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► **To cite this version:**

Andreas Bergwall. Conceptualizing reasoning-and-proving opportunities in textbook expositions: Cases from secondary calculus. CERME 10, Feb 2017, Dublin, Ireland. hal-01873227

HAL Id: hal-01873227

<https://hal.science/hal-01873227>

Submitted on 13 Sep 2018

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Conceptualizing reasoning-and-proving opportunities in textbook expositions: Cases from secondary calculus

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Several recent textbook studies focus on opportunities to learn reasoning-and-proving. They typically investigate the extent to which justifications are general proofs and what opportunities exist for learning important elements of mathematical reasoning. In this paper, I discuss how a particular analytical framework for this might be refined. Based on an in-depth analysis of certain textbook passages in upper secondary calculus textbooks, I make an account for analytical issues encountered during this process and identify aspects of reasoning-and-proving in textbooks that might be missed unless the framework is refined. Among them there are characterizations of generality, use of different representations, logical and mathematical structure, and ordering of material and student activities. Finally, implications beyond textbook research are discussed.

Keywords: Reasoning-and-proving, mathematics textbook, upper secondary calculus.

Introduction and background

Almost two decades ago, Hanna and de Bruyn (1999) pointed out that textbook research with specific focus on reasoning and proving was rare. Even though a number of papers with such a focus have been published in prominent journals since then, the field is still young. While the ultimate goal is to come up with well-founded prescriptions for textbook design, research is still striving to describe the current state of the art for reasoning-and-proving in textbooks (Stylianides, 2014).

Several studies have focused on (potential) opportunities to learn reasoning-and-proving (RP). Textbooks from different stages in mathematics education, from different educational contexts, and from different content areas have been studied (e.g., Davis, Smith, Roy, & Bilgic, 2014; Nordström & Löfwall, 2006; Otten, Gilbertson, Males, & Clark, 2014; Stacey & Vincent, 2009; Stylianides, 2009; Thompson, Senk, & Johnson, 2012). They typically include one or several of the following aspects of RP: generality (are statements justified with proofs or specific cases?), elements of proof-related reasoning (are students asked to make and investigate conjectures, find and correct errors, design counter examples?), proof methods (direct, indirect, by contradiction), purposes of proof (conviction, verification, discovery etc.), levels of formalism, and mathematical structure.

The variety of analytical frameworks developed for textbook studies can make it difficult to compare findings. However, some researchers have purposefully chosen to use frameworks and methods developed by others. For instance, the framework by Thompson et al. (2012) has been used with slight modification by Otten et al. (2014) and Bergwall and Hemmi (2017), and it was the basis for Bergwall (2015). Their framework is similar to the one developed by Stylianides (2009), which also has been used by Davis et al. (2014). While this simplifies comparison of findings, there is a risk that certain aspects of RP always are missed in the analysis. The purpose of this paper is to examine such potential aspects in relation to the framework by Thompson et al. (2012) and to contribute to a more refined conceptualization of opportunities to learn RP in mathematics textbooks.

Theory and analytical framework

Mathematics textbooks are widely used in classrooms around the world and are important links between national curricula and student learning (e.g., Stein, Remillard, & Smith, 2007). Tasks and expository sections, as they appear in a textbook, are potential sources for opportunities to learn RP. The concept of RP goes beyond formal proof and includes proving elements such as developing, outlining, or correcting an argument; deriving a formula; making or testing a conjecture; and providing a counterexample.

In this paper, I will focus on opportunities to learn RP through justifications in expository sections. I will use the framework and analytical procedure by Thompson et al. (2012). They employ a four item framework for justifications: A general proof is named a *general justification* (G); a deductive justification based on a generic case is named a *specific justification* (S); if the authors explicitly ask the student to provide a rationale it is referred to as *justification left to student* (L); and otherwise there is *no justification* (N). As in Bergwall and Hemmi (2017), I include all non-proof arguments in the S-category.

Stylianides (2009) uses a more refined framework with a separate category for specific justifications that are not generic. Otten et al. (2014) made modifications to the framework by Thompson et al. (2012) and distinguish between specific and general statements. They also have additional categories for justifications that only outline the general proof and for justifications that can be found in past or future lessons. We have adopted Thompson et al. (2012)'s methodology for the present and other studies (Bergwall, 2015; Bergwall & Hemmi, 2017). It has been put forward that mathematics education research needs more of cumulative research (Lesh & Sriraman, 2010) and we want to compare with – and build on – Thompson et al.'s extensive results on US upper secondary textbooks.

Textbook sample and analytical procedure

Cases for the present paper are chosen from the two most commonly used textbooks in Sweden and the only Finnish textbook available in Swedish (for Finland's Swedish speaking minority): Alfredsson, Bråting, Erixon, and Heikne (2012); Szabo, Larson, Viklund, Dufåker, and Marklund (2012); and Kontkanen, Lehtonen, Luosto, Savolainen, and Lillhonga (2008). I refer to them as SW1, SW2, and FI1 respectively.

In Bergwall and Hemmi (2017), we report our findings from an analysis of all expository sections and students' tasks on integral calculus in these textbooks (and others). In that study, we identified all mathematical statements presented as results and categorized their justifications using the framework described above. Like Thompson et al. (2012), we also checked if there were opportunities for the students to conjecture the result, how the statements were labeled, and what proving methods were used. Like researchers always do during such processes, we encountered a number of analytical difficulties. In the present paper, I will focus on these difficulties and on other issues that became apparent when the textbooks were compared to each other. I consider them a relevant base for discussing the development of frameworks for RP opportunities.

An upper secondary textbook cannot present a general theory for integral calculus. Thus its authors face the problem of what kind of justifications to include. This makes this topic relevant when examining frameworks for opportunities to learn RP. I will illustrate my findings with an analysis of the sections where students first encounter the definition of primitive function, the statement of the

representation formula $F(x) + C$ for all primitive functions to F' , and the justification of this result. This particular choice was made since it includes a complete definition-theorem-proof chain for a central concept and a non-trivial result. Furthermore, the textbooks present this particular content quite differently.

Analysis and results

The analysis and results are presented as follows. I give a condensed description of how each textbook treats primitive functions, following the chronology of that textbook. This description will include all details needed to: (1) make an analysis according to the Thompson et al. (2012) framework, (2) describe analytical difficulties, and (3) make my points about the need to further develop the framework. Aside from the textbook's *definition*, *justification* and *statement*, I describe material placed immediately *before*, *after*, and *in between* them if such exists. This is followed by my *analysis* and description of *analytical difficulties* and *other issues*. Finally, I make a short *summary* of aspects of RP opportunities that could be better incorporated in the framework.

For easier reference, the descriptions of the justifications are presented as numbered lists. Note that the representation formula can be expressed as an equivalence. Therefore the (trivial) statement that $F(x) + C$ is a primitive function to $F'(x)$ will be referred to as 'the sufficiency', while the (non-trivial) statement that *all* primitive functions have this form is referred to as 'the necessity'.

SW1 (Alfredsson et al., 2012, pp. 173-174)

Before. There is one exercise where the student, based on graphical representations, shall identify which function has a certain derivative, and another where the student shall draw two different graphs with the same derivative. This is followed by a short note that it now is time to turn the problem of finding the derivative around.

Definition. The following text is framed and labelled 'Primitive function': "A function F is called a primitive function to f if $F'(x) = f(x)$."

In between. The authors write about three questions that need to be answered: How to find *one* primitive function, *all* primitive functions, and *the* primitive function satisfying a certain condition?

Justification.

1. x^2 and $x^2 + 5$ are presented as examples of functions with derivative $2x$ and the reader is told that "whatever constant C we add to x^2 we get a primitive function to $f(x) = 2x$ ".
2. There are plots of the graphs to $x^2 + 1$, x^2 , $x^2 - 1$ and $x^2 - 2$, and the authors write: "Obviously, graphs to functions with the same derivative must for every x -value have the same slope. Hence the graphs have the same form, they are only translated in the y -direction".
3. The authors continue: "This means that if $f(x) = 2x$ then every function $F(x) = x^2 + C$, where C is a constant, is a primitive function to $f(x)$ ".
4. The authors ask if there are other functions with derivative $2x$ and immediately answer that it can be proven that there are no such functions.

Statement. The following text is framed and labelled 'Summary': "If $F(x)$ is a primitive function to $f(x)$ then $F(x) + C$, where C is a constant, denotes all primitive functions to $f(x)$ ".

After. There are two worked examples illustrating how primitive functions are determined, a table with some elementary primitive functions and then a student exercise set.

Analysis. (1) provides two specific cases for the sufficiency (x^2 and $x^2 + 5$), and it is said in words (without explanation) that any additive constant works. The necessity is touched upon in (2). This might be meant as an intuitive argument. But it is merely a formulation in words of the statement itself with no further warrants for the conclusion. The authors also chose to return to the sufficiency in (3) before they return to the necessity in (4), but once again without any argument. This means that in relation to the framework by Thompson et al. (2012) the sufficiency is justified with a specific case (S) and that there is no justification (N) for the necessity.

Analytical difficulties. The first difficulty was to decide if this justification should be counted as one or two. In Bergwall and Hemmi (2017), we chose the second alternative. However, if the unit of analysis is the justification of the statement as it is formulated in the textbook one could also choose the first. Then there are at least two alternatives: the justification receives the code N (since there are not justifications for both directions) or the code S (since there is a specific case justification for at least one direction).

The second difficulty was whether (2) should be counted as an intuitive justification of the necessity and receive the code S instead of N, since it seems to have a convincing purpose.

Other issues. Even a specific case such as $x^2 + C$ has some generality to it: the identity $(x^2 + C)' = 2x$ holds for *all* x . This indicates that when dealing with functions there is room for a more nuanced way of describing justifications than the categories G and S admit. Also, if the textbook statement had been that $x^2 + C$ denotes all primitive functions to $2x$, then the justification offered for the sufficiency is a general proof.

Summary. The analytical framework/method should be developed to better account for opportunities to learn: *the difference between an equivalence and an implication and how such are justified; the roles of different kinds of non-proof justifications, such as intuitive arguments based on visual impressions from a drawing of an “arbitrary” case; and that justifications can be specific in different ways when statements include several kinds of variables (dependent and independent), and that whether a justification is general or not also depends on how general the statement is.*

SW2 (Szabo et al., 2012, pp. 154-155)

Before. The authors demonstrate how velocity can be obtained by differentiating the distance function and then state that the opposite problem can be solved by asking which function has a certain derivative. In the margin there is a table with some elementary derivatives.

Definition. The following text is framed and labelled ‘Primitive function’: “A function F is a primitive function to f if $F'(x) = f(x)$.”

Justification.

1. x^2 , $x^2 + 5$, and $x^2 - 4$ are given as examples of functions with derivative $2x$ and in the margin it is emphasised that the derivative of a constant term is 0.
2. The authors write: “You can add and subtract any constant to a primitive function without altering its derivative. Thus a given function has an infinite number of primitive functions”.

Statement. The following text is framed and labelled ‘All primitive functions’: “If $F'(x) = f(x)$ then $F(x) + C$, where C is a constant, gives all primitive functions to $f(x)$.”

After. There are two worked examples illustrating how primitive functions are determined followed by a student exercise set.

Analysis. The sufficiency is justified with three specific functions in (1). That any constant C can be added/subtracted is explained in (2). However, it is not clear if the first sentence cited in (2) refers to a primitive function to *any* function or to a primitive function to $2x$. In the former case, the argument could have been expressed symbolically as $(F(x) + C)' = F'(x)$, which most teachers and mathematicians would have accepted as a proof. In the latter case, the sufficiency is only justified with a specific case. Concerning the necessity, there is neither a justification nor a remark that there is something more to prove. Summing up, this means that there is an ambivalence concerning the sufficiency ((S) or (G)) and that there is no justification (N) for the necessity.

Analytical difficulties. The question arises whether (2) is a general proof or not. There are two issues here: The use of words instead of algebraic symbols, and clarity in what the authors refer to.

Other issues. When comparing SW1 and SW2, we see at least three differences even though the classifications of the justifications are the same. First, SW1 discusses the necessity and states that it can be shown that there are no other primitive functions, which SW2 does not. But neither textbook clearly expresses the representation formula as an equivalence. Second, SW1 uses graphic representations and describes the meaning of the statement in terms of slope and form which SW2 does not. Third, SW2 is less vague in its labelling and formulations. While SW1 labels the statement “summary” and expresses that $F(x) + C$ “denotes” all primitive functions, SW2 uses the label “All primitive functions” and expresses that $F(x) + C$ “gives” all primitive functions.

Summary. The analytical framework/method should be developed to better account for opportunities to learn: *what needs to be justified, what has been left out of a certain justification, or if a justification is a proof or not; the role of different forms of representations; and the structure of mathematics, i.e. what part of a mathematics text that is a definition, a statement, and a proof, and what their different roles are.*

FI1 (Kontkanen et al., 2008, pp. 7-8)

Definition. The following text is framed and labelled ‘Primitive function’: “Assume that the functions f and F are defined in the open interval I . The function F is a primitive function to f for every $x \in I$, if $F'(x) = f(x)$.”

In between. In worked examples, the authors demonstrate how one checks if a certain function is a primitive function to another given function. In one of these examples, it turns out that two different functions can be primitive functions to the same function. However, the algebraic descriptions of these functions are not such that it is obvious that they only differ by an additive constant.

Statement. The following text is framed and labelled “theorem”: “Assume that F_0 is a primitive function to f . Then all functions of the type $F(x) = F_0(x) + C$ are primitive functions to f . The function f has no other primitive functions.”

Justification. The justification is labelled “proof” and divided in two steps. First the sufficiency is justified by differentiation of $F(x) = F_0(x) + C$. Then the necessity is justified using the fact that if

a derivative is 0 everywhere the function is constant. For this fact, there is a reference to a theory section at the end of the book.

After. It is pointed out and illustrated in a diagram that the additive constant C corresponds to a vertical translation of the graph. The notation $\int f(x)dx$ is introduced. This is followed by three worked examples on calculation of primitive functions and a set of student exercises.

Analysis. The sufficiency and the necessity are both justified with general proofs (G).

Analytical difficulties: There are none that have not been mentioned so far.

Other issues: In FI1 it is clear that the statement contains two parts even though it is not formulated as an equivalence. The justification is labelled proof (SW1 and SW2 have no labels on their justifications). The justification comes after the statement (not before as in the Swedish books). There is a graphical interpretation of the statement but it is put after the proof (not before as in SW1) and it seems to have the purpose of illustrating the meaning of the statement (and not to justify it as in SW1). FI1 is the only textbook that emphasizes that being a primitive function actually is a global property (i.e. that $F'(x) = f(x)$ should hold for *all* x in an interval). However, as in SW1 and SW2 the definition is phrased using the word ‘if’ even though it should be interpreted as ‘if and only if’.

SW1 and SW2 have activities and/or worked examples before the definition which together with their justifications give the student an opportunity to discover and conjecture the statement. In FI1 the section starts with the definition. The indefinite integral notation is used throughout FI1 but is completely avoided in SW1 and SW2.

Summary. The analytical framework/method should be developed to better account for opportunities to learn: *mathematical formalism, detail and notation; different purposes with different forms of representation; the conjecturing as well as the verifying nature of mathematical work; and the importance of clear definitions.*

Discussion

When opportunities to learn RP are studied in textbooks there are several aspects to take into account and there is always a risk that important aspects are left out. The examples mentioned above illustrate a number of such aspects identified when a specific analytical framework was applied to a few textbook passages on primitive functions. Here I chose to discuss the importance of four such aspects of RP and their relevance in a refined framework for RP.

The first aspect is *generality* and relates to opportunities to learn what makes a justification a proof. Students’ difficulties with understanding the difference between a general proof and an example are well-established (e.g., Harel & Sowder, 2007). However, justifications can have different levels of generality, or ‘scope of variation’, which opens up for sub-categories of non-proof justifications (e.g., Bergwall, 2015). Also, a justification must be judged in relation to the statement’s formulation and the level of detail in relevant definitions. Thus an analysis of textbook justifications should include an analysis of statements (which Otten et al. (2014) do) and definitions.

The second aspect concerns *forms of representation* and relates to opportunities to learn how proofs are communicated. Sometimes a justification is better expressed in words but often algebraic symbols bring more precision and detail to the argument. Graphical representations may be used to illustrate

meaning as well as the idea behind an argument. Frameworks should take the use of different forms of representation and their roles and purposes into account.

The third aspect is *structure* and relates to opportunities to learn the role of proof in mathematical theory. Here I include the logical structure of individual definitions, statements and justifications as well as the overall structure of the mathematical theory, with its definitions, theorems and proofs, and the connections between them. To some extent this is captured in an analysis of labeling (as in Thompson et al. (2012)) and references to other lessons (as in Otten et al. (2014)).

The fourth aspect is about *ordering* of the material, including student exercises and worked examples, and relates to opportunities to learn different purposes of proof, and to how justifications can serve different educational purposes. Student investigations, specific cases and intuitive arguments placed before a statements can emphasize the creative and conjecturing side of mathematical work, while formal general proofs placed after the statement can emphasize the verifying and organizing side.

All four aspects have one thing in common. They concern proofs and justifications as objects and not only as processes (e.g., Sfard, 1991). To analyze if textbooks offer opportunities to understand proofs and justifications as objects, the analytical frameworks and methods need to focus on opportunities to learn object properties of proofs and justifications. Generality, forms of representation, structure, and ordering are examples of such properties.

Finally, development of frameworks and methods that better capture important aspects of RP are of importance not only for textbook analysts and textbook authors. Similar frameworks can be used for analyzing lecture scripts and teaching episodes. Hence they can also aid teachers when they plan their lectures and teaching elements. A detailed framework risks being of limited analytical use but is an important contribution when conceptualizing opportunities to learn RP.

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