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# SEM-DG Approximation for elasto-acoustics

Hélène Barucq<sup>1</sup>, Henri Calandra<sup>2</sup>, Aurélien Citrain<sup>3,1</sup>, Julien Diaz<sup>1</sup> and Christian Gout<sup>3</sup>

<sup>1</sup> Team project Magique.3D, INRIA, E2S UPPA, CNRS, Pau, France.

<sup>2</sup> TOTAL SA, CSTJF, Pau, France.

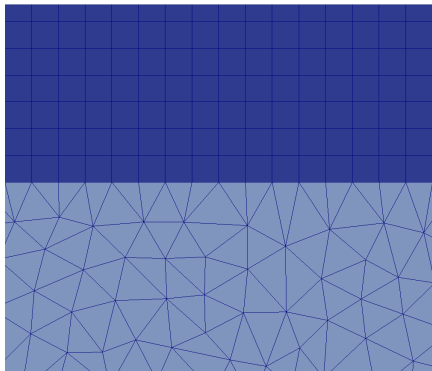
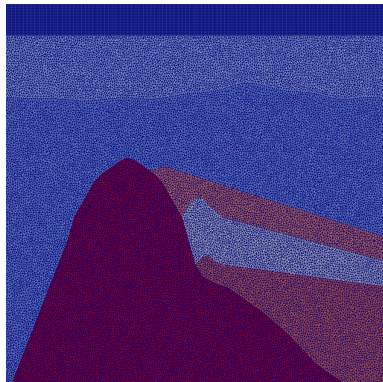
<sup>3</sup> INSA Rouen-Normandie Université, LMI EA 3226, 76000, Rouen.

MATHIAS 2018 October 22-24



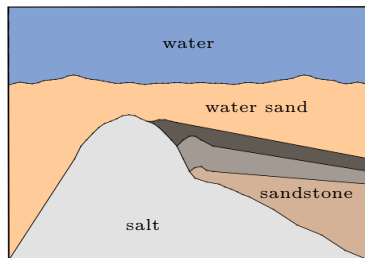
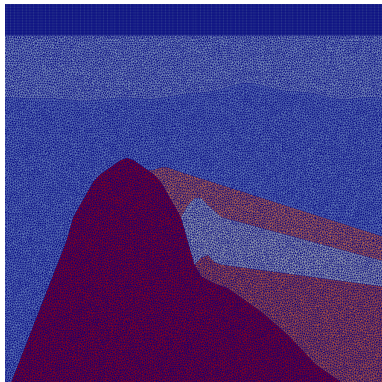
*The authors thank the M2NUM project which is co-financed by the European Union with the European regional development fund (ERDF, HN0002137) and by the Normandie Regional Council.*

## Why using hybrid meshes?



- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water).
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy.

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$x \in \Omega \subset \mathbb{R}^d$ ,  $t \in [0, T]$ ,  $T > 0$  :

$$\begin{cases} \rho(x) \frac{\partial v}{\partial t}(x, t) &= \nabla \cdot \underline{\underline{\sigma}}(x, t), \\ \frac{\partial \underline{\underline{\sigma}}}{\partial t}(x, t) &= \underline{\underline{C}}(x) \underline{\underline{\epsilon}}(v(x, t)). \end{cases}$$

With:

- $\rho(x)$  the density,
- $\underline{\underline{C}}(x)$  the elasticity tensor,
- $\underline{\underline{\epsilon}}(x, t)$  the deformation tensor,
- $v(x, t)$ , the wavespeed,
- $\underline{\underline{\sigma}}(x, t)$  the strain tensor.

Software written in **Fortran** for wave propagation simulation in the **time domain**

## Features

Simulation:

- on various types of meshes (**unstructured triangles and tetrahedra**),
  - on **heterogeneous media (acoustic, elastic and elasto-acoustic)**.
- 
- **Discontinuous Galerkin (DG)** based on **unstructured triangles and unstructured tetrahedra**,
  - with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**,
  - with **multi-order** computation(**p-adaptivity**)...

- 1 Numerical Methods
- 2 Comparison DG/SEM on structured quadrangle mesh
- 3 DG/SEM coupling
- 4 3D extension

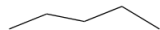
- 1 Numerical Methods
  - Discontinuous Galerkin Method (DG)
  - Spectral Element Method (SEM)
  - Advantages of each method



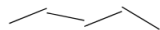
Use discontinuous functions :



mesh

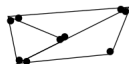


continuous

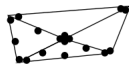


discontinuous

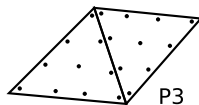
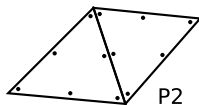
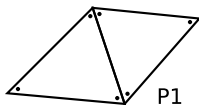
$h$  adaptivity :



$p$  adaptivity :

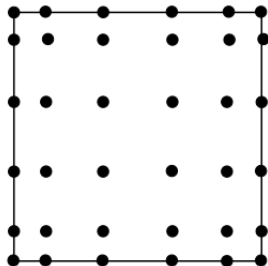


Degrees of freedom necessary on each cell :



## General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature,
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on  $L^2$ -norm).



- $\int f(x) dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j),$
- $\varphi_i(\xi_j) = \delta_{ij}.$

## DG

- Element per element computation ( *hp*-adaptivity).
- Time discretization quasi explicit (block diagonal mass matrix).
- Simple to parallelize.
- Robust to brutal changes of physics and geometry

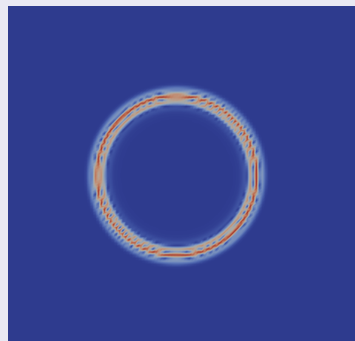
## SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).

## 2 Comparison DG/SEM on structured quadrangle mesh

- Description of the test cases
- Comparative tables

## Physical parameters



<i>P wavespeed</i>	$1000 \text{ m.s}^{-1}$
<i>Density</i>	$1 \text{ kg.m}^{-3}$

Second order **Ricker Source** in *P*wave  
( $f_{peak} = 10\text{Hz}$ )

## General context

- **Acoustic homogeneous** medium.
- Four different meshes : **10000 cells**, **22500 cells**, **90000 cells**, **250000 cells**.
- CFL computed using **power iteration** method.
- **Leap-Frog** time scheme.
- **Eight threads** parallel execution with **OpenMP**.

- Error computed as the difference between an analytical and a numerical solution for each method.
- Three cases considered : DG without penalization terms, DG with penalization terms and SEM.

	CFL	L2-error	CPU-time	Nb of time steps
DG( $\alpha = 0$ )	3.18e-3	2e-1	5.13	629
SEM	4.9e-3	5e-2	0.80	409

Figure: DG not penalized and SEM comparison on the 10000 cells case

	CFL	L2-error	CPU-time(s)	Nb of time steps
DG( $\alpha = 0$ )	2.12e-3	7e-1	18.11	943
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Figure: DG not penalized and SEM comparison on the 20000 cells case

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Figure: DG penalized and SEM comparison using the same CFL on a 10000 thousands cells mesh

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## DG

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## SEM

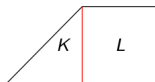
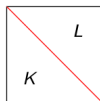
- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method.
- Simplifies the mass and stiffness matrices (mass matrix diagonal).
- **Reduces the computational costs on structured quadrangle cells in comparison with DG**



- 3 DG/SEM coupling
  - Hybrid meshes structures
  - Variational formulation

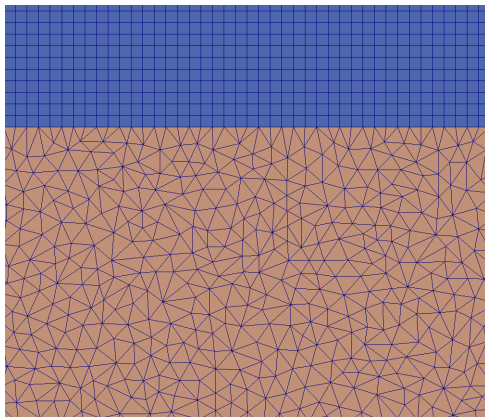
- Aim at coupling  $P_k$  and  $Q_k$  structures.
- Need to extend or split some structures (e.g. neighbour indexes).
- Define new face matrices:

$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L.$$



## Global context

- Domain in two parts :  $\Omega_{h,1}$  (**structured quadrangles + SEM**),  $\Omega_{h,2}$  (**unstructured triangles + DG**).



SEM variational formulation :

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1, \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C\xi_1 \mathbf{n}_1) \cdot v_1. \end{cases}$$

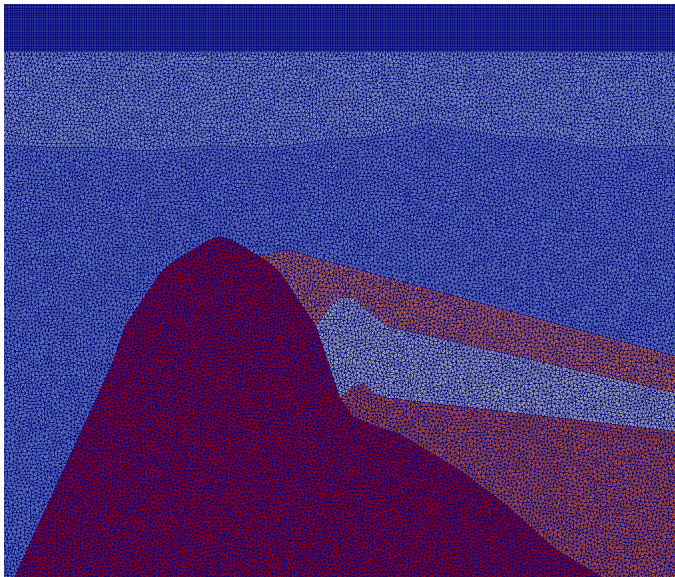
DG variational formulation :

$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [[w_2]] \cdot \mathbf{n}_2, \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C\xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} [[C\xi_2]] \cdot \mathbf{n}_2. \end{cases}$$

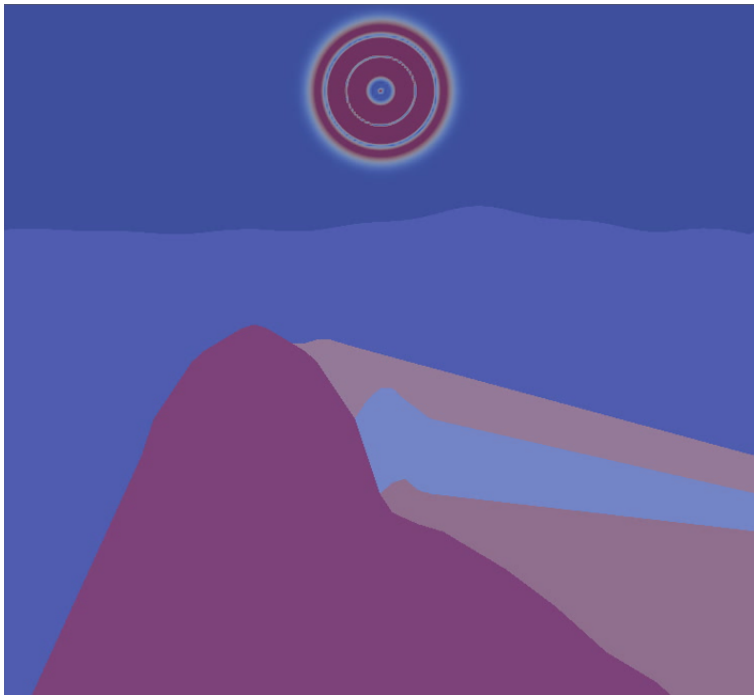
$$\left\{ \begin{aligned}
 & \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 + \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 \\
 & + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [ [ w_2 ] ] \cdot \mathbf{n}_2 \\
 & + \int_{\Gamma_{1/2}} [ [ \sigma w ] ] \cdot \mathbf{n}, \\
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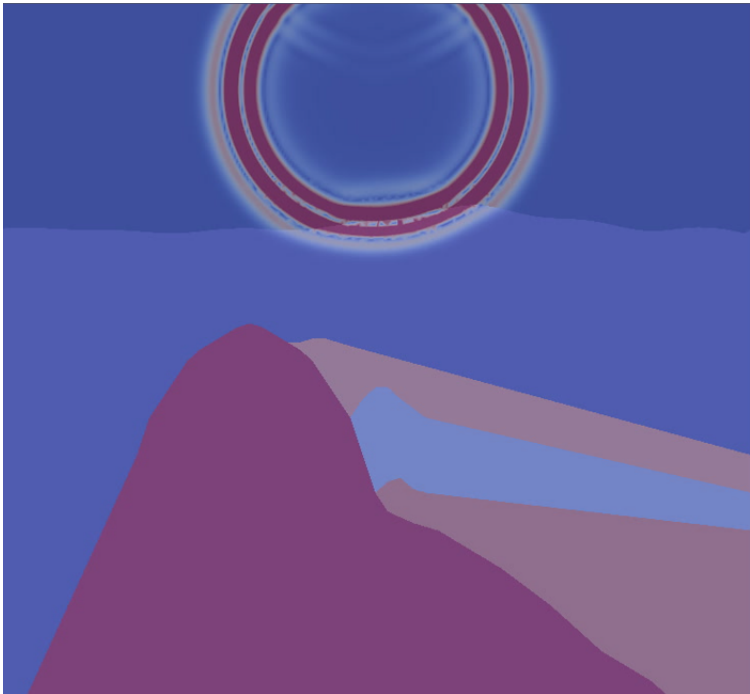
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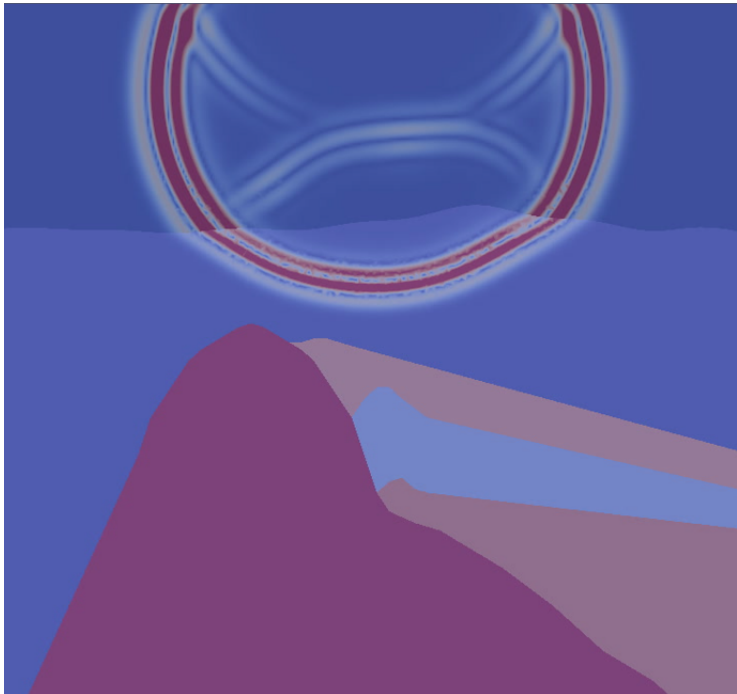
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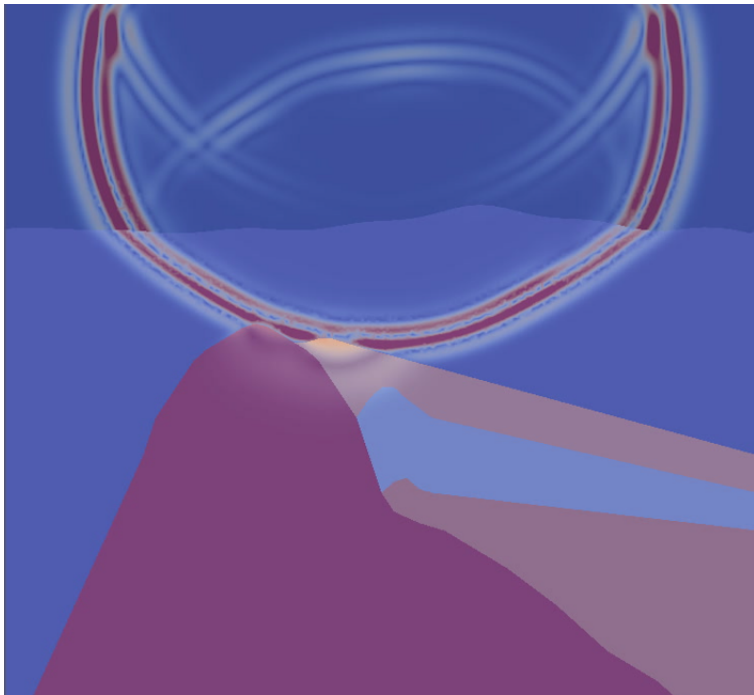


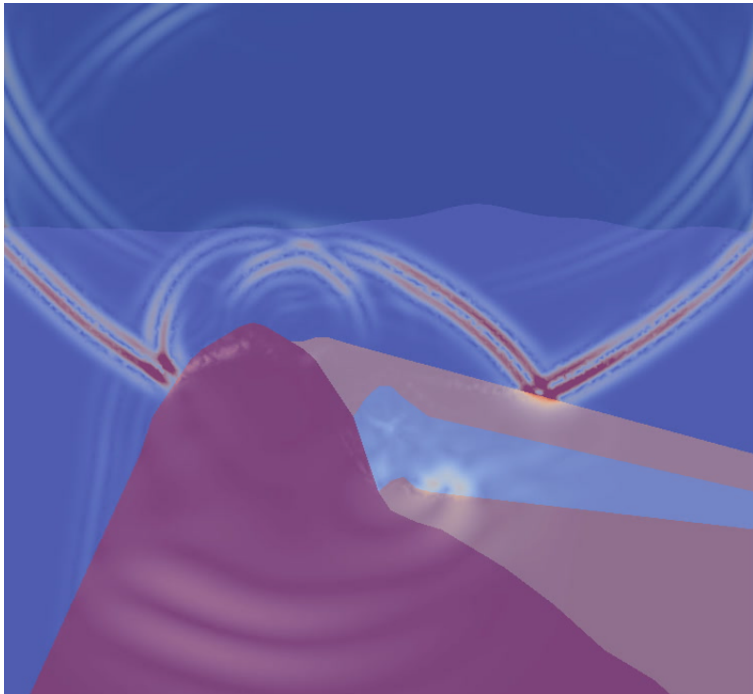












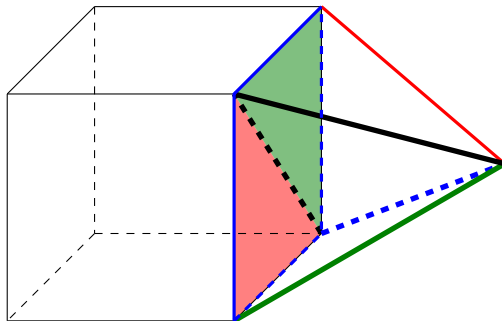


Figure: Hexa/Tet boundary configuration

- Only deal with a simple case of 3D hybrid meshes : one hexahedron has only two tetrahedra as neighbour.
- Extend SEM in 3D (basis functions...).
- Require introducing a new matrix which handles the rotation cases between two elements.

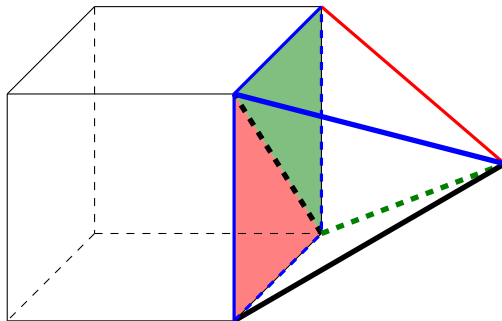


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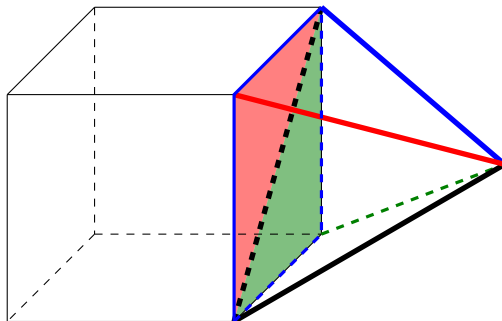


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- Extend SEM in 3D (basis functions...).
- Require introducing a new matrix which handles the rotation cases between two elements.



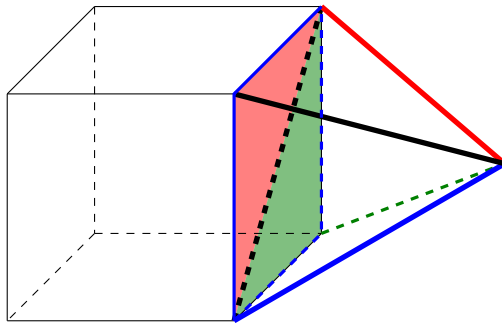
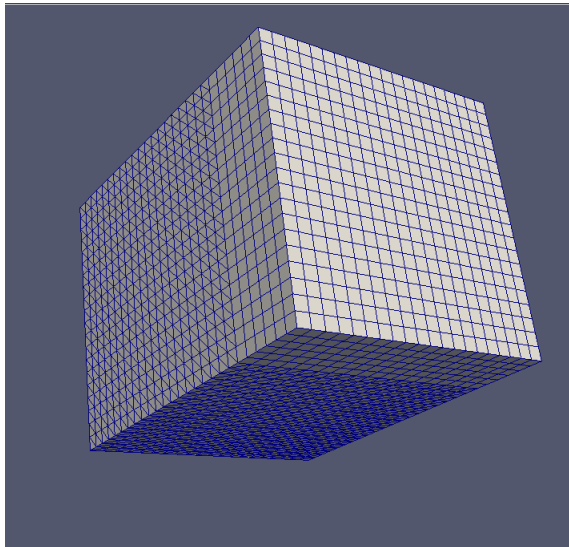
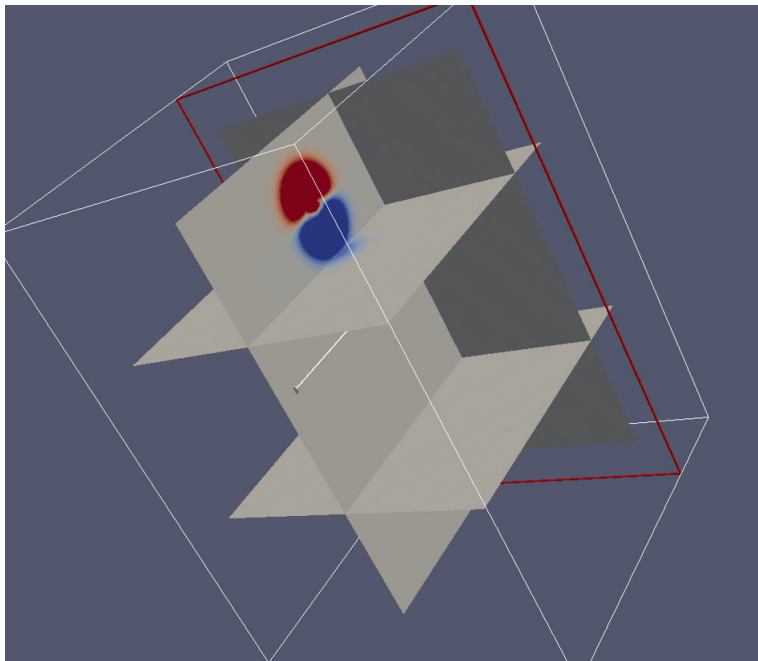
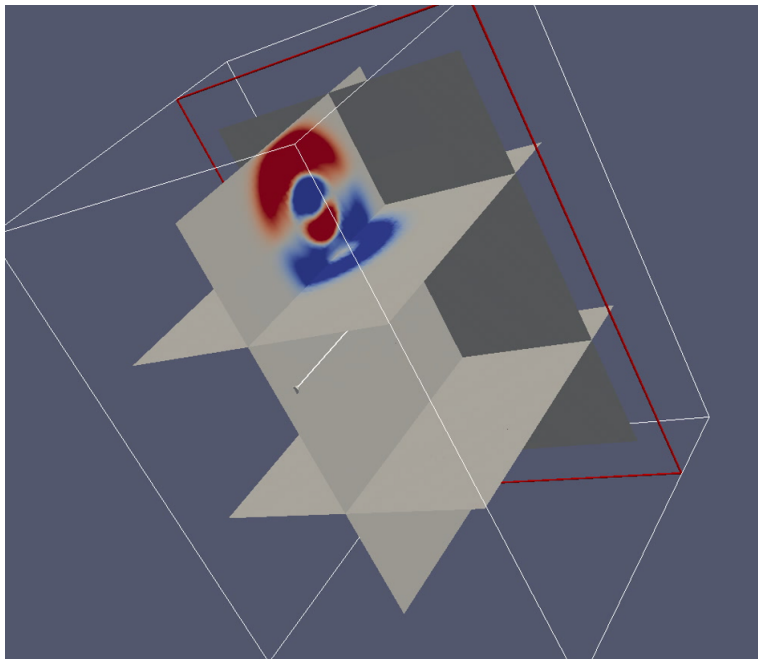


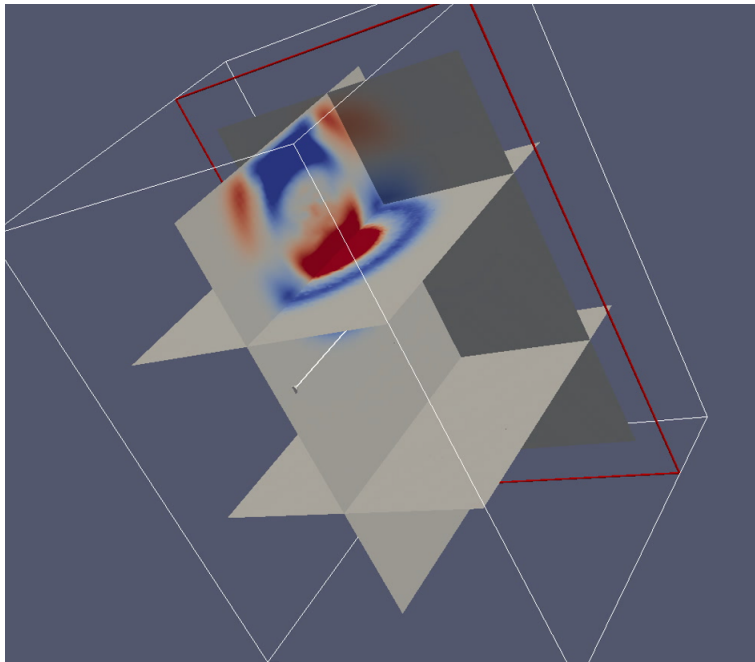
Figure: Hexa/Tet boundary configuration

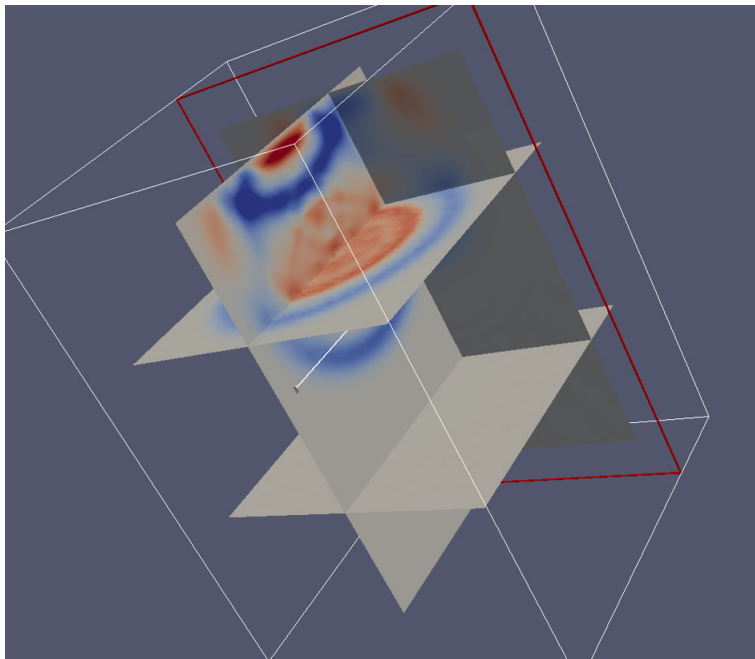
- Only deal with a simple case of 3D hybrid meshes : one hexahedron has only two tetrahedra as neighbour.
- Require introducing a new matrix which handles the rotation cases between two elements.

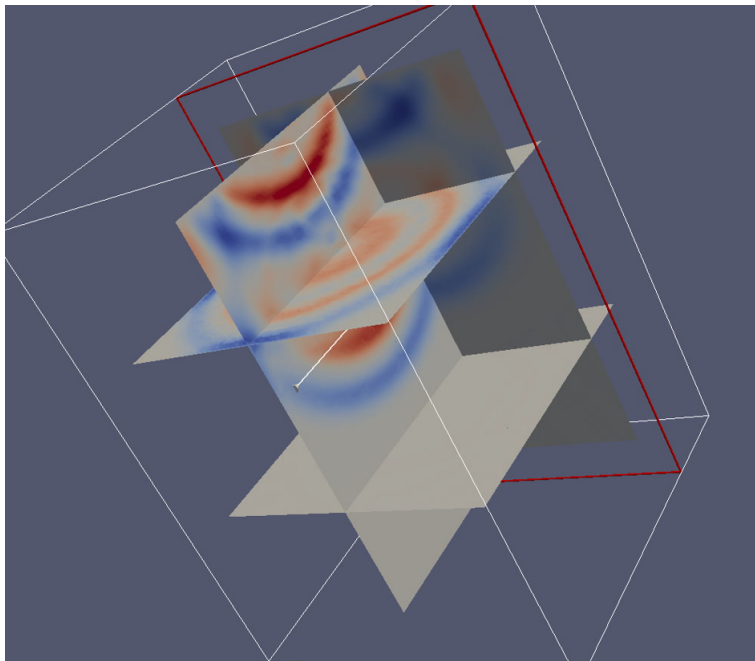


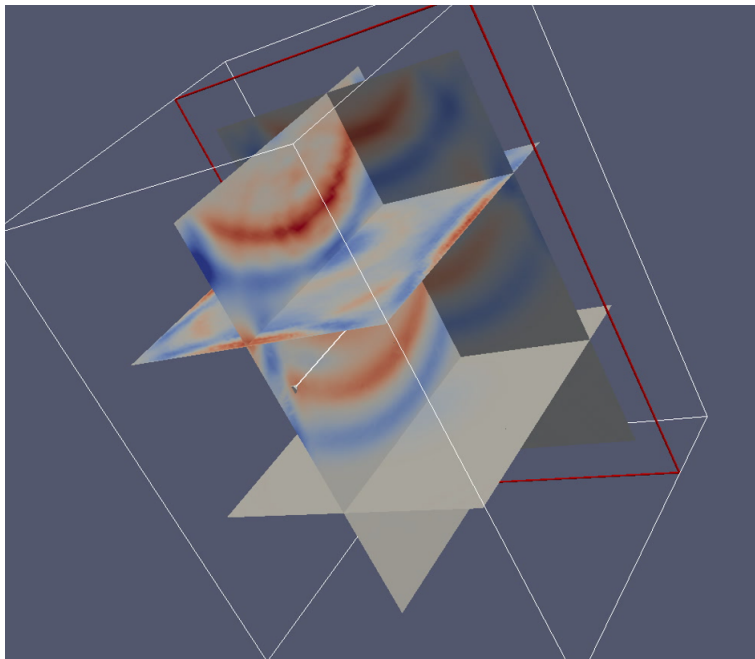




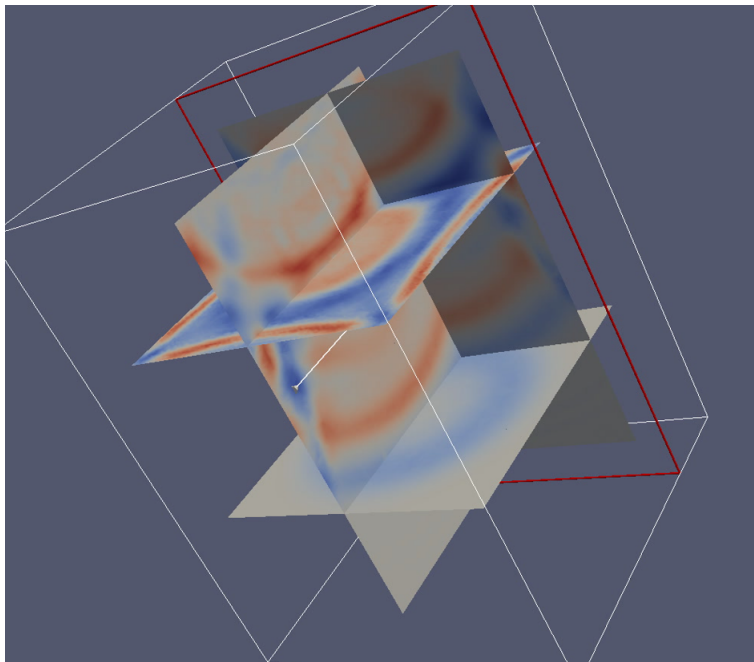












## Conclusion

- 1 SEM is more efficient on structured quadrangle mesh than DG
- 2 Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
- 3 Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

## Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Develop PML in the hexahedral part
- Add a local time-stepping scheme

*Thank you for your attention !*

Questions?

# Error-order graphic

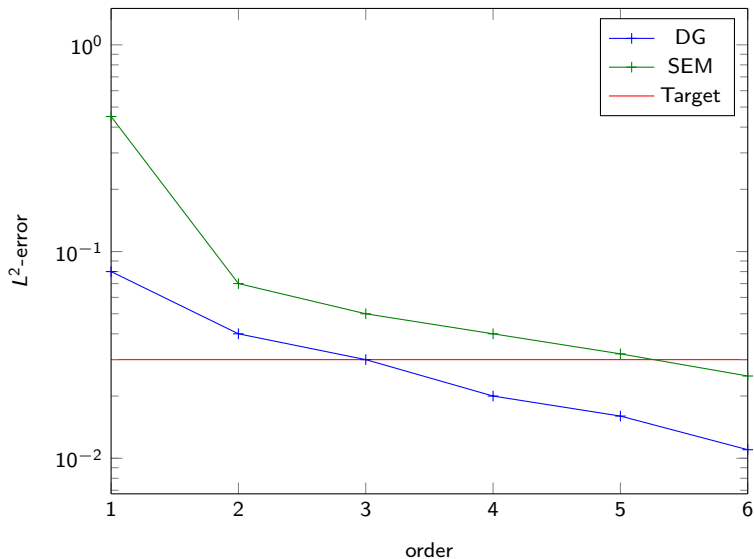


Figure:  $L^2$ -error comparison on a 10000 cells mesh

# Error-order graphic

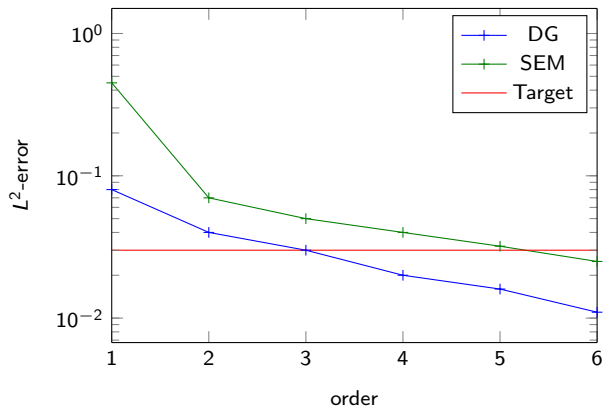


Figure:  $L^2$ -error comparison on a 10000 cells mesh

	CFL	L2-error	CPU-time	Nb of time steps
DG	2e-3	3e-2	7.93	1000
SEM(order five)	2.13e-3	3e-2	9.06	943

Figure: SEM and DG comparison with fixed error