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Robust switching control design for uncertain discrete-time switched affine systems [★]

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Abstract

This paper focusses on the design of a robust switching control law for an uncertain discrete-time switched affine system. In order to cope with model uncertainties, a novel control law is introduced and its parameters result from an optimization problem, aiming at reducing the volume of the attractive and invariant set, where the solutions of the closed-loop systems converge to. The design is based on a quadratic Lyapunov function and guarantees global practical stability and robustness with respect to parameter variations. Our method and the associated relaxed control law are then compared with existing conditions from the literature and are validated through numerical examples.

Key words: Discrete-time switched affine systems, robust control, Lyapunov analysis, LMI.

1 Introduction

Switched systems [15] are encountered in many applications including mixing of fluids, DC-DC power conversion [24,25], event-triggered control [12], viral mutation in HIV treatment [13], mobile sensor networks, damping of vibrating structures, and several others, see [3] for references to these applications. The area of linear switched systems received wide attention. There exist many contributions in the literature aiming at guaranteeing asymptotic stabilization of the operating point of linear switched systems, see, e.g., [9,10,16,27], and also [15, Section 3.4] and [26, Chapter 5,6]. A vast literature exists on the topic and the reader may refer to [16,23] for recent overviews.

Interestingly, the class of switched affine systems, which we consider in this paper, received less attention. The set of operating points of this class of systems are given for a dynamic averaging, obtaining solutions in the generalized sense of Krasovskii. Many works found in the literature control the switched affine systems in continu-

ous time by a min-projection control strategy [2,7,8,19], and even for systems with a general nonlinear form [17,18]. In these works the provided controllers may lead to arbitrarily fast switching control. Some solutions to this problem can be found in the literature, as [4,21,25], where, the authors aim at ensuring a minimum dwell time associated with an admissible chattering around the operating point. Nevertheless, [4] proves a minimum dwell time associated to a spacial regularization. [25] is focussed on a specific electronic architecture, a boost converter and, [21] does not provide a complete stability proof. On the other hand, in [5], the authors present an open-loop stabilization strategy based on dwell-time computation, [1] proposes a minimum dwell-time with a space and time regularization, and [11] provides a minimum and maximum dwell-time by solving optimization problems. These solutions present a common characteristic: systems are controlled in an aperiodic manner.

In many occasions, the control law is implemented periodically, so that the previous references cannot apply straightforwardly. In order to deal with this issue, a solution consisting in the discretization of the continuous-time model with a fixed periodic sampling time was provided in [6,11]. The authors of [6] present a controller based on a Lyapunov function synthesized by solving an optimization problem, whose objective is to reduce the volume around the equilibrium, where the solutions converge. However, extension of these Lyapunov-based control laws to robustness is not easy to conduct. On

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the other hand, in [11], the authors design sampled-data switching control laws for a relaxed system guaranteeing robustness, and estimating a tight positive invariant set by using a Lyapunov function. One of this control solution is based on Lyapunov function. An attempt to relax these kind of Lyapunov-based controllers was given in [?]. However, this result neither guarantees an invariant set around the equilibrium, nor robustness with respect to parameter variations.

While the existing control strategies are based on the selection of the mode that minimizes the decrease of the Lyapunov function, this paper suggests an alternative and more relaxed structure of controllers, than the one presented in [?]. Moreover, the control parameters result from an optimization process. This relaxation is inspired from recent event-triggered control strategies deployed in [20,22]. The method provides a minimum invariant set with a simple computational algorithm, guaranteeing a global and practical asymptotic stabilization. It is also demonstrated that this relaxed control law includes Lyapunov-base controllers, in particular cases. We illustrate this potential, by addressing the same control problem given in [6]. For this, we show that the optimal solution given in [6] is included in our solution. In addition, the numerical results show that the solution of [6] is more conservative than our method. Another byproduct of our method relies on the possibility of providing a robust stabilization in the situation of parameter variations of uncertainties. Some simulations validate our contribution.

The paper is organized as follow. The problem formulation is stated in Section 2. The design method is presented in Section 3, and Section 4 proposes the extension of the method to achieve robustness with respect to parameter uncertainty. In Section 5 some numerical results are shown. The paper closes with a conclusion section.

Notation: Throughout the paper, \mathbb{N} denotes the set of natural, \mathbb{R} real numbers, \mathbb{R}^n the n -dimensional Euclidean space and $\mathbb{R}^{n \times m}$ the set of all real $n \times m$ matrices. The set composed by the first N positive integers, namely $\{1, 2, \dots, N\}$, is denoted by \mathbb{K} . Matrix $0_{n,m}$ denote the null matrix of $\mathbb{R}^{n \times m}$. When no confusion are possible, the subscripts of this matrix that precise the dimension, will be omitted. For any matrix M of $\mathbb{R}^{n \times n}$, the notation $M \succ 0$, ($M \prec 0$) means that M is symmetric positive (negative) definite and $\det(M)$ represents its. Co is the convex hull of a set.

2 Problem formulation

2.1 System data

Inspired by the work in [6], we focus on the following class of discrete switched affine systems:

$$z_{k+1} = A_\sigma z_k + \mathcal{B}_\sigma \quad (1)$$

where $z_k \in \mathbb{R}^n$ is the state, and A_σ and \mathcal{B}_σ present suited dimensions. The control action is performed through the switching signal $\sigma \in \mathbb{K} := \{1, 2, \dots, N\}$, which may be only modified at instant $k \in \mathbb{N}$.

This paper focuses on the design problem of a feedback law for the periodic switching signal σ , in such a way as to ensure suitable practical convergence properties of the plant state z_k to a neighborhood of the target given by z_e , which is not necessarily an equilibrium of the dynamics in (1), but can be obtained as an equilibrium for the switched system with arbitrary switching. A necessary and sufficient condition characterizing this equilibrium is then represented by the following standard assumption (see [1,6]).

Assumption 1 *There exists $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_N]$ satisfying $\sum_{i \in \mathbb{K}} \lambda_i = 1$ and $\lambda_i \geq 0$, for all $i \in \mathbb{K}$, such that the following equality holds:*

$$\sum_{i \in \mathbb{K}} \lambda_i ((A_i - I)z_e + \mathcal{B}_i) = (A_\lambda - I)z_e + \mathcal{B}_\lambda = 0. \quad (2)$$

where $A_\lambda := \sum_{i \in \mathbb{K}} \lambda_i A_i$ and $\mathcal{B}_\lambda := \sum_{i \in \mathbb{K}} \lambda_i \mathcal{B}_i$.

Remark 1 It is emphasized that Assumption 1 is both necessary and sufficient for the existence of a suitable switching signal ensuring forward invariance of the point z_e (namely inducing an equilibrium at z_e) when understanding solutions in the generalized sense of Krasovskii or Filippov. Indeed, under (2), we can conclude that the error equation of (1):

$$x_{k+1} = A_\sigma x_k + B_\sigma, \quad (3)$$

with $x_k := z_k - z_e$ and $B_\sigma := (A_\sigma - I)x_e + \mathcal{B}_\sigma$, has $x = 0$ as an equilibrium when the convex combination of Assumption 1 is used, since

$$B_\lambda := \sum_{i \in \mathbb{K}} \lambda_i B_i = 0. \quad (4)$$

┘

2.2 Control objectives

When considering such switching affine systems, asymptotic stability to zero is in general not possible. Therefore one has to relax the control objectives and to consider attractor sets, which are not necessarily reduced to the equilibrium. In this paper, we will consider an estimation of the attractor, which is defined by a level set of a Lyapunov function given by

$$V(x, x_c) = (x - x_c)^\top P(x - x_c), \quad (5)$$

where P is a positive definite matrix, and x_c is a vector that allows to shift the center of the level set. Then, we

introduce attractor given by

$$\mathcal{A} := \{x \in \mathbb{R}^n, \quad V(x, x_c) \leq 1\}. \quad (6)$$

In the sequel, the objective is to ensure the invariance and attractiveness of \mathcal{A} .

3 Switching control

3.1 Main result

In this section, we propose an novel stabilization theorem based on a relaxed control law, which notably differs from the classical Lyapunov-based min-projection control law developed in [6,14,19], for instance. This is formalized in the following theorem.

Theorem 1 For the linear combination λ related to Assumption 1 and for any given parameter $0 < \mu < 1$, assume that matrices $P \in \mathbb{R}^{n \times n} \succ 0$, $h \in \mathbb{R}^n$, and $N_i = N_i^\top \in \mathbb{R}^{(n+1) \times (n+1)}$ are the solution to the convex optimization problem

$$\min_{P, h, N_i} -\log(\det(P)), \quad (7)$$

$$\text{s.t. } P \succ 0, \quad (8)$$

$$\begin{bmatrix} \Psi_i + N_\lambda - N_i & - \begin{bmatrix} 0 & 0 \\ * & \mu \end{bmatrix} \\ * & -\mu P \end{bmatrix} \prec 0, \forall i \in \mathbb{K}, \quad (9)$$

where

$$\Psi_i := \begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix}^\top \begin{bmatrix} P & h \\ h^\top & 0 \end{bmatrix} \begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} P & h \\ h^\top & 0 \end{bmatrix}. \quad (10)$$

$$N_\lambda := \sum_{i \in \mathbb{K}} \lambda_i N_i. \quad (11)$$

Then, the switching control law given by

$$\sigma_k = \operatorname{argmin}_{i \in \mathbb{K}} \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top N_i \begin{bmatrix} x_k \\ 1 \end{bmatrix} \quad (12)$$

ensures the following properties for system (3)

- (i) if x_k is not in \mathcal{A} at a given time instant k , then the forward increment of the Lyapunov function, $\Delta V(x_k)$, is negative definite.
- (ii) The equilibrium $x = 0$ belongs to \mathcal{A} .
- (iii) If x_k belongs to \mathcal{A} at a given time instant k , then x_{k+1} also belongs to \mathcal{A} under the dynamics of (3).

□

Proof 1 *Proof of (i):* Assume that there exist a positive definite matrix P , a vector h and symmetric matrices N_i , $i \in \mathbb{K}$ that are solution to the convex problem (7), (9) for a given positive parameter $\mu \in (0, 1)$. Then, consider the Lyapunov function (5) with $x_c = -P^{-1}h$ given by

$$V(x, P^{-1}h) := (x + P^{-1}h)^\top P(x + P^{-1}h). \quad (13)$$

It is clear that the assumption on the positive definiteness of matrix P ensures the positive definiteness of V . The computation of the forward increment of the Lyapunov function is given by

$$\begin{aligned} \Delta V(x_k, -P^{-1}h) &= V(x_{k+1}, -P^{-1}h) - V(x_k, -P^{-1}h) \\ &= (x_{k+1} + P^{-1}h)^\top P(x_{k+1} + P^{-1}h) \\ &\quad - (x_k + P^{-1}h)^\top P(x_k + P^{-1}h) \\ &= x_{k+1}^\top P x_{k+1} - x_k^\top P x_k + 2(x_{k+1} - x_k)^\top h. \end{aligned}$$

Replacing x_{k+1} by its expression in (3), the forward increment of the Lyapunov function can be rewritten as follows

$$\Delta V(x_k, -P^{-1}h) = \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \Psi_{\sigma_k} \begin{bmatrix} x_k \\ 1 \end{bmatrix},$$

where $\sigma_k \in \mathbb{K}$, corresponds to the active mode at instant k and where Ψ_{σ_k} is defined in (10) with $i = \sigma_k$. In order to include the selection of the control law (12) and the fact that $x_k \notin \mathcal{A}$, two additional steps are required. First, the switching control law (12) ensures that the inequality

$$\forall x_k \in \mathbb{R}^n, \quad \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top (N_j - N_{\sigma_k}) \begin{bmatrix} x_k \\ 1 \end{bmatrix} \geq 0$$

holds for any $j \in \mathbb{K}$. Therefore, for any convex combination, and more especially for the particular case λ , we have, for any k ,

$$\begin{aligned} \sum_{j \in \mathbb{K}} \lambda_j (N_j - N_{\sigma_k}) &= \sum_{j \in \mathbb{K}, j \neq \sigma_k} \lambda_j (N_j - N_{\sigma_k}) \\ &= N_\lambda - \lambda_{\sigma_k} N_{\sigma_k} - (1 - \lambda_{\sigma_k}) N_{\sigma_k} \\ &= N_\lambda - N_{\sigma_k}, \end{aligned}$$

which, together with the previous inequality, yields

$$\sum_{j \in \mathbb{K}} \lambda_j \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top (N_j - N_{\sigma_k}) \begin{bmatrix} x_k \\ 1 \end{bmatrix} = \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top (N_\lambda - N_{\sigma_k}) \begin{bmatrix} x_k \\ 1 \end{bmatrix} \geq 0.$$

The second step is related to the fact that the negative definiteness of ΔV is only required outside \mathcal{A} , i.e. for any

$x \in \mathbb{R}^n$, such that $V(x, P^{-1}h) \geq 1$, which can be easily rewritten as follows

$$\forall x_k \notin \mathcal{A}, \quad \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \begin{bmatrix} P & h \\ h^\top & h^\top P^{-1}h - 1 \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix} \geq 0.$$

Hence, to summarize the previous computations, (i) holds if, for some parameter $\mu \in (0, 1)$ and λ satisfying Assumption 1, there exist matrices P, h and N_i such that

$$\begin{aligned} & \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \Psi_i \begin{bmatrix} x_k \\ 1 \end{bmatrix} < 0, \quad \forall (x_k, i) \in \mathbb{R}^n \times \mathbb{K} \\ \text{when} & \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \begin{bmatrix} P & h \\ h^\top & h^\top P^{-1}h - 1 \end{bmatrix} \begin{bmatrix} x_k \\ 1 \end{bmatrix} \geq 0 \\ \text{and} & \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top (N_\lambda - N_i) \begin{bmatrix} x_k \\ 1 \end{bmatrix} \geq 0. \end{aligned}$$

Using two successive S -procedures, which introduce unavoidably some conservatism, this problem is recast into the existence of a parameter $\mu > 0$ such that

$$\Psi_i + N_\lambda - N_i + \mu \begin{bmatrix} P & h \\ h^\top & h^\top P^{-1}h - 1 \end{bmatrix} < 0, \quad i \in \mathbb{K}. \quad (14)$$

The proof is concluded by noting that matrix $\begin{bmatrix} P & h \\ h^\top & h^\top P^{-1}h \end{bmatrix}$ can be rewritten as $\begin{bmatrix} P \\ h^\top \end{bmatrix} P^{-1} \begin{bmatrix} P \\ h^\top \end{bmatrix}^\top$, so that the application of the Schur complement to this term leads to condition (9).

Proof of (ii): The objective is here to prove that the linear matrix inequality (LMI) (9) ensures that the equilibrium $x = 0$ belongs to \mathcal{A} . More precisely, this means that

$$V(0, P^{-1}h) = h^\top P^{-1}h \leq 1.$$

To proceed with this proof, let us compute the linear combination of (14), weighted by λ . This yields

$$\sum_{i \in \mathbb{K}} \lambda_i \left(\Psi_i + N_\lambda - N_i + \mu \begin{bmatrix} P & h \\ h^\top & h^\top P^{-1}h - 1 \end{bmatrix} \right) < 0.$$

Using the condition $\sum_{i \in \mathbb{K}} \lambda_i = 1$ and the fact that $\sum_{i \in \mathbb{K}} \lambda_i N_i = N_\lambda$, the previous inequality leads to

$$\sum_{i \in \mathbb{K}} \lambda_i \Psi_i + \mu \begin{bmatrix} P & h \\ h^\top & h^\top P^{-1}h - 1 \end{bmatrix} < 0.$$

A necessary condition for the satisfaction of this inequality is that the last diagonal term located at position $(n+1, n+1)$ has to be negative definite. However, looking

at the expression of the Ψ 's matrices, the last diagonal term of $\sum_{i \in \mathbb{K}} \lambda_i \Psi_i$ is equal to zero. Hence, a necessary condition for (9) to hold is that $h^\top P^{-1}h - 1 < 0$, which corresponds exactly to the condition $V(0, -P^{-1}h) < 1$, concluding the proof of (ii).

Proof of (iii): From (9), we know that, for any $k \in \mathbb{N}$

$$\begin{aligned} & V(x_{k+1}, -P^{-1}h) - V(x_k, -P^{-1}h) \\ & + \mu (V(x_k, -P^{-1}h) - 1) + C_{\sigma_k} < 0, \end{aligned}$$

where C_{σ_k} refers to the controller (12) and is given by

$$C_{\sigma_k} = \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top (N_\lambda - N_{\sigma_k}) \begin{bmatrix} x_k \\ 1 \end{bmatrix} \geq 0.$$

Hence, we have

$$\begin{aligned} V(x_{k+1}, -P^{-1}h) & \leq (1 - \mu)V(x_k, -P^{-1}h) + \mu - C_{\sigma_k} \\ & \leq (1 - \mu)V(x_k, -P^{-1}h) + \mu. \end{aligned}$$

Using the assumption that x_k belongs to \mathcal{A} , we know, by definition, that $V(x_k, -P^{-1}h) \leq 1$. Since the parameter μ has been selected in $(0, 1)$, then $1 - \mu$ is positive. This leads to the following inequality

$$V(x_{k+1}, -P^{-1}h) \leq (1 - \mu) + \mu = 1,$$

which means that x_{k+1} also belongs to \mathcal{A} , and concludes the proof.

Remark 2 The minimization problem presented in Theorem 1 refers to the one presented in [6]. The main motivation of maximizing $\log(\det(P))$ is related to the minimization of the volume of the invariant set \mathcal{A} through the minimization of eigenvalues of the positive definite matrix P . Indeed, the volume of the ellipsoid characterized by $V(x, x_c) \leq 1$ is proportional to $(\det(P))^{-1/2}$. In the sequel, the evaluation of performance will be based on the minimum value of $(\det(P))^{-1/2}$, obtained in several situations. \square

3.2 Comparison with the optimal control design of [6]

In this section, we show as the solution given in [6] for System (1) is included in the solutions given in Section before. Let us first recall the statement of [6, Theorem 3], which has been reformulated in order to be consistent with the presentation of the present paper.

Theorem 2 [6] For the linear combination λ related to Assumption 1, assume that matrices $P, W \in \mathbb{R}^{n \times n} \succ 0$ and parameter $\beta > 0$ are the solution to the convex

optimization problem

$$\min_{P, W, \beta} -\log(\det(P)), \quad (15)$$

$$\text{s.t. } P \succ 0, \quad (16)$$

$$\sum_{i=1}^{\mathbb{K}} \lambda_i A_i^\top P A_i - P < -W \quad (17)$$

$$\sum_{i \in \mathbb{K}} \lambda_i \begin{bmatrix} \beta W & 0 & -P \\ * & 1 - \beta B_i^\top P B_i & -h_*^\top \\ * & * & P \end{bmatrix} > 0 \quad (18)$$

where $h_* = (1 - A_\lambda^\top)^{-1} \sum_{i \in \mathbb{K}} \lambda_i A_i^\top P B_i$. Then, the switching control law, given by

$$\sigma_k^* = \operatorname{argmin}_{i \in \mathbb{K}} V(A_i x_k + B_i, P^{-1} h) \quad (19)$$

where V corresponds to the Lyapunov function (13), ensures the same properties (i), (ii) and (iii) as the one stated in Theorem 1. \square

Remark 3 The interest of this theorem relies on the optimality of a minmax problem that provides the smallest upper bound of the forward increment of V . This optimal value of the shifting vector h is given by the expression $h_* = (1 - A_\lambda^\top)^{-1} \sum_{i \in \mathbb{K}} \lambda_i A_i^\top P B_i$. However, this selection of h_* is performed regardless the minimization of volume of the invariant set \mathcal{A} . \dashv

In order to validate the potential of Theorem 1 over this already optimal theorem from [6], there is a need to understand if the optimal solution to this theorem is a particular solution provided in Theorem 1. This property is summarized in the following proposition.

Proposition 1 *The solution, (P^*, h_*, β) , of Theorem 2 is also solution of the minimization problem of Theorem 1.*

Proof 2 *We first note that the control law (19) provided in [6] can be expressed as the control law (12) provided in Theorem 1 using the Ψ_i matrices defined in (10), with $P = P^*$, since we have*

$$\sigma_k^* = \operatorname{argmin}_{i \in \mathbb{K}} \begin{bmatrix} x_k \\ 1 \end{bmatrix}^\top \left(\Psi_i + \begin{bmatrix} P^* & h_* \\ h_*^\top & 0 \end{bmatrix} \right) \begin{bmatrix} x_k \\ 1 \end{bmatrix}, \quad (20)$$

with $N_i = \Psi_i + \begin{bmatrix} P^* & h_* \\ h_*^\top & 0 \end{bmatrix}$. Hence re-injecting this partic-

ular selection of N_i in (9) yields

$$\Gamma_i := \begin{bmatrix} \Psi_i + \Psi_\lambda - \Psi_i - \begin{bmatrix} 0 & 0 \\ * & \mu \end{bmatrix} & \mu \begin{bmatrix} P^* \\ h^\top \end{bmatrix} \\ * & -\mu P^* \end{bmatrix}, \quad i \in \mathbb{K}$$

$$:= \begin{bmatrix} \Psi_\lambda - \begin{bmatrix} 0 & 0 \\ * & \mu \end{bmatrix} & \mu \begin{bmatrix} P^* \\ h^\top \end{bmatrix} \\ * & -\mu P^* \end{bmatrix}, \quad i \in \mathbb{K}, \quad (21)$$

where $\Psi_\lambda = \sum_{j=1}^{\mathbb{K}} \lambda_j \Psi_j$. We already note from the previous expression, that the expression of Γ_i does not depend on the mode i . Therefore, one can omit the subscript i and only consider $\Gamma_i = \Gamma$, for all i in \mathbb{K} . Let us now compute this matrix Ψ_λ . From its definition, we have

$$\Psi_\lambda = \sum_{i=1}^{\mathbb{K}} \lambda_i \left(\begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix}^\top \begin{bmatrix} P^* & h_* \\ h_*^\top & 0 \end{bmatrix} \begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} P^* & h_* \\ h_*^\top & 0 \end{bmatrix} \right)$$

$$= \sum_{i=1}^{\mathbb{K}} \lambda_i \left(\begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix}^\top \begin{bmatrix} P^* & h_* \\ h_*^\top & 0 \end{bmatrix} \begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix} \right) - \begin{bmatrix} P^* & h_* \\ h_*^\top & 0 \end{bmatrix}$$

$$= \sum_{i=1}^{\mathbb{K}} \lambda_i \begin{bmatrix} A_i^\top P^* A_i - P^* & A_i^\top P^* B_i - (I - A_i) h_* \\ * & B_i^\top P^* B_i + B_i h_* + h_*^\top B_i \end{bmatrix}.$$

Note that the definition of h_* ensures that the non diagonal entry of the previous matrix verifies

$$\sum_{i=1}^{\mathbb{K}} \lambda_i (A_i^\top P^* B_i - (I - A_i) h_*)$$

$$= \sum_{i=1}^{\mathbb{K}} \lambda_i A_i^\top P^* B_i - (I - A_\lambda) h_* = 0$$

and, we use the assumption stated that $B_\lambda = 0$ to guarantee that

$$\sum_{i=1}^{\mathbb{K}} \lambda_i (B_i^\top P^* B_i + B_i h_* + h_*^\top B_i)$$

$$= \sum_{i=1}^{\mathbb{K}} \lambda_i B_i^\top P^* B_i + B_\lambda h_* + h_*^\top B_\lambda = \sum_{i=1}^{\mathbb{K}} \lambda_i B_i^\top P^* B_i.$$

Therefore, from (17), the following upper bound of Γ is derived

$$\Gamma \prec -\mu \underbrace{\begin{bmatrix} \mu^{-1} W & 0 & -P \\ * & 1 - \mu^{-1} \sum_{i=1}^{\mathbb{K}} \lambda_i B_i^\top P B_i & -h_* \\ * & * & P \end{bmatrix}}_{\succ 0 \text{ with } \mu = \beta^{-1}} \prec 0,$$

In order to be complete, it remains to prove that parameter $\mu = \beta^{-1}$ belongs to the interval $(0, 1)$. To do that, let us note that a necessary condition for condition (18) to hold is that matrix $\begin{bmatrix} \beta W & -P \\ -P & P \end{bmatrix}$ is positive definite or, equivalently, $W \succ \beta^{-1}P$, which together with (17) ensures

$$\sum_{i \in \mathbb{K}} \lambda_i A_i^\top P A_i - (1 - \beta^{-1})P \prec 0.$$

This inequality cannot be satisfied unless $\mu = \beta^{-1}$ belongs to $(0, 1)$, which concludes the proof.

The previous proposition demonstrates that the optimal solution presented in [6] is also a solution of the optimization problem (7)–(9). This means that Theorem 1 provides, at least, the same optimal solution as the solution of Theorem 2. Moreover, since the control law defined in Theorem 1 has more degrees of freedom, one may expect that it may even relax some constraints and provide a less conservative solution. Another advantage of our unstructured control matrices N_i 's over the Lyapunov-based control law relies on the fact that matrices N_i 's are decoupled from the Lyapunov matrix P . This means that there is no need to introduce an additional parameter to the LMI when applying the S-procedure to account for control law (12).

4 Robust controlled system

Another interest of our approach over [6] concerns the design of a robust switching control law for switched affine systems. In this situation matrices A_i and B_i may be subject to parameter uncertainties. Therefore, it is not possible to define a control law as in (19), which depends explicitly on the precise knowledge of the model. However, the conditions of Theorem 1 can be straightforwardly extended to the robust stabilization of switched affine systems as provided below.

Let us now assume that the matrices A_i, B_i are subject to parameter uncertainties, that are, for the sake of simplicity, expressed as polytopic type of uncertainties. This means that

$$[A_i, B_i] \in \text{Co} \left([A_i^j, B_i^j] \right)_{j \in \mathbb{D}}, \quad \forall i \in \mathbb{K}, \quad (22)$$

where \mathbb{D} is a bounded subset of \mathbb{N} , and where matrices A_i^j and B_i^j , for any $i \in \mathbb{K}$ and any $j \in \mathbb{D}$, are constant and known. Note that other types of uncertainties can be easily undertaken, as norm-bounded ones. Indeed, since the condition of Theorem 1 is convex respect to A_i and B_i several manipulations would allow one to derive a robust stabilizing theorem dedicated to this situation. The motivation for considering polytopic-type uncertainties arises from the fact that it requires less technical manipulations.

It also has to be noted that the equilibrium and the convex combination parameters λ are defined for a given nominal system included in the polytope. In this context, the following theorem holds.

Theorem 3 For the linear combination λ related to Assumption 1 and for any given parameter $0 < \mu < 1$, assume that matrices $P \in \mathbb{R}^{n \times n} \succ 0$, $h \in \mathbb{R}^n$, and $N_i = N_i^\top \in \mathbb{R}^{n+1 \times n+1}$ are the solution the convex optimization problem

$$\min_{P, h, N_i} -\log(\det(P)), \quad (23)$$

$$\text{s.t. } P \succ 0, \quad (24)$$

$$\begin{bmatrix} \Psi_i^j + N_\lambda - N_i - \begin{bmatrix} 0 & 0 \\ * & \mu \end{bmatrix} & \mu \begin{bmatrix} P \\ h^\top \end{bmatrix} \\ * & -\mu P \end{bmatrix} \prec 0, \quad (25)$$

holds for any $(i, j) \in \mathbb{K} \times \mathbb{D}$, where

$$\Psi_i^j := \begin{bmatrix} A_i^j & B_i^j \\ 0 & 1 \end{bmatrix}^\top \begin{bmatrix} P & h \\ h^\top & 0 \end{bmatrix} \begin{bmatrix} A_i^j & B_i^j \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} P & h \\ h^\top & 0 \end{bmatrix}. \quad (26)$$

Then, the switching control law given by (12) ensures the following properties for system (3) and (22)

- (i) if x_k is not in \mathcal{A} at a given time instant k , then the forward increment of the Lyapunov function, $\Delta V(x_k)$, is negative for system (3) with the control law (12).
- (ii) The equilibrium $x = 0$ belongs to \mathcal{A} .
- (iii) If x_k belongs to \mathcal{A} at a given time instant k , then x_{k+1} also belongs to \mathcal{A} , under the dynamics of (3).

□

Proof 3 The proof of Theorem 3 is based on the convexity of the quadratic term (26), which can be summarized as follows

$$\begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix}^\top \begin{bmatrix} P & h \\ h^\top & 0 \end{bmatrix} \begin{bmatrix} A_i & B_i \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} P & h \\ h^\top & 0 \end{bmatrix} \leq \sum_{j \in \mathbb{D}} \bar{\lambda}_j \Psi_i^j$$

provided that the matrices $[A_i \ B_i]$ can be rewritten as $\sum_{j \in \mathbb{D}} \bar{\lambda}_j [A_i^j \ B_i^j]$, where the scalar coefficients $\bar{\lambda}_j$ are positive and verify $\sum_{j \in \mathbb{D}} \bar{\lambda}_j = 1$. The remainder of the proof directly follows the lines of the proof of Theorem 1.

Remark 4 Note that the system with matrices A_i and B_i fulfilling (22) is more constrained than the nominal one, and consequently the size of \mathcal{A} given in Theorem 1 will be equal or smaller than the one computed with Theorem 3. ┘

Th.	μ	$10^2 \cdot P$	$(\det(P))^{-\frac{1}{2}}$	x_c	$10^2 \cdot N_1$	$10^2 \cdot N_2$
[6]	0.0833	$\begin{bmatrix} 7.77 & 5.80 & 4.99 \\ 5.80 & 10.44 & 6.25 \\ 4.99 & 6.25 & 10.10 \end{bmatrix}$	59.9697	$\begin{bmatrix} -0.4930 \\ -0.1930 \\ -0.1771 \end{bmatrix}$	$\begin{bmatrix} 4.00 & 3.49 & -0.12 & 2.40 \\ 3.49 & 8.22 & 4.47 & 1.26 \\ -0.12 & 4.47 & 5.85 & -0.18 \\ 2.40 & 1.26 & -0.18 & -3.21 \end{bmatrix}$	$\begin{bmatrix} 7.77 & 7.95 & 6.37 & 10.99 \\ 7.95 & 10.98 & 7.70 & 13.07 \\ 6.37 & 7.70 & 7.41 & 13.91 \\ 10.99 & 13.07 & 13.91 & 21.84 \end{bmatrix}$
Th. 1	0.0977	$\begin{bmatrix} 10.32 & 7.72 & 6.53 \\ 7.72 & 13.95 & 8.31 \\ 6.53 & 8.31 & 13.46 \end{bmatrix}$	38.7049	$\begin{bmatrix} -0.4804 \\ -0.1993 \\ -0.2914 \end{bmatrix}$	$\begin{bmatrix} 2.88 & 2.90 & 4.89 & 6.20 \\ 2.90 & 1.81 & 2.09 & 8.13 \\ 4.89 & 2.09 & 1.82 & 9.42 \\ 6.20 & 8.13 & 9.42 & 18.78 \end{bmatrix}$	$-N_1$

Table 1

Comparison of the numerical results of Theorem 1 and [6]. For the optimal value of μ , this table reports, the Lyapunov matrix, P and the inverse of the square root of its determinant, the center of the invariant set \mathcal{A} and the control matrices N_1 and N_2 .

Remark 5 Note that the solutions of Theorem 3 constrained by the structured Lyapunov-based control matrices $N_i = \Psi_i + \begin{bmatrix} P^* & h_* \\ h_*^\top & 0 \end{bmatrix}$, for any i in \mathbb{K} , corresponding to the control law (19) given in [6] are again included in the solutions of the original conditions of Theorem 3. Therefore, the relaxed control law employed in Theorem 3 can only improve the performance compared to Lyapunov based-controllers. \square

5 Numerical validations

5.1 Nominal case and comparison to [6]

In this section, we take the example 1 given in [6] in order to compare the controller proposed here with the controller proposed in [6]. This example considers system (1) composed by two unstable subsystems, defined by:

$$A_i = e^{F_i T}, \quad \mathcal{B}_i = \int_0^T e^{F_i \tau} d\tau g_i, \quad i = 1, 2,$$

where matrices F_1, F_2 and g_1, g_2 are given by

$$F_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, \quad g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and where parameter T , representing the sampling period of the associated continuous-time system, is equal to 1. The desired equilibrium is $z_e = [5/3 \quad -0.6 \quad -0.4]^\top$, which satisfies Assumption 1 under $\lambda = [0.6 \quad 0.4]$.

Table 1 presents the numerical results obtained by solving the conditions of Theorem 1 for a given value of μ . These values have to be regarded in comparison with the ones issued from the application of the performances of the controller developed in [6] for a given value of μ , corresponding to β^{-1} in this paper. This table shows the value of parameter μ , that minimizes the volume of the

invariant set, which is characterized by Lyapunov matrix P , more precisely, by $\det(P)^{-\frac{1}{2}}$. Then the associated control matrices N_1 and N_2 are presented.

It is worth noting that the Lyapunov matrix P provided by Theorem 1 generates a notably smaller volume (referring to the quantity $\det(P)^{-\frac{1}{2}}$) than the one proposed in [6]. Indeed the difference with respect to the proposed in [6] is 32.07%. It also has to be noted that the center x_c of the invariant set \mathcal{A} provided in both cases are very close to each other. In addition, the resulting values of h obtained by Theorem 1 and in Remark 3 are $h = [0.0840 \quad 0.0891 \quad 0.0871]^\top$ and $h^* = [0.0599 \quad 0.0614 \quad 0.0560]^\top$, respectively. These values of h are notably different and one can conclude that the one provided by our method does not follow the optimality criteria given in [6].

Another advantage of the unstructured control law (12) over the one developed in [6] expressed in (19) or in (20), is that, on the one hand, it does not depend on the system matrices and on the other hand it only requires the computation of a single matrix N_1 since $N_1 = -N_2$. This simple structure is particular to the case of two modes since, in this situation, $\lambda_2 = 1 - \lambda_1$ and, consequently, in LMI (9), we note that $N_\lambda - N_1 = (1 - \lambda_1)(N_2 - N_1)$ and $N_\lambda - N_2 = \lambda_1(N_2 - N_1)$, which means that matrices N_1 and N_2 are linked. This demonstrates the potential of employing the proposed method and controller over the ones presented in [6].

5.2 Robust switching control of affine systems

Consider now the uncertain system driven by (1) composed by three functioning modes:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0.15 + \delta \\ -0.35 & -1 \end{bmatrix}, & \mathcal{B}_1 &= \begin{bmatrix} 1 \\ 0.35 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.24 & 0.15 + \delta \\ -2.35 & -1 \end{bmatrix}, & \mathcal{B}_2 &= \begin{bmatrix} -1 \\ -0.35 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -0.24 & 0.15 + \delta \\ -2.35 & -0.5 \end{bmatrix}, & \mathcal{B}_3 &= \begin{bmatrix} 0.05 \\ 1.5 \end{bmatrix}, \end{aligned} \quad (27)$$

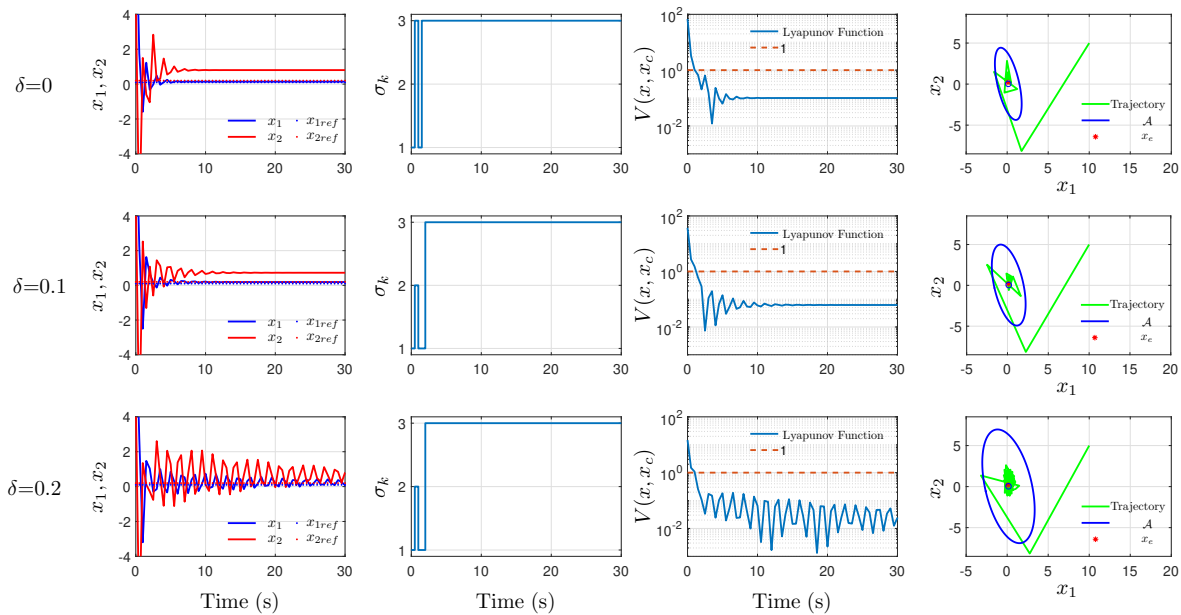


Fig. 1. Simulation results of uncertain switched affine system (1), (27) for three different values of $\bar{\delta}$, from top to bottom. From left to right, the figure shows the evolution of the state variables (x_1, x_2) , the control input σ and the Lyapunov function $V(x, x_c)$ (in a logarithmic scale), with respect to time. The last column shows the trajectories of the state in the phase portrait.

where δ is an unknown, possibly time-varying perturbation to the system, which is only assumed to be bounded by a known parameter $\bar{\delta}$, such that

$$|\delta| \leq \bar{\delta}.$$

The desired equilibrium is $z_e = [0.1 \ 0.2]^T$ with $\lambda = [0.36 \ 0.3 \ 0.34]$.

From Theorem 3, we compute the corresponding matrices P , N_i with $i = 1, 2, 3$ and μ , for several values of $\bar{\delta} \in [0, 0.2]$. Figure 1, shows the evolution of the volume of \mathcal{A} with respect to δ with control (12) and with the control given in [6], summarized in (20). It can first be seen that increasing upper bound of the uncertainty $\bar{\delta}$ implies a natural increase of the size of $(\det(P))^{-\frac{1}{2}}$, which, again, represents the volume of the invariant set \mathcal{A} . Moreover, it can be seen that the use of the relaxed and unstructured control matrices N_i provides a lower volume than the one of the Lyapunov-based controller. This demonstrates again the potential of our approach and validates both Proposition 1 and Remark 5.

Finally, Figure 2 depicts three simulations of uncertain system (1), (27) in the situation where the uncertainty δ is taken constant and equal to the upper bounds $\bar{\delta} = 0, 0.1, 0.2$, respectively. The figure exposes the state and σ evolutions, and the attractor in the state plane. Note that the states converge to a point close to the desired equilibrium which enters lies in the invariant set. The plot of the Lyapunov function shows that once the system enters to \mathcal{A} , i.e. V is below the threshold of value 5,

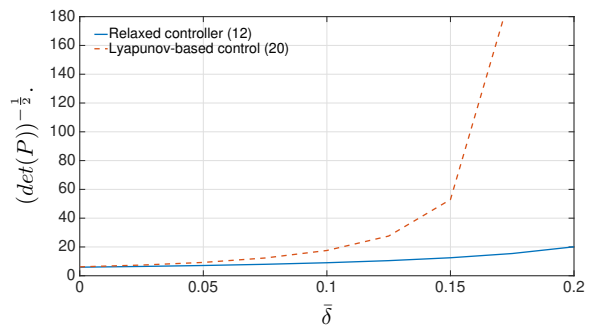


Fig. 2. Evolution of $(\det(P))^{-\frac{1}{2}}$ with unstructured matrices N_i 's (solid) in (12), and with the Lyapunov-based control law (20) (dashed).

it never leaves. Likewise, note that the size of \mathcal{A} increases when $\bar{\delta}$ increases, as is stated in Remark 4. Therefore, the practical stability established by Theorems 1 and 3 holds.

6 Conclusions and perspectives

This paper presents a robust switching controller for discrete time switched affine systems. The main result presented here provides a relaxed control which is not based on the computation of the Lyapunov function as usually considered in the literature. It is demonstrated, by theoretical and numerical validations, that the solution of the optimization problem provided in [6] is included in the solutions of the one in this paper. Numerical applications even show a notable improvement of the

guarantees on an example. Besides, the potential of our method compared to the existing solution from the literature is the possibility to apply the same control design methodology in the case of uncertain system, which is not straightforward using Lyapunov-based control laws because it requires the exact knowledge of the system matrices. In future work we aim at exploiting the same methodology to design robust switching control laws in the continuous-time case.

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