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Pre-service middle school mathematics teachers' interpretation of logical equivalence in proof by contrapositive

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The purposes of this study are to investigate pre-service middle school mathematics teachers' interpretations of logical equivalence in proof by contrapositive and the reasons for their incorrect interpretations. Data analysis indicated that pre-service middle school mathematics teachers were considerably unsuccessful in interpreting logical equivalence of statements. Lack of knowledge related to indirect proof methods, accepting a true statement as false, suggesting to apply direct proof instead of selecting given choices, and thinking contrapositive statements as unrelated could be regarded as the reasons for their incorrect interpretations.

Keywords: Contrapositive, logical equivalence, pre-service middle school mathematics teachers.

Introduction

Proof does not have simple roles in mathematics and mathematics education; it is a fundamental component and includes different forms and methods (Jones, 1997). A review of the literature indicated that there are limited number of studies focusing on particular proof methods (Antonini & Mariotti, 2008; Baccaglini-Frank, Antonini, Leung, & Mariotti, 2013; Bedros, 2003; Stylianides, Stylianides, & Philippou, 2004). According to Stylianides, Stylianides and Philippou (2004), the least attention has been given to proof by contrapositive compared to other proof methods such as mathematical induction, proof by contradiction, and direct proof. Thus, in this study the focus is given to proof by contrapositive. According to Bedros (2003), proof by contrapositive is a method of indirect reasoning. Since a conditional statement $p \Rightarrow q$ and its contrapositive $q' \Rightarrow p'$ are logically equivalent, in order to prove a given statement $p \Rightarrow q$, the statement $q' \Rightarrow p'$ can be proved by using direct proof (Bloch, 2000). In other words, when a statement is proved, its contrapositive is also proved (Antonini, 2004). This study focused on the logical equivalence of contrapositive statements, which is the key idea of proof by contrapositive method.

According to Baştürk (2010), students have difficulty in deciding which proof method to use and in applying the selected method. Moreover, students have many more difficulties in indirect proof methods rather than direct proof methods (Antonini & Mariotti, 2008). For example, Dickerson (2008) commented that undergraduate and graduate students have difficulty in understanding the language and logic of indirect proof methods. In the study by Stylianides, Stylianides and Philippou (2004), it was stated that some undergraduate students had difficulty in understanding logical equivalence in contrapositive and used incorrect equivalences such as $p \Rightarrow q \equiv p' \Rightarrow q'$ in their explanations. Similarly, many students could not distinguish proof by contradiction from proof by contrapositive (Goetting, 1995).

As seen, indirect proofs such as proof by contrapositive have the potential to reveal many difficulties that students possess in relation to proof (Bedros, 2003). Teachers' knowledge of proof plays an important role in developing students' understanding in proof. For instance, when mathematics

teachers present various proof methods in the class, it helps students to enhance their logical thinking and proof abilities (Altıparmak & Öziş, 2005). Therefore, mathematics teachers should have necessary knowledge and experience concerning different proof methods. Since pre-service middle school mathematics teachers are future teachers, their interpretations related to the logic of particular proof methods such as proof by contrapositive are important to investigate. Thus, to examine pre-service middle school mathematics teachers' interpretation of logical equivalence in proof by contrapositive and the reasons for their incorrect interpretations were determined as the purposes of the present study. Moreover, in the teacher education program, pre-service teachers take various mathematics courses and their ability in interpreting proof related concepts might depend on these mathematics courses since some of which place more importance on proof. In relation to this, how pre-service teachers' success levels differ by year level in the program was also investigated. By considering these purposes, the research questions were stated as follows:

1. To what extent are Turkish pre-service middle school mathematics teachers successful in interpreting logical equivalence in proof by contrapositive, and how does their success differ by year level?
2. What are the reasons for Turkish pre-service middle school mathematics teachers' incorrect interpretations?

Method

Since data were collected at just one point in time from a selected sample in order to describe certain characteristics of the population by asking questions (Fraenkel & Wallen, 2005), this study was designed as a cross-sectional survey. Using convenience sampling methods, the sample for this study was determined as 115 pre-service middle school mathematics teachers attending a state university in Ankara, Turkey. In terms of their year level, 19 were freshmen (16.5%), 25 were sophomores (21.7%), 39 were juniors (33.9%), and 32 were seniors (27.8%).

In Turkey, the middle school mathematics teacher education programs offer mathematics courses such as Calculus, Algebra; mathematics education courses involving Methods of Teaching Mathematics, Practicum; education courses such as Classroom Management; general courses involving Academic Oral Presentation Skills, and elective courses. The first two years of the program mainly consist of mathematics courses while the last two years put more emphasis on education, mathematics education, and elective courses.

This study was conducted as part of a larger study focusing on pre-service middle school mathematics teachers' interpretation of the logic behind proof methods. In this study, the answers given by pre-service teachers to three questions related to the logical equivalence of contrapositive statements were analyzed. These questions were prepared by reviewing the related literature (Knuth, 1999; Saeed, 1996). In more detail, Question 1 (Q1) and Question 2 (Q2) were prepared by the researchers by considering the format of the multiple choice questions in the study undertaken by Knuth (1999). The students were asked to select the correct statement that can be used to start to prove the given statement and explain their answers. The correct choice involves the proposition $q \Rightarrow p$ as the starting point to prove the proposition $p \Rightarrow q$ which is known as proof by contrapositive. The other choices were not appropriate to start any proof. The correct choices were identified as (d) for Q1 and (c) for Q2. Questions 1 and 2 are presented below.

<p>Q1. Assume that m and n are positive integers. If $mn=100$, then $m \leq 10$ or $n \leq 10$. To prove the given statement, which statement can you begin with?</p> <p>a) Assume that m and n are positive integers. If $m > 10$ or $n > 10$, then $mn=100$. b) Assume that m and n are positive integers. If $m \leq 10$ or $n \leq 10$, then $mn=100$. c) Assume that m and n are positive integers. If $m \leq 10$ and $n \leq 10$, then $mn \neq 100$. d) Assume that m and n are positive integers. If $m > 10$ and $n > 10$, then $mn \neq 100$. e) None of the above</p> <p>- Why? State your reasons.</p>	<p>Q2. Assume that a, b and c are real numbers and $a > b$. If $ac \leq bc$, then $c \leq 0$. To prove the given statement, which statement can you begin with?</p> <p>a) Assume that a, b and c are real numbers and $a > b$. If $c > 0$, then $ac \leq bc$. b) Assume that a, b and c are real numbers and $a > b$. If $c \leq 0$, then $ac \leq bc$. c) Assume that a, b and c are real numbers and $a > b$. If $c > 0$, then $ac > bc$. d) Assume that a, b and c are real numbers and $a > b$. If $c \geq 0$, then $ac \geq bc$. e) None of the above</p> <p>- Why? State your reasons.</p>
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Figure 1: Question 1 and Question 2

Question 3 (Q3) was adapted from the study of Saeed (1996) and involves a discussion about the proofs of two contrapositive statements. In the question, the participants were asked to select the person they agreed with and explain the reasons for their choice. The students' answers were accepted as incorrect if they agreed with Pinar and correct if they agreed with Ahmet.

<p>Q3. Statement A: If n^2 is an odd integer, then n is an odd integer. Statement B: If n is an even integer, then n^2 is even integer.</p> <p>Ahmet: I think statement A is true, Pinar. Pinar: Let me see, if $n^2=9$, then $n=\pm 3$ is odd; if $n^2=25$, then $n=\pm 5$ is odd. So, statement A seems to be true Ahmet. Ahmet: I also think that statement B is true, Pinar. Pinar: Why? Ahmet: Since n is even, then $n=2k$ where k is some integer. Therefore, $n^2 = 4k^2 = 2(2k^2)$ is also even. Pinar: But Ahmet, this only show the statement B is true, but does not show that statement A is true. Ahmet: This argument also shows that statement A is correct.</p> <p>Questions; - Considering the discussion above, who do you agree with? Ahmet _____ Pinar _____ - Why? Explain your reasons.</p>
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Figure 2: Question 3

To investigate the research questions, descriptive statistics and item-based analysis were conducted. Firstly, pre-service middle school mathematics teachers' interpretations of logical equivalence in proof by contrapositive were analyzed based on the rubric given in Table 1. Then, the reasons for their incorrect interpretations were examined qualitatively by generating themes.

	Answer types in Q1 and Q2	Answer types in Q3
	No answer	No answer
Incorrect answer	Incorrect choice was marked, no explanation was stated	Agreed with no one or both of them Agreed with Pinar, no explanation was stated
	Incorrect choice was marked, explanation was stated	Agree with Pinar, explanation was stated
Correct answer	Correct choice was marked, no explanation was stated	Agreed with Ahmet, no explanation was stated
	Correct choice was marked, explanation was given but not referring to the logical equivalence	Agreed with Ahmet, explanation was given but not referring to the logical equivalence
	Correct choice was marked, explanation was given referring to the logical equivalence	Agreed with Ahmet, explanation was given referring to the logical equivalence

Table 1: Rubric for questions

Findings

In order to investigate the first research question, pre-service middle school mathematics teachers' answers to Q1 and Q2 were analyzed. The results of 115 pre-service middle school mathematics teachers' answers are presented in Table 2.

Answer types		Question 1		Question 2	
No answer		4 (3.5%)		4 (3.5%)	
Incorrect answer	Incorrect choice was marked, no explanation was stated	12 (10.4%)	50 (43.4%)	12 (10.4%)	55 (47.8%)
	Incorrect choice was marked, explanation was stated	38 (33.0%)		43 (37.4%)	
Correct answer	Correct choice was marked, no explanation was stated	43 (37.4%)	61 (53.1%)	33 (28.7%)	56 (48.7%)
	Correct choice was marked, explanation was given but not referring to the logical equivalence	7 (6.1%)		9 (7.8%)	
	Correct choice was marked, explanation was given referring to the logical equivalence	11 (9.6%)		14 (12.2%)	

Table 2: Frequencies of the answers to Q1 and Q2

Table 2 shows that 4 students (3.5%) did not answer to Q1 and Q2. When the answers of the students to Q1 were investigated, it was seen that 50 students (43.4%) answered incorrectly and 61 students (53.1%) selected the correct choice. In addition, 43 students (37.4%) marked the correct choice without stating their reasons and the answers of 7 students (6.1%) were correct but their explanations were not related to logical equivalence. The remaining 11 students (9.6%) answered correctly by providing an explanation based on logical equivalence of contrapositive statements. In terms of year level in the program, freshmen (73.7%) had the highest percentage of correct answers and seniors (40.6%) had the lowest percentage of correct answers in Q1. As an example of a correct answer with an explanation referring to logical equivalence, Participant 52 stated as follows:

$$\begin{aligned}
 p: mn=100 & \quad p': mn \neq 100 & \quad p \Rightarrow q \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p' \\
 q: m \leq 10 \vee n \leq 10 & \quad q': m > 10 \wedge n > 10 \\
 q' \Rightarrow p' & \text{ (If } m > 10 \text{ and } n > 10, \text{ then } mn \neq 100) \text{ (Participant 52, junior)}
 \end{aligned}$$

The analysis of the answers to Q2 showed that 55 students (47.8%) answered incorrectly whereas 56 students (48.7%) answered correctly. Thirty-three students (28.7%) marked the correct choice in the question but did not substantiate their ideas. Moreover, 9 students (7.8%) answered correctly without referring to contrapositive statements, and 14 students (12.2%) answered correctly by referring to the logical equivalence of contrapositive statements. While sophomores (64.0%) had the highest percentage of correct answers, freshmen (36.8%) and seniors (37.4%) had the lowest percentages of correct answers in Q2. To illustrate, Participant 97 answered correctly and explained by referring to logical equivalence in proof by contrapositive.

$$\begin{aligned}
 p: ac \leq bc & \quad q: c \leq 0 \\
 \text{Then, proof by contrapositive, } p \Rightarrow q & \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p' \text{ (Participant 97, senior)}
 \end{aligned}$$

Since Q3 has a different rubric from the multiple choice questions, pre-service middle school mathematics teachers' answers to Q3 are presented in Table 3.

Answer types		Question 3	
No answer		4 (3.5%)	
Agreed with no one or both of them		3 (2.6%)	
Incorrect answer	Agreed with Pinar, no explanation was stated	16 (13.9%)	75 (65.2%)
	Agree with Pinar, explanation was stated	59 (51.3%)	
Correct answer	Agreed with Ahmet, no explanation was stated	5 (4.3%)	33 (28.7%)
	Agreed with Ahmet, explanation was given but not referring to the logical equivalence	24 (20.9%)	
	Agreed with Ahmet, explanation was given referring to the logical equivalence	4 (3.5%)	

Table 3: Frequencies of the answers to Q3

According to Table 3, 4 students (3.5%) did not answer Q3. The answers of 3 students (2.6%) showed that they agreed with neither Pinar nor Ahmet but did not explain their rationale. Moreover, 75 students (65.2%) agreed with Pinar, which is accepted as incorrect answer and 33 students (28.7%) agreed with Ahmet, which is accepted as correct answer. Five students (4.3%) agreed with Ahmet without giving any explanation, 21 students (20.9%) agreed with Ahmet and explained without referring to logical equivalence, and 4 students (3.5%) explained their agreement with Ahmet by referring to logical equivalence of contrapositive statements. Moreover, juniors (38.4%) had the highest percentage of correct answers and sophomores (4.0%) had the lowest percentage of correct answers to Q3. An example of a correct answer, Participant 52 agreed with Ahmet and her explanation was related to logical equivalence used in proof by contrapositive.

p : n is even q : n^2 is even

$p \Rightarrow q$ was proved

$p \Rightarrow q \equiv p' \vee q \equiv q \vee p' \equiv q' \Rightarrow p'$

Thus, if n^2 is odd then n is odd. Therefore, Ahmet is right. (Participant 52, junior)

For the second research question, pre-service middle school mathematics teachers' explanations for their incorrect answers were analyzed. As presented in Tables 2 and 3, 50 students (43.4%) answered Q1 incorrectly and 38 of them (33.0%) gave explanations for their answers. Fifty-five students (47.8%) answered Q2 incorrectly, of whom 43 (37.4%) explained their answer. Lastly, 75 students (65.2%) answered Q3 incorrectly and 59 of them (51.3%) suggested explanations for their answers. Table 4 shows the reasons behind the students' incorrect interpretations grouped under four categories.

Reasons	Q1	Q2	Q3
R1 Lack of knowledge related to indirect proof methods	30 (26.1%)	35 (30.4%)	-
R2 Accepting a true statement as false	5 (4.3%)	-	-
R3 Suggesting to apply direct proof instead of selecting given choices	3 (2.6%)	8 (7.0%)	-
R4 Thinking that contrapositive statements are unrelated	-	-	59 (51.3%)
Total	38 (33.0%)	43 (37.4%)	59 (51.3%)

Table 4: Reasons for students' incorrect interpretations

The first reason for the incorrect interpretations is students' lack of knowledge related to indirect proof methods. As a result of this inadequacy, students thought that one of the choices in the question was related to contradiction or contrapositive; however, this choice was not related to these methods.

For example, in Q2, Participant 7 selected one of the incorrect choices and explained it as an assumption for contradiction.

To prove by contradiction, we have to prove the converse situation. The choice b can be used in this situation. (Participant 7, freshman)

The second reason behind students' incorrect interpretations is that they accepted the given statement as false even though it was true and tried to find counterexamples to refute it. For instance, in Q1, Participant 114 could not see that the given statement was true.

The given statement 'Assume that m and n are positive integers. If $mn=100$, then $m \leq 10$ or $n \leq 10$.' is not true.

As counterexamples, $m=12$ and $n=12$ can be used.

Then, $mn=12 \cdot 12=144 \neq 100$

Therefore, 'if $mn=100$ then $m \leq 10$ and $n \leq 10$ ' is a true statement. (Participant 114, senior)

The third reason is that students mentioned using direct proof instead of selecting one of the given choices. For instance, the answer of Participant 106 to Q1 is given below:

Firstly, we can assume that $mn=100$; we can try to deduce $m \leq 10$ or $n \leq 10$. We cannot start with the sentences given above. (Participant 106, senior)

The last reason for incorrect interpretations is that students thought that there was no relation between the given contrapositive statements A and B. For example, in Q3, Participant 30 cited that statements A and B were different.

Because the statements are different, one of them starts with an even number and the other one starts with an odd number. The proof of statement A can't be the same with the proof of statement B. (Participant 30, sophomore)

Discussion

According to the results of pre-service middle school mathematics teachers' answers to questions, it was found that nearly half of the sample answered Q1 and Q2 correctly and almost one third answered Q3 correctly. In other words, students' achievement levels in interpreting logical equivalence in proof by contrapositive were found to be considerably low. The findings revealed that freshmen had the highest achievement level for Q1, sophomores had the highest achievement level for Q2, and juniors had the highest achievement level for Q3. Although seniors were expected to have been the most successful group by considering the number of mathematics courses they took in the program, they were not the most successful in terms of all questions. This result might stem from the fact that seniors did not take any mathematics course in their last year of the program. Therefore, seniors might not remember the details of the logical equivalence used in proof by contrapositive. To avoid this situation, teacher educators could offer elective courses related to logic and proof to enhance prospective teachers' reasoning skills.

Four reasons for the incorrect interpretations were detected from three questions. The first reason is preservice teachers' lack of knowledge related to indirect proof methods. This finding is consistent with the results of Atwood (2001), who stated that students had difficulty in using the words converse, contrapositive, contradiction, and counterexample, and that they might use them interchangeably, which is not correct. Moreover, in the case that where students generally memorize proof methods

instead of understanding the structure of the proof might cause them to have difficulty in related proof methods. Therefore, the participants in this study might use proof by contrapositive and proof by contradiction inaccurately and interchangeably. The second reason why students answer incorrectly is accepting a true statement as false and trying to find counterexamples based on this idea. Some of the terms and signs involved in the given statement in Q1 such as ‘or’ and ‘ \leq ’ might cause students to misunderstand the statement. Thus, students might have had trouble in deciding whether the given statement was true or false and evaluate it as false. The third reason is that students suggested proving the given statement with direct proof instead of selecting one of the given choices in the question. This situation may result from the fact that the majority of the proofs in the textbooks are given as direct proofs (Atwood, 2001). Therefore, students may have a tendency to use direct proofs since they are more familiar with this method. The last reason is that students thought that statements A and B given in Q3 were unrelated. In this study, students might fail to see the relation between proofs of given two contrapositive statements. Therefore, they might think that statement A which involves $p \Rightarrow q$ and statement B which involves $q' \Rightarrow p'$ should be proven separately.

In mathematics teacher education programs, proof should be considered as an important theme. Thus, the content or place of mathematics courses in teacher education programs might be revised and developed in order to enhance preservice teachers’ understanding of reasoning, proof, and logical rules behind proof methods. For example, mathematics courses might be taught by paying attention to logical rules behind proof methods. This study pointed out the importance of having knowledge of logical rules in reading and interpreting a given proof statement or conducting proof by using particular proof methods. Moreover, similar findings related to the interpretation of logical equivalence used in proof by contrapositive might be achieved with pre-service mathematics teachers in different countries. Therefore, to compare and to gain a global perspective about pre-service mathematics teachers’ understanding of logical rules behind proof methods, cross-cultural studies could be conducted. Based on the findings of such studies, teacher educators might develop strategies to overcome pre-service mathematics teachers’ current difficulties in logic and proof by considering the characteristics of their teacher education programs.

The results of the study are limited to the data collected with three questions. For further studies, pre-service middle school mathematics teachers’ interpretations of logical equivalence used in proof by contrapositive might be investigated by using alternative questions in various formats. An investigation of the effect of pre-service mathematics teachers’ knowledge of logic on their ability to prove might also be undertaken. Moreover, to analyze the answers of the pre-service mathematics teachers and to determine the reasons for their incorrect interpretations regarding logic in-depth, follow-up interviews might be conducted in future studies.

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