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## SEM-DG Approximation for elasto-acoustics

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# On the coupling of Spectral Element Method with Discontinuous Galerkin approximation for elasto-acoustic problems

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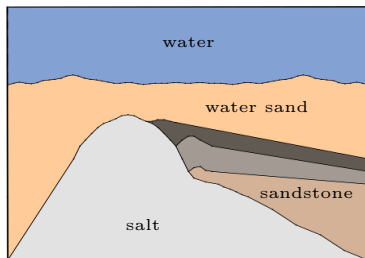
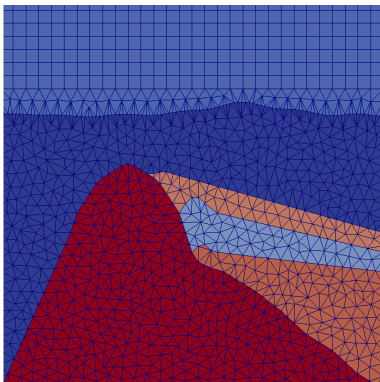
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## Why using hybrid meshes?



- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water)
- Well suited for the coupling of numerical methods in order to reduce the computational cost and improve the accuracy

## Elastodynamic system

$$\begin{cases} \rho(x) \frac{\partial \underline{v}}{\partial t}(x, t) & = \nabla \cdot \underline{\underline{\sigma}}(x, t) \\ \frac{\partial \underline{\underline{\sigma}}}{\partial t}(x, t) & = \underline{\underline{C}}(x) \underline{\underline{\epsilon}}(\underline{v}(x, t)) \end{cases}$$

With :

- $\rho(x)$  the density
- $\underline{\underline{C}}(x)$  the elasticity tensor
- $\underline{\underline{\epsilon}}(x, t)$  the deformation tensor
- $\underline{v}(x, t)$ , the wavespeed
- $\underline{\underline{\sigma}}(x, t)$  the strain tensor

# Elasticus software

Written in **Fortran** for wave propagation simulation in the **time domain**

## Features

- Using various types of meshes (**unstructured triangle and tetrahedra**)
- Modelling of various physics (**acoustic, elastic and elasto-acoustic**)

- **Discontinuous Galerkin Method (DG)** based on **unstructured triangle and unstructured tetrahedra**
- with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**
- with **p-adaptivity, multi-order** computation...

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- 5 3D extension

## 1 Numerical Methods

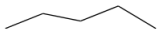
- Discontinuous Galerkin Method (DG)
- Spectral Element Method (SEM)
- Advantages of each method

# Discontinuous Galerkin Method

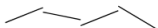
Use discontinuous functions :



mesh

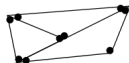


continuous

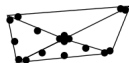


discontinuous

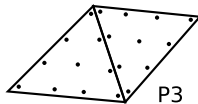
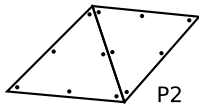
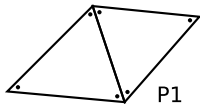
$h$  adaptivity :



$p$  adaptivity :



Degrees of freedom on each cell :

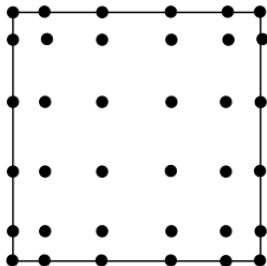




# Spectral Element Method

## General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature
- Gauss-Lobatto points as degrees of freedom ( exponential convergence on  $L^2$ -norm)



- $\int f(x) dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j)$
- $\varphi_i(\xi_j) = \delta_{ij}$

# Spectral Element Method

## Main change with DG

- DG discontinuous, SEM continuous
- Need of defining local to global numbering
- Global matrices required by SEM
- Basis functions computed differently

# Advantages of each method

## DG

- Element per element computation ( *hp*-adaptivity)
- Time discretization quasi explicit (block diagonal mass matrix)
- Simple to parallelize

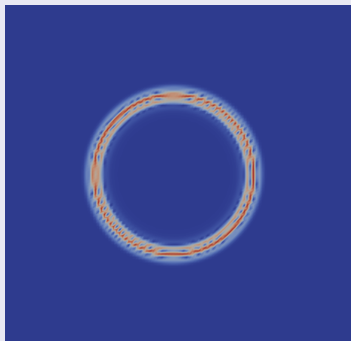
## SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method
- Reduces the computational cost when using structured meshes in comparison with DG

- 2 Comparison DG/SEM on structured quadrangle mesh
  - DG/SEM comparison on quadrangle mesh
  - Description of the test cases
  - Comparative tables

# Description of the test cases

## Physical parameters



<i>P</i> wavespeed	$1000 \text{ m.s}^{-1}$
Density	$1 \text{ kg.m}^{-3}$

Second order **Ricker Source** in *P*wave  
( $f_{peak} = 10\text{Hz}$ )

## General context

- **Acoustic homogeneous** medium
- Four different meshes : **10000 cells**, **22500 cells**, **90000 cells**, **250000 cells**
- CFL computed using **power iteration** method
- **Leap-Frog** time scheme
- **Four threads** parallel execution with **OpenMP**

# Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method

Quadrangle mesh 10000 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.99e-3	2.5e-2	19.30	500
SEM	4.9e-3	1.3e-1	0.36	204

Quadrangle mesh 22500 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	100.48	750
SEM	3.26e-3	7e-2	1.19	306

# Comparative tables

- Error computed as the difference between an analytical and a numerical solution for each method

Quadrangle mesh 10000 elements:

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DG	1.99e-3	2.5e-2	19.30	500
SEM	4.9e-3	1.3e-1	0.36	204
SEM(DG CFL)	1.99e-3	4.8e-2	0.73	502

Quadrangle mesh 22500 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	100.48	750
SEM	3.26e-3	7e-2	1.19	306
SEM(DG CFL)	1.33e-3	1.2e-2	2.82	751

SEM fifty time faster than DG on a mesh with 22500 cells

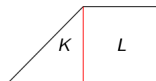
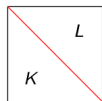
- 3 DG/SEM coupling
  - Hybrid meshes structures
  - Variational formulation
  - Space discretization



# Hybrid meshes structures

- Aim at coupling  $P_k$  and  $Q_k$  structures.
- Need to extend or split some of the structures (e.g. neighbour indexes)
- Define new face matrices

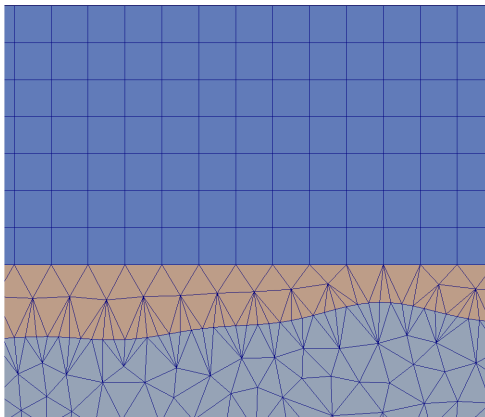
$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L$$



# Variational formulation

## Global context

- Domain in two parts :  $\Omega_{h,1}$  (structured quadrangle + SEM),  $\Omega_{h,2}$  (unstructured triangle + DG)



# Variational formulation

SEM variational formulation :

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out,1}} (\sigma_1 \mathbf{n}_1) \cdot w_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out,1}} (C\xi_1 \mathbf{n}_1) \cdot v_1 \end{cases}$$

DG variational formulation :

$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out,2}} (\sigma_2 \mathbf{n}_2) \cdot w_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [ [ w_2 ] ] \cdot \mathbf{n}_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out,2}} (C\xi_2 \mathbf{n}_2) \cdot v_2 + \int_{\Gamma_{int}} \{ \{ v_2 \} \} [ [ C\xi_2 ] ] \cdot \mathbf{n}_2 \end{cases}$$

# Variational formulation

Add the average of the solution of each part at the interface + put  $\sigma_* \mathbf{n}_* = 0$

$$\begin{cases} \int_{\Omega_{h,1}} \rho \partial_t \mathbf{v}_1 \cdot \mathbf{w}_1 = - \int_{\Omega_{h,1}} \sigma_1 \cdot \nabla \mathbf{w}_1 + \frac{1}{2} \int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2) \mathbf{n}_1 \cdot \mathbf{w}_1 \\ \int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = - \int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot \mathbf{v}_1 + \frac{1}{2} \int_{\Gamma_{1/2}} (C\xi_1 \mathbf{n}_1) \cdot (\mathbf{v}_1 + \mathbf{v}_2) \end{cases}$$

$$\begin{cases} \int_{\Omega_{h,2}} \rho \partial_t \mathbf{v}_2 \cdot \mathbf{w}_2 = - \int_{\Omega_{h,2}} \sigma_2 \cdot \nabla \mathbf{w}_2 + \int_{\Gamma_{int}} \{ \{ \sigma_2 \} \} [[\mathbf{w}_2]] \cdot \mathbf{n}_2 - \frac{1}{2} \int_{\Gamma_{1/2}} (\sigma_1 + \sigma_2) \mathbf{n}_1 \cdot \mathbf{w}_2 \\ \int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = - \int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot \mathbf{v}_2 + \int_{\Gamma_{int}} \{ \{ \mathbf{v}_2 \} \} [[C\xi_2]] \cdot \mathbf{n}_2 \\ - \frac{1}{2} \int_{\Gamma_{1/2}} (C\xi_2 \mathbf{n}_1) \cdot (\mathbf{v}_1 + \mathbf{v}_2) \end{cases}$$

# Energy study

Goal : Show that our coupling preserves the energy

- We set  $\xi_1 = \sigma_1, \xi_2 = \sigma_2, w_1 = v_1, w_2 = v_2$
- We add the equations of the two parts variational formulation

$$\frac{d}{dt} \mathbb{E} = 0$$

## Space discretization : SEM part

- $\varphi_i$  : SEM basis functions
- $\psi_j$  : DG basis functions

$$\begin{cases} M_{\mathbf{v}_1} \partial_t \mathbf{v}_{h,1} + R_{\sigma_1} \sigma_{h,1} + R_{\sigma_2}^{2,1} \sigma_{h,2} = 0 \\ M_{\sigma_1} \partial_t \sigma_{h,1} + R_{\mathbf{v}_1} \mathbf{v}_{h,1} + R_{\mathbf{v}_2}^{2,1} \mathbf{v}_{h,2} = 0 \end{cases}$$

$$\bullet M_{ij} = \int_{\Omega} \varphi_i \varphi_j \approx \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \sum_{k=1}^{(r+1)^d} \omega_k \varphi_i(\xi_k) \varphi_j(\xi_k) = \sum_{e \in \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j)} \omega_i \delta_{i,j} \quad \text{the}$$

mass matrix

$$\bullet R_{p_{ij}} = \int_{\Omega} \varphi_i \frac{\partial \varphi_j}{\partial p} \quad \text{stiffness matrix}$$

Matrix of DG/SEM coupling :

$$R_{\sigma_2, ij}^{2,1} = \frac{1}{2} \int_{\partial\Omega_1 \cap \partial\Omega_2} \psi_i \varphi_j$$

## Space discretization : DG part

$$\begin{cases} \rho M_{v_2} \partial_t \mathbf{v}_{h,2} + R_{\sigma_2} \boldsymbol{\sigma}_{h,2} - R_{\sigma_1}^{1,2} \boldsymbol{\sigma}_{h,1} = 0 \\ M_{\sigma_2} \partial_t \boldsymbol{\sigma}_{h,2} + R_{v_2} \mathbf{v}_{h,2} - R_{v_1}^{1,2} \mathbf{v}_{h,1} = 0 \end{cases}$$

- $M_{ij}^K = \int_K \psi_i^K \psi_j^K$  mass matrix,
- $R_{p_{ij}}^K = \int_K \psi_i^K \frac{\partial \psi_j^K}{\partial p}$  stiffness matrix,
- $R_{p_{ij}}^{K,L} = \int_{\partial K \cap \partial L} \psi_i^K \psi_j^L \mathbf{n}_K \cdot \mathbf{e}_p$  the mass-face matrix

Two new matrices which come from the DG/SEM coupling  $R_{\star}^{1,2}$ . Block composed :

$$R_{v_1}^{1,2} = R_{\sigma_1}^{1,2} = -\frac{1}{2} \int_{\partial \Omega_2 \cap \partial K_1} \psi_j^{K_2} \varphi_i \quad (1)$$

- 4 Comparison between DG/SEM and DG on hybrid meshes
  - Experimentation context
  - Comparative tables



## Context

- Acoustic homogeneous medium
- 54000 triangles
- 21000 quadrangles
- Using Leap-Frog time scheme
- Parallel computation using OpenMP
- Done with different orders of discretization

# Comparative tables

$P_1 - Q_3$  computation :

	CFL	$L_2$ -error	CPU-time
DG	1e-5	0.03	7343.92
DG/SEM	1e-5	0.03	823.22

$P_2 - Q_3$  computation :

	CFL	$L_2$ -error	CPU-time
DG	1e-5	0.002	9452.73
DG/SEM	1e-5	0.003	1393.80

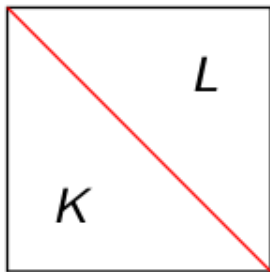
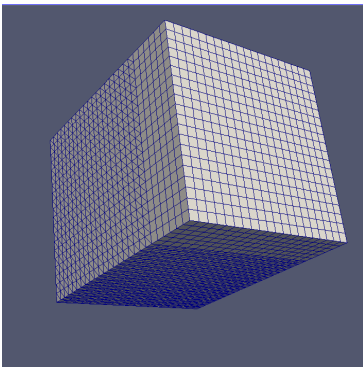
$P_3 - Q_1$  computation :

	CFL	$L_2$ -error	CPU-time
DG	3e-5	0.009	3078.15
DG/SEM	3e-5	0.01	2951

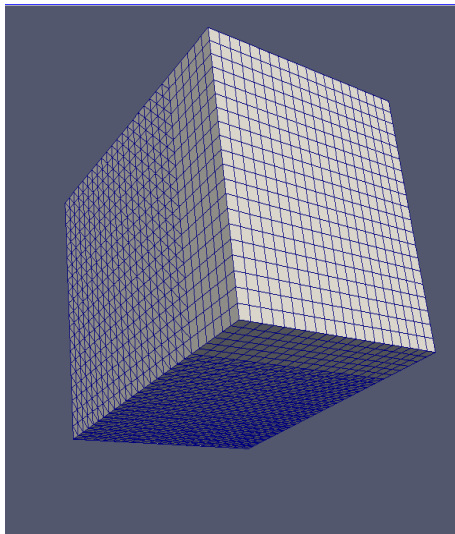
$P_3 - Q_2$  computation :

	CFL	$L_2$ -error	CPU-time
DG	1e-5	5.4e-4	9951.60
DG/SEM	1e-5	0.007	3122

## General settings



- Only deal with a simple case of 3D hybrid meshes : one hexahedra has only two tetrahedra as neighbour
- Extend SEM in 3D (basis functions...)
- Require introducing a new matrix which handles the rotation cases between two elements



# Conclusion and perspectives

## Conclusion

- 1 Build a variational formulation for DG/SEM coupling and find a CFL condition that ensures stability
- 2 As expected, SEM is more efficient on structured quadrangle mesh than DG
- 3 Show the utility of using hybrid meshes and method coupling (reduce computational cost,...)

## Perspectives

- Implement DG/SEM coupling on the code (2D) ✓
- Develop DG/SEM coupling in 3D ✓
- Add a local time-stepping scheme
- Develop PML in the hexahedral part

*Thank you for your attention !*

Questions?