $\text{KK}_G(C^*G, C(B))$ .*

*Proof.* By Proposition 2.1, the representation  $\pi$  is  $G$ -equivariant. For any  $\varphi \in C^\infty(G)$ , it is easy to see that  $\pi(\varphi)$  preserves the domain of  $P$  and by Remark 2.8, we have  $[\pi(\varphi), P] = 0$ . It remains to check that  $(1 + P^2)^{-1} \circ \pi(\varphi) \in k(\mathcal{E})$ . We may take for our operator  $\Delta_G$  for instance the Casimir operator, which is a differential operator along the fibers of order 2. The operator  $1 + P^2 + \Delta_G$  is elliptic along the fibers since  $\sigma(P_b)$  is invertible in the transversal direction to the orbits and  $\sigma(\Delta_G)$  is invertible in the orbits direction, see [1]. We hence deduce that the resolvent  $(1 + P^2 + \Delta_G)^{-1}$  is a compact operator in  $\mathcal{E}$ .

We now show that for any  $\varphi \in C^\infty(G)$ , the operator

$$(1 + P^2 + \Delta_G)^{-1} \circ \pi(\varphi) - (1 + P^2)^{-1} \circ \pi(\varphi),$$

is compact, which will insure that  $(1 + P^2) \circ \pi(\varphi)$  is also compact. Denote again by  $\Delta$  the Laplacian on  $G$  viewed as a riemannian  $G$ -manifold. Using that  $\Delta_G \pi(\varphi) = \pi(\Delta \varphi)$  and that  $[\pi(\varphi), P] = 0$ , we have:

$$\begin{aligned} (1 + P^2 + \Delta_G)^{-1} \circ \pi(\varphi) - (1 + P^2)^{-1} \circ \pi(\varphi) &= -(1 + P^2 + \Delta_G)^{-1} \Delta_G (1 + P^2)^{-1} \circ \pi(\varphi) \\ &= -(1 + P^2 + \Delta_G)^{-1} \pi(\Delta \varphi) (1 + P^2)^{-1}. \end{aligned}$$

Now since  $\pi(\Delta \varphi)$  and  $(1 + P^2)^{-1}$  are adjointable operators and since  $(1 + P^2 + \Delta_G)^{-1}$  is compact, we deduce that  $(1 + P^2 + \Delta_G)^{-1} \circ \pi(\varphi) - (1 + P^2)^{-1} \circ \pi(\varphi)$  is a compact operator on  $\mathcal{E}$ .  $\square$

**Definition 2.19.** The index class  $\text{Ind}_G^{\text{MB}}(P_0)$  of a  $G$ -invariant family  $P_0$  of  $G$ -transversally elliptic pseudodifferential operators of positive order, is the class in  $\text{KK}_G(C^*G, C(B))$  of the  $(C^*G, C(B))$ -unbounded Kasparov cycle  $(\mathcal{E}, \pi, P)$ , i.e.

$$\text{Ind}_G^{\text{MB}}(P_0) = [\mathcal{E}, \pi, P] \in \text{KK}_G(C^*G, C(B)).$$

**Remark 2.20.** The relation with the bounded version is made using the Woronowicz transformation, see [4]. More precisely, if  $P_0$  is a  $G$ -invariant family of  $G$ -transversally elliptic pseudodifferential operators of order 1, then the triple  $(\mathcal{E}, \pi, P(1 + P^2)^{-1/2})$  is a (bounded) Kasparov cycle and the index class coincides with  $[\mathcal{E}, \pi, P(1 + P^2)^{-1/2}]$ .

## 2.4 K-theory Multiplicity

In the present paragraph, we shall only need the bounded version of the index class. Recall that we also have the index class  $\text{Ind}^{\text{MB}}(P)$  in  $\text{KK}(C^*G, C(B))$  obtained by forgetting the  $G$ -action on  $C^*G$ . For any irreducible unitary representation  $V$  of  $G$ , we now define a class  $m_P(V) \in K(B)$  corresponding to the multiplicity of  $V$  in the index class  $\text{Ind}^{\text{MB}}(P)$ , and prove its compatibility with a second definition using the obvious cup-product.

So let us fix an irreducible unitary representation  $V$  of  $G$ . Let  $\mathcal{E}_V^G$  be the Hilbert  $C(B)$ -submodule of  $V \otimes \mathcal{E}$  composed of the  $G$ -invariant elements, i.e.  $\mathcal{E}_V^G := (V \otimes \mathcal{E})^G$ . In term of continuous fields of Hilbert spaces, the Hilbert module  $\mathcal{E}_V^G$  corresponds to the field  $((L^2(M_b, V \otimes E_b)^G)_{b \in B})$  with the space of continuous sections being the space of  $G$ -invariant continuous sections of the field associated with  $V \otimes E$ . If  $C_{v,w}$  is the coefficient of the representation  $V$  which corresponds to the vectors  $v, w \in V$ , then the inner product satisfies the formula

$$\langle v \otimes s, w \otimes t \rangle = \langle s, \pi(C_{v,w})t \rangle.$$

Indeed, we have

$$\begin{aligned} \langle v \otimes s, w \otimes t \rangle(b) &= \int_{M_b} \left\langle \int_G (g \cdot (v \otimes s))(m) dg, \int_G (h \cdot (w \otimes t))(m) dh \right\rangle_{(\underline{V \otimes E})_m} d\mu_b(m) \\ &= \int_{M_b} \int_G \int_G \langle g \cdot v, h \cdot w \rangle_V \langle (g \cdot s)(m), (h \cdot t)(m) \rangle_{E_m} dg dh d\mu_b(m) \\ &= \int_{M_b} \langle s(m), \pi(C_{v,w})t(m) \rangle_{E_m} d\mu_b(m) \\ &= \langle s, \pi(C_{v,w})t \rangle(b). \end{aligned}$$

By restricting the operator  $\text{id}_V \otimes P$  to the  $G$ -invariant elements, we get the adjointable operator  $P_V^G \in \mathcal{L}_{C(B)}(\mathcal{E}_V^G)$ , i.e.

$$P_V^G := (\text{id}_V \otimes P)|_{\mathcal{E}_V^G}.$$

**Lemma 2.21.**  $(\mathcal{E}_V^G, P_V^G)$  is an even Kasparov cycle which defines a class in  $\text{KK}(\mathbb{C}, C(B))$ .

*Proof.* We only need to show that  $(P_V^G)^2 - 1$  is compact. We have  $(P_V^G)^2 - 1 = (\text{id}_V \otimes (P^2 - 1))|_{\mathcal{E}_V^G}$ . Let  $(e_1, \dots, e_n)$  be an orthonormal basis of  $V$ . Let  $s = \sum_i e_i \otimes s_i$  be an element of  $\mathcal{E}_V^G$ . Then we have  $\int_G g \cdot s = s$  so that an easy computation gives

$$s = \sum_{i,j} e_j \otimes \pi(C_{e_j, e_i})s_i.$$

But

$$\begin{aligned}
((P_V^G)^2 - 1)s &= (\text{id}_V \otimes (P^2 - 1))(\sum_{i,j} e_j \otimes \pi(C_{e_j, e_i})s_i) \\
&= \sum_{i,j} e_j \otimes (P^2 - 1) \circ \pi(C_{e_j, e_i})s_i \\
&= \sum_{t,j} e_j \otimes e_t^* \otimes (P^2 - 1) \circ \pi(C_{e_j, e_t})(s).
\end{aligned}$$

Thus it follows that  $(P_V^G)^2 - 1 = \sum_{t,j} e_j \otimes e_t^* \otimes (P^2 - 1) \circ \pi(C_{e_j, e_t})$ . Moreover, we know that  $(\dim V)C_{e_j, e_j} \star C_{e_j, e_t} = C_{e_j, e_t}$  and that  $(P^2 - 1) \circ \pi(C_{e_j, e_j})$  are compact operators. Let  $(R_n^j)_n$  be a sequence of finite rank operators on  $\mathcal{E}$  which converges to  $(P^2 - 1) \circ \pi(C_{e_j, e_j})$ . The sequence of finite rank operators  $(e_j \otimes R_n^j \circ \pi(C_{e_j, -}))$  then converges to  $e_j \otimes (P^2 - 1) \circ \pi(C_{e_j, -})$  for the operator norm induced by the inner product on  $\mathcal{E}_V^G$ . Indeed, we have:

$$\begin{aligned}
&\langle e_j \otimes ((P^2 - 1) \circ \pi(C_{e_j, e_j}) - R_n^j) \circ \pi(C_{e_j, -})(s), s' \rangle_{(V \otimes \mathcal{E})^G} \\
&= \left\langle \left( (P^2 - 1) \circ \pi(C_{e_j, e_j}) - R_n^j \right) \sum_k \pi(C_{e_j, e_k})s_k, \sum_t \pi(C_{e_j, e_t})s'_t \right\rangle_{\mathcal{E}},
\end{aligned}$$

□

**Definition 2.22.** The K-multiplicity  $m_P(V)$  of the irreducible unitary  $G$ -representation  $V$  in the index class  $\text{Ind}^{\text{M|B}}(P_0)$  is the image of the class  $[(\mathcal{E}_V^G, P_V^G)] \in \text{KK}(\mathbb{C}, C(B))$  under the isomorphism  $\text{KK}(\mathbb{C}, C(B)) \cong \text{K}(B)$ . So  $m_P(V)$  is the class of a virtual vector bundle over  $B$ , an element of the topological K-theory group  $\text{K}(B)$ .

Recall that the representation  $V$  defines a class in the K-theory of the  $C^*$ -algebra  $C^*G$ , and hence an element, denoted  $[V]$ , of the Kasparov group  $\text{KK}(\mathbb{C}, C^*G)$ . See [17] for the details.

**Proposition 2.23.** The class  $[(\mathcal{E}_V^G, P_V^G)]$  coincides (as expected) with the Kasparov product

$$[V] \otimes_{C^*G} \text{Ind}^{\text{M|B}}(P_0) \in \text{KK}(\mathbb{C}, C(B)).$$

Thus the K-multiplicity of  $V$  in the index of  $P_0$  is the element of  $\text{K}(B)$  corresponding to the Kasparov product  $[V] \otimes_{C^*G} \text{Ind}^{\text{M|B}}(P_0)$ , through the isomorphism  $\text{KK}(\mathbb{C}, C(B)) \cong \text{K}(B)$ .

*Proof.* The class  $[V]$  is given by the cycle  $[(V, 0)] \in \text{KK}(\mathbb{C}, C^*G)$ . The operator  $\text{id}_V \otimes_{C^*G} P$  makes sense because  $P$  commute with the action of  $C^*G$ . Using the  $C^*G$ -module structure on  $V$ , it follows that  $\forall v \in V, v = \sum_k e_k \cdot C_{e_k, v}$ . Therefore  $\forall s = \sum_i v_i \otimes s_i \in V \otimes_{C^*G} \mathcal{E}$ , we have  $s = \sum_k e_k \otimes \sum_i \pi(C_{e_k, v_i})s_i$ . Using again the relation  $(\dim V)C_{e_j, e_j} \star C_{e_j, v} = C_{e_j, v}$ , we deduce again that  $(\text{id}_V \otimes_{C^*G} P)^2 - 1$  is compact. Moreover, the  $C(B)$ -modules  $V \otimes_{C^*G} \mathcal{E}$  and  $(V \otimes \mathcal{E})^G$  are isomorphic. The isomorphism is given by

$$\begin{aligned}
V \otimes_{C^*G} \mathcal{E} &\longrightarrow (V \otimes \mathcal{E})^G \\
\sum v_i \otimes s_i &\longmapsto \sum_i \int_G g \cdot (v_i \otimes s_i) dg.
\end{aligned}$$

This map is well defined because we have

$$\int_G g \cdot (v \cdot \varphi \otimes s) dg = \int_G g \cdot (v \otimes \pi(\varphi)s) dg.$$

But we have already seen that  $\int_G g \cdot (v \otimes s) dg = \sum_k e_k \otimes \pi(C_{e_k, v})s$ . Therefore we have proved that  $[\mathcal{E}_V^G, P_V^G] = [V] \otimes_{C^*G} \text{Ind}_G^{\text{M|B}}(P_0)$ .  $\square$

### 3 The properties of the index class

As before, we denote by  $T_G^V M := T^V M \cap T_G M$ .

#### 3.1 The index map

**Proposition 3.1.** *The index class  $\text{Ind}_G^{\text{M|B}}(P)$  only depends on the class  $[\sigma(P)] \in K_G(T_G^V M)$  of the principal symbol  $\sigma(P)$ , and the index map induces a group homomorphism:*

$$\text{Ind}_G^{\text{M|B}} : K_G(T_G^V M) \rightarrow \text{KK}(C^*G, C(B)).$$

More precisely, the map  $[\sigma] \mapsto \text{Ind}_G^{\text{M|B}}(P(\sigma))$  is well defined, when we choose some quantization  $P(\sigma)$  of  $\sigma$ .

*Proof.* This is classical and we follow [2] and [1]. Let  $\mathcal{C}(T_G^V M)$  be the semigroup of homotopy classes of transversally elliptic symbols of order 0 and  $\mathcal{C}_\phi(T_G^V M) \subset \mathcal{C}(T_G^V M)$  the subspace composed of classes of symbols whose restriction to the sphere bundle of  $T_G^V M$ , is induced by a bundle isomorphism over  $M$ . We know that  $K_G(T_G^V M) := \mathcal{C}(T_G^V M)/\mathcal{C}_\phi(T_G^V M)$ , [2]. It thus suffices to show that:

1. if  $[\sigma] = [\sigma']$  in  $\mathcal{C}(T_G^V M)$  then  $\text{Ind}_G^{\text{M|B}} \sigma = \text{Ind}_G^{\text{M|B}} \sigma'$ ;
  2. for  $P$  and  $Q$  families of  $G$ -transversally elliptic operators, we have  $\text{Ind}_G^{\text{M|B}}(P \oplus Q) = \text{Ind}_G^{\text{M|B}}(P) + \text{Ind}_G^{\text{M|B}}(Q)$ ,
  3. if  $P$  has symbol which is induced by a bundle isomorphism over  $M$  then  $\text{Ind}_G^{\text{M|B}}(P) = 0$ .
1. Let  $\sigma_t$  be a homotopy between  $\sigma$  and  $\sigma'$  composed of transversally elliptic symbols along the fibers, the quantization of this homotopy give an operator homotopy, and hence we end up with the same class in  $\text{KK}_G(C^*G, C(B))$ , by definition of the Kasparov group  $\text{KK}_G(C^*G, C(B))$ .
  2. If  $P : C^{\infty,0}(M, E) \rightarrow C^{\infty,0}(M, E)$  and  $Q : C^{\infty,0}(M, F) \rightarrow C^{\infty,0}(M, F)$  are families of  $G$ -transversally elliptic operators, then  $\text{Ind}_G^{\text{M|B}}(P \oplus Q) = [(\mathcal{E} \oplus \mathcal{F}, \pi_{\mathcal{E}} \oplus \pi_{\mathcal{F}}, P \oplus Q)] = [(\mathcal{E}, \pi_{\mathcal{E}}, P)] + [(\mathcal{F}, \pi_{\mathcal{F}}, Q)] = \text{Ind}_G^{\text{M|B}}(P) + \text{Ind}_G^{\text{M|B}}(Q)$ .
  3. This is clear, see for instance [3].  $\square$



### 3.2 The case of free actions

Let  $G$  and  $H$  be compact Lie groups. Let  $p : M \rightarrow B$  be a  $G \times H$ -equivariant fibration of compact manifolds. We suppose that the action of  $G \times H$  is trivial on  $B$  and that  $H$  acts freely on  $M$ . Denote by  $\pi^H : M \rightarrow M/H$  the  $G$ -equivariant projection. By [1], we know that  $\pi^H$  induces an isomorphism

$$\pi^{H*} : K_G(T_G^V(M/H)) \rightarrow K_{G \times H}(T_{G \times H}^V M)$$

which identifies the classes of symbols of  $G$ -invariant families of  $G$ -transversally elliptic operators on  $M/H \rightarrow B$  with those of symbols of  $G \times H$ -invariant families of  $G \times H$ -transversally elliptic operators on  $M \rightarrow B$ . Denote by  $\pi : T^V M \rightarrow M$  and by  $\bar{\pi} : T^V(M/H) \rightarrow M/H$  the projections. The inverse map of  $\pi_H^*$  is given as follow. Let  $\sigma : \pi^* E^+ \rightarrow \pi^* E^-$  be a  $G \times H$ -transversally elliptic symbol along the fibers on  $M$ . Then we define a  $G$ -equivariant symbol  $\bar{\sigma} : \bar{\pi}^*(E^+/H) \rightarrow \bar{\pi}^*(E^-/H)$  on  $T^V(M/H)$  which is  $G$ -transversally elliptic along the fibers by setting  $\bar{\sigma}(\bar{m}, \bar{\xi})(\bar{v}) = \sigma(m, \xi)(v)$ , where  $\bar{m} = \pi^H(m) \in M/H$ ,  $\bar{\xi} = \pi^H(\xi) \in T_{\bar{m}}^V(M/H)$ ,  $\bar{v} = \pi^H(v) \in (E/H)_{\bar{m}}$ .

Let  $\hat{H}$  be the space of isomorphism classes of unitary irreducible representations of  $H$ . Denote by  $\chi_\alpha$  the character associated with the representation  $\alpha$  and  $W_\alpha$  the corresponding vector space. We have a natural map:

$$\begin{array}{ccc} R(H \times G) & \rightarrow & K_G(M/H) \\ V & \mapsto & \underline{V}^* \end{array}$$

where  $\underline{V}^* = M \times_H V^*$  is the homogeneous  $G$ -vector bundle associated to  $V^*$  on  $M/H$ .

**Theorem 3.2.** [1] *The following diagram is commutative:*

$$\begin{array}{ccccc} K_G(T_G^V(M/H)) & \xrightarrow{\pi^{H*}} & K_{G \times H}(T_{G \times H}^V M) & \xrightarrow{\text{Ind}^{M|B}} & KK(C^*(G \times H), C(B)) \\ \downarrow \underline{W}_\alpha^* \otimes & & & & \downarrow \chi_\alpha \otimes_{C^*H} \\ K_G(T_G^V(M/H)) & \xrightarrow{\text{Ind}^{M/H|B}} & & & KK(C^*G, C(B)). \end{array}$$

In other words, if  $a \in K_G(T_G^V(M/H))$  then

$$\text{Ind}^{M/H|B}(\underline{W}_\alpha^* \otimes a) = \chi_\alpha \otimes_{C^*H} \text{Ind}^{M|B}(\pi^{H*} a),$$

**Remark 3.3.** In particular, as an element of  $\text{Hom}(R(H), KK(C^*G, C(B)))$  we have

$$\text{Ind}^{M|B}(\pi^{H*} a) = \sum_{\alpha \in \hat{H}} \hat{\chi}_\alpha \otimes \text{Ind}^{M/H|B}(\underline{W}_\alpha^* \otimes a),$$

where  $\hat{\chi}_\alpha$  is the element associated to  $W_\alpha$  in  $KK(C^*H, \mathbb{C})$ .

*Proof.* Let  $\tilde{A}_0 : C^{\infty,0}(M, \pi^{H*} E^+) \rightarrow C^{\infty,0}(M, \pi^{H*} E^-)$  be a  $G \times H$ -invariant family of  $G \times H$ -transversally elliptic pseudodifferential operators associated to  $\pi^{H*} a \in K_{G \times H}(T_{G \times H}^V M)$ . Denote by  $\pi^{H*} \mathcal{E}$  the  $C^*$ -completion of  $C^{\infty,0}(M, \pi^{H*} E)$  as a Hilbert  $C(B)$ -module. We have:

$$\text{Ind}^{M|B}(\pi^{H*} a) = [\pi^{H*} \mathcal{E}, \pi_{G \times H}, \tilde{A}] = \sum_{\alpha \in \hat{H}} \hat{\chi}_\alpha \otimes [(\underline{W}_\alpha \otimes \pi^{H*} \mathcal{E})^H, \pi_G, (id_{W_\alpha^*} \otimes \tilde{A})_{|G\text{-inv}}],$$

where  $\tilde{A}$  is the self-adjoint operator associated with  $\begin{pmatrix} 0 & \tilde{A}_0^* \\ \tilde{A}_0 & 0 \end{pmatrix}$  as in Proposition 2.17. We shall use the following isomorphism:

$$\begin{array}{ccc} C^{\infty,0}(M, W_\alpha^* \otimes \pi^{H^*} E)^H & \longrightarrow & C^{\infty,0}(M/H, \underline{W}_\alpha^* \otimes E) \\ s & \longmapsto & s_{M/H}, \end{array}$$

given by  $s_{M/H}(x) = [m, s(m)]$ , for  $x = \pi^H(m)$  which allows to identify  $(W_\alpha^* \otimes \pi^{H^*} \mathcal{E})^H$  with  $\overline{C^{\infty,0}(M/H, \underline{W}_\alpha^* \otimes E)}^{C(B)}$ .

It remains to show that for a good choice of  $id_{W_\alpha^*} \otimes \tilde{A}$ , the family of operators  $id_{\underline{W}_\alpha^*} \otimes A$  is a family of pseudodifferential operators associated to  $\underline{W}_\alpha^* \otimes a \in K_G(T_G^{*V}(M/H))$ .

Let  $A_1 : C^{\infty,0}(M/H, \pi^{H^*} E^+) \rightarrow C^{\infty,0}(M/H, \pi^{H^*} E^-)$  be a family of pseudodifferential operators of order 1 associated to  $a$ . Let us choose a trivializing system  $\{O_j\}$  of  $\pi^H : M \rightarrow M/H$  and a subordinate partition of unity  $\{\phi_j^2\}$ . Let  $M_j = (\pi^H)^{-1}(O_j) \simeq O_j \times H$ , we denote by  $A_1^j = \phi_j A_1 \phi_j$  the induced family of pseudodifferential operators of order 1 over  $O_j$ , it lifts to a family of pseudodifferential operators  $\tilde{A}_1^j \in \mathcal{P}^1(M_j, \pi^{H^*} E)$  on  $M_j$  [3]. Thanks to the partition  $\phi_j$ , this family of pseudodifferential operators can be gathered into a family of pseudodifferential operators on  $M$ . By averaging the sum of the  $\tilde{A}_1^j$  over  $G$ , we obtain:

$$\tilde{A} = \int_G g \cdot (\sum_j \tilde{A}_1^j) dg,$$

which is a family of pseudodifferential operators of order 1 on  $M$ . Using that the symbol commutes with lifting and averaging, we see that  $\sigma(\tilde{A})$  represents  $\pi^{H^*} a$ . Moreover, the restriction of  $\tilde{A}$  to the  $H$ -invariant sections is just the  $G$ -invariant family  $A = \int_G g \cdot (\sum_j A_1^j) dg$  of order 1.

Therefore the family  $A : C^{\infty,0}(M/H, E^+) \rightarrow C^{\infty,0}(M/H, E^-)$  of pseudodifferential operators of order 1 satisfies  $[\sigma(A)] = a$ , and lifts to a family of pseudodifferential operators  $\tilde{A}$ , of order 1 with  $[\sigma(\tilde{A})] = \pi^{H^*} a$ . By restricting the map  $\tilde{A}$  into the map  $\tilde{A} : C^{\infty,0}(M, \pi^{H^*} E^+) \rightarrow C^{\infty,0}(M, \pi^{H^*} E^-)$ , we recover  $A$ . So we have  $[(\pi^{H^*} \mathcal{E})^H, \pi_G, \tilde{A}|_{G\text{-inv}}] = \text{Ind}^{\text{M|B}}(a)$ , by replacing  $a$  by  $\underline{W}_\alpha^* \otimes a$ , we get the result.  $\square$

**Corollary 3.4.** *Let  $p : M \rightarrow B$  be a  $G \times H$ -equivariant fibration. Suppose that the action of  $G \times H$  is trivial on  $B$  and that the action of  $H$  is free on  $M$ . Then for  $a \in K_G(T^{*V}(M/H))$ , we have the following equality:*

$$\chi_\alpha \otimes_{C^*G} \text{Ind}_G^{\text{M|B}}(\pi^{H^*} a) = \text{Ind}_G^{\text{M/H|B}}(\underline{W}_\alpha^* \otimes a),$$

where  $\pi^H : M \rightarrow M/H$  is the projection and  $\text{Ind}_G^{\text{M/H|B}}(\underline{W}_\alpha^* \otimes a)$  is given by the families index along the fibers of  $M/H \rightarrow B$ .

In particular, as an element of  $\text{Hom}(R(H), \text{KK}(\mathbb{C}, C(B)))$  we have:

$$\text{Ind}_G^{\text{M|B}}(\pi^{H^*} a) = \sum_{\alpha \in \hat{H}} \hat{\chi}_\alpha \otimes \text{Ind}_G^{\text{M/H|B}}(\underline{W}_\alpha^* \otimes a),$$

where  $\hat{\chi}_\alpha$  is the element associated to  $W_\alpha$  in  $\text{KK}(C^*H, \mathbb{C})$ .

### 3.3 Multiplicative property

Let  $G$  and  $H$  be two compact Lie groups. Let  $p : M \rightarrow B$  be a  $G$ -equivariant fibration of compact manifolds and let  $p' : M' \rightarrow B'$  be a  $G \times H$ -equivariant fibration of compact manifolds. We suppose that  $G$  acts trivially on  $B$  and that  $G$  and  $H$  act trivially on  $B'$ . If  $B = B'$  then the fiber product  $M \times_B M'$  is a  $G \times H$ -equivariant fibration on  $B$ . The purpose of this section is to compute the index of the product  $a \sharp b \in K_{G \times H}(T_{G \times H}^{*V}(M \times_B M'))$  of  $a \in K_G(T_G^{*V}M)$  and  $b \in K_{G \times H}(T_H^{*V}M')$  in terms of the index of  $a$  and the index of  $b$ . Recall that the product of  $a$  and  $b$  is given by the sharp-product  $a \sharp b = \begin{pmatrix} a \otimes 1 & -1 \otimes b^* \\ 1 \otimes b & a^* \otimes 1 \end{pmatrix}$ , that we restrict to  $T_{G \times H}^{*V}(M \times_B M')$  [1], see too [2]. This product is well defined since any element of  $K_G(T_G^{*V}M)$  can be represented by a symbol on  $T^V M$  which is invertible on  $T_G^V M \setminus M$ , see again [1].

Let  $\Delta_B : B \rightarrow B \times B$  be the diagonal map. We denote by  $[\Delta_B]$  the Kasparov element in  $KK(C(B) \otimes C(B), C(B))$  associated to  $(C(B), \Delta_B^*, 0)$ .

Recall the Kasparov descent [20]. Let  $A$  and  $D$  be  $G$ - $C^*$ -algebras. For a given  $G$ - $C^*$ -algebra  $A$  we recall the crossed product  $C^*$ -algebra  $A \rtimes G$ . More precisely, the space  $C(G, A)$  is an involutive algebra for the rules

$$a_1 \cdot a_2(t) = \int_G a_1(s) \cdot s(a_2(st)) ds \text{ and } a^*(t) = t(a(t^{-1}))^*, \text{ where } a, a_1, a_2 \in C(G, A).$$

The completion of the image of  $C(G, A)$  by the left regular representation,  $C(G, A) \rightarrow \mathcal{L}(L^2(G, A))$  given by

$$(a \cdot l)(t) = \int_G t^{-1}(a(s)) \cdot l(s^{-1}t) ds$$

where  $a \in C(G, A)$ ,  $l \in L^2(G, A)$ , is the  $C^*$ -algebra  $A \rtimes G$ . Let  $\mathcal{E}$  be a  $G$ - $D$ -Hilbert module. Define a right action and a scalar product by

$$e \cdot d(s) = \int_G e(t) t(d(t^{-1}s)) dt \text{ and } \langle e_1, e_2 \rangle(s) = \int_G t^{-1}(\langle e_1(t), e_2(ts) \rangle_{\mathcal{E}}) dt,$$

where  $e, e_1, e_2 \in C(G, \mathcal{E})$  and  $d \in C(G, D)$  then the completion of  $C(G, \mathcal{E})$  with respect to this Hilbert structure defines a  $D \rtimes G$ -Hilbert module. If  $\pi : A \rightarrow \mathcal{L}_D(\mathcal{E})$  is a  $*$ -morphism then the map  $\pi^{\rtimes G} : A \rtimes G \rightarrow \mathcal{L}_D(\mathcal{E} \rtimes G)$  given by  $(\pi^{\rtimes G}(a)e)(t) = \int_G \pi(a(s)) s(e(s^{-1}t)) ds$  is a  $*$ -morphism. If  $T \in \mathcal{L}_D(\mathcal{E})$  then  $T$  induces an operator  $\tilde{T} \in \mathcal{L}_{D \rtimes G}(\mathcal{E} \rtimes G)$  defined by  $(\tilde{T}e)(s) = T(e(s))$ .

**Definition 3.5.** The Kasparov descent map  $j^G : KK_G(A, D) \rightarrow KK(A \rtimes G, D \rtimes G)$  is given by  $[(\mathcal{E}, \pi, T)] \rightarrow [(\mathcal{E} \rtimes G, \pi^{\rtimes G}, \tilde{T})]$ .

**Theorem 3.6.** Let  $a \in K_G(T_G^{*V}M)$  and  $b \in K_{G \times H}(T_H^{*V}M')$ . When  $B = B'$  we denote by  $i : M \times_B M' \hookrightarrow M \times M'$  the inclusion.

1. The index of  $a \sharp b \in K_{G \times H}(T_{G \times H}^{*V}(M \times M'))$  is given by the following product:

$$\text{Ind}^{M \times M' | B \times B'}(a \sharp b) = j^G \text{Ind}_G^{M|B}(b) \otimes_{C^*G} \text{Ind}^{M'|B'}(a) \in KK(C^*G \otimes C^*H, C(B) \otimes C(B')).$$

The inclusion  $T_{G \times H}^{*V}(M \times_B M') \hookrightarrow T_{G \times H}^{*V}(M \times M')$  is a  $(G \times H)$ -equivariant proper inclusion.

2. The index of  $i^*(a \sharp b) \in K_{G \times H}(T_{G \times H}^{*V}(M \times_B M'))$  is given by:

$$\text{Ind}^{M \times_B M' | B}(i^* a \sharp b) = \text{Ind}^{M \times M' | B}(a \sharp b) \otimes_{C(B \times B)} [\Delta_B].$$

*Proof.*

1. Recall that if  $P_0$  is a family of pseudodifferential operators of positive order then we denote by  $\hat{P}$  the family  $\begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$  and by  $P$  the closure of  $\hat{P}$ .

Let  $A_0 : C^{\infty,0}(M, E_2^+) \rightarrow C^{\infty,0}(M, E_2^-)$  be a  $G$ -invariant family of  $G$ -transversally elliptic operators of order 1 associated to  $a$  and  $(\mathcal{E}_2, \pi_G, A)$  the  $(C^*G, C(B'))$  unbounded Kasparov cycle associated (see Definition 2.19). Let  $B_0 : C^{\infty,0}(M, E_1^+) \rightarrow C^{\infty,0}(M, E_1^-)$  be a  $G \times H$ -invariant family of  $H$ -transversally elliptic operators of order 1 associated to  $b$  and  $(\mathcal{E}_1, \pi_H, B)$  the associated  $(C^*H, C(B))$  unbounded Kasparov cycle.

Denote by  $Q(T) = T(1 + T^2)^{-1}$  the Woronowicz transformation of a selfadjoint regular operator  $T$ . Recall that the index classes associated respectively to  $(\mathcal{E}_1, \pi_G, A)$  and  $(\mathcal{E}_2, \pi_H, B)$  are respectively  $\text{Ind}^{M' | B'}(A) = [(\mathcal{E}_2, \pi_G, Q(A))]$  et  $\text{Ind}_G^{M | B}(b) = [(\mathcal{E}_2, \pi_H, Q(B))]$ . Notice that the operator  $1 \otimes_{\pi_G} A$  is well defined since  $A$  commutes with  $\pi_G$ . We will show that the cycle  $(\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2, \pi_H^{\rtimes G} \otimes_{\pi_G} 1, Q(\tilde{B}_1 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A_1))$  is the Kasparov product of  $j^G \text{Ind}_G^{M | B}(B)$  and  $\text{Ind}^{M' | B'}(A)$ .

Notice that  $B \otimes 1 + 1 \otimes A$  is the operator used to define the index class as in Definition 2.19 because  $\hat{B} \otimes 1 + 1 \otimes \hat{A}$  is essentially selfadjoint. We have to check that  $\text{Ind}_G^{M \times M' | B \times B'}(A \sharp B) = [(\mathcal{E}_1 \otimes \mathcal{E}_2, \pi_{G \times H}, B \otimes 1 + 1 \otimes A)]$  is given by  $[(\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2, \pi_H \rtimes G \otimes_{\pi_G} 1, Q(\tilde{B} \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A))]$ .

Let  $U : C^\infty(G, C^{\infty,0}(M, E_1)) \otimes C^{\infty,0}(M', E_2) \rightarrow C^{\infty,0}(M \times M', E_1 \boxtimes E_2)$  be the map defined by  $U(s \otimes s') = \int_G s(t) \otimes g \cdot s' dg$ . We easily check that  $U(s \cdot \varphi \otimes s') = U(s \otimes \pi_G(\varphi)s')$ ,  $\forall s \in C^\infty(G, C^{\infty,0}(M, E_1))$ ,  $\varphi \in C^\infty(G)$  and  $s' \in C^{\infty,0}(M', E_2)$ , so  $U$  induces a map still denoted by  $U$  from  $C^\infty(G, C^{\infty,0}(M, E_1)) \otimes_{\pi_G} C^{\infty,0}(M', E_2)$  to  $C^{\infty,0}(M \times M', E_1 \boxtimes E_2)$ . This map extends to a unitary isomorphism from  $\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2$  to  $\mathcal{E}_1 \otimes \mathcal{E}_2$ . Indeed,  $U(C^\infty(G, C^{\infty,0}(M, E_1)) \otimes_{\pi_G} C^{\infty,0}(M', E_2))$  is dense in  $\mathcal{E}_1 \otimes \mathcal{E}_2$  and we have the equality  $\langle U(s_1 \otimes_{\pi_G} s_2), U(s'_1 \otimes_{\pi_G} s'_2) \rangle_{\mathcal{E}_1 \otimes \mathcal{E}_2} = \langle s_1 \otimes_{\pi_G} s_2, s'_1 \otimes_{\pi_G} s'_2 \rangle_{\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2}$ ,  $\forall s_1, s'_1 \in \mathcal{E}_1 \rtimes G$  and  $s_2, s'_2 \in \mathcal{E}_2$ . Let us check this equality. We have:

$$\begin{aligned} \langle U(s_1 \otimes_{\pi_G} s_2), U(s'_1 \otimes_{\pi_G} s'_2) \rangle_{\mathcal{E}_1 \otimes \mathcal{E}_2} &= \int_G \int_G \langle s_1(h) \otimes h \cdot s_2, s'_1(k) \otimes k \cdot s'_2 \rangle_{\mathcal{E}_1 \otimes \mathcal{E}_2} dh dk \\ &= \int_G \int_G \langle s_1(h), s'_1(k) \rangle_{\mathcal{E}_1} \langle s_2, h^{-1}k \cdot s'_2 \rangle_{\mathcal{E}_2} dh dk. \end{aligned}$$

By substituting  $g = h^{-1}k$ , we get:

$$\begin{aligned} \langle U(s_1 \otimes_{\pi_G} s_2), U(s'_1 \otimes_{\pi_G} s'_2) \rangle_{\mathcal{E}_1 \otimes \mathcal{E}_2} &= \int_G \int_G \langle s_1(h), s'_1(hg) \rangle_{\mathcal{E}_1} \langle s_2, g \cdot s'_2 \rangle_{\mathcal{E}_2} dh dg \\ &= \int_G \int_G \langle s_2, \langle s_1(h), s'_1(hg) \rangle_{\mathcal{E}_1} g \cdot s'_2 \rangle_{\mathcal{E}_2} dh dg \\ &= \langle s_1 \otimes_{\pi_G} s_2, s'_1 \otimes_{\pi_G} s'_2 \rangle_{\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2}. \end{aligned}$$

It thus remains to check that  $U$  intertwines representations and operators. Let  $\psi \in C^\infty(H)$  and  $\varphi \in C^\infty(G)$ ,  $s_1 \in C^\infty(G, C^{\infty,0}(M, E_1))$  and  $s_2 \in C^{\infty,0}(M', E_2)$ . We have:

$$\begin{aligned}\pi_{H \times G}(\psi \otimes \phi)(U(s_1 \otimes s_2)) &= \pi_{H \times G}(\psi \otimes \varphi)\left(\int_G s_1(t) \otimes t \cdot s_2 dt\right) \\ &= \int_{H \times G} \int_G \psi(h) \varphi(g)(h, g) \cdot (s_1(t)) \otimes (gt) \cdot s_2 dt dh dg.\end{aligned}$$

Setting  $gt = u$ , we obtain:

$$\begin{aligned}\pi_{H \times G}(\psi \otimes \phi)(U(s_1 \otimes s_2)) &= \int_{H \times G} \int_G \psi(h) \varphi(g)(h, g) \cdot (s_1(g^{-1}u)) \otimes (u) \cdot s_2 du dh dg \\ &= \int_G \int_G \pi_H(\psi \otimes \varphi(g))g \cdot (s_1(g^{-1}u)) \otimes (u) \cdot s_2 du dg \\ &= \int_G (\pi_H^{\times G}(\psi \otimes \varphi)(s_1))(u) \otimes (u) \cdot s_2 du \\ &= U(\pi_H^{\times G}(\psi \otimes \varphi) \otimes_{\pi_G} 1(s_1 \otimes_{\pi_G} s_2)),\end{aligned}$$

so  $U$  intertwines representations. Let us now compute  $(B_1 \otimes 1 + 1 \otimes A_1)(U(s_1 \otimes_{\pi_G} s_2))$ . We have:

$$\begin{aligned}(B_1 \otimes 1 + 1 \otimes A_1)(U(s_1 \otimes_{\pi_G} s_2)) &= (B_1 \otimes 1 + 1 \otimes A_1)\left(\int_G s_1(t) \otimes t \cdot s_2 dt\right) \\ &= \int_G B_1(s_1(t)) \otimes t \cdot s_2 + s_1(t) \otimes t \cdot A_1(s_2) dt \\ &= \int_G (\tilde{B}_1 s_1)(t) \otimes t \cdot s_2 + s_1(t) \otimes t \cdot A_1(s_2) dt \\ &= U((\tilde{B}_1 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A_1)(s_1 \otimes_{\pi_G} s_2)),\end{aligned}$$

so that  $U$  intertwines operators as well.

We now show that the operator  $Q(\tilde{B} \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A)$  is a  $Q(A)$ -connection on  $\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2$ . Let us write

$$Q(\tilde{B} \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A) = M^{1/2} Q(\tilde{B}) \otimes_{\pi_G} 1 + N^{1/2} 1 \otimes_{\pi_G} Q(A),$$

where  $M = \frac{1 + \tilde{B}^2 \otimes_{\pi_G} 1}{1 + \tilde{B}^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2}$  and  $N = \frac{1 + 1 \otimes_{\pi_G} \tilde{A}^2}{1 + \tilde{B}^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2}$ . Notice that the operators  $M$  and  $N$  are selfadjoint and bounded. Moreover  $M$  and  $N$  commute and we have  $M + N = 1 + (1 + \tilde{B}^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2)^{-1}$ . To show that the operator  $Q(\tilde{B} \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A)$  is a  $Q(A)$ -connection on  $\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2$  it is sufficient to show that  $M$  is a 0-connection and that  $N$  is a 1-connexion. We obtain then by a classical result on connections that  $M^{1/2}$  is a 0-connection and that  $N^{1/2}$  is a 1-connection, so that  $Q(\tilde{B} \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A)$  is a  $Q(A)$ -connection since  $1 \otimes_{\pi_G} Q(A)$  is a  $Q(A)$ -connection.

Let us check that  $M$  is a 0-connection. Let  $\varphi \in C^\infty(G)$  and  $s \in C^{\infty,0}(M, E_1)$ , we have:

$$\begin{aligned}M \circ T_{s \otimes \varphi} &= \frac{1 + B^2 \otimes_{\pi_G} 1}{1 + B^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2} \circ T_s \circ \pi_G(\varphi) \\ &= \frac{1 + 1 \otimes_{\pi_G} A}{1 + B^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2} \circ T_{(1+B^2)s} \circ (1 + A^2)^{-1} \circ \pi_G(\varphi) \\ &= N \circ T_{(1+B^2)s} \circ (1 + A^2)^{-1} \circ \pi_G(\varphi).\end{aligned}$$

As  $(1+A^2)^{-1} \circ \pi_G(\varphi)$  is compact, we deduce that  $M \circ T_{s \otimes \varphi}$  is compact because  $N \in \mathcal{L}(\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2)$  and  $T_{(1+B^2)s} \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2)$ . Moreover, the map which associates to  $s \in \mathcal{E}_1 \rtimes G$  the operator  $M \circ T_s$  is continuous so we obtain that  $\forall s \in \mathcal{E}_1 \rtimes G$ ,  $M \circ T_s$  is compact which shows that  $M$  is a 0-connection. To show that  $N$  is a 1-connection it is sufficient to check that  $(1+B^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2)^{-1}$  is a 0-connection since  $M + N = 1 + (1+B^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2)^{-1}$ . This is obtained in a way similar to the previous computation, since  $(1+B^2 \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A^2)^{-1} = N \circ (1 + 1 \otimes_{\pi_G} A^2)^{-1}$ . It remains finally to check the positivity condition. But we have:

$$[Q(B \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A), Q(B) \otimes_{\pi_G} 1] = 2MQ(B)^2(\theta) \otimes_{\pi_G} 1$$

so  $\pi_H^{\times G}(\theta) \otimes_{\pi_G} 1 [Q(B \otimes_{\pi_G} 1 + 1 \otimes_{\pi_G} A), Q(B) \otimes_{\pi_G} 1] \pi_H^{\times G}(\theta) \otimes_{\pi_G} 1$  is positive modulo  $k(\mathcal{E}_1 \rtimes G \otimes_{\pi_G} \mathcal{E}_2)$ .

2. For the last equality, the product of  $\text{Ind}^{M \times M' | B}(a \sharp b)$  with  $[\Delta_B]$  on the right allows to see the Hilbert  $C(B \times B)$ -module  $\mathcal{E}_1 \otimes \mathcal{E}_2$  as an Hilbert  $C(B)$ -module.  $\square$

### 3.4 Excision property

In this section, we show an excision property which allows to compute the index of a family of  $G$ -transversally elliptic operators on an open subset of  $M$  as the index of a family of  $G$ -transversally elliptic operators on the compact fibration  $p : M \rightarrow B$ .

We start with a lemma from [9] which contains the ideas used in the proof of the excision theorem.

**Lemma 3.7.** [9] *Let  $(\mathcal{E}, \pi, F)$  be an  $(A, B)$  Kasparov bimodule. Let  $\mathcal{K}_1$  be the  $C^*$ -subalgebra of  $k(\mathcal{E})$  generated by  $[\pi(a), F]$ ,  $\pi(a)(F^2 - 1)$  and  $\pi(a)(F - F^*)$  for  $a \in A$  and the multiples by  $A$ ,  $F$  and  $F^*$ . Let  $\mathcal{E}_1$  be the closed  $D$ -submodule of  $\mathcal{E}$  generated by  $\mathcal{K}_1 \mathcal{E}$ . This module  $\mathcal{E}_1$  is called the support of  $(\mathcal{E}, \pi, F)$ , it is stable under the actions of  $A$  and  $F$ . Let  $F_1$  be the restriction of  $F$  to  $\mathcal{E}_1$ . Then the classes of  $(\mathcal{E}_1, \pi, F_1)$  and  $(\mathcal{E}, \pi, F)$  coincide in  $\text{KK}(A, B)$ .*

Let  $G$  be a compact Lie group. Let  $p : M \rightarrow B$  be  $G$ -equivariant compact fibration and assume that the action of  $G$  on  $B$  is trivial. We will use the following lemma shown in [1, lemma 3.6] in the case where  $B$  is the point, see also [2]. The proof being identical, it is omitted. Let  $U$  be a  $G$ -invariant open subspace of a compact fibration  $p : M \rightarrow B$ . Denote by  $j : U \hookrightarrow M$  the inclusion. Let  $\pi : T^V U \rightarrow U$  be the projection of the vertical tangent bundle defined by  $T^V U = \ker((p \circ j)_*)$ ,  $T^V U$  is an open subspace of  $T^V M$ . We denote by  $T_G^V U = T^V U \cap T_G M$ .

**Lemma 3.8.** *Each element  $a \in K_G(T_G^V U)$  can be represented by a homogeneous complex on  $T^V U$  of degree zero*

$$0 \longrightarrow \pi^* E^0 \xrightarrow{a} \pi^* E^1 \longrightarrow 0$$

*such that, outside a compact set in  $U$ , the bundles  $E^0$  and  $E^1$  are trivial and  $a$  is the identity.*

We can now state our theorem.

**Theorem 3.9.** *Let  $j : U \hookrightarrow M$  be an open  $G$ -embedding. We assume that  $p : M \rightarrow B$  is a compact  $G$ -fibration and that the action of  $G$  is trivial on  $B$ . Then the composition*

$$K_G(T_G^V U) \xrightarrow{j_*} K_G(T_G^V M) \xrightarrow{\text{Ind}^{M|B}} \text{KK}(C^*G, C(B))$$

does not depend on  $j$ . Said differently, if  $j' : U \hookrightarrow M'$  is another  $G$ -embedding with  $M' \rightarrow B$  a compact  $G$ -fibration then

$$\text{Ind}^{M'|B} \circ j'_* = \text{Ind}^{M|B} \circ j_*.$$

*Proof.* Let  $a \in K_G(T_G^V U)$ . We denote by  $\pi^V : T^V U \rightarrow U$  the projection. We start by representing  $a$  by a symbol of order 0 on  $T^V U$

$$0 \longrightarrow \pi^{V*} E^+ \xrightarrow{\sigma} \pi^{V*} E^- \longrightarrow 0,$$

trivial outside a compact set  $K$  of  $U$ , by using Lemma 3.8. We denote by  $E = E^+ \oplus E^-$ . The inclusion  $j$  defines a map  $j_*$  in equivariant K-theory:

$$j_* : K_G(T_G^V U) \rightarrow K_G(T_G^V M).$$

The map  $j_*$  associates to  $[\sigma]$  the class of  $\sigma$  trivially extended outside  $U$  on  $M$  using the trivializations.

Denote by  $j_* E$  the vector bundle  $E$  trivially extended outside  $U$ ,  $\mathcal{E} = \overline{C^{\infty,0}(M, j_* E)}^{C(B)}$ . We choose a family  $P_0$  of pseudodifferential operators associated to  $\sigma$  on  $U$  such that  $P_0$  is the identity outside the support  $L$  of  $\sigma$ , that is if  $s \in C_c^{\infty,0}(U \setminus L, E^+)$  then  $P_0 s \in C_c^{\infty,0}(U \setminus L, E^-)$  and it is given by

$$P_0(s)(x) = (\psi_x^-)^{-1} \circ \psi_x^+(s(x)),$$

where  $\psi^\pm$  are the trivializations of  $E^\pm$  outside  $L$ . Denote by  $P$  the operator  $\begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ . Let

$\theta \in C_c^{\infty,0}(U, [0, 1])$  be a function equal to 1 on  $L$  with support  $L'$ . With the help of such a function  $\theta$ , we extend by the identity the family  $P$  on  $M$ , and we denote by  $j_* P$  this operator on  $M$ . The symbol of this family  $j_* P$  is  $j_* \sigma$  and it is  $G$ -transversally elliptic along the fibers of  $p : M \rightarrow B$ . Let  $\mathcal{E}$  be the completion of  $C^{\infty,0}(M, j_* E)$  as a  $C(B)$ -module defined as in section 2.1. We then obtain an index class  $\text{Ind}^{M|B}(j_* P) = [\mathcal{E}, \pi, j_* P]$ . The map  $j$  allows to see  $C_c^{\infty,0}(U, E)$  as a submodule of  $\mathcal{E}$ . Denote by  $\mathcal{E}_{U \subset M}$  the completion of  $C_c^{\infty,0}(U, E)$  in  $\mathcal{E}$ . The operator  $j_* P$  preserves the closed submodule  $\mathcal{E}_{U \subset M}$ . Indeed, if  $s \in C_c^{\infty,0}(U, E)$  then  $j_* P(s) = P(s) \in C_c^{\infty,0}(U, E)$ . Moreover, the representation  $\pi$  of  $C^*G$  preserves  $\mathcal{E}_{U \subset M}$ . Indeed,  $\pi$  preserves  $C_c^{\infty,0}(U, E)$  since  $U$  is a  $G$ -invariant open subspace in  $M$ , so also  $\mathcal{E}_{U \subset M}$ . We then get a well defined class  $[\mathcal{E}_{U \subset M}, \pi, j_* P|_{\mathcal{E}_{U \subset M}}]$  in  $\text{KK}(C^*G, C(B))$ . The cycle thus obtained is also a representative for the index class  $\text{Ind}^{M|B}(j_* P)$  of  $j_* P$ . Indeed, by Lemma 3.7 above, it is sufficient to show that the two cycles have the same support, since then the operators will coincide as we will see. But if  $\chi \in C_c^{\infty,0}(U, [0, 1])$  is a function which is equal to 1 on the support of  $\theta$  then  $((j_* P)^2 - \text{id})(s) = ((j_* P)^2 - \text{id})(\chi s)$  and hence the equality of the supports.

Now let  $U'$  be a  $G$ -invariant open subspace of a compact fibration  $p' : M' \rightarrow B$  which is diffeomorphic to  $U$ , denote by  $j' : U' \hookrightarrow M'$  the inclusion. Denote by  $f : U \rightarrow U'$  a given diffeomorphism, and assume that  $p \circ j = (p \circ j') \circ f$ . The class of the element  $a \in K_G(T_G^V U)$  seen as an element of  $K_G(T_G^V U')$  using  $f$  can be represented by  $(f^{-1})^* \sigma$ . We associate to  $(f^{-1})^* \sigma$  the operator  $(f^{-1})^* P f^*$  that we extend by the identity on  $M'$  to an operator  $j'_* P$ . We thus get (as for  $U$ ) an index class  $\text{Ind}^{M'|B}(j'_* P) = [\mathcal{E}', \pi', j'_* P]$  by taking the completion of  $C^{\infty,0}(M', j'_*(f^{-1})^* E)$ . We define a closed submodule  $\mathcal{E}'_{U' \subset M'}$  in the same way as for  $\mathcal{E}_{U \subset M}$ . We obtain that  $\text{Ind}^{M'|B}(j'_* P) =$

$[\mathcal{E}'_{U' \subset M'}, \pi', j'_* P_{|\mathcal{E}'_{U' \subset M'}}]$ . But the cycle  $(\mathcal{E}'_{U' \subset M'}, \pi', j'_* P_{|\mathcal{E}'_{U' \subset M'}})$  is then unitary equivalent to the cycle  $(\mathcal{E}_{U \subset M}, \pi, j_* P_{|\mathcal{E}_{U \subset M}})$ . Indeed, if  $(\mu_b)_{b \in B}$  and  $(\mu'_b)_{b \in B}$  are respectively the family of measures of Lebesgue type on respectively  $M$  and  $M'$  used in the definition of the previous index classes then we denote by  $h = (h_b)_{b \in B}$  the continuous family of Radon-Nikodym derivatives associated on  $U$  identified to  $U'$  using  $f$ . We then get a unitary  $V : \mathcal{E}_{U \subset M} \rightarrow \mathcal{E}_{U' \subset M'}$  given by  $s \rightarrow \sqrt{h}(f^{-1})^* s$ . Furthermore,  $V^{-1}(f^{-1})^* P f^* V = \frac{1}{\sqrt{f^* h}} P \sqrt{f^* h}$  and  $V^{-1} \pi'(\varphi) V = \frac{1}{\sqrt{f^* h}} \pi(\varphi) \sqrt{f^* h}$  which give a unitarily equivalent cycle of  $(\mathcal{E}_{U \subset M}, \pi, j_* P_{|\mathcal{E}_{U \subset M}})$ . So the index class does not depend on the embedding  $j$  chosen.  $\square$

**Remark 3.10.** The equivalence between the two cycles  $(\mathcal{E}, \pi, j_* P)$  and  $(\mathcal{E}_{U \subset M}, \pi, j_* P_{|\mathcal{E}_{U \subset M}})$  can be deduced immediately by rewriting on the proof of Lemma 3.7, see appendix A of [9].

### 3.5 The naturality and induction theorems

We recall here some constructions from [1]. Let  $G$  be a compact Lie group. Let  $H$  be a closed subgroup of  $G$ . Denote by  $i : H \hookrightarrow G$  the inclusion. The group  $G$  is a  $G \times H$ -manifold for the action given by  $(g, h) \cdot g' = gg'h^{-1} \forall g, g' \in G$  and  $h \in H$ . Let  $p : M \rightarrow B$  be a  $G$ -equivariant fibration. We still assume that the action of  $G$  on  $B$  is trivial. We have the following product [1] (see also the previous section):

$$K_H(T_H^V M) \otimes K_{G \times H}(T_G^* G) \rightarrow K_{G \times H}(T_{G \times H}^V(G \times M)).$$

As  $H$  acts freely on  $G \times M$ , the quotient space  $Y = G \times_H M$  is a fibration on  $B$  and

$$K_{G \times H}(T_{G \times H}^V(G \times M)) \cong K_G(T_G^V Y).$$

The operator  $0 : C^\infty(G) \rightarrow 0$  is  $G$ -transversally elliptic and  $T_G^* G = G \times \{0\}$ . Furthermore,  $[\sigma(0)] \in K_{G \times H}(T_G^* G) \cong R(H)$  is the class of the trivial bundle. So we can define a map  $i_* : K_H(T_H^V M) \rightarrow K_G(T_G^V Y)$  by the composition of the product by  $[\sigma(0)]$  with the isomorphism  $K_{G \times H}(T_{G \times H}^V(G \times M)) \cong K_G(T_G^V Y)$ .

Recall now the definition of the element  $i^* \in KK(C^*G, C^*H)$  from [18]. Let us fix the Haar measures on  $H$  and  $G$ . We consider on the space  $C(G)$  of continuous functions on  $G$  with values in  $\mathbb{C}$  the right  $L^1(H)$ -module structure induced by the right action of  $H$  on  $G$ , i.e.

$$\varphi \cdot \psi(g) = \int_H \varphi(gh^{-1}) \psi(h) dh, \quad \forall \varphi \in C(G) \text{ and } \psi \in L^1(H)$$

and the following hermitian structure with values in  $L^1(H)$ :

$$\langle f_1, f_2 \rangle(h) = \int_G \bar{f}_1(g) f_2(gh) dg.$$

Denote by  $J(G)$  the completion of  $C(G)$  for the norm  $\|f\| = \|\langle f, f \rangle\|_{C^*H}^{1/2}$  induced by the hermitian structure. Denote by  $\pi_G : C^*G \rightarrow \mathcal{L}_{C^*H}(J(G))$  the natural representation of  $C^*G$  induced by the left  $G$  action on itself.



**Definition 3.11.** [18] The element  $i^* \in \text{KK}(C^*G, C^*H)$  is defined as the class of the cycle  $(J(G), \pi_G, 0)$ .

**Lemma 3.12.** *The element  $i^*$  is the image of the trivial representation of  $H$  seen as a trivial  $G$ -equivariant vector bundle on  $G/H$  by the Morita equivalence between  $C(G/H) \rtimes G$  and  $C^*H$ .*

*Proof.* In  $\text{KK}_G(\mathbb{C}, C(G/H))$  the trivial representation of  $H$  is given by the cycle  $(C(G/H), 0)$ . Its image by the Kasparov descent  $j^G([C(G/H), 0]) = [C(G/H) \rtimes G, \pi_G, 0]$  is an element of  $\text{KK}(C^*G, C(G/H) \rtimes G)$ . The cycle given by the Morita equivalence is  $[J(G), \rho, 0]$ , where  $\rho : C(G/H) \rtimes G \rightarrow \mathcal{L}_{C^*H}(J(G))$  is given by  $\rho(\theta)s(g) = \int_G \theta(t, g)s(t^{-1}g)dt$  for all  $\theta \in C(G, C(G/H))$  and  $s \in C(G)$ .  $\square$

We also denote by  $i_* : \text{KK}(C^*H, C(B)) \rightarrow \text{KK}(C^*G, C(B))$  the left Kasparov product by  $i^* \in \text{KK}(C^*H, C^*G)$ .

**Theorem 3.13.** [1] *The following diagram is commutative.*

$$\begin{array}{ccc} K_H(T_H^V M) & \xrightarrow{i_*} & K_G(T_G^V Y) \\ \text{Ind}^{M|B} \downarrow & & \downarrow \text{Ind}^{Y|B} \\ \text{KK}(C^*H, C(B)) & \xrightarrow{i_*} & \text{KK}(C^*G, C(B)) \end{array}$$

*Proof.* Let  $a \in K_H(T_H^V M)$ . From the multiplicative property of the index, we have:

$$\text{Ind}^{G \times M|B}(a \cdot [\sigma(0)]) = j^H(\text{Ind}^{G|*}([\sigma(0)])) \otimes_{C^*H} \text{Ind}^{M|B}(a).$$

Furthermore, the isomorphism  $K_{G \times H}(T_{G \times H}^V(G \times M)) \cong K_G(T_G^V Y)$  send  $a \cdot [\sigma(0)]$  on  $i_*(a)$  by definition of  $i_*$ . By Theorem 3.2, we know that  $\text{Ind}^{Y|B}(i_*(a)) = \chi_0^H \otimes_{C^*H} \text{Ind}^{G \times M|B}(a \cdot [\sigma(0)])$ . From this we deduce that:

$$\text{Ind}^{Y|B}(i_*(a)) = \chi_0^H \otimes_{C^*H} [j^H(\text{Ind}^{G|*}([\sigma(0)])) \otimes_{C^*H} \text{Ind}^{M|B}(a)].$$

It is easy to see that  $i_* = \chi_0^H \otimes_{C^*H} j^H(\text{Ind}^{G|*}([\sigma(0)]))$  coincides with  $i^*$ . Indeed, the action of  $H$  on  $C^*G$  is trivial here so this follows from a trivial generalization of the Green-Julg theorem, see [17]. We get the result by associativity of the Kasparov product.  $\square$

**Remark 3.14.** The previous theorem allows for instance to reduce the index problem to the case of connected Lie groups, since any compact Lie group can be embedded in a unitary group.

In order to reduce the computation of the index map to the case of torus action, we need to investigate the induction map. Let us first note that if  $M$  is a compact  $G$ -fibration then the map  $(g, m) \rightarrow (gH, g \cdot m)$  is a  $G$ -equivariant diffeomorphism between the  $G$ -fibrations  $Y = G \times_H M$  and  $G/H \times M$  over  $B$ . Let us assume that  $G$  is connected and that  $H$  is a maximal torus so that the homogeneous space  $G/H$  is a complex manifold, and hence is  $K$ -oriented. Then we

have a Dolbeault operator  $\bar{\partial}$  on  $G/H$  whose  $G$ -equivariant index is  $\text{Ind}(\bar{\partial}) = 1 \in R(G)$ . By the multiplicative property of the index, we have the following commutative diagram:

$$\begin{array}{ccccc} K_G(T_G^V M) & \otimes & K_G(T^*(G/H)) & \longrightarrow & K_G(T_G^V Y) \\ \text{Ind}^{M|B} \downarrow & & \downarrow \text{Ind}^{G/H|*} & & \downarrow \text{Ind}^{Y|B} \\ KK(C^*G, C(B)) & \otimes & KK_G(\mathbb{C}, \mathbb{C}) & \longrightarrow & KK(C^*G, C(B)). \end{array}$$

Multiplication by the symbol  $[\sigma(\bar{\partial})] \in K_G(T^*(G/H))$  induces a morphism

$$k : K_G(T_G^V M) \rightarrow K_G(T_G^V Y)$$

which preserves the index map. More precisely:

**Theorem 3.15.** *Let  $H$  be a maximal torus of a connected compact Lie group  $G$ . Denote by  $r = (i_*)^{-1} \circ k : K_G(T_G^V M) \rightarrow K_H(T_H^V M)$  the composition of the morphism  $k$  with the inverse of  $i_*$ . The following diagram then commutes:*

$$\begin{array}{ccc} K_G(T_G^V M) & \xrightarrow{r} & K_H(T_H^V M) \\ \text{Ind}^{M|B} \downarrow & & \downarrow \text{Ind}^{M|B} \\ KK(C^*G, C(B)) & \xleftarrow{i_*} & KK(C^*H, C(B)). \end{array}$$

*Proof.* The following diagram is commutative:

$$\begin{array}{ccc} K_G(T_G^V M) & \xrightarrow{k} & K_G(T_G^V Y) \\ \text{Ind}^{M|B} \downarrow & & \downarrow \text{Ind}^{Y|B} \\ KK(C^*G, C(B)) & \xrightarrow{=} & KK(C^*G, C(B)). \end{array}$$

Indeed, if  $a \in K_G(T_G^V M)$ , we have:

$$\begin{aligned} \text{Ind}^{Y|B}(k(a)) &= \text{Ind}^{Y|B}(a \cdot [\sigma(\bar{\partial})]) \\ &= j^G(\text{Ind}(\bar{\partial})) \otimes_{C^*G} \text{Ind}^{M|B}(a) \\ &= j^G(1) \otimes_{C^*G} \text{Ind}^{M|B}(a) \\ &= 1 \otimes_{C^*G} \text{Ind}^{M|B}(a) \\ &= \text{Ind}^{M|B}(a). \end{aligned}$$

By Theorem 3.13, we get that the following diagram is commutative:

$$\begin{array}{ccccc} K_G(T_G^V M) & \xrightarrow{k} & K_G(T_G^V Y) & \xleftarrow{i_*} & K_H(T_H^V M) \\ \text{Ind}^{M|B} \downarrow & & \downarrow \text{Ind}^{Y|B} & & \downarrow \text{Ind}^{M|B} \\ KK(C^*G, C(B)) & \xrightarrow{=} & KK(C^*G, C(B)) & \xleftarrow{i_*} & KK(C^*H, C(B)). \end{array}$$

□

We end this section with the statement of Theorem 3.16 below which shows the compatibility of the index map with the Gysin map. Let  $j : M \hookrightarrow M'$  be an inclusion of  $G$ -fibrations on  $B$  such that  $p' \circ j = p$  where  $p : M \rightarrow B$  and  $p' : M' \rightarrow B$  are the projections. Let us assume that  $M$  is compact. Following [1], we will define a morphism of  $R(G)$ -modules  $j_! : K_G(T_G^V M) \rightarrow K_G(T_G^V M')$ . Let  $N$  be a  $G$ -invariant tubular neighbourhood of  $M$  in  $M'$ . Then  $N$  is a  $G$ -manifold and it can be identified with the normal bundle of  $M$  in  $M'$ . The vector bundle  $T^V M$  is a closed  $G$ -submanifold of  $T^V M'$  and the tubular neighbourhood  $U$  of  $T^V M$  in  $T^V M'$  can be identified with the vector bundle given by lifting the vector bundle  $N \oplus N$  by the projection  $\pi : T^V M \rightarrow M$ . The exterior algebra of  $\pi^*(N \otimes \mathbb{C})$  defines a complex on  $T^V N$  which is exact outside the zero section  $T^V M \subset T^V N$ . We denote it by  $\Lambda(T^V N)$ . Denote by  $q : T_G^V N \rightarrow T_G^V M$  the projection. If  $E$  is a complex on  $T_G^V M$  with compact support then the product  $\Lambda(T^V N)_{|T_G N} \otimes q^* E$  has compact support in  $T_G^V N$ . The product by  $\Lambda(T^V N)$  defines a morphism of  $R(G)$ -modules

$$\phi : K_G(T_G^V M) \rightarrow K_G(T_G^V N),$$

called the Thom homomorphism. As  $T_G^V N$  is an open subspace of  $T_G^V M'$ , the inclusion induces a map

$$k_* : K_G(T_G^V N) \rightarrow K_G(T_G^V M').$$

The composition of these two maps gives the desired Gysin map

$$j_! : K_G(T_G^V M) \rightarrow K_G(T_G^V M').$$

**Theorem 3.16.** [1] *Let  $j : M \hookrightarrow M'$  be a  $G$ -embedding over  $B$  with  $M$  compact. The following diagram is commutative:*

$$\begin{array}{ccc} K_G(T_G^V M) & \xrightarrow{j_!} & K_G(T_G^V M') \\ \text{Ind}^{M|B} \downarrow & & \downarrow \text{Ind}^{M'|B} \\ KK(C^*G, C(B)) & = & KK(C^*G, C(B)). \end{array}$$

*Proof.* By the excision theorem, the index commutes with  $k_*$ . It is thus sufficient to show the theorem in the case of a real  $G$ -vector bundle  $N$  on  $M$  and with  $j_! : K_G(T_G^V M) \rightarrow K_G(T_G^V N)$  being the Thom homomorphism. We can write  $N = P \times_{O(n)} \mathbb{R}^n$ , where  $P$  is a  $O(n)$ -principal bundle on  $M$ . The  $G$ -action on  $P$  commutes with the  $O(n)$ -action and the  $G$ -action is trivial on  $\mathbb{R}^n$ . We have the following product:

$$K_{G \times O(n)}(T_{G \times O(n)}^V P) \otimes K_{G \times O(n)}(T\mathbb{R}^n) \rightarrow K_{G \times O(n)}(T_{G \times O(n)}^V (P \times \mathbb{R}^n)),$$

and the following isomorphisms:

$$\begin{aligned} q_1^* : K_G(T_G^V M) &\rightarrow K_{G \times O(n)}(T_{G \times O(n)}^V P) \\ q_2^* : K_G(T_G^V N) &\rightarrow K_{G \times O(n)}(T_{G \times O(n)}^V (P \times \mathbb{R}^n)). \end{aligned}$$

Therefore, we obtain a product:

$$K_G(T_G^V M) \otimes K_{G \times O(n)}(T\mathbb{R}^n) \rightarrow K_G(T_G^V N).$$

The inclusion of the origin in  $\mathbb{R}^n$  induces a Bott morphism  $i_! : R(O(n)) \rightarrow K_{O(n)}(T\mathbb{R}^n)$  and we have  $\text{Ind}(i_!(1)) = 1 \in \text{KK}_{O(n)}(\mathbb{C}, \mathbb{C})$  by [2]. As  $G$  acts trivially on  $\mathbb{R}^n$ , we get that  $\text{Ind}(i_!(1)) = 1 \in \text{KK}_{G \times O(n)}(\mathbb{C}, \mathbb{C})$ . The product by  $i_!(1)$  makes the following diagram commutative

$$\begin{array}{ccc} K_G(T_G^V M) & \xrightarrow{j_!} & K_G(T_G^V N) \\ q_1^* \downarrow & & \downarrow q_2^* \\ K_{G \times O(n)}(T_{G \times O(n)}^V P) & \xrightarrow{i_!(1)} & K_{G \times O(n)}(T_{G \times O(n)}^V (P \times \mathbb{R}^n)) \end{array}$$

from this it follows that

$$\text{Ind}^{P \times \mathbb{R}^n | B}(q_2^*(j_!(a))) = \text{Ind}^{P \times \mathbb{R}^n | B}(q_1^*(a) \cdot i_!(1)), \quad \forall a \in K_G(T_G^V M).$$

By the multiplicative property of the index, we obtain:

$$\text{Ind}^{P \times \mathbb{R}^n | B}(q_2^*(j_!(a))) = j^{G \times O(n)}(1) \otimes_{C^*G \otimes C^*O(n)} \text{Ind}^{P | B}(q_1^*(a)).$$

Furthermore, we have the following equalities by Theorem 3.2:

$$\text{Ind}^{M | B}(a) = \chi_0^{O(n)} \otimes_{C^*O(n)} \text{Ind}^{P | B}(q_1^*(a)),$$

$$\text{Ind}^{N | B}(j_!(a)) = \chi_0^{O(n)} \otimes_{C^*O(n)} \text{Ind}^{P \times \mathbb{R}^n | B}(q_2^*(j_!(a))),$$

where  $\chi_0^{O(n)}$  is the  $O(n)$  trivial representation. Finally we conclude that  $\text{Ind}^{M | B}(a) = \text{Ind}^{N | B}(j_!(a))$  since  $j^{G \times O(n)}(1) = 1$ . □

**Remark 3.17.** The properties of the index proved above allow to reduce the computation of the index map to the case of a fibration  $B \times \mathbb{R}^n \rightarrow B$  equipped with a torus action on  $\mathbb{R}^n$ .

## 4 Cup product of the index class with an elliptic operator on the base

In the present section, we insist that  $B$  is a manifold. This was not rigorously needed in the previous sections. We compute now the Kasparov product of the index class of family of  $G$ -transversally elliptic operators with the K-homology class of an elliptic operator on the base  $B$ . We begin by recalling some definitions and results from [14].

**Proposition 4.1.** [14, Proposition A.8] *Let  $E_1$  be a hermitian bundle on  $M$  and  $E_2$  be a hermitian bundle on  $B$ . Let  $Q : C^\infty(B, E_2) \rightarrow C^\infty(B, E_2)$  be a pseudodifferential operator on  $B$  of order 0. Let  $Q' : C^\infty(M, E_1 \otimes E_2) \rightarrow C^\infty(M, E_1 \otimes E_2)$  be a pseudodifferential operator of order 0 such that*

$$\sigma_{|p^*T^*B}(Q') = 1_{E_1} \hat{\otimes} p^*\sigma(Q),$$

*where  $p^*\sigma(Q)(m, \xi) = \sigma(Q)(p(m), p_*\xi)$ ,  $\forall (m, \xi) \in T^*M$ . Then  $Q'$  is a  $Q$ -connection for  $\mathcal{E}_1$ , where  $\mathcal{E}_1$  is the  $C(B)$ -module associated to  $E_1$  defined in Section 2.1.*

**Proposition 4.2.** [14, proposition A.10.2] Let  $P : C^{\infty,0}(M, E_1) \rightarrow C^{\infty,0}(M, E_1)$  be a family of pseudodifferential operators of order 0 on  $M$ . Let  $Q'$  be a pseudodifferential operator of order 0 on  $M$  such that

$$\sigma|_{p^*T^*B}(Q') = 1_{E_1} \hat{\otimes} \sigma',$$

where  $\sigma'$  is a symbol of order 0 on  $B$ . Then  $[P \hat{\otimes} 1, Q'] : C^\infty(M, E_1 \hat{\otimes} E_2) \rightarrow C^\infty(M, E_1 \hat{\otimes} E_2)$  is a pseudodifferential operator of order 0 and

$$\sigma([P \hat{\otimes} 1, Q']) = [\sigma(P) \hat{\otimes} 1, \sigma(Q')].$$

**Remark 4.3.** In the previous two propositions, all operators and all symbols can obviously be taken  $G$ -invariant.

Let us check that the product, between  $G$ -transversally elliptic symbols along the fibers and elliptic symbols on the base, gives  $G$ -transversally elliptic symbols on  $M$ .

**Lemma 4.4.** The map

$$K_G(T_G^{*V}M) \otimes K(T^*B) \rightarrow K_G(T_G^*M)$$

defined by

$$(\sigma \hat{\otimes} 1 + 1 \hat{\otimes} \sigma')(m, \xi) = \sigma(m, \xi - p_*\xi) \hat{\otimes} 1 + 1 \hat{\otimes} \sigma'(p(m), p_*\xi),$$

for  $\sigma \in K_G(T_G^{*V}M)$ ,  $\sigma' \in K(T^*B)$  and  $(m, \xi) \in T_G^*M$  induces a well defined product in  $K$ -theory.

*Proof.* On the one hand, the map  $K_G(T_G^{*V}M) \otimes K(T^*B) \rightarrow K_G(T_G^{*V}M \times T^*B)$  defined by  $(\sigma, \sigma') \mapsto \sigma \hat{\otimes} 1 + 1 \hat{\otimes} \sigma'$  induces a well defined product (this is a Kasparov product). On the other hand,  $T_G^*M = T_G^{*V}M \times_M p^*T^*B$  so by restriction of the product in  $T_G^{*V}M \times p^*T^*B$  to  $T_G^{*V}M \times_M p^*T^*B$ , we obtain a product in  $K$ -theory with values in  $K_G(T_G^*M)$ .  $\square$

**Theorem 4.5.** Let  $P : C^{\infty,0}(M, E_1) \rightarrow C^{\infty,0}(M, E_1)$  be a  $G$ -invariant selfadjoint family of  $G$ -transversally elliptic pseudodifferential operators of order 0. Let  $Q : C^\infty(B, E_2)$  be an elliptic pseudodifferential operator on  $B$  of order 0. The Kasparov product of the class  $[\mathcal{E}_1, \pi, P] = \text{Ind}^{M|B}(P)$  by  $[Q] \in \text{KK}(C(B), \mathbb{C})$  is given by  $[\mathcal{E}_1 \hat{\otimes}_{C(B)} \mathcal{E}_2, Q']$ , where  $Q'$  is any  $G$ -invariant pseudodifferential operator of order 0 which is  $G$ -transversally elliptic on  $M$  with symbol given by

$$\sigma(Q') = \sigma(P) \hat{\otimes} 1 + 1 \hat{\otimes} p^*\sigma(Q).$$

*Proof.* Notice first that if  $Q'$  is a  $G$ -invariant and  $G$ -transversally elliptic operator on  $M$  then  $\text{Ind}^M(Q') \in \text{KK}(C^*G, \mathbb{C})$  is well defined and only depends on its symbol  $[\sigma(Q')]$ , see [1]. Let us check that such  $Q'$  is a  $Q$ -connection for  $\mathcal{E}_1$  and that  $\forall \varphi \in C^*G$ ,  $\pi(\varphi)[P \hat{\otimes} 1, Q']\pi(\varphi)^*$  is positive modulo compact operators. Since  $\sigma(Q')|_{p^*T^*B} = 1 \hat{\otimes} p^*\sigma(Q)$ , we get by Proposition 4.1 that  $Q'$  is a  $Q$ -connection for  $\mathcal{E}_1$ . On the other hand, we obtain by Proposition 4.2 that  $[P \hat{\otimes} 1, Q']$  is a pseudodifferential operator on  $M$  and that its symbol is given by

$$\sigma([P \hat{\otimes} 1, Q']) = [\sigma(P) \hat{\otimes} 1, \sigma(Q')] = \sigma(P)^2 \hat{\otimes} 1,$$

which is positive. Moreover,  $Q'$  and  $P \hat{\otimes} 1$  are  $G$ -invariant so they commute with  $\pi(\varphi)$ . It remains to show that  $[P \hat{\otimes} 1, Q']$  is positive modulo compact operators. But  $\sigma([P \hat{\otimes} 1, Q']) \geq 0$  so  $[P \hat{\otimes} 1, Q'] \geq 0$  modulo compact operators.  $\square$

**Corollary 4.6.** *The pairing of the index class of a  $G$ -invariant family  $P$  of  $G$ -transversally elliptic pseudodifferential operators of order 0 with an element of the  $K$ -homology  $\mathrm{KK}(C(B), \mathbb{C})$  of  $B$  is given by the index class of a  $G$ -invariant pseudodifferential operator which is  $G$ -transversally elliptic on  $M$ .*

*Proof.* As any element of  $\mathrm{KK}(C(B), \mathbb{C})$  can be represented by an elliptic operator on  $B$ , this is a consequence of Theorem 4.5.  $\square$

**Corollary 4.7.** *If  $\alpha$  is an element of the  $K$ -homology of  $B$  then the class  $[\mathcal{E}, \pi, P] \otimes_{C(B)} \alpha \in \mathrm{KK}(C^*G, \mathbb{C}) \simeq \mathrm{Hom}(R(G), \mathbb{C})$  is given by the distributional index of Atiyah [1], i.e. the multiplicities of  $\mathrm{Ind}^{\mathrm{M|B}}(P) \otimes_{C(B)} \alpha$  are summable in the sense of distributions on  $G$ . Denote by  $m([V] \otimes_{C^*G} \mathrm{Ind}^{\mathrm{M|B}}(P) \otimes_{C(B)} \alpha)$  the integer associated to the multiplicity  $[V] \otimes_{C^*G} \mathrm{Ind}^{\mathrm{M|B}}(P) \otimes_{C(B)} \alpha$  of  $V$  in  $\mathrm{Ind}^{\mathrm{M|B}}(P) \otimes_{C(B)} \alpha$ . We have that for any  $\varphi \in C^\infty(G)$ : the sum*

$$\sum_{V \in \hat{G}} m([V] \otimes_{C^*G} \mathrm{Ind}^{\mathrm{M|B}}(P) \otimes_{C(B)} \alpha) \langle \chi_V, \varphi \rangle_{L^2(G)}$$

*is convergent.*

*Proof.* Indeed, by the previous proposition  $\mathrm{Ind}^{\mathrm{M|B}}(P) \otimes_{C(B)} \alpha$  is represented by the index class of a  $G$ -transversally elliptic pseudodifferential operator on  $M$ . By [1], we know that the distributional index of a  $G$ -transversally elliptic operator  $Q$  is tempered on  $G$  and that it is totally determined by its multiplicities, that is to say  $\mathrm{Ind}^M(Q) = \sum_{V \in \hat{G}} m_Q(V) \chi_V$  is defined as distribution.  $\square$

## A Some results on families of pseudodifferential operators

In this appendix, we recall some classical results on families of pseudodifferential operators. We refer the reader to [23, 24, 25, 26, 27, 28, 6].

For the following two well known results see [15, Theorem 2.2.1 and Proposition 2.2.2.] (see also [29, Theorem 6.2 and Proposition 6.1] and [21, Theorem 3.2 and Proposition 3.3]) for the standard results.

**Theorem A.1.** [21, Theorem 3.2] *Let  $P$  be a  $G$ -invariant family of pseudodifferential operators of order 0. Assume that its principal symbol  $\sigma_P$  is bounded at infinity in the vertical cotangent direction by a constant  $C > 0$ , i.e.  $\forall K \subset M$  compact*

$$\overline{\lim_{\substack{\xi \rightarrow \infty \\ \xi \in T^V M^*}}} \|\sigma_P(x, \xi)\| := \lim_{t \rightarrow \infty} \sup_{\substack{|\xi| \geq t \\ x \in K}} \|\sigma_P(x, \xi)\| < C.$$

*Then there exist a  $G$ -invariant family of pseudodifferential operators  $S$  of order 0 and a  $G$ -invariant family of integral operators  $R$  with continuous kernel such that  $P^*P + S^*S - C^2 = R$ .*

**Proposition A.2.** [21, Proposition 3.3] *Let  $Q$  be a  $G$ -invariant family of pseudodifferential operators of order 0 such that  $Q^* = Q$ . Assume that its principal symbol  $\sigma_Q$  satisfies that for any*

compact subset  $K \subset M$ :

$$\lim_{\substack{\xi \rightarrow \infty \\ \xi \in T^V M^*}} \operatorname{Re}(\sigma_Q(x, \xi)) := \lim_{t \rightarrow 0} \inf_{\substack{|\xi| \geq t \\ x \in K}} \operatorname{Re}(\sigma_Q(x, \xi)) > 0.$$

Then there exists a  $G$ -invariant family of operators  $S$  of order 0 such that

$$R = S^* S - Q$$

have continuous kernel.

**Lemma A.3.** *Let  $R$  be a continuous family of integral operators with continuous kernels.*

1. *The family  $R$  extends to a bounded operator on  $\mathcal{E}$ .*
2. *We have  $R \in k(\mathcal{E})$ .*

**Proposition A.4.** *Let  $P_0 : C^{\infty,0}(M, E^+) \rightarrow C^{\infty,0}(M, E^-)$  be a family of  $G$ -invariant pseudodifferential operators of order 0. We denote by  $P : C^{\infty,0}(M, E) \rightarrow C^{\infty,0}(M, E)$ , the family of operators  $\begin{pmatrix} 0 & P_0^* \\ P_0 & 0 \end{pmatrix}$ . The family  $P$  extends to a  $G$ -invariant bounded operator of  $\mathcal{L}_{C(B)}(\mathcal{E})$ .*

**Theorem A.5.** *Let  $P$  be a family of pseudodifferential operators of order 0. Assume that*

$$\overline{\lim_{\substack{\xi \rightarrow \infty \\ \xi \in T^V M^*}}} \|\sigma_P(m, \xi)\| = 0$$

*then  $P \in k(\mathcal{E})$ .*

**Corollary A.6.** *Let  $P$  be a family of negative order. Then  $P$  extends to a compact operator on  $\mathcal{E}$ .*

## References

- [1] M.F. Atiyah. *Elliptic operators and compact groups*. Lecture notes in mathematics. Springer Verlag, 1974.
- [2] M.F. Atiyah and I.M. Singer. The index of elliptic operators I. *Ann. Math.*, 87:484–530, 1968.
- [3] M.F. Atiyah and I.M. Singer. The index of elliptic operators IV. *Ann. Math.*, 93:119–138, 1971.
- [4] Saad Baaj and Pierre Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les  $C^*$ -modules hilbertiens. *CR Acad. Sci. Paris Sér. I Math*, 296(21):875–878, 1983.
- [5] Alexandre Baldare. The index of  $G$ -transversally elliptic families in cohomology. 2018.
- [6] Alexandre Baldare. *Théorie de l'indice pour les familles d'opérateurs  $G$ -transversalement elliptiques*. PhD thesis, Université de Montpellier, 2018.
- [7] Nicole Berline and Michèle Vergne. The Chern character of a transversally elliptic symbol and the equivariant index. *Inventiones mathematicae*, 124(1):11–49, 1996.
- [8] Nicole Berline and Michèle Vergne. L'indice équivariant des opérateurs transversalement elliptiques. *Inventiones mathematicae*, 124(1):51–101, 1996.

- [9] A. Connes and G. Skandalis. The longitudinal index theorem for foliations. *Publications RIMS Kyoto Univ*, 20:135–179, 1984.
- [10] Jacques Dixmier and Adrien Douady. Champs continus d’espaces hilbertiens et de  $C^*$ -algèbres. *Bulletin de la Société mathématique de France*, 91:227–284, 1963.
- [11] Jeffrey Fox and Peter Haskell. The index of transversally elliptic operators for locally free actions. *Pacific Journal of Mathematics*, 164(1):41–85, 1994.
- [12] Michel Hilsum. Fonctorialité en  $K$ -théorie bivariante pour les variétés lipschitziennes. *K-theory*, 3(5):401–440, 1989.
- [13] Michel Hilsum. Bordism invariance in  $KK$ -theory. *Mathematica Scandinavica*, 107(1):73–89, 2010.
- [14] Michel Hilsum and Georges Skandalis. Morphismes  $K$ -orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes). 20(3):325–390, 1987.
- [15] Lars Hörmander. Fourier integral operators. I. *Acta Mathematica*, 127(1):79–183, 1971.
- [16] P Julg. Indice relatif et  $K$ -théorie bivariante de Kasparov. *CR Acad. Sci. Paris*, 307:243–248, 1988.
- [17] Pierre Julg.  $K$ -théorie équivariante et produits croisés. *CR Acad. Sci. Paris Sér. I Math*, 292:629–632, 1981.
- [18] Pierre Julg. Induction holomorphe pour le produit croisé d’une  $C^*$ -algebre par un groupe de Lie compact. *CR Acad. Sci. Paris Sér. I Math*, 294(5):193–196, 1982.
- [19] Gennadi Kasparov. THE OPERATOR  $K$ -FUNCTOR AND EXTENSIONS OF  $C^*$ -ALGEBRAS. *Mathematics of the USSR-Izvestiya*, 16(3):513, 1981.
- [20] Gennadi Kasparov. Equivariant  $KK$ -theory and the Novikov conjecture. *Inventiones mathematicae*, 91(1):147–202, 1988.
- [21] Gennadi Kasparov. Elliptic and transversally elliptic index theory from the viewpoint of  $KK$ -theory. *Journal of Noncommutative Geometry*, 10(4):1303–1378, 2016.
- [22] E.C. Lance. *Hilbert  $C^*$ -Modules: A Toolkit for Operator Algebraists*. Lecture note series / London mathematical society. Cambridge University Press, 1995.
- [23] Robert Lauter, Bertrand Monthubert, and Victor Nistor. Pseudodifferential analysis on continuous family groupoids. *Doc. Math*, 5:625–655, 2000.
- [24] Robert Lauter, Bertrand Monthubert, and Victor Nistor. Spectral invariance for certain algebras of pseudodifferential operators. *Journal of the Institute of Mathematics of Jussieu*, 4(3):405–442, 2005.
- [25] Robert Lauter and Victor Nistor. *Pseudodifferential analysis on groupoids and singular spaces*. Johannes-Gutenberg-Univ., Fachbereich Mathematik, 1999.
- [26] Bertrand Monthubert and François Pierrot. Indice analytique et groupoïdes de lie. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 325(2):193–198, 1997.
- [27] Victor Nistor, Alan Weinstein, and Ping Xu. Pseudodifferential operators on differential groupoids. *Pacific journal of mathematics*, 189(1):117–152, 1999.
- [28] Alan L. T. Paterson. The equivariant analytic index for proper groupoid actions, 2007.
- [29] M.A. Shubin. *Pseudodifferential Operators and Spectral Theory*. Pseudodifferential Operators and Spectral Theory. Springer Berlin Heidelberg, 2001.
- [30] G. Skandalis.  *$C^*$ -algèbre, algèbre de von Neumann, exemples*.