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Hong-Wei Zhang

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Abstract. This paper is devoted to study the dispersive properties of the linear Klein-Gordon equation on a class of locally symmetric spaces. As a consequence, we obtain the Strichartz estimate and prove global well-posedness results for the corresponding semilinear equation with low regularity data as on real hyperbolic spaces.

1. Introduction

Let M be a Riemannian manifold and denote by Δ the Laplace-Beltrami operator on M. The theory is well established for the following wave equation on $M=\mathbb{R}^n$,

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) = F(t,x), \\ u(0,x) = f(x),\ \partial_t|_{t=0} u(t,x) = g(x), \end{cases} \tag{1.1}$$

where the solutions u satisify the Strichartz estimates: [1]

$$\|\nabla_{\mathbb{R}\times\mathbb{R}^n} u\|_{L^p(I;H^{-\sigma,q}(\mathbb{R}^n))} \lesssim \|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\tilde{p}'}(I;H^{\tilde{\sigma},\tilde{q}'}(\mathbb{R}^n))},$$

on any interval $I\subseteq\mathbb{R}$ under the assumptions

$$\sigma = \frac{n+1}{2}\left(\frac{1}{2}-\frac{1}{q}\right),\ \widetilde{\sigma} = \frac{n+1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{q}}\right),$$

and the couples $(p,q),(\tilde{p},\tilde{q})\in(2,+\infty]\times[2,2\frac{n-1}{n-3})$ fulfill the admissibility conditions:

$$\frac{1}{p} = \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{q}\right),\ \frac{1}{\tilde{p}} = \frac{n-1}{2}\left(\frac{1}{2}-\frac{1}{\tilde{q}}\right).$$

These estimates serve as a tool for finding minimal regularity conditions on the initial data ensuring well-posedness for corresponding semilinear wave equations, which is addressed in [21], and almost fully answered in [11, 16, 22, 24].

Analogous results have been found for the Klein-Gordon equation

$$\begin{cases} \partial_t^2 u(t,x) - \Delta u(t,x) + cu(t,x) = F(t,x), \\ u(0,x) = f(x),\ \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{1.2}$$

with $c=1$, see [7, 17, 27, 28].

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[1] The symbol $\lesssim$, let us recall, means precisely that there exists a constant $0<C<+\infty$ such that $\|\nabla_{\mathbb{R}\times\mathbb{R}^n} u\|_{L^p(I;H^{-\sigma,q}(\mathbb{R}^n))} \le C\left(\|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^{\tilde{p}'}(I;H^{\tilde{\sigma},\tilde{q}'}(\mathbb{R}^n))}\right)$.

Given the rich Euclidean theory, it is natural to look at the corresponding equations on more general manifolds. We consider in the present paper a class of noncompact locally symmetric spaces M, on which we study the Klein-Gordon equation (1.2) with $c \geq -\rho^2$, where ρ is a positive constant depending on the structure of M and defined in next section. Due to large-scale dispersive effects in negative curvature, we expect stronger results than in the Euclidean setting, as on real hyperbolic space, see [4, 5].

In the critical case $c = -\rho^2$, (1.2) is called the shifted wave equation. To our knowledge, it was first considered in [13, 14] in low dimensions $n = 2$ and $n = 3$. In [5, 6], a detailed analysis of the shifted wave equation was carried out on real hyperbolic spaces and on Damek-Ricci spaces, which contains all rank one symmetric spaces of noncompact type. In the non-shifted case $c > -\rho^2$, similar results on real hyperbolic spaces were obtained in [4].

In the recent paper [15], the Schrödinger equation was considered on certain locally symmetric spaces. In the present paper, we study the wave and Klein-Gordon equations in the same spirit.

1.1. **Notations.**

Let G be a semisimple Lie group, connected, noncompact, with finite center, and K be a maximal compact subgroup of G. The homogenous space $X = G/K$ is a Riemannian symmetric space of the noncompact type, whose dimension is denoted by n. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of its Lie algebra. The Killing form of $\mathfrak{g}$ induces a K-invariant inner product on $\mathfrak{p}$, and hence a G-invariant Riemannian metric on G/K.

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. The symmetric space X is said to have rank one if $\dim \mathfrak{a} = 1$. Denote by $\mathfrak{a}^*$ the real dual of $\mathfrak{a}$, let $\Sigma \subset \mathfrak{a}^*$ be the root system of $(\mathfrak{g}, \mathfrak{a})$ and denote by W the Weyl group associated to Σ. Choose a set Σ^+ of positive roots, let $\mathfrak{a}^+ \subset \mathfrak{a}$ be the corresponding positive Weyl chamber and $\overline{\mathfrak{a}+}$ its closure. Denote by ρ the half sum of positive roots counted with their multiplicites:

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha,$$

where m_α is the dimension of root space $\mathfrak{g}_\mathfrak{a} = \{Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y, \forall H \in \mathfrak{a}\}$.

Let Γ be a discrete torsion-free subgroup of G. The locally symmetric space $M = \Gamma \backslash X$, equipped with the Riemannian structure inherited from X becomes a Riemannian manifold. We say that M has rank one if X has rank one. Moreover Γ is called convex cocompact if the quotient group $\Gamma \backslash \mathrm{Conv}(\Lambda_\Gamma)$ is compact, where $\mathrm{Conv}(\Lambda_\Gamma)$ is the convex hull of the limit set Λ_Γ of Γ. We denote by Δ the Laplace-Beltrami operator, by $d(\cdot, \cdot)$ the Riemannian distance, and by dx the associated measure, both on X and M. Consider the Poincaré series

$$P(s; x, y) = \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma y)}, \ s > 0, \ x, y \in X,$$

and denote by $\delta(\Gamma)$ its critical exponent:

$$\delta(\Gamma) = \inf \{s > 0 \mid P(s; x, y) < +\infty\}.$$

1.2. **Assumptions.**

In this paper, $M = \Gamma \backslash X$ is a rank one locally symmetric space such that Γ is convex cocompact and $\delta(\Gamma) < \rho$.

Let us comment a few words on these assumptions. Wave type equations on non-compact rank one symmetric spaces are well understood. Sharp pointwise estimates of wave kernels on X(see Section 2.2), which were obtained in [4, 5], allow us to deal with wave kernels on locally symmetric space M. Notice that such information is lacking in higher rank.

The rank one symmetric spaces of the noncompact type are the hyperbolic spaces $H^n(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $H^2(\mathbb{O})$. In particular, we have $\mathfrak{a}^* = \mathfrak{a}$ and $\mathfrak{a}^+ \cong \mathbb{R}_+^*$, hence ρ is just a positive constant depending on the structure of X. Specifically, as a direct consequence of the assumption $\delta(\Gamma) < \rho$, the series (3.1) defining the wave kernel on M is absolutely convergent, see Proposition 3.1. In addition, according to [9], the bottom λ_0 of the L^2-spectrum of $-\Delta$ on M is equal to ρ^2, as on X. Consequently, we obtain an analogous L^2 Kunze-Stein phenomenon on M without further assumptions, see Proposition 3.2. Notice that $\lambda_0 = \rho^2 > 0$ implies $\mathrm{Vol}(M) = +\infty$, because $\lambda_0 = 0$ if M is a lattice.

At last, the convex cocompactness assumption implies a uniform upper boundary of the Poincaré series, see Lemma 3.3, which is crucial for the $L^1 \to L^\infty$ boundedness of wave propagators on M.

Remark 1.1. The Schrödinger equation is studied in [15] under slightly different assumptions, our well-posedness results hold also in that setting.

1.3. **Statement of the results.**

Consider the operator $D = \sqrt{-\Delta - \rho^2 + \kappa^2}$ with $\kappa > 0$, then the Klein-Gordon equations (1.2) becomes

$$\begin{cases} \partial_t^2 u(t,x) + D_x^2 u(t,x) = F(t,x), \\ u(0,x) = f(x),\ \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{1.3}$$

with $c = \kappa^2 - \rho^2 > -\rho^2$. Notice that (1.3) is the wave equation when $\kappa = \rho$ and becomes the shifted wave equation in the limit case $\kappa = 0$. Consider another operator $\widetilde{D} = \sqrt{-\Delta - \rho^2 + \widetilde{\kappa}^2}$ with $\widetilde{\kappa} > \rho$. We denote by ω_t^σ the radial convolution kernel of the wave operator $W_t^\sigma := \widetilde{D}^{-\sigma} e^{itD}$ on the symmetric space X:

$$W_t^\sigma f(x) = f * \omega_t^\sigma(x) = \int_G \omega_t^\sigma(y^{-1}x) f(y) dy, \tag{1.4}$$

where f is any reasonable function on X, see Section 2.2 for more details. By bi-K-invariance of the kernel ω_t^σ, we deduce that $W_t^\sigma f$ is left Γ-invariant and right K-invariant if f is defined on the locally symmetric space M. Thus the wave operator on M, denoted by $\widehat{W_t^\sigma}$, is also defined by (1.4). Consider the wave kernel $\widehat{\omega_t^\sigma}$ on M, which is given by

$$\widehat{\omega_t^\sigma}(x,y) = \sum_{\gamma \in \Gamma} \omega_t^\sigma(y^{-1}\gamma x),\ \forall x, y \in X.$$

Then the wave operator $\widehat{W_t^\sigma}$ on M is an integral operator:

$$\widehat{W_t^\sigma} f(x) = \int_M \widehat{\omega_t^\sigma}(x,y) f(y) dy,$$

see Proposition 3.1. The aim of this paper is to prove the following dispersive properties:

Theorem 1.2. *For $n \geq 3$, $2 < q < +\infty$ and $\sigma \geq (n+1)\left(\frac{1}{2} - \frac{1}{q}\right)$,*

$$\left\| \widehat{W_t^\sigma} \right\|_{L^{q'}(M) \to L^q(M)} \lesssim \begin{cases} |t|^{-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)} & if\ 0 < |t| < 1, \\ |t|^{-\frac{3}{2}} & if\ |t| \geq 1. \end{cases} \tag{1.5}$$

Remark 1.3. In dimension $n = 2$, there is an additional logarithmic factor in the small time bound, which becomes $|t|^{-\left(\frac{1}{2}-\frac{1}{q}\right)}(1 - \log|t|)^{1-\frac{2}{q}}$, see Theorem 2.1 in the next section.

Remark 1.4. At the endpoint $q = 2$, $t \mapsto e^{itD}$ is a one-parameter group of unitary operators on $L^2(M)$.

By applying the classical TT^* method and by using the previous dispersive properties, we obtain the Strichartz estimate

$$\|\nabla_{\mathbb{R}\times M} u\|_{L^p(I;H^{-\sigma,q}(M))} \lesssim \|f\|_{H^1(M)} + \|g\|_{L^2(M)} + \|F\|_{L^{\tilde{p}'}(I;H^{\tilde{\sigma},\tilde{q}'}(M))}$$

for the solutions u of (1.3), see Section 4 for more information about the Sobolev spaces $H^{-\sigma,q}(M)$. Here $I \subset \mathbb{R}$ is any time interval, possibly unbounded,

$$\sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right),\ \widetilde{\sigma} \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right),$$

and the couples (p, q) and $(\tilde{p}, \tilde{q})$ are admissible, which means that $\left(\frac{1}{p}, \frac{1}{q}\right)$, $\left(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}\right)$ belong, in dimension $n \geq 4$ (see Section 4 for the lower dimensions) to the triangle

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) \;\middle|\; \frac{1}{p} \geq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right\} \bigcup \left\{\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{n-1}\right)\right\}.$$

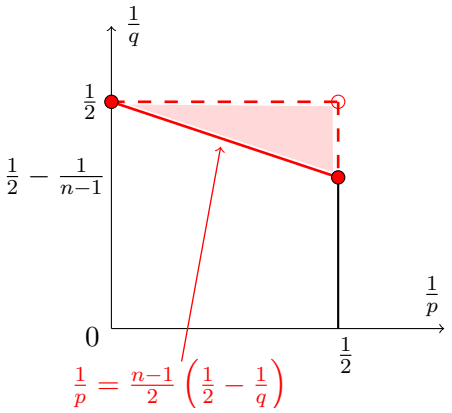

FIGURE 1. Admissibility in dimension $n \geq 4$.

Notice that the admissible set for M is larger than the admissible set for $\mathbb{R}^n$ which corresponds only to the lower edge of the triangle. For the Schrödinger equation on real hyperbolic spaces, see [3] and [16]. In comparison with X, we loose the right edge of the triangle, which corresponds to the critical case $\frac{1}{p} = \frac{1}{2}$ and $\frac{1}{q} > \frac{1}{2} - \frac{1}{n-1}$, this will be explained in Section 4. Notice that we obtain nevertheless the same well-posedness results as on X.

This paper is organized as follows. In Section 2, we review spherical analysis on noncompact symmetric spaces, and recall pointwise estimates of wave kernels on rank one symmetric space obtained in [4]. In Section 3, after proving the necessary lemmas, we prove the dispersive estimate by an interpolation argument. As a consequence, we deduce the Strichartz estimate and obtain well-posedness results for the semilinear Klein-Gordon equation in Section 4.

2. Preliminaries

2.1. **Spherical analysis on noncompact symmetric spaces.**

We review in this section some elementary facts about noncompact symmetric spaces. We refer to [1, 2, 12, 18] for more details.

Recall that $\overline{\mathfrak{a}+}$ is the closure of the positive Weyl chamber $\mathfrak{a}+$. Denote by $\mathfrak{n}=\sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ the nilpotent Lie subalgebra of $\mathfrak{g}$ associated with Σ^+, and by N the corresponding Lie subgroup of G. Then we have the following two decompositions of G:

$$\begin{cases} G=N\,(\exp\mathfrak{a})\,K & (Iwasawa),\\ G=K\,(\exp\overline{\mathfrak{a}+})\,K & (Cartan). \end{cases}$$

In the Cartan decomposition, the Haar measure on G writes

$$\int_G f(g)dg=const.\int_K dk_1\int_{\mathfrak{a}_+}\prod_{\alpha\in\Sigma^+}(\sinh\alpha(H))^{m_\alpha}\,dH\int_K f(k_1(\exp H)k_2)dk_2. \tag{2.1}$$

In the rank one case, which we consider in this paper,

$$\int_{\mathfrak{a}_+}\prod_{\alpha\in\Sigma^+}(\sinh\alpha(H))^{m_\alpha}\,dH=const.\int_0^{+\infty}(\sinh r)^{m_\alpha}\,(\sinh 2r)^{m_{2\alpha}}\,dr,$$

where

$$(\sinh r)^{m_\alpha}\,(\sinh 2r)^{m_{2\alpha}}\lesssim e^{2\rho r},\quad \forall r>0. \tag{2.2}$$

Denote by $\mathcal{S}(K\backslash G/K)$ the Schwartz space of bi-K-invariant functions on G. The spherical Fourier transform $\mathcal{H}$ is defined by

$$\mathcal{H}f(\lambda)=\int_G f(x)\varphi_\lambda(x)dx,\ \forall\lambda\in\mathfrak{a}^*\cong\mathbb{R},\ \forall f\in\mathcal{S}(K\backslash G/K).$$

Here $\varphi_\lambda\in\mathcal{C}^\infty(K\backslash G/K)$ is a spherical function, which can be characterized as a radial eigenfunction of the negative Laplace-Beltrami operator $-\Delta$ satisfying

$$\begin{cases} -\Delta\varphi_\lambda(x)=\left(\lambda^2+\rho^2\right)\varphi_\lambda(x),\\ \varphi_\lambda(e)=1. \end{cases} \tag{2.3}$$

In the noncompact case, the spherical function is characterized by

$$\varphi_\lambda(x)=\int_K e^{(i\lambda+\rho)A(kg)}dk,\ \lambda\in\mathfrak{a}_{\mathbb{C}}^*, \tag{2.4}$$

where $A(kg)$ is the unique $\mathfrak{a}$-component in the Iwasawa decomposition of kg.

Denote by $\mathcal{S}(\mathfrak{a}^*)^W$ the subspace of W-invariant functions in the Schwartz space $\mathcal{S}(\mathfrak{a}^*)$. Then $\mathcal{H}$ is an isomorphism between $\mathcal{S}(K\backslash G/K)$ and $\mathcal{S}(\mathfrak{a}^*)^W$. The inverse spherical Fourier transform is defined by

$$f(x) = const. \int_{\mathfrak{a}^*} \mathcal{H}f(\lambda)\varphi_{-\lambda}(x)|\mathbf{c}(\lambda)|^{-2}d\lambda, \ \forall x \in G, \ \forall f \in \mathcal{S}(\mathfrak{a}^*)^W,$$

where $\mathbf{c}(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function.

2.2. **Pointwise estimates of the wave kernel on symmetric spaces.**

We recall in this section the pointwise wave kernel estimates on rank one symmetric space obtained in [4] and [6]. Via the spherical Fourier transform and (2.3), the negative Laplace-Beltrami operator $-\Delta$ corresponds to $\lambda^2+\rho^2$, hence the operators $D = \sqrt{-\Delta - \rho^2 + \kappa^2}$ and $\widetilde{D} = \sqrt{-\Delta - \rho^2 + \widetilde{\kappa}^2}$ to

$$\sqrt{\lambda^2 + \kappa^2} \quad and \quad \sqrt{\lambda^2 + \tilde{\kappa}^2}.$$

By the inverse spherical Fourier transform, the radial convolution kernel ω_t^σ of $W_t^\sigma = \widetilde{D}^{-\sigma}e^{itD}$ on X is given by

$$\omega_t^\sigma(r) = const. \int_{-\infty}^{+\infty} (\lambda^2 + \tilde{\kappa}^2)^{-\frac{\sigma}{2}} e^{it\sqrt{\lambda^2+\kappa^2}}\varphi_\lambda(r)|\mathbf{c}(\lambda)|^{-2}d\lambda$$

for suitable exponents $\sigma \in \mathbb{R}$. Consider smooth even cut-off functions χ_0 and χ_∞ on $\mathbb{R}$ such that

$$\begin{cases} \chi_0(\lambda) + \chi_\infty(\lambda) = 1, \\ \chi_0(\lambda) = 1, \quad \forall |\lambda| \leq 1, \\ \chi_\infty(\lambda) = 1, \quad \forall |\lambda| \geq 2. \end{cases}$$

Let us split up

$$\begin{aligned} \omega_t^\sigma(r) =& \omega_t^{\sigma,0}(r) + \omega_t^{\sigma,\infty}(r) \\ =& \ const. \int_{-\infty}^{+\infty} \chi_0(\lambda)(\lambda^2 + \tilde{\kappa}^2)^{-\frac{\sigma}{2}} e^{it\sqrt{\lambda^2+\kappa^2}}\varphi_\lambda(r)|\mathbf{c}(\lambda)|^{-2}d\lambda \\ &+ const. \int_{-\infty}^{+\infty} \chi_\infty(\lambda)(\lambda^2 + \tilde{\kappa}^2)^{-\frac{\sigma}{2}} e^{it\sqrt{\lambda^2+\kappa^2}}\varphi_\lambda(r)|\mathbf{c}(\lambda)|^{-2}d\lambda. \end{aligned}$$

As the kernel $\omega_t^{\sigma,\infty}$ has a logarithmic singularity on the sphere $r = t$ when $\sigma = \frac{n+1}{2}$, we consider the analytic family of operators

$$\widetilde{W}_t^{\sigma,\infty} := \frac{e^{\sigma^2}}{\Gamma\left(\frac{n+1}{2} - \sigma\right)} \chi_\infty(D)\tilde{D}^{-\sigma}e^{itD}, \tag{2.5}$$

in the vertical strip $0 \leq \operatorname{Re}\sigma \leq \frac{n+1}{2}$, and their kernels

$$\widetilde{\omega}_t^{\sigma,\infty}(r) = const. \frac{e^{\sigma^2}}{\Gamma\left(\frac{n+1}{2} - \sigma\right)} \int_{-\infty}^{+\infty} \chi_\infty(\lambda)(\lambda^2 + \tilde{\kappa}^2)^{-\frac{\sigma}{2}} e^{it\sqrt{\lambda^2+\kappa^2}}\varphi_\lambda(r)|\mathbf{c}(\lambda)|^{-2}d\lambda.$$

The following pointwise estimates of the kernels $\omega_t^{\sigma,0}$ and $\widetilde{\omega}_t^{\sigma,\infty}$, which were obtained in [4] for real hyperbolic spaces, extend straightforwardly to all rank one Riemannian symmetric spaces of the noncompact type.

Theorem 2.1. *For all $\sigma \in \mathbb{R}$, the kernel $\omega_t^{\sigma,0}$ satisfies*

$$|\omega_t^{\sigma,0}(r)| \lesssim \begin{cases} \varphi_0(r), & \forall t \in \mathbb{R},\ \forall r \geq 0, \\ |t|^{-\frac{3}{2}}(1+r)\varphi_0(r), & \forall |t| \geq 1,\ \forall\ 0 \leq r \leq \frac{|t|}{2}. \end{cases}$$

For all $\sigma \in \mathbb{C}$ with $\operatorname{Re}\sigma = \frac{n+1}{2}$, and for every $r \geq 0$, the following estimates hold for the kernel $\widetilde{\omega}_t^{\sigma,\infty}$:

$$|\widetilde{\omega}_t^{\sigma,\infty}(r)| \lesssim \begin{cases} |t|^{-\frac{n-1}{2}} e^{-\rho r}, & \forall 0 < |t| < 1,\ if\ n \geq 3, \\ |t|^{-N}(1+r)^N \varphi_0(r), & \forall |t| \geq 1,\ \forall N \in \mathbb{N}. \end{cases}$$

In the 2-dimensional case, the small time estimate of $\widetilde{\omega}_t^{\sigma,\infty}$ reads

$$|\widetilde{\omega}_t^{\sigma,\infty}(r)| \lesssim |t|^{-\frac{1}{2}}(1 - \log|t|)e^{-\frac{r}{2}}, \quad \forall 0 < |t| < 1.$$

3. Dispersive estimates for the wave operator on locally symmetric spaces

In this section, we prove our main result, namely Theorem 1.2. Let us first describe the wave operator $\widehat{W_t^\sigma}$ on locally symmetric space M. Recall that the wave kernel on M is given by

$$\widehat{\omega_t^\sigma}(x,y) = \sum_{\gamma \in \Gamma} \omega_t^\sigma(y^{-1}\gamma x),\ \forall x, y \in X. \tag{3.1}$$

Proposition 3.1. *The series (3.1) is convergent for every $x, y \in X$, and the wave operator on M is given by*

$$\widehat{W_t^\sigma} f(x) = \int_M \widehat{\omega_t^\sigma}(x,y) f(y) dy,$$

for any reasonable function f on M.

Proof. According to the Cartan decomposition of G, we can write $y^{-1}\gamma x = k_\gamma(\exp H_\gamma)k'_\gamma$ with $H_\gamma \in \mathbb{R}_+$ and $k_\gamma, k'_\gamma \in K$. Notice that $H_\gamma = d(x, \gamma y)$. Then, by the bi-$K$-invariance of ,$\omega_t^\sigma$, we have

$$|\widehat{\omega_t^\sigma}(x,y)| = \left| \sum_{\gamma \in \Gamma} \omega_t^\sigma(\exp H_\gamma) \right| \lesssim \sum_{\gamma \in \Gamma} |\omega_t^{\sigma,0}(\exp H_\gamma)| + \sum_{\gamma \in \Gamma} |\widetilde{\omega}_t^{\sigma,\infty}(\exp H_\gamma)|.$$

For the first part, Theorem 2.1 implies that for all $H_\gamma \geq 0$,

$$\sum_{\gamma \in \Gamma} |\omega_t^{\sigma,0}(\exp H_\gamma)| \lesssim \sum_{\gamma \in \Gamma} \varphi_0(\exp H_\gamma) \lesssim \sum_{\gamma \in \Gamma} (1 + H_\gamma) e^{-\rho H_\gamma}.$$

By choosing $0 < \varepsilon < \rho - \delta(\Gamma)$, we obtain

$$\sum_{\gamma \in \Gamma} |\omega_t^{\sigma,0}(\exp H_\gamma)| \lesssim \sum_{\gamma \in \Gamma} e^{-(\delta(\Gamma)+\varepsilon)d(x,\gamma y)} = P_{\delta(\Gamma)+\varepsilon}(x,y) < +\infty.$$

The second part is handled similarly and thus omitted. Hence the serie (3.1) is convergent. According to (1.4), we know that

$$\widehat{W_t^\sigma} f(x) = \int_G \omega_t^\sigma(y^{-1}x) f(y) dy = \int_X \omega_t^\sigma(y^{-1}x) f(y) dy,$$

since ω_t^σ is bi-K-invariant and f is right K-invariant. By using Weyl's formula and the fact that f is left Γ-invariant, we deduce that

$$\widehat{W_t^\sigma f}(x) = \int_{\Gamma\backslash X} \left(\sum_{\gamma\in\Gamma} f(\gamma y)\omega_t^\sigma(y^{-1}\gamma x) \right) dy = \int_M \widehat{\omega_t^\sigma}(x,y) f(y) dy.$$

□

Next, we introduce the following version of the L^2 Kunze-Stein phenomenon on locally symmetric space M, which plays an essential role in the proof of the dispersive estimate.

Proposition 3.2. *Let ψ be a reasonable bi-K-invariant functions on G, e.g., in the Schwartz class. Then*

$$\|.*\psi\|_{L^2(M)\to L^2(M)} \leq \int_G |\psi(x)|\varphi_0(x)dx. \tag{3.2}$$

The Kunze-Stein phenomenon is a remarkable convolution property on semisimple Lie groups and symmetric spaces (see e.g., [23], [19], [10] and [20]), which was extended to some classes of locally symmetric spaces in [25] and [26]. Let us prove Proposition 3.2 along the lines of [26] in our setting where $\operatorname{rank} X = 1$, $\delta(\Gamma) < \rho$ and with no additional assumption.

Proof. We denote by $\widehat{G}$ the unitary dual of G and by $\widehat{G}_K$ the spherical subdual. Decompose

$$L^2(\Gamma\backslash G) \cong \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi d\nu(\pi)$$

and

$$L^2(M) \cong \int_{\widehat{G}_K}^{\oplus} (\mathcal{H}_\pi)^K d\nu(\pi) \tag{3.3}$$

accordingly, where ν is a positive measure on $\widehat{G}$. Recall that $(\mathcal{H}_\pi)^K = \mathbb{C}e_\pi$ is one-dimensional for every $\pi \in \widehat{G}_K$. Recall moreover that, in rank one, $\widehat{G}_K$ is parametrized by a subset of $\mathbb{C}/\pm 1$. Specifically, $\widehat{G}_K$ consists of

- the unitary spherical principal series $\pi_{\pm\lambda}$ $(\lambda \in \mathbb{R}/\pm 1)$,
- the trivial representation $\pi_{\pm i\rho} = 1$,
- the complementary series $\pi_{\pm i\lambda}$ $(\lambda \in I)$, where

$$I = \begin{cases} (0,\rho) & if\ X = H^n(\mathbb{R})\ or\ H^n(\mathbb{C}), \\ \left(0, \frac{m_\alpha}{2}+1\right] & if\ X = H^n(\mathbb{H})\ or\ H^2(\mathbb{O}). \end{cases}$$

Under the assumption $\delta(\Gamma) \leq \rho$, we know that $\lambda_0 = \rho^2$ is the bottom of the spectrum of $-\Delta$ on $L^2(M)$. As $-\Delta$ acts on $(\mathcal{H}_\pi)^K$ by multiplication by $\lambda^2+\rho^2$, we deduce that (3.3) involves only tempered representations, i.e., representations π_λ with $\lambda \in \mathbb{R}/\pm 1$. Moreover, as the right convolution by $\psi \in \mathcal{S}(K\backslash G/K)$ acts on $(\mathcal{H}_{\pi_\lambda})^K$ by multiplication by

$$\mathcal{H}f(\lambda) = \int_G f(x)\varphi_\lambda(x)dx,$$

where $\varphi_\lambda(x) = \langle \pi_\lambda(x) e_{\pi_\lambda}, e_{\pi_\lambda} \rangle$ is the spherical function (2.3), we deduce from (2.4) that

$$\|. * \psi\|_{L^2(M)\to L^2(M)} \leq \sup_{\lambda\in\mathbb{R}} \left| \int_G \psi(x)\varphi_\lambda(x)dx \right| \leq \int_G |\psi(x)|\varphi_0(x)dx.$$

□

The following two lemmas are used in the proof of dispersive estimates.

Lemma 3.3. *If Γ is convex cocompact, then there exists a constant $C > 0$ such that for all $x, y \in X$,*

$$P(s; x, y) \leq C P(s; \mathbf{0}, \mathbf{0}),$$

where $\mathbf{0} = eK$ denotes the origin of X.

Proof. Let $\mathrm{Conv}(\Lambda_\Gamma)$ be the convex hull of the limit set Λ_Γ of Γ. Recall that Γ is said to be convex cocompact if $\Gamma\backslash \mathrm{Conv}(\Lambda_\Gamma)$ is compact. Let F be a compact fundamental domain containing $\mathbf{0}$ for the action of Γ on $\mathrm{Conv}(\Lambda_\Gamma)$. Then, for each $z \in \mathrm{Conv}(\Lambda_\Gamma)$, there exists $\gamma \in \Gamma$ and $z' \in F$ such that $z = \gamma z'$.

The orthogonal projection $\pi_\perp : X \to \mathrm{Conv}(\Lambda_\Gamma)$ is defined as follows. For every x in X, $\pi_\perp(x)$ is the unique point in $\mathrm{Conv}(\Lambda_\Gamma)$ such that

$$d(x, \pi_\perp(x)) = d(x, \mathrm{Conv}(\Lambda_\Gamma)) := \inf_{y\in \mathrm{Conv}(\Lambda_\Gamma)} d(x, y).$$

Then, for all $x, y \in X$, we have (see [8], Chap II, Proposition 2.4.)

$$d(\pi_\perp(x), \pi_\perp(y)) \leq d(x, y).$$

On the other hand, for all $x \in X$ and $\gamma \in \Gamma$,

$$d(\gamma x, \pi_\perp(\gamma x)) = d(\gamma x, \mathrm{Conv}(\Lambda_\Gamma)) = d(x, \mathrm{Conv}(\Lambda_\Gamma)),$$

since Λ_Γ and $\mathrm{Conv}(\Lambda_\Gamma)$ are Γ-invariant. Thus

$$d(\gamma x, \pi_\perp(\gamma x)) = d(x, \pi_\perp(x)) = d(\gamma x, \gamma\pi_\perp(x)),$$

which implies that, for all $x \in X$ and $\gamma \in \Gamma$, $\pi_\perp(\gamma x) = \gamma\pi_\perp(x)$. Therefore, for every $x, y \in X$ and $s > 0$, the Poincaré series satisfies:

$$\begin{aligned} P(s; x, y) = \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} &\leq \sum_{\gamma\in\Gamma} e^{-sd(\pi_\perp(x),\pi_\perp(\gamma y))} \\ &= \sum_{\gamma\in\Gamma} e^{-sd(\pi_\perp(x),\gamma\pi_\perp(y))} = P(s; \pi_\perp(x), \pi_\perp(y)), \end{aligned} \tag{3.4}$$

with $\pi_\perp(x), \pi_\perp(y) \in \mathrm{Conv}(\Lambda_\Gamma)$. Moreover, as there exist $\gamma_1, \gamma_2 \in \Gamma$ and $x', y' \in F$ such that $\pi_\perp(x) = \gamma_1 x'$, $\pi_\perp(y) = \gamma_2 y'$, we have

$$P(s; \pi_\perp(x), \pi_\perp(y)) = \sum_{\gamma\in\Gamma} e^{-sd(x', \gamma_1^{-1}\gamma\gamma_2 y')} = \sum_{\gamma'\in\Gamma} e^{-sd(x',\gamma' y')} = P(s; x', y'). \tag{3.5}$$

Since $x', y' \in F$, the triangular inequality yields

$$d(\mathbf{0}, \gamma'\mathbf{0}) \leq d(\mathbf{0}, x') + d(x', \gamma' y') + d(\gamma' y', \gamma'\mathbf{0}) \leq d(x', \gamma' y') + 2\,\mathrm{diam}(F).$$

Hence

$$P(s; x', y') \leq e^{2s\,\mathrm{diam}(F)} P(s; \mathbf{0}, \mathbf{0}). \tag{3.6}$$

We conclude by combining (3.4), (3.5) and (3.6). □

Consider the radial weight function defined by

$$\mu(x) = e^{(\delta(\Gamma)+\varepsilon)d(x,\mathbf{0})},$$

with $0 < \varepsilon < \rho - \delta(\Gamma)$. We prove the following lemma by applying previous results.

Lemma 3.4. *Let f be a reasonable function on M, and g be a radial reasonable function on X. Then the bilinear operator $B(f,g) := f * (\mu^{-1}g)$ satisfies the following estimate:*

$$\|B(\cdot,g)\|_{L^{q'}(M)\to L^q(M)} \leq C_q \left(\int_G \varphi_0(x)\mu^{-1}(x)|g(x)|^{q/2}dx\right)^{2/q},$$

for all $2 \leq q \leq \infty$.

Proof. According to Proposition 3.2,

$$\|B(\cdot,g)\|_{L^2(M)\to L^2(M)} \leq \int_G \varphi_0(x)\mu^{-1}(x)|g(x)|dx. \tag{3.7}$$

Since f is left Γ-invariant, we can rewrite

$$\begin{aligned} B(f,g)(x) &= \int_X \left(\mu^{-1}g\right)\left(y^{-1}x\right) f(y)dy \\ &= \int_{\Gamma\backslash X} \left(\sum_{\gamma\in\Gamma} \left(\mu^{-1}g\right)\left(y^{-1}\gamma x\right)\right) f(y)dy, \end{aligned}$$

with

$$\left|\sum_{\gamma\in\Gamma} \left(\mu^{-1}g\right)\left(y^{-1}\gamma x\right)\right| \leq ||g||_\infty \sum_{\gamma\in\Gamma} e^{-(\delta(\Gamma)+\varepsilon)d(x,\gamma y)}.$$

According to the previous lemma, the last sum is uniformly bounded. Hence

$$\|B(\cdot,g)\|_{L^1(M)\to L^\infty(M)} = \sup_{x,y\in G}\left|\sum_{\gamma\in\Gamma} \left(\mu^{-1}g\right)\left(y^{-1}\gamma x\right)\right| \leq C||g||_\infty. \tag{3.8}$$

We conclude by standard interpolations between (3.7) and (3.8). □

We prove now our main result.

Proof of Theorem 1.2. We split up the proof into two parts, depending whether the time t is small or large.

Dispersive estimate for small time

Assume that $0 < |t| < 1$. On the one hand, by using the previous lemma with $g(x) = \mu(x)\omega_t^{\sigma,0}(x)$, we have

$$\|\cdot * \omega_t^{\sigma,0}\|_{L^{q'}(M)\to L^q(M)} \leq C_q \left(\int_G \varphi_0(x)\mu(x)^{\frac{q}{2}-1}|\omega_t^{\sigma,0}(x)|^{\frac{q}{2}}dx\right)^{\frac{2}{q}}.$$

Notice that the ground spherical function φ_0, the weight μ and the kernel $\omega_t^{\sigma,0}$ are all bi-K-invariant. By using the expression (2.1) of the Haar measure in the Cartan decomposition, together with the estimate (2.2), we obtain fisrt

$$\int_G \varphi_0(x)\mu(x)^{\frac{q}{2}-1}|\omega_t^{\sigma,0}(x)|^{\frac{q}{2}}dx \lesssim \int_0^{+\infty} \varphi_0(r)\mu(r)^{\frac{q}{2}-1}|\omega_t^{\sigma,0}(r)|^{\frac{q}{2}}e^{2\rho r}dr.$$

As $|\omega_t^{\sigma,0}(r)| \lesssim \varphi_0(r)$, according to Theorem 2.1, and [2] $\varphi_0(r) \asymp (1+r)e^{-\rho r}$, we obtain next

$$\int_0^{+\infty} \varphi_0(r)\mu(r)^{\frac{q}{2}-1}|\omega_t^{\sigma,0}(r)|^{\frac{q}{2}}e^{2\rho r}dr \lesssim \int_0^{+\infty}(1+r)^{\frac{q}{2}+1}e^{-\left(\frac{q}{2}-1\right)(\rho-\delta(\Gamma)-\varepsilon)r}dr.$$

Since $\rho-\delta(\Gamma)-\varepsilon>0$, the last integral is finite of any $2<q<+\infty$. By using Lemma (3.4), we conclude that

$$\sup_{0<|t|<1}\|\cdot *\omega_t^{\sigma,0}\|_{L^{q'}(M)\to L^q(M)}<+\infty.$$

On the other hand, consider the analytic family of operators $\widetilde{W}_t^{\sigma,\infty}$ defined by (2.5). When $\mathrm{Re}\,\sigma=0$, the spectral theorem yields

$$\left\|\widetilde{W}_t^{\sigma,\infty}\right\|_{L^2(M)\to L^2(M)} \lesssim \left\|e^{itD}\right\|_{L^2(M)\to L^2(M)}=1, \tag{3.9}$$

for all $t\in\mathbb{R}^*$. When $\mathrm{Re}\,\sigma=\frac{n+1}{2}$, Theorem 2.1 yields

$$\left\|\widetilde{W}_t^{\sigma,\infty}\right\|_{L^1(M)\to L^\infty(M)}=\sup_{x,y\in M}\left\|\widehat{\tilde{\omega}_t^\sigma}(x,y)\right\| \lesssim \|\mu\tilde{\omega}_t^\sigma(x,y)\|_\infty \lesssim |t|^{-\frac{n-1}{2}}$$

in dimension $n\geq 3$. By applying Stein's interpolation theorem for an analytic family of operators, we obtain

$$\left\|\widetilde{W}_t^{\frac{n+1}{2}(1-\theta),\infty}\right\|_{L^{q'}(M)\to L^q(M)} \lesssim |t|^{-\frac{n-1}{2}(1-\theta)},$$

where $\theta=\frac{2}{q}$, that is

$$\|\cdot *\widetilde{\omega}_t^{\sigma,\infty}\|_{L^{q'}(M)\to L^q(M)} \lesssim |t|^{-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)},$$

with $\sigma=(n+1)(\frac{1}{2}-\frac{1}{q})$. In conclusion,

$$\left\|\widehat{W_t^\sigma}\right\|_{L^{q'}(M)\to L^q(M)} \lesssim |t|^{-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)},\ \forall 0<|t|<1$$

for $n\geq 3$, $\sigma=(n+1)(\frac{1}{2}-\frac{1}{q})$ and $2<q<\infty$. In dimension $n=2$, the same arguments yield

$$\left\|\widehat{W_t^\sigma}\right\|_{L^{q'}(M)\to L^q(M)} \lesssim |t|^{-\left(\frac{1}{2}-\frac{1}{q}\right)}(1-\log|t|)^{1-\frac{2}{q}},\ \forall 0<|t|<1$$

for $\sigma=3\left(\frac{1}{2}-\frac{1}{q}\right)$ and $2<q<+\infty$.

Dispersive estimate for small time

Assume now that $|t|\geq 1$. We proceed as before after splitting up the kernel as follows:

$$\omega_t^\sigma=\mathbb{1}_{B\left(0,\frac{|t|}{2}\right)}\omega_t^{\sigma,0}+\mathbb{1}_{X\setminus B\left(0,\frac{|t|}{2}\right)}\omega_t^{\sigma,0}+\omega_t^{\sigma,\infty}.$$

[2] The symbol $\asymp$ means that there exist two constants $0<C_1\leq C_2<+\infty$ such that

$$C_1\leq\frac{\varphi_0(r)}{(1+r)e^{-\rho r}}\leq C_2,\ \forall r\geq 0.$$

By using Lemma 3.4 and Theorem 2.1, we obtain

$$\begin{aligned}\left\|\cdot * \mathbb{1}_{B\left(0,\frac{|t|}{2}\right)}\omega_t^{\sigma,0}\right\|_{L^{q'}(M)\to L^q(M)} &\lesssim \left\{\int_0^{\frac{|t|}{2}} \varphi_0(r)\mu(r)^{\frac{q}{2}-1}|\omega_t^{\sigma,0}(r)|^{\frac{q}{2}}e^{2\rho r}dr\right\}^{\frac{2}{q}}\\ &\lesssim |t|^{-\frac{3}{2}}\underbrace{\left\{\int_0^{\frac{|t|}{2}}(1+r)^{1+q}e^{-\left(\frac{q}{2}-1\right)(\rho-\delta(\Gamma)-\varepsilon)r}dr\right\}^{\frac{2}{q}}}_{<+\infty}\end{aligned}$$

and

$$\begin{aligned}\left\|f * \mathbb{1}_{X\setminus B\left(0,\frac{|t|}{2}\right)}\omega_t^{\sigma,0}\right\|_{L^{q'}(M)\to L^q(M)} &\lesssim \left\{\int_{\frac{|t|}{2}}^{+\infty} \varphi_0(r)\mu(r)^{\frac{q}{2}-1}|\omega_t^{\sigma,0}(r)|^{\frac{q}{2}}e^{2\rho r}dr\right\}^{\frac{2}{q}}\\ &\lesssim \underbrace{\left\{\int_{\frac{|t|}{2}}^{+\infty}(1+r)^{\frac{q}{2}+1}e^{-\left(\frac{q}{2}-1\right)(\rho-\delta(\Gamma)-\varepsilon)r}dr\right\}^{\frac{2}{q}}}_{\lesssim |t|^{-\infty}}.\end{aligned}$$

Instead of $\omega_t^{\sigma,\infty}$, we consider again the kernel $\widetilde{\omega}_t^{\sigma,\infty}$. By Theorem 2.1, the associated operators satisfy

$$\left\|\widetilde{W}_t^{\sigma,\infty}\right\|_{L^1(M)\to L^\infty(M)} \lesssim |t|^{-N},\ \forall N\in\mathbb{N}$$

when $\operatorname{Re}\sigma = \frac{n+1}{2}$. By using again Stein's interpolation theorem and by summing up these estimates, we obtain finally

$$\left\|\widehat{W_t^\sigma}\right\|_{L^{q'}(M)\to L^q(M)} \lesssim |t|^{-\frac{3}{2}},\ \forall |t|\geq 1$$

for $n\geq 2$, $\sigma=(n+1)(\frac{1}{2}-\frac{1}{q})$ and $2<q<\infty$. □

4. Strichartz estimate and applications

Let $\sigma\in\mathbb{R}$ and $1<q<\infty$. Recall that the Sobolev space $H^{\sigma,q}(M)$ is the image of $L^q(M)$ under the operator $(-\Delta)^{-\frac{\sigma}{2}}$, equipped with the norm

$$\|f\|_{H^{\sigma,q}(M)} = \|(-\Delta)^{\frac{\sigma}{2}}f\|_{L^q(M)}.$$

If $\sigma=N$ is a nonnegative integer, then $H^{\sigma,q}(M)$ coincides with the classical Sobolev space

$$W^{N,q}(M) = \left\{f\in L^q(M)\mid \nabla^j f\in L^q(M),\ \forall 1\leq j\leq N\right\},$$

defined by means of covariant derivatives. The following Sobolev embedding theorem is used in next subsection:

Theorem 4.1. *Let $1<q_1,q_2<\infty$ and $\sigma_1,\sigma_2\in\mathbb{R}$ such that $\sigma_1-\sigma_2\geq\frac{n}{q_1}-\frac{n}{q_2}\geq 0$. Then*

$$H^{\sigma_1,q_1}(M)\subset H^{\sigma_2,q_2}(M). \tag{4.1}$$

We refer to [29] for more details about function spaces on Riemannian manifolds. Let us state next the Strichartz estimate and some applications. The proofs are straightforwardly adapted from [4] and are therefore omitted.

4.1. **Strichartz estimate.**

Recall the linear inhomogenous Klein-Gordon equation on M:

$$\begin{cases} \partial_t^2 u(t,x) + D^2 u(t,x) = F(t,x), \\ u(0,x) = f(x),\ \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{4.2}$$

whose solution is given by Duhamel's formula:

$$u(t,x) = (\cos tD)\, f(x) + \frac{\sin tD}{D} g(x) + \int_0^t \frac{\sin(t-s)D}{D} F(s,x) ds.$$

We consider first the case $n \geq 4$ and discuss the 2-dimensional and 3-dimensional cases in the final remarks. Recall that a couple (p,q) is called admissible if $\left(\frac{1}{p}, \frac{1}{q}\right)$ belongs to the triangle

$$\left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) \;\middle|\; \frac{1}{p} \geq \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) \right\} \bigcup \left\{ \left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{n-1}\right) \right\}.$$

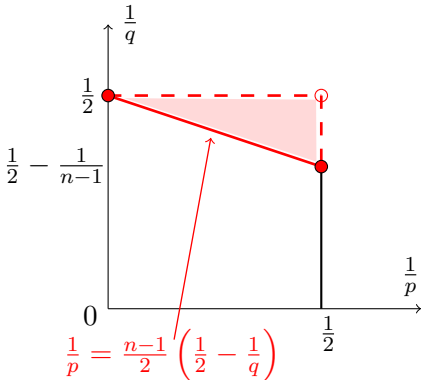

FIGURE 2. Admissibility in dimension $n \geq 4$.

Theorem 4.2. *Let (p,q) and $(\tilde{p},\tilde{q})$ be two admissible couples, and let*

$$\sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) \text{ and } \widetilde{\sigma} \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{\tilde{q}}\right).$$

Then all solutions u to the Cauchy problem (4.2) satisfy the following Strichartz estimate:

$$\|\nabla_{\mathbb{R}\times M} u\|_{L^p(I;H^{-\sigma,q}(M))} \lesssim \|f\|_{H^1(M)} + \|g\|_{L^2(M)} + \|F\|_{L^{\tilde{p}'}(I;H^{\widetilde{\sigma},\tilde{q}'}(M))}. \tag{4.3}$$

Remark 4.3. In comparison with hyperbolic spaces , observe that we loose the right edge of the admissible triangle. The reason is that the standard TT^* method used to prove the Strichartz estimate breaks down in the critical case where $p = 2$ and $q < 2\frac{n-1}{n-3}$. The dyadic decomposition method carried out in [22] takes care of the endpoints, but it requires a stronger dispersive property than Theorem 1.2 in small time, which reads

$$\left\| \widehat{W_t^\sigma} \right\|_{L^{q'}(M) \to L^q(M)} \lesssim |t|^{-(n-1)\max\left(\frac{1}{2}-\frac{1}{q}, \frac{1}{2}-\frac{1}{\tilde{q}}\right)}, \ \forall\ 0 < |t| < 1$$

for $n \geq 3$, $2 < q, \tilde{q} < \infty$ and $\sigma \geq (n+1)\max\left(\frac{1}{2} - \frac{1}{q}, \frac{1}{2} - \frac{1}{\tilde{q}}\right)$. Such an estimate would follow from

$$\left\|\widehat{\omega_t^\sigma}\right\|_{L^q(M)} \lesssim |t|^{-\frac{n-1}{2}}, \ \forall 0 < |t| < 1$$

for $\sigma \geq \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$ and $2 < q < +\infty$, which is unknown so far.

However, these critical points are not relevant for the following well-posedness problems, hence we obtain the same results as on real hyperbolic spaces. The admissible range in (4.3) can be widen by using the Sobolev embedding theorem.

Corollary 4.4. *Let (p, q) and $(\tilde{p}, \tilde{q})$ be two couples corresponding to the square*

$$\left[0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) \bigcup \left\{\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2} - \frac{1}{n-1}\right)\right\},$$

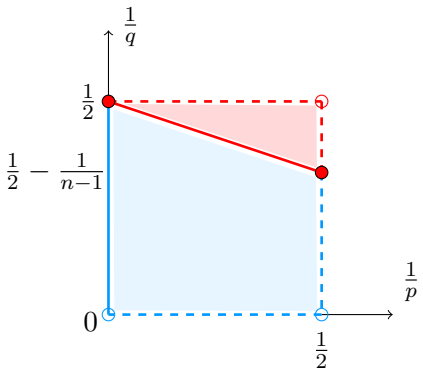

FIGURE 3. Case $n \geq 4$

Let $\sigma, \tilde{\sigma} \in \mathbb{R}$ such that $\sigma \geq \sigma(p, q)$, where

$$\sigma(p, q) = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) + \max\left\{0, \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{p}\right\},$$

and similarly $\tilde{\sigma} \geq \sigma(\tilde{p}, \tilde{q})$. Then the Strichartz estimate (4.3) holds for all solutions to the Cauchy problem (4.2).

Remark 4.5. Theorem 4.2 and Corollary 4.4 still hold true in lower dimension $n = 3$ and $n = 2$ with similar proofs. In particular, the endpoint $(p, q) = (2, \infty)$ is excluded and the admissible set in dimension 2 becomes

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right) \ \middle| \ \frac{1}{p} > \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right\} \bigcup \left\{\left(0, \frac{1}{2}\right)\right\},$$

and the region in Corollary 4.4 is

$$\left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in \left(0, \frac{1}{4}\right) \times \left(0, \frac{1}{2}\right) \ \middle| \ \frac{1}{p} \leq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)\right\}.$$

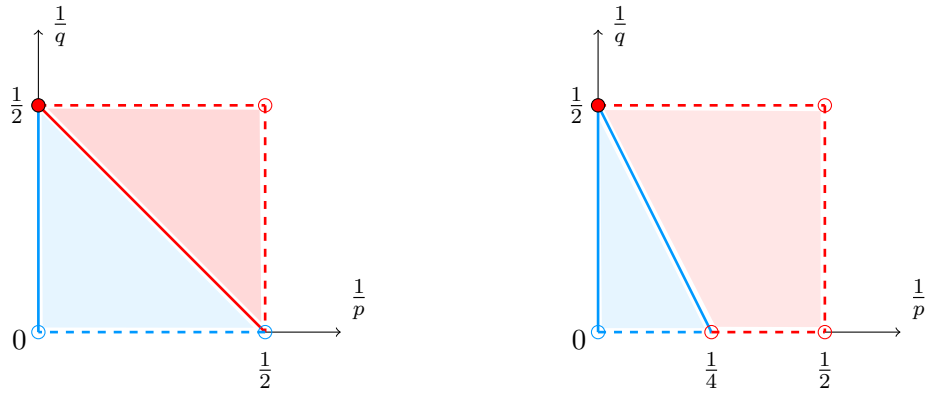

FIGURE 4. Cases $n = 3$ and $n = 2$

4.2. **Global well-posedness in $L^p(\mathbb{R}, L^q(M))$.**

We refer to [4] for more detailed proofs of the following well-posedness results. By using the classical fixed point scheme with the previous Strichartz estimates, one obtains the global well-posedness for the semilinear equation

$$\begin{cases} \partial_t^2 u(t,x) + D^2 u(t,x) = F(u(t,x)), \\ u(0,x) = f(x),\ \partial_t|_{t=0} u(t,x) = g(x). \end{cases} \tag{4.4}$$

on M with power-like nonlinearities F satisfying

$$|F(u)| \le C|u|^\gamma, \quad |F(u) - F(v)| \le C\left(|u|^{\gamma-1} + |v|^{\gamma-1}\right)|u - v|, \quad \gamma > 1.$$

and small initial data f and g. Assume that $n \ge 3$, and consider the following powers

$$\gamma_1 = 1 + \frac{3}{n},\ \gamma_2 = 1 + \frac{2}{\frac{n-1}{2} + \frac{2}{n-1}},\ \gamma_c = 1 + \frac{4}{n-1},$$

$$\gamma_3 = \begin{cases} \frac{\frac{n+6}{2} + \frac{2}{n-1} + \sqrt{4n + \left(\frac{6-n}{2} + \frac{2}{n-1}\right)^2}}{n} & if\ n \le 5, \\ 1 + \frac{2}{\frac{n-1}{2} - \frac{1}{n-1}} & if\ n \ge 6, \end{cases}$$

$$\gamma_4 = \begin{cases} 1 + \frac{4}{n-2} & if\ n \le 5, \\ \frac{n-1}{2} + \frac{3}{n+1} - \sqrt{\left(\frac{n-3}{2} + \frac{3}{n+1}\right)^2 - 4\frac{n-1}{n+1}} & if\ n \ge 6, \end{cases}$$

and the following curves

$$\sigma_1(\gamma) = \frac{n+1}{4} - \frac{(n+1)(n+5)}{8n}\frac{1}{\gamma - \frac{n+1}{2n}},$$

$$\sigma_2(\gamma) = \frac{n+1}{4} - \frac{1}{\gamma - 1}, \quad \sigma_3(\gamma) = \frac{n}{2} - \frac{2}{\gamma - 1}.$$

Denote by 0^+ any small positive constant. In dimension $n \geq 3$, the equation (4.4) is globally well-posed for small initial data in $H^{\sigma,2}(M) \times H^{\sigma-1,2}(M)$ provided that

$$\begin{cases} \sigma = 0^+, & if\ 1 < \gamma \leq \gamma_1, \\ \sigma = \sigma_1(\gamma), & if\ \gamma_1 < \gamma \leq \gamma_2, \\ \sigma = \sigma_2(\gamma), & if\ \gamma_2 < \gamma \leq \gamma_c, \\ \sigma = \sigma_3(\gamma), & if\ \gamma_c < \gamma \leq \gamma_4, \end{cases} \tag{4.5}$$

Similar results hold in dimension 2, see [4]. Observe that one obtains the same global well-posedness results on M as on real hyperbolic spaces, without further assumptions. In comparison with the Euclidean setting, this is a consequence of the larger admissible set for the Strichartz estimate.

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