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Cognitive unity of theorems, theories and related rationalities

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The construct of cognitive unity of theorems was introduced twenty years ago to identify suitable conditions for students' "smooth" approach to proving. In this paper the Habermas' construct of rationality, adapted to mathematics education in previous research, is used to identify some factors in the activation of cognitive unity of theorems. In particular, I consider the dependence of cognitive unity on the specific rationality (e.g. analytic geometry rationality, or synthetic geometry rationality) according to which a conjecturing and proving problem is dealt with. The analysis of some examples will provide evidence for it, together with hints for further research.

Keywords: Theorems, conjecturing and proving, cognitive unity, proving as rational behavior.

Introduction

"Cognitive unity of theorems" (CUTHE) is a construct introduced in Garuti, Boero, Lemut & Mariotti (1996) to account for a phenomenon detected in a grade 8 (13-years-old students) classroom engaged in a conjecturing and proving activity, concerning a theorem of space geometry contextualized and verbally expressed in terms of Sun rays (instead of straight lines) and Sun shadows (instead of shapes projected on a plane according to parallel projection rules). The conjecturing task (see Garuti et al, 1996) may be shortly reported this way: "Is it possible that the Sun shadows of two non-parallel sticks are parallel on the ground? If yes, under which conditions?". After comparison and standard re-phrasing ("if… then…") of their conjectures, students were asked to validate their statements by "general reasoning". We observed that, while trying to validate their conjectures, several students resumed some pieces of personal reasoning (e.g. ways of looking at the Sun rays and the Sun shadows) developed during the production of the conjecture and the search for reasons for its validity, and arranged them in a deductive chain of statements. The ways of looking at the Sun rays and the space relationships had been different for different students; those ways corresponded to the different ways of proving the theorem by them. After having found other theorems (in geometry, and in elementary arithmetic) for which students behaved in a similar way, we defined "cognitive unity of theorem" (CUTHE) what happens for some theorems when:

> during the production of the conjecture, the student progressively works out his/her statement through an intensive argumentative activity functionally intermingled with the justification of the plausibility of his/her choices. During the subsequent statement-proving stage, the student links up with this process in a coherent way, organizing some of the previously produced arguments according to a logical chain (Garuti, Boero, & Lemut, 1998, p. 345).

The CUTHE construct was also extended to the case of the relationships between the exploratory phase of proving a theorem, and the subsequent construction of a proof for that theorem (Garuti et al., 1998): indeed, the exploratory phase of proving shares some common aspects with conjecturing (as re-construction of the meaning, and appropriation, of a statement; and identification of elements for its validity). The construct of cognitive unity resulted in various research developments. Pedemonte (2007, 2008) performed studies in which (given a theorem for which CUTHE is accessible to students) the mechanism of arranging arguments produced in the exploratory phase

does not result in a proof for some students, due to their difficulty of re-arranging inductive or abductive arguments into deductive arguments. These difficulties are not likely to emerge in algebraic conjecturing and proving (Pedemonte, 2008), while they frequently emerge in the case of plane geometry (Pedemonte, 2007). Leung and Lopez Real (2003) investigated CUTHE in the case of computer-based learning environments, which change the nature of students' exploration and make CUTHE difficult to activate, finding out ways of activating it in the new situation. Fujita, Jones and Kunimune (2010) studied conditions under which CUTHE may be activated in the field of synthetic geometry: they "analyze the circumstances when students *unite*, or not, their conjecture production and proof construction"; the potential of geometrical constructions for the activation of CUTHE was explored. The quoted studies suggest the opportunity of investigating the conditions for the activation of CUTHE for a given theorem. Boero et al. (1998) started a discussion on it, taking into account both the student (her skills, her knowledge and expertise in a given field of mathematics) and the field of mathematics in which a given statement is dealt with. Douek (1998) analyzed the individual variety of exploration strategies and their effects on conjecturing and proving; at present (personal communication) she is further deepening the idea of subject-relativity of CUTHE, together with the relationships between the quality of student's exploration (including its semiotic features) and the construction of the proof. In this paper I will try to identify some aspects of the relativity of CUTHE referred to a given system of discursive practices that concern the truth of statements, the ways of producing and validating them, and the ways of communicating with others - i.e. a "rationality", according to Habermas (1998).

Theoretical assumptions

Mathematical theory

It is possible to define a mathematical theory (shortly, a theory) by its characteristic components: primitive notions, and definitions related to them; postulates; inference rules to get true statements from the postulates and other statements proved as true. 'Characteristic components' depend on the historical period and, in a given historical period, on epistemological assumptions that may be different, according to different fields of mathematics. The case of Euclidean geometry before and after Hilbert's Grundlagen der geometrie is a well-known paradigmatic example of historical change in the ways of considering the requirements of a mathematical theory. In this paper, we will consider the following theories: Synthetic geometry (in particular, Euclidean geometry); Analytic geometry (including the algebraic treatment of conic sections); Elementary, verbal-semantic number theory (evidence for truth and inference rules rely on properties of the concept of number and its representations); Elementary, algebraic-formal number theory (evidence for truth of statements comes from the interpretation of an algebraic expression derived, through suitable syntactic transformations, from the algebraic expression which represents the problem situation).

Theorem

Mariotti (2001) defines a theorem as a statement and its proof with reference to a theory (and related inference rules). The definition results in the possibility of considering different theorems with the same statement (in particular, when different proofs referring to different theories are available). The definition encompasses theorems related to various kinds of theories and related inference rules: Euclid's as well as Hilbert's geometry; analytic geometry; graph theory, with its

reference to visual objects; 19th-century probability theory as well as Kolmogorov's axiomatic theory, etc.; and the different ways of considering proof since the Greeks, including verbal-semantic proofs (like in Euclid) and modern algebraic-formal proofs of arithmetic statements.

CUTHE and Habermas' rationality

In this paper we are interested in CUTHE, one possible aspect of the conjecturing and proving process, in order to identify for which theorems (in Mariotti's sense) it may be easily activated, thus we need a comprehensive frame to deal with the process of proving and its relationships with the product (proof) to be built up in a given theory. Habermas (1998, pp. 310–317) deals with the complexity of discursive practices according to three *interrelated* components, concerning: knowledge at play, and the answer to "why is it true" questions in a given cultural context (*epistemic rationality*); action and its goals, and strategies to achieve them, to be evaluated (*teleological rationality*); communication and related, intentional choices in a given social context on a given subject (*communicative rationality*). In Boero & Planas (2014) a detailed elaboration of the reasons for adapting Habermas' construct to mathematics education is presented, with references to how it has been used in different studies. In the case of proof and proving, according to Mariotti's definition of theorem, the adaptation of the Habermas' construct concerns:

- criteria for validity of inferences and truth of statements within a theory, and their dependence on historical periods, mathematical domains, and institutions and cultures. Inferences may rely on visual evidence, or conceptual meaning, or syntactic transformations, etc.;
- problem solving strategies that may be adopted to reach the goal of proving, along with their effectiveness; strategies may use analogies, abduction, and so on. Strategies and exploration are not constrained within the border of the reference theory;
- the choice and use of appropriate communication means for proof in a given context,

together with the relationships among them, taking into account the goal of the proving process — a proof, conforming to requirements specified for the first and the third components. The expression 'rationality frame' will be used to put into evidence the system of epistemic constraints, strategies and forms of communication, which works as reference for proving and proof in a given theory.

Examples

Moving to the school, the role of the following examples is to provide evidence for the hypothesis that CUTHE depends, for the same statement, on the specific rationality frame in which a conjecturing and proving problem is dealt with by the student; and also to provide elements for further investigation. The examples will include some excerpts from students' think aloud solving processes. *Italic* is for written texts. (…) is for omitted sentences. … is for a pause in oral speech.

Example 1

The same conjecturing and proving problem was proposed in grades VIII and IX: "Consider all the products of three consecutive natural numbers. What is their GCD? Prove that it is their GCD". S-A is a grade VIII (13-years-old) student not yet familiar with the use of letters to prove:

Student S-A: *1·2·3=6* *2·3·4=24* *3·4·5= 60* *10·11·12=1320;* it is evident that 6 is the GCD of the first three products, because it is the greatest divisor of the first product and a divisor of the other products. Is it a divisor of 1320? … Yes, 1320 is

an even number divisible by 3 because the sum of its digits is a multiple of 3. Then 6 might be the divisor of all the other products too. But why? Probably, by looking at these four products, all the products are even… But why? OK, one factor is always even! Even numbers go two by two, thus among three numbers one number … one number at least is even, and they may be two, like in the case of 2·3·4. Look at, three is there! And a multiple of three is in the last product! Why? In the case of 2, multiples go two by two … In the case of 3, numbers go three by three. That is the reason! Now I try to write down the general reasoning: *The greatest common divisor is 6 because every product is divisible by 6 because every three consecutive numbers contain one even number (multiple of 2) and one multiple of 3, because multiples of 2 go two by two, and multiples of 3 go three by three* (The teacher writes the following question: *Why greatest*?) (after a while) *Because the first product is divisible by 6, and no greater divisor is there*.

S-A resumes the examples, which conjecturing was based on, to identify *general* reasons for the truth of the conjecture. The intention of proving is related to the emerging conjecture, through "But why?" self-posed questions of epistemic relevance. A narrow intertwining between epistemic, teleological and communicative components of rationality allows the student to move continuously from exploration to the production of the conjecture, to proof construction by exploiting relevant elements got during the exploration, and then to proof writing. We may consider S-A's solution as an example of CUTHE in the rationality frame of verbal-semantic elementary theory of numbers.

S-B is a grade IX student who tries to solve the problem after some classroom work (about 10 hours) on the use of letters to prove in an algebraic way. Note that he would be free (according to the didactic contract) to choose another way of solving the problem, as other schoolmates do:

Student S-B: $(n+1)(n+2)(n+3)=(n2+2n+n+2)(n+3)=(n^2+3n+2)(n+3)=n^3+3n^2+2n+3n2+9n+6$ $=n^3+6n^2+11n+6$. I do not see anything. But if I consider, for instance, 2·3·4=24 3·4·5= 60 5·6·7=210 I see that… Yes, I see that 6 is always a divisor, because I see it as 2·3, as one half of 3·4, as 6 in the products. The same for 13·14·15. (…) 24 is also divisible by 12, and by 8, but 60 is not divisible by 8, but it is divisible by 12. Let us see 210: (…) not divisible by 12, thus 6 is the only remained candidate! With algebra: $n^3+6n^2+11n+6=6(n^2+1)+n(n^2+11)$. I do not see anything. Perhaps it is not true! 16·17·18 (…) not a good counter-example! Because 18 is divisible by 6. 21·22·23= (the student uses his cellular phone to make calculations; the product is divisible by 6). Perhaps it is easier by considering: $(n-1)n(n+1)=n(n^2-1)$.… I see nothing! I am not able to prove it!

S-B tries to solve the conjecturing and proving problem in the rationality frame of elementary algebraic theory of numbers; the difficulty to produce a conjecture in that frame is overcome by moving to the rationality frame of verbal-semantic theory, where afterwards he will also try to dispel a doubt on the truth of the conjecture by considering a further, more elaborated example. Differently from S-A, no effort is addressed to find general numerical regularities that might be exploited to build up a verbal-semantic proof. In terms of rational behavior, this is an example of lack of connection between two different strategies (teleological aspect): to produce the conjecture and afterwards to provide some empirical evidence for it; and to produce a general reasoning for

proving. As a consequence, CUTHE does not work in the rationality frame where it could have been activated (verbal-semantic theory). The same happened with the other students who tried to build up an algebraic - formal proof. Note that an algebraic - formal validation of the statement may be performed either in combinatorics, or in modular arithmetic. Some schoolmates get the conjecture in the rationality frame of verbal-semantic theory of numbers, then they consider the products $(n+1)(n+2)(n+3)$ or $(n-1)n(n+1)$ and realize that in these products one number is divisible by three and at least one number is divisible by two; thus proving still relies on semantic considerations related to the number line and the positions of multiples of 2 and 3 in it, like in the case of S-A. The algebraic expression of the product is only a device to favor the transition to a general reasoning. CUTHE works thanks to the intention of finding general regularities and a proof in the same rationality frame of verbal-semantic theory, where the conjecture had been produced.

Example 2

A conjecturing and proving problem was proposed by the same teacher in grade XI, in three parallel classes, as an individual task: “Among the triangles with a given side and the same perimeter, find the triangle with the greatest area”. Those classes were familiar with conjecturing and proving in number theory (both in a verbal-semantic way and in an algebraic way), and in Euclidean geometry.

The first class at that moment was familiar only with proving in plane Euclidean geometry; according to the conjecturing style of Euclidean geometry, some students (one third of that class) got the conjecture (the solution of the problem is the isosceles triangle) by considering that, after drawing some triangles, an isosceles triangle looks as the “widest” one (students say: “the fattest”) among the drawn triangles (but three students got the conjecture of a right-angled triangle with the same considerations); a few students got the conjecture through a “limit & symmetry” consideration related to the fact that, when the triangle becomes strongly asymmetric, the surface within it becomes very “small”, if we want to keep the same perimeter. During the discussion on the produced conjectures, after disproving (through measures) the conjecture concerning the right angled triangle, some students proposed to consider another triangle with the same height of the isosceles triangle (thus with the same area), and to try to prove that its perimeter is longer than in the case of the isosceles triangle. But a rigorous proof is not easy to build up, and in fact no student built it up, in spite of a long time spent for it in the classroom, by working in small groups (and then at home as well!); a relatively easy proof needs an auxiliary construction and the use of related theorems. The exploration to get the conjecture only suggests a first step of a proving process, and does not provide the ingredients to build up the proof: CUTHE does not work.

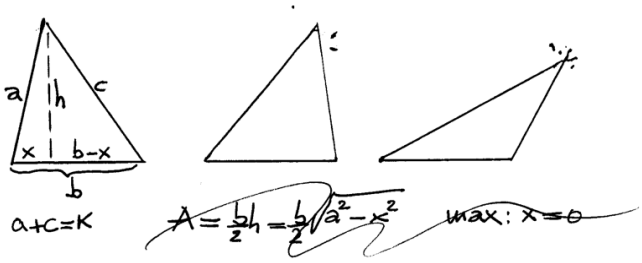
The second class had already met conic sections in synthetic geometry (they knew that an ellipse is the locus of points whose sum of distances from two given points is constant, and its basic properties concerning symmetry, axes, etc.). In this class, the conjecture was produced in a similar way as in the first class; but one fourth of students, thanks to the drawings of some triangles with approximately the same perimeter, arrived also to make a link with the ellipse in synthetic geometry. Students shared what had been discovered; then (by working in small groups) four groups out of six were able to solve the proving problem by considering the properties of an ellipse in synthetic geometry. The exploration provided students with a visual link with the ellipse in synthetic geometry, thus bridging conjecturing with proving – even if proving did not rely on the

considerations ("fatness" of triangles) that has generated the conjecture (and thus CUTHE did not work). Here is an excerpt from S-C's think aloud process:

Student S-C: (…) Now I have a reasonable conjecture. How to prove it? (student C draws three more triangles, with the same side in common with the four previously drawn triangles, and approximately the same perimeter). It is even more evident that the isosceles triangle has the largest area. But it seems to me that all those triangles have something in common. Their free edges are …Yes! I understand: the same perimeter means that the free edges are on an ellipse. Thus I may try to see if I succeed to build the proof by using the ellipse. (…)

The third class had already constructed, under the teacher's guide, the equations of a circumference and a parabola by translating into algebraic equations the characteristic conditions of those geometric loci. They had not yet met the equation of an ellipse, or the notion of an ellipse in synthetic geometry. The teacher suggested to use algebra to solve the problem. Student S-D is a representative of those students (about one third of the class) who succeeded in finding the conjecture and proving it. S-D draws three triangles with (approximately) the same perimeter:

Student S-D: I must maximize an expression for the area of the triangle, when x changes:



The maximum is when x=0. Perhaps this is the solution! But I have not considered the condition a+c=K. And what I found is … it is obvious: x=0 means the rectangle triangle. Obvious: in that case the side of length a is vertical, namely, maximum height of the triangle. But that side has always the same length. But in this problem a is related to c. I should find how to take the condition a+c=K into account. Perhaps it should be good to compare two expressions for the height of the triangle, perhaps… in order to get the area depending only on b and K.

$$h=\sqrt{a^2-x^2}=\sqrt{c^2-(b-x)^2} \qquad a=K-c \qquad \cancel{K^2-2K(K-a)-x^2=} \qquad c=K-a$$

$$a^2-x^2=K^2-2Ka+a^2-b^2+2bx-x^2 \qquad 2Ka=K^2-b^2+2bx \qquad a=\frac{K^2-b^2+2bx}{2K}$$

$$a=\frac{T+2bx}{2K} \qquad T>0 \qquad A=\frac{b}{2}\sqrt{\left(\frac{T+2bx}{2K}\right)^2-x^2}$$

Good! Given K and b, the area depends only on x (algebraic calculations follow)

$$A=\frac{b}{4K}\sqrt{T(-4x^2+4bx+T)}$$

Now it works: I see the equation of a parabola; … if x=b/2 I get the the vertex of the parabola, it means the maximum… the maximum of the area. (S-D draws an isosceles triangle) OK, it looks fine: the isosceles triangle looks as the widest one!

We may observe how (as it usually happens in analytic geometry) conjecturing and proving are dealt with at the same time, thus CUTHE works. Exploration is driven by the goal to be attained through algebra, thus the initial figures are not exploited to get a conjecture. The first trial is abandoned after interpretation of the algebraic result, the second one develops and brings to the conclusion. Epistemic control works on formalization, choice of syntactic transformations, and interpretation of results (see Morselli & Boero, 2011, pp. 455–456).

Conclusion and discussion

The aim of this paper was to elaborate the idea of cognitive unity of theorems (CUTHE) by relating it to the rationality frame available to (or chosen by) students to solve a conjecturing and proving problem. Through the examples (particularly Example 1, S-A and also Example 2, S-D) we have seen how the same statement may be produced in a particular rationality frame and then proved in the same rationality frame by exploiting some elements produced during the conjecturing phase, in a continuous process where the intention to achieve the conjecture and ascertain why it is true drives the attention of the student to relevant aspects of the problem situation, useful to build up the proof. While the same statement of Example 1 resists S-B's effort of proving it in another rationality frame. The same for the statement of Example 2 in the frame of Euclidean geometry.

This paper brings some elements of novelty in the field of research, which deals with the relationships between the exploratory phase of conjecturing and proving (or of proving a given statement), and the phase of proof construction. Through the use of the rationality construct, the hypothesis of dependence of activation of CUTHE on the theory chosen as reference for conjecturing and proving, already briefly presented in Garuti et al (1998), is further elaborated, with a counterpart in some examples from classroom activities. The rationality perspective provides a lens to compare (and distinguish between) different rationalities in mathematics, with different opportunities to validate the same statement by activating CUTHE. The chosen examples (particularly in the case of S-B if compared with S-A and with some S-B's schoolmates) also suggest to move to a deeper consideration of the relationships between the student's intention (i.e. the teleological component of her rational behavior) and the production of those elements, which might be arranged in a deductive chain in order to get a proof. Another, possible research development (related to Douek's present work) concerns a connection with what is called "semantic proof production" in Weber (2005, p. 356–357): in his reported example the student produces a visual-graphic representation of the sequence $(a_n)=(1,0,1,0,1, \ldots)$ and a horizontal band, which 'shows' that the sequence is not convergent to a limit; that "informal representation" suggests and guides "the formal inferences that (she) would draw". CUTHE does not work: elements produced during the exploration are not resumed as steps of the construction of the proof in the rationality frame of formal Calculus. But those elements allow to bridge the exploration of the proving situation with the construction of a proof in terms of the teleological component of rationality, with some analogy with the case of S-C (in Example 2); both cases suggest to widen the idea of CUTHE by including that kind of productive relationships between exploration and proof construction.

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