



HAL
open science

SEM-DG Approximation for elasto-acoustics

Hélène Barucq, Henri Calandra, Aurélien Citrain, Julien Diaz, Christian Gout

► **To cite this version:**

Hélène Barucq, Henri Calandra, Aurélien Citrain, Julien Diaz, Christian Gout. SEM-DG Approximation for elasto-acoustics. MATHIAS 2018 Computational Science Engineering & Data Science by TOTAL, Oct 2018, Serris, Paris, France. hal-01872812v1

HAL Id: hal-01872812

<https://hal.science/hal-01872812v1>

Submitted on 12 Sep 2018 (v1), last revised 29 Oct 2018 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the coupling of Spectral Element Method with Discontinuous Galerkin approximation for elasto-acoustic problems

Hélène Barucq¹, Henri Calandra², Aurélien Citrain^{3,1}, Julien Diaz¹ and Christian Gout³

¹ Team project Magique.3D, INRIA.UPPA.CNRS, Pau, France.

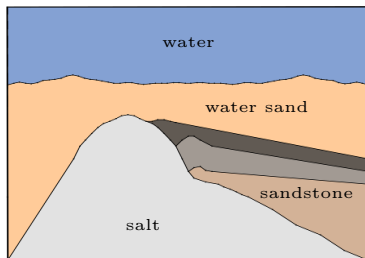
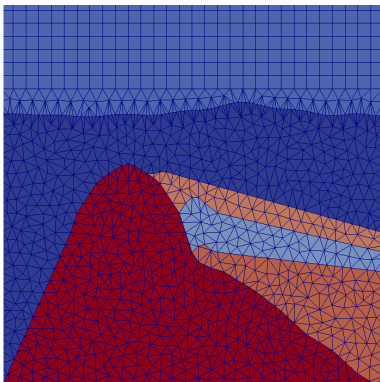
² TOTAL SA, CSTJF, Pau, France.

³ INSA Rouen-Normandie Université, LMI EA 3226, 76000, Rouen.

WCCM Congress - New York 25/07/2018



Why using hybrid meshes?



- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water)
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy

Elastodynamic system

$$\begin{cases} \rho(x) \frac{\partial \underline{v}}{\partial t}(x, t) & = \nabla \cdot \underline{\underline{\sigma}}(x, t) \\ \frac{\partial \underline{\underline{\sigma}}}{\partial t}(x, t) & = \underline{\underline{C}}(x) \underline{\underline{\epsilon}}(v(x, t)) \end{cases}$$

With :

- $\rho(x)$ the density
- $C(x)$ the elasticity tensor
- $\epsilon(x, t)$ the deformation tensor
- $v(x, t)$, the wavespeed
- $\underline{\underline{\sigma}}(x, t)$ the strain tensor

Elasticus software

Written in **Fortran** for wave propagation simulation in the **time domain**

Features

- Using various types of meshes (**unstructured triangle and tetrahedra**)
- Modelling of various physics (**acoustic, elastic and elasto-acoustic**)

- **Discontinuous Galerkin Method (DG)** based on **unstructured triangle and unstructured tetrahedra**
- with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**
- with **p-adaptivity, multi-order** computation...

Table of contents

- 1 Numerical Methods
- 2 Comparison DG/SEM on structured quadrangle mesh
- 3 DG/SEM coupling
- 4 Comparison between DG/SEM and DG on hybrid meshes
- 5 3D extension

1 Numerical Methods

- Discontinuous Galerkin Method (DG)
- Spectral Element Method (SEM)
- Advantages of each method

Discontinuous Galerkin Method

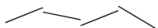
Use discontinuous functions :



mesh

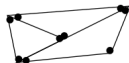


continuous

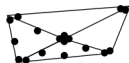


discontinuous

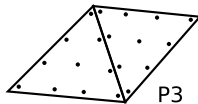
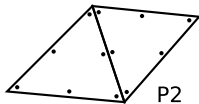
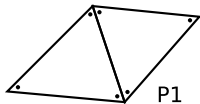
h adaptivity :



p adaptivity :



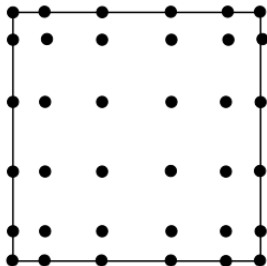
Degrees of freedom on each cell :



Spectral Element Method

General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature
- Gauss-Lobatto points as degrees of freedom (exponential convergence on L^2 -norm)



- $\int f(x) dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j)$
- $\varphi_i(\xi_j) = \delta_{ij}$

Spectral Element Method

Main change with DG

- DG discontinuous, SEM continuous
- Need of defining local to global numbering
- Global matrices required by SEM
- Basis functions computed differently

Advantages of each method

DG

- Element per element computation (*hp*-adaptivity)
- Time discretization quasi explicit (block diagonal mass matrix)
- Simple to parallelize

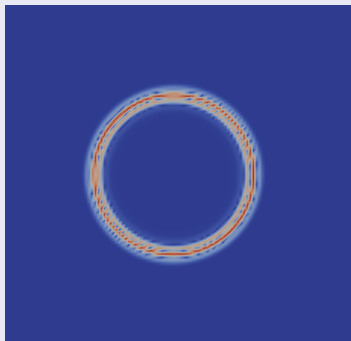
SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method
- Reduces the computational cost when using structured meshes in comparison with DG

- 2 Comparison DG/SEM on structured quadrangle mesh
 - DG/SEM comparison on quadrangle mesh
 - Description of the test cases
 - Comparative tables

Description of the test cases

Physical parameters



<i>P wavespeed</i>	1000 m.s^{-1}
<i>Density</i>	1 kg.m^{-3}

Second order **Ricker Source** in *P*wave
($f_{peak} = 10\text{Hz}$)

General context

- **Acoustic homogeneous** medium
- Four different meshes : **10000 cells**, **22500 cells**, **90000 cells**, **250000 cells**
- CFL computed using **power iteration** method
- **Leap-Frog** time scheme
- **Four threads** parallel execution with **OpenMP**

Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method

Quadrangle mesh 10000 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.99e-3	2.5e-2	19.30	500
SEM	4.9e-3	1.3e-1	0.36	204

Quadrangle mesh 22500 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	100.48	750
SEM	3.26e-3	7e-2	1.19	306

Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method

Quadrangle mesh 10000 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.99e-3	2.5e-2	19.30	500
SEM	4.9e-3	1.3e-1	0.36	204
SEM(DG CFL)	1.99e-3	4.8e-2	0.73	502

Quadrangle mesh 22500 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	100.48	750
SEM	3.26e-3	7e-2	1.19	306
SEM(DG CFL)	1.33e-3	1.2e-2	2.82	751

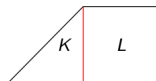
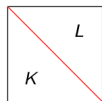
SEM fifty time faster than DG on a mesh with 22500 cells

- 3 DG/SEM coupling
 - Hybrid meshes structures
 - Variational formulation
 - Space discretization

Hybrid meshes structures

- Aim at coupling P_k and Q_k structures.
- Need to extend or split some of the structures (e.g. neighbour indexes)
- Define new face matrices

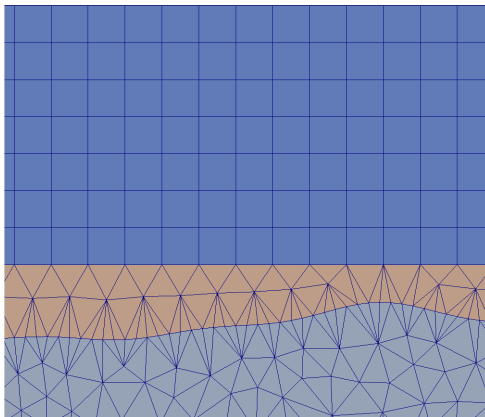
$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L$$



Variational formulation

Global context

- Domain in two parts : $\Omega_{h,1}$ (structured quadrangle + SEM), $\Omega_{h,2}$ (unstructured triangle + DG)



Variational formulation

SEM variational formulation :

$$\left\{ \begin{array}{l} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C\xi_1 \mathbf{n}_1) \cdot v_1 \end{array} \right.$$

DG variational formulation :

$$\left\{ \begin{array}{l} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [[w_2]] \cdot \mathbf{n}_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C\xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} [[C\xi_2]] \cdot \mathbf{n}_2 \end{array} \right.$$

Variational formulation

Add the average of the solution of each part at the interface + put $\sigma_* \mathbf{n}_* = 0$

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t \mathbf{v}_1 \cdot \mathbf{w}_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla \mathbf{w}_1 + \frac{1}{2} \int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2) \mathbf{n}_1 \cdot \mathbf{w}_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot \mathbf{v}_1 + \frac{1}{2} \int_{\Gamma_{1/2}} (C\xi_1 \mathbf{n}_1) \cdot (\mathbf{v}_1 + \mathbf{v}_2) \end{cases}$$

$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t \mathbf{v}_2 \cdot \mathbf{w}_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla \mathbf{w}_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [[\mathbf{w}_2]] \cdot \mathbf{n}_2 - \frac{1}{2} \int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2) \mathbf{n}_1 \cdot \mathbf{w}_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot \mathbf{v}_2 + \int_{\Gamma_{int}} \{ \{ \mathbf{v}_2 \} \} [[C\xi_2]] \cdot \mathbf{n}_2 \\ - \frac{1}{2} \int_{\Gamma_{1/2}} (C\xi_2 \mathbf{n}_1) \cdot (\mathbf{v}_1 + \mathbf{v}_2) \end{cases}$$

Energy study

Goal : Show that our coupling preserves the energy

- We set $\xi_1 = \sigma_1, \xi_2 = \sigma_2, w_1 = v_1, w_2 = v_2$
- We add the equations of the two parts variational formulation

$$\frac{d}{dt} \mathbb{E} = 0$$

Space discretization : SEM part

- φ_i : SEM basis functions
- ψ_j : DG basis functions

$$\begin{cases} M_{\mathbf{v}_1} \partial_t \mathbf{v}_{h,1} + R_{\sigma_1} \sigma_{h,1} + R_{\sigma_2}^{2,1} \sigma_{h,2} = 0 \\ M_{\sigma_1} \partial_t \sigma_{h,1} + R_{\mathbf{v}_1} \mathbf{v}_{h,1} + R_{\mathbf{v}_2}^{2,1} \mathbf{v}_{h,2} = 0 \end{cases}$$

$$\text{■ } M_{ij} = \int_{\Omega} \varphi_i \varphi_j \approx \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \sum_{k=1}^{(r+1)^d} \omega_k \varphi_i(\xi_k) \varphi_j(\xi_k) = \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \omega_i \delta_{i,j} \quad \text{the}$$

mass matrix

$$\text{■ } R_{p_{ij}} = \int_{\Omega} \varphi_i \frac{\partial \varphi_j}{\partial p} \quad \text{stiffness matrix}$$

Matrix of DG/SEM coupling :

$$R_{\sigma_2, ij}^{2,1} = \frac{1}{2} \int_{\partial\Omega_1 \cap \partial\Omega_2} \psi_i \varphi_j$$

Space discretization : DG part

$$\begin{cases} \rho M_{v_2} \partial_t \mathbf{v}_{h,2} + R_{\sigma_2} \boldsymbol{\sigma}_{h,2} - R_{\sigma_1}^{1,2} \boldsymbol{\sigma}_{h,1} = 0 \\ M_{\sigma_2} \partial_t \boldsymbol{\sigma}_{h,2} + R_{v_2} \mathbf{v}_{h,2} - R_{v_1}^{1,2} \mathbf{v}_{h,1} = 0 \end{cases}$$

- $M_{ij}^K = \int_K \psi_i^K \psi_j^K$ mass matrix,
- $R_{p_{ij}}^K = \int_K \psi_i^K \frac{\partial \psi_j^K}{\partial p}$ stiffness matrix,
- $R_{p_{ij}}^{K,L} = \int_{\partial K \cap \partial L} \psi_i^K \psi_j^L n_K \cdot e_p$ the mass-face matrix

Two new matrices which come from the DG/SEM coupling $R_{\star}^{1,2}$. Block composed :

$$R_{v_1}^{1,2} = R_{\sigma_1}^{1,2} = -\frac{1}{2} \int_{\partial \Omega_2 \cap \partial K_1} \psi_j^{K_2} \varphi_i \quad (1)$$

- 4 Comparison between DG/SEM and DG on hybrid meshes
 - Experimentation context
 - Comparative tables

Context

- Acoustic homogeneous medium
- 54000 triangles
- 21000 quadrangles
- Using Leap-Frog time scheme
- Parallel computation using OpenMP
- Done with different orders of discretization

Comparative tables

$P_1 - Q_3$ computation :

	CFL	L_2 -error	CPU-time
DG	1e-5	0.03	7343.92
DG/SEM	1e-5	0.03	823.22

$P_2 - Q_3$ computation :

	CFL	L_2 -error	CPU-time
DG	1e-5	0.002	9452.73
DG/SEM	1e-5	0.003	1393.80

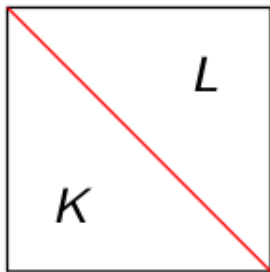
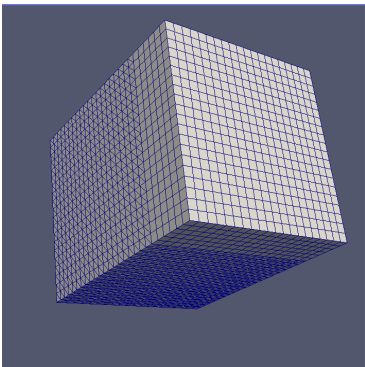
$P_3 - Q_1$ computation :

	CFL	L_2 -error	CPU-time
DG	3e-5	0.009	3078.15
DG/SEM	3e-5	0.01	2951

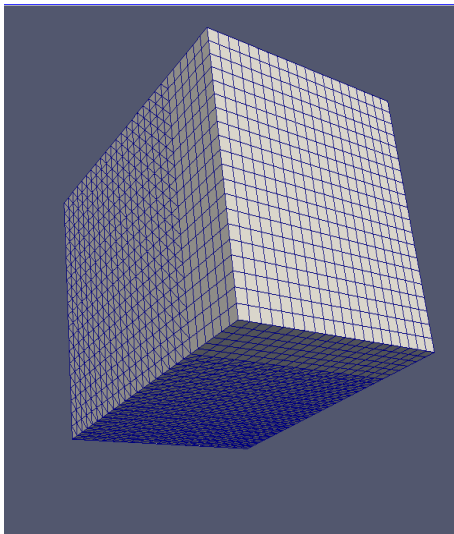
$P_3 - Q_2$ computation :

	CFL	L_2 -error	CPU-time
DG	1e-5	5.4e-4	9951.60
DG/SEM	1e-5	0.007	3122

General settings



- Only deal with a simple case of 3D hybrid meshes : one hexahedra has only two tetrahedra as neighbour
- Extend SEM in 3D (basis functions...)
- Require introducing a new matrix which handles the rotation cases between two elements



Conclusion and perspectives

Conclusion

- 1 Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
- 2 As expected, SEM is more efficient on structured quadrangle mesh than DG
- 3 Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Add a local time-stepping scheme
- Develop PML in the hexahedral part

Thank you for your attention !

Questions?