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# NON UNIFORM ROTATING VORTICES AND PERIODIC ORBITS FOR THE TWO-DIMENSIONAL EULER EQUATIONS

CLAUDIA GARCÍA, TAOUFIK HMIDI, AND JUAN SOLER

ABSTRACT. This paper concerns the study of some special ordered structures in turbulent flows. In particular, a systematic and relevant methodology is proposed to construct non trivial and non radial rotating vortices with non necessarily uniform densities and with different  $m$ -fold symmetries,  $m \geq 1$ . In particular, a complete study is provided for the truncated quadratic density  $(A|x|^2 + B)\mathbf{1}_{\mathbb{D}}(x)$ , with  $\mathbb{D}$  the unit disc. We exhibit different behaviors with respect to the coefficients  $A$  and  $B$  describing the rarefaction of bifurcating curves.

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## 1. INTRODUCTION

The main goal of this paper is to investigate the emergence of some special ordered structures in turbulent flows. The search for Euler and Navier-Sokes solutions is a classical problem of permanent relevance that seeks to understand the complexity and dynamics of certain singular structures in Fluid Mechanics. Only a few solutions are known without much information about their dynamics.

We will focus on the two-dimensional Euler equations that can be written in the velocity–vorticity formulation as follows

$$(1.1) \quad \begin{cases} \omega_t + (v \cdot \nabla)\omega = 0, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ v = K * \omega, & \text{in } [0, +\infty) \times \mathbb{R}^2, \\ \omega(t = 0, x) = \omega_0(x), & \text{with } x \in \mathbb{R}^2. \end{cases}$$

The second equation links the velocity to the vorticity through the so called Biot–Savart law, where  $K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$ . This is a Hamiltonian system that develops various interesting behaviors at different levels, which are in the center of intensive research activities. Lots of studies have been devoted to the existence and stability of relative equilibria (in general, translating and rotating steady-state solutions called V-states). We point out that despite the complexity of the motion and the deformation process that the vorticity undergoes, some special vortices subsist without any deformation and keep their shape during the motion. These fascinating and intriguing structures illustrate somehow the emergence of the order from disordered and chaotic motion. The first known example in the literature goes back to Kirchhoff who discovered that a vorticity uniformly distributed inside an elliptic shape performs a uniform rotation about its center with constant angular velocity. Notice that the solutions of vortex patch type (solutions with piecewise constant vorticity) have motivated important mathematical achievements in recent years. For example, the existence of global solutions in this setting is rigorously obtained by Yudovich [61]. The  $L^1$  assumption can be replaced by a m–fold condition of symmetry thanks to the work of Elgindi and Jeong [25]. The main feature of the vortex patch problem is the persistence of this structure due to the transportation of the vorticity by the flow. However, the regularity persistence of the boundary with  $\mathcal{C}^{1,\alpha}$ -regularity is delicate and was first shown by Chemin [14], and then via different techniques by Bertozzi and Constantin [6] and Serfati [57]. Coming back to the emergence of relative equilibria, uniformly rotating m-fold patches with lower symmetries generalizing Kirchhoff ellipses were discovered numerically by Deem and Zabusky [20]. Having this kind of V-states solutions in mind, Burbea [7] designed a rigorous approach to generate them close to a Rankine vortex through complex analytical tools and bifurcation theory. Later this idea was improved and extended to different directions: regularity of the boundary, various topologies, effects of the rigid boundary, and different nonlinear transport equations. For the first subject, the regularity of the contour was analyzed in [10, 11, 35]. There, it was proved that close to the unit disc the boundary of the rotating patches are not only  $\mathcal{C}^\infty$  but analytic. As to the second point, similar results with richer structures have been obtained for doubly connected patches [22, 32]. The existence of small loops in the bifurcation diagram has been achieved very recently in [36]. For disconnected patches, the existence of co-rotating and counter-rotating vortex pairs was discussed in [34]. We mention that the bifurcation theory is so robust that partial results have been extended to different models such as

the generalized surface quasi-geostrophic equations [11, 29] or Shallow-water quasi-geostrophic equations [21], but the computations turn out to be much more involved in those cases.

It should be noted that the particularity of the rotating patches is that the dynamics is reduced to the motion of a finite number of curves in the complex plane, and therefore the implementation of the bifurcation is straightforward. However, the construction of smooth rotating vortices is much more intricate due to the size of the kernel of the linearized operator, which is in general infinite dimensional because it contains at least every radial function. Some strategies have been elaborated in order to capture some non trivial rotating smooth solutions. The first result amounts to Castro, Córdoba and Gómez-Serrano [13] who established for the SQG and Euler equations the existence of 3-fold smooth rotating vortices using a reformulation of the equation through the level sets of the vorticity. However the spectral study turns to be highly complex and they use computer-assisted proofs to check the suitable spectral properties. In a recent paper [12] the same authors removed the computer assistance part and proved the existence of  $\mathcal{C}^2$  rotating vorticity with  $m$ -fold symmetry, for any  $m \geq 2$ . The proof relies on the desingularization and bifurcation from the vortex patch problem. We point out that the profile of the vorticity is constant outside a very thin region where the transition occurs, and the thickness of this region serves as a bifurcation parameter. Remark that different variational arguments were developed in [8, 27].

The main objective of this paper is to construct a systematic scheme which turns to be relevant to detect non trivial rotating vortices with non uniform densities, far from the patches but close to some known radial profiles. Actually, we are looking for compactly supported rotating vortices in the form

$$(1.2) \quad \omega(t, x) = \omega_0(e^{-it\Omega}x), \quad \omega_0 = (f \circ \Phi^{-1})\mathbf{1}_D, \quad \forall x \in \mathbb{R}^2,$$

where  $\Omega$  is the angular velocity,  $\mathbf{1}_D$  is the characteristic function of a smooth simply connected domain  $D$ , the real function  $f : \mathbb{D} \rightarrow \mathbb{R}$  denotes the density profile and  $\Phi : \mathbb{D} \rightarrow D$  is a conformal mapping from the unit disc  $\mathbb{D}$  into  $D$ . It is a known fact that an initial vorticity  $\omega_0$  with velocity  $v_0$  generates a rotating solution, with constant angular velocity  $\Omega$ , if and only if

$$(1.3) \quad (v_0(x) - \Omega x^\perp) \cdot \nabla \omega_0(x) = 0, \quad \forall x \in \mathbb{R}^2.$$

Thus the ansatz (1.2) is a solution of Euler equations (1.1) if and only if the following equations

$$(1.4) \quad (v(x) - \Omega x^\perp) \cdot \nabla (f \circ \Phi^{-1})(x) = 0, \quad \text{in } D,$$

$$(1.5) \quad (v(x) - \Omega x^\perp) \cdot (f \circ \Phi^{-1})(x)\vec{n}(x) = 0, \quad \text{on } \partial D,$$

are simultaneously satisfied, where  $\vec{n}$  is the upward unit normal vector to the boundary  $\partial D$ . Regarding its relationship with the issue of finding vortex patches, the problem presented here exhibits a greater complexity. While a rotating vortex patch solution can be described by the boundary equation (1.5), here we also need to work with the corresponding coupled density equation (1.4). One major problem that one should face in order to make the bifurcation argument useful is related to the size of the kernel of the linearized operator which is in general infinite-dimensional. In the vortex patch framework we overcome this difficulty using the contour dynamics equation and by imposing a suitable symmetry on the V-states: they should be invariant by the dihedral group  $D_m$ . In this manner we guarantee that the linearized operator becomes a Fredholm operator with zero index. In the current context, we note that all smooth radial functions belong to the kernel. One possible strategy that one could implement is to filter those non desirable functions from the structure of the function spaces by removing the mode zero. However, this attempt fails because the space will not be stable by the nonlinearity especially for the density equation (1.4): the frequency zero can be obtained from a resonant regime, for example the square of a non vanishing function on the disc generates always the zero mode. Even though, if we assume that we were able to solve this technical problem by some special fine tricks, a second but more delicate one arises with the formulation (1.3). The

linearized operator around any radial solution is not of Fredholm type: it is smoothing in the radial component. In fact, if  $\omega_0$  is radial, then the linearized operator associated with the nonlinear map

$$F(\omega)(x) = (v(x) - \Omega x^\perp) \cdot \nabla \omega(x),$$

is given in polar coordinates by

$$\mathcal{L}h = \left( \frac{v_\theta^0}{r} - \Omega \right) \partial_\theta h + K(h) \cdot \nabla \omega_0, \quad K(h)(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} h(y) dy.$$

The loss of information in the radial direction can not be compensated by the operator  $K$  which is compact. This means that when using standard function spaces, the range of the linearized operator will be of infinite codimension. This discussion illustrates the limitation of working directly with the model (1.3). Thus, we should first proceed with reformulating differently the equation (1.3) in order to avoid the preceding technical problems and capture non radial solutions by a bifurcation argument. We point out that the main obstacle comes from the density equation (1.4) and the elementary key observation is that a solution to this equation means that the density is constant along the level sets of the relative stream function. This can be guaranteed if one looks for solutions to the restricted problem,

$$(1.6) \quad G(\Omega, f, \Phi)(z) \triangleq \mathcal{M}(\Omega, f(z)) + \frac{1}{2\pi} \int_D \log |\Phi(z) - \Phi(y)| f(y) |\Phi'(y)|^2 dy - \frac{1}{2} \Omega |\Phi(z)|^2 = 0,$$

for every  $z \in \mathbb{D}$ , and for some suitable real function  $\mathcal{M}$ . The free function  $\mathcal{M}$  can be fixed so that the radial profile is a solution. For instance, as it will be shown in Section 4, for the radial profile

$$(1.7) \quad f_0(r) = Ar^2 + B$$

we get the explicit form

$$\mathcal{M}(\Omega, s) = \frac{4\Omega - B}{8A} s - \frac{1}{16A} s^2 + \frac{3B^2 + A^2 + 4AB - 8\Omega B}{16A}.$$

Moreover, with this reformulation, we can ensure that no other radial solution can be captured around the radial profile except for a singular value, see Proposition 4.3.

Before stating our result we need to introduce the following set, which is nothing but the singular set introduced later, see (1.10), in the case of the quadratic profile,

$$\mathcal{S}_{\text{sing}} = \left\{ \frac{A}{4} + \frac{B}{2} - \frac{A(n+1)}{2n(n+2)} - \frac{B}{2n}, \quad n \in \mathbb{N}^* \cup \{+\infty\} \right\}.$$

The main result of this paper concerning the quadratic profile is the following.

**Theorem 1.1.** *Let  $A > 0$ ,  $B \in \mathbb{R}$  and  $m$  a positive integer. Then the following results hold true.*

- (1) *If  $A + B < 0$ , then there is  $m_0 \in \mathbb{N}$  (depending only on  $A$  and  $B$ ) such that for any  $m \geq m_0$ , there exists a branch of non radial rotating solutions with  $m$ -fold symmetry for the Euler equation, bifurcating from the radial solution (1.7) at some given  $\Omega_m > \frac{A+2B}{4}$ .*
- (2) *If  $B > A$ , then for any integer  $m \in [1, \frac{B}{A} + \frac{1}{8}]$  or  $m \in [1, \frac{2B}{A} - \frac{9}{2}]$  there exists a branch of non radial rotating solutions with  $m$ -fold symmetry for the Euler equation, bifurcating from the radial solution (1.7) at some given  $0 \leq \Omega_m < \frac{B}{2}$ . However, there is no bifurcation for any symmetry  $m \geq \frac{2B}{A} + 2$ .*
- (3) *If  $B > 0$  or  $B \leq -\frac{A}{1+\epsilon}$  for some  $0,0581 < \epsilon < 1$ , then there exists a branch of non radial 1-fold symmetry rotating solutions for the Euler equation, bifurcating from the radial solution (1.7) at  $\Omega_1 = 0$ .*
- (4) *If  $-\frac{A}{2} < B < 0$  and  $\Omega \notin \mathcal{S}_{\text{sing}}$ , then there is no solutions to (1.6) close to the quadratic profile.*

- (5) *In the frame of the rotating vortices constructed in (1), (2) and (3), the particle trajectories inside their supports are concentric periodic orbits around the origin.*

This theorem will be fully detailed in Theorem 5.6, Theorem 7.3 and Theorem 8.2. Before giving some details about the main ideas of the proofs, we wish to draw some useful comments:

- The upcoming Theorem 8.2 states that the orbits associated with (1.4) are periodic with smooth period, and at any time the flow is invariant by a rotation of angle  $\frac{2\pi}{m}$ . Moreover, it generates a group of diffeomorphisms of the closed unit disc.
- The V-states constructed in the above theorem have the form  $f \circ \Phi^{-1} \mathbf{1}_D$ . Also, it is proved that the density  $f$  is  $\mathcal{C}^{1,\alpha}(\mathbb{D})$  and the boundary  $\partial D$  is  $\mathcal{C}^{2,\alpha}$  with  $\alpha \in (0, 1)$ . We believe that by implementing the techniques used in [11] it could be shown that the density and the domain are analytic. An indication supporting this intuition is provided by the generator of the kernel associated with the density equation, see (5.34), which is analytic up to the boundary.
- The dynamics of the 1-fold symmetric V-states is rich and very interesting. The branch can survive even in the region where no other symmetry is allowed. It should be noted that in the context of vortex patches the bifurcation from the disc or the annulus occurs only with symmetry  $m \geq 2$  and never with the symmetry 1. The only examples that we know in the literature about the emergence of the symmetry one is the bifurcation from Kirchhoff ellipses [11, 33] or the presence of the boundary effects [23].
- From the homogeneity of Euler equations the transformation  $(A, B, \Omega) \mapsto (-A, -B, -\Omega)$  leads to the same class of solutions in Theorem 1.1. This observation allows including in the main theorem the case  $A < 0$ .
- The assumptions on  $A$  and  $B$  seen in Theorem 1.1 – (1) – (2) about the bifurcation cases imply that the radial profile  $f_0$  is not changing the sign in the unit disc. However in the point (3) the profile can change the sign.
- The bifurcation with  $m$ -fold symmetry,  $m \geq 1$ , when  $B \in (-A, -\frac{A}{2})$  is not well understood. We only know that we can obtain a branch of 1-fold symmetric solutions bifurcating from  $\Omega_1 = 0$  for  $B \in (-A, -\frac{A}{1+\epsilon})$  for some  $\epsilon \in (0, 1)$ , nothing is known for other symmetries. We expect that similarly to the result of Theorem 1.1-(2), they do exist but only for lower frequencies, and the bifurcation curves are rarefied when  $B$  approaches  $-\frac{A}{2}$ .
- Let us remark the existence of solutions with lower  $m$ -fold symmetry coming from the second point of the above Theorem. Fixing  $A$ , the number of allowed symmetries increases when  $|B|$  increases. We guess that there is a smooth curve when passing from one symmetry to another one, see Fig.1.

Let us briefly outline the general strategy we follow to prove the main result and that could be implemented for more general profiles. Using the conformal mapping  $\Phi$  we can translate the equations (1.4)–(1.5) into the disc  $\mathbb{D}$  and its boundary  $\mathbb{T}$ . Equations (1.4)–(1.5) depend functionally on the parameters  $(\Omega, f, \Phi)$ , so that we can write them as

$$(1.8) \quad \begin{cases} G(\Omega, f, \Phi)(z) = 0, & \forall z \in \mathbb{D}, \\ F(\Omega, f, \Phi)(w) = 0, & \forall w \in \mathbb{T}, \end{cases}$$

with

$$F(\Omega, f, \Phi)(w) \triangleq \operatorname{Im} \left[ \left( \Omega \overline{\Phi(w)} - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y) \right) \Phi'(w)w \right] = 0, \quad \forall w \in \mathbb{T},$$

where the functional  $G$  is described in (1.6). The aim is to parametrize the solutions in  $(\Omega, f, \Phi)$  close to some initial radial solution  $(\Omega, f_0, \operatorname{Id})$ , with  $f_0$  being a radial profile and  $\operatorname{Id}$  the identity

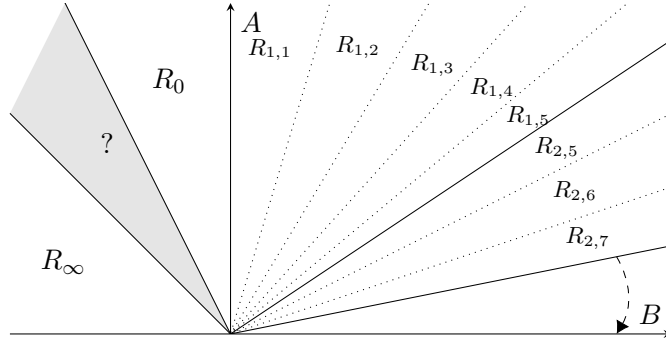


FIGURE 1. This diagram shows the different bifurcation regimes given in Theorem 1.1, with  $A > 0$ . In the case  $B > 0$ , we can find only a finite number of eigenvalues  $\Omega_m$  for which it is possible to obtain a branch of non radial  $m$ -fold symmetric solutions of the Euler equation. Here,  $m \geq 1$  increases as  $B$  does. The region  $R_{i,j}$  admits solutions with  $m$ -fold symmetry for  $1 \leq m \leq i$ . In addition, solutions with  $m$ -fold symmetry for  $m > j$  are not found. The transition between  $m = i$  and  $m = j$  is not known. Notice that in the region  $R_\infty$  the bifurcation occurs with an infinite countable family of eigenvalues. However, the bifurcation is not possible in the region  $R_0$  but the transition between  $R_0$  and  $R_\infty$  is not well-understood due to some spectral problem concerning the linearized operator. We only know the existence of 1-fold symmetric solutions in a small region.

map. Then, we will deal with the unknowns  $g$  and  $\phi$  defined by

$$(1.9) \quad f = f_0 + g, \quad \Phi = \text{Id} + \phi.$$

Thus, the equations in (1.8) are parametrized in the form  $G(\Omega, g, \phi)(z) = 0$  and  $F(\Omega, g, \phi)(w) = 0$ , where  $G(\Omega, 0, 0)(z) = 0$  and  $F(\Omega, 0, 0)(w) = 0$ . The idea is to start by solving the boundary equation, which would reduce a variable through a mapping  $(\Omega, f) \mapsto \Phi = \mathcal{N}(\Omega, f)$ , i.e. to prove that under some restrictions  $F(\Omega, g, \phi) = 0$  is equivalent to  $\phi = \mathcal{N}(\Omega, g)$ . However, the argument stumbles when we realize that this can only be done outside a set of singular values

$$(1.10) \quad \mathcal{S}_{\text{sing}} \triangleq \left\{ \Omega : \partial_\phi F(\Omega, 0, 0) \text{ is not an isomorphism} \right\},$$

for which the Implicit Function Theorem can be applied. Then, we prove that there exists an open interval  $I$  for  $\Omega$  such that  $\bar{I} \subset \mathbb{R} \setminus \mathcal{S}_{\text{sing}}$  and  $\mathcal{N}$  is well-defined in appropriated spaces, which will be subspaces of Hölder-continuous functions. Under the hypothesis that  $\Omega \in I$ , the problem of finding solutions of (1.8) is reduced to solve

$$(1.11) \quad \widehat{G}(\Omega, g)(z) \triangleq G(\Omega, g, \mathcal{N}(\Omega, g))(z) = 0, \quad \forall z \in \mathbb{D}.$$

In order to find time dependent non radial rotating solutions to (1.1) we use the procedure developed in [7] that suggests the bifurcation theory as a tool to generate solutions from a stationary one via the Crandall-Rabinowitz Theorem. The values  $\Omega$  that could lead to the bifurcation to non trivial solutions are located in the dispersion set

$$(1.12) \quad \mathcal{S}_{\text{disp}} \triangleq \left\{ \Omega : \text{Ker } D_g \widehat{G}(\Omega, 0) \neq \{0\} \right\}.$$

The problem then consists in verifying that the singular (1.10) and dispersion (1.12) sets are well-separated, for a correct definition of the interval  $I$ . Achieving this objective together with the analysis of the dimension properties of the kernel and the codimension of the range of

$D_g \widehat{G}(\Omega, 0)$ , as well as verifying the transversality property requires a complex and precise spectral and asymptotic analysis. Although our discussion is quite general, we focus our attention on the special case of quadratic profiles (1.7). In this case we obtain a compact representation of the dispersion set. Indeed, as we shall see in Section 5, the resolution of the kernel equation leads to a Volterra type integro-differential equation that one may solve through transforming it into an ordinary differential equation of second order with polynomial coefficients. Surprisingly, the new equation can be solved explicitly through variation of the constant and is connected to Gauss hypergeometric functions. The structure of the dispersion set is very subtle and appears to be very sensitive to the parameters  $A$  and  $B$ . Our analysis allows us to highlight some special regimes on  $A$  and  $B$ , see Proposition 6.6 and Proposition 6.7.

Let us emphasize that the techniques developed in the quadratic profile are robust and could be extended to other profiles. In this direction, we first provide in Section 4 the explicit expression of the function  $\mathcal{M}$  when the density admits a polynomial or Gaussian distribution. In general, the explicit resolution of the kernel equations may turn out to be a very challenging problem. Second, we will notice in Remark 5.9 that when  $f_0 = Ar^{2m} + B$  with  $m \in \mathbb{N}^*$ , explicit formulas are expected through some elementary transformations and the kernel elements are linked also to hypergeometric equations.

In Section 8 we shall be concerned with the proof of the point (5) of Theorem 1.1 concerning the planar trajectories of the particles located inside the support of the rotating vortices. We analyze the properties of periodicity and symmetries of the solutions via the study of the associated dynamical Hamiltonian structure in Eulerian coordinates, which was highlighted by Arnold [2]. This Hamiltonian nature of the Euler equations has been the idea behind the study of conservation laws in the hydrodynamics of an ideal fluid [2, 45, 48, 50], as well as in a certain sense to justify Boltzmann's principle from classical mechanics [60].

We shall give in Theorem 8.2 a precise statement and prove that close to the quadratic profile all the trajectories are periodic orbits located inside the support of the V-states, enclosing a simply connected domain containing the origin, and are symmetric with respect to the real axis. In addition, every orbit is invariant by a rotation of angle  $\frac{2\pi}{m}$ , as it has been proved for the branch of bifurcated solutions, where the parameter  $m$  is determined by the spectral properties. The periodicity of the orbits follows from the Hamiltonian structure of the autonomous dynamical system,

$$(1.13) \quad \partial_t \psi(t, z) = W(\Omega, f, \Phi)(\psi(t, z)), \quad \psi(0, z) = z \in \overline{\mathbb{D}},$$

where

$$W(\Omega, f, \Phi)(z) = \left( \frac{i}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(z) - \Phi(y)} |\Phi'(y)|^2 dy - i\Omega\Phi(z) \right) \overline{\Phi'(z)}, \quad z \in \mathbb{D}.$$

Notice that  $W$  is nothing but the pull-back of the vector field  $v(x) - \Omega x^\perp$  by the conformal mapping  $\Phi$ . This vector field remains Hamiltonian and is tangential to the boundary  $\mathbb{T}$ . Moreover, we check that close to the radial profile, it has only one critical point located at the origin which must be a center. As a consequence, the trajectories near the origin are organized through periodic orbits. Since the trajectories are located in the level sets of the energy functional given by the relative stream function, then using simple arguments we show the limit cycles are excluded and thus all the trajectories are periodic enclosing the origin which is the only fixed point, which, together with the trajectories defined above, is a way of solving the hyperbolic system (1.4). This allows to define the period map  $z \in \mathbb{D} \mapsto T_z$ , whose regularity will be at the same level as the profiles. As a by-product we find the following equivalent reformulation of the density equation

$$(1.14) \quad f(z) - \frac{1}{T_z} \int_0^{T_z} f(\psi(\tau, z)) d\tau = 0, \quad \forall z \in \overline{\mathbb{D}}.$$



Other approaches to study stationary solutions, and thus to explore the possibilities of bifurcating them, have been proposed through the study of characteristic trajectories associated with stationary velocities

$$\frac{\partial X(t)}{\partial t} = v(X(t)), \quad X(0) = x \in \mathbb{R}^2,$$

in connection with the elliptic equation  $-\Delta\psi = \omega = \tilde{F}(\psi)$  ( $\psi$  being the current function  $\nabla^\perp\psi = v$ ), see [43]. In this context, the idea of looking for smooth stationary solutions was first developed by Nadirashvili in [49], where the geometry (curvature) of streamlines was studied. Luo and Shvydkoy [42] provide a classification of some kind of homogeneous stationary smooth solutions with locally finite energy, where new solutions having hyperbolic, parabolic and elliptic structure of streamlines appear. Choffrut and Šverák [15] showed analogies of finite-dimensional results in the infinite-dimensional setting of Euler's equations, under some non-degeneracy assumptions, proving a local one-to-one correspondence between steady-states and co-adjoint orbits. Then, Choffrut and Székelyhidi [16] showed, using an h-principle [24], that there is an abundant set of weak, bounded stationary solutions in the neighborhood of any smooth stationary solution. Kiselev and Šverák [39] construct an example of initial data in the disc such that the corresponding solutions for the 2D Euler equation exhibit double exponential growth in the gradient of vorticity, which is related to the lack of Lipschitz regularity, and to an example of the singular stationary solution provided by Bahouri and Chemin [3], which produces a flow map whose Hölder regularity decreases in time.

We finally comment on three recent approaches to the analysis of rotating solutions. The first one concerns rotating vortex patches. In [30], Hassania, Masmoudi and Wheeler construct continuous curves of rotating vortex patch solutions, where the minimum along the interface of the angular fluid velocity in the rotating frame becomes arbitrarily small, which agrees with the conjecture about singular limiting patches with  $90^\circ$  corners [9, 51]. In the second contribution [12], it was studied the existence of smooth rotating vortices desingularized from a vortex patch, as it was mentioned before. The techniques are based on the analysis of the level sets of the vorticity of a global rotating solution. Since the level sets  $z(\alpha, \rho, t)$  rotate with constant angular velocity, they satisfy  $\omega(z(\alpha, \rho, t), t) = f(\rho)$ . Thus, in [12] is studied the problem of bifurcating it for some specific choice of  $f$ . In a broad sense, this result connects with that developed in this paper about the study of orbits and their periodicity. Finally, Bedrossian, Coti Zelati and Vicol in [5] analyze the incompressible 2D Euler equations linearized around a radially symmetric, strictly monotone decreasing vorticity distribution. For sufficiently regular data, inviscid damping of the  $\theta$ -dependent radial and angular velocity fields is proved. In this case, the vorticity weakly converges back to radial symmetry as  $t \rightarrow \infty$ , a phenomenon known as vortex axisymetrization. Also they show that the  $\theta$ -dependent angular Fourier modes in the vorticity are ejected from the origin as  $t \rightarrow \infty$ , resulting in faster inviscid damping rates than those possible with passive scalar evolution (vorticity depletion).

The results and techniques presented in this paper are powerful enough to be extended to other situations and equations such as SQG equations, co-rotating time-dependent solutions, solutions depending only on one variable,...

## 2. PRELIMINARIES AND STATEMENT OF THE PROBLEM

The aim of this section is to formulate the equations governing general rotating solutions of the Euler equations. We will also set down some of the tools that we use throughout the paper such as the functional setting or some properties about the extension of Cauchy integrals.

**2.1. Equation for rotating vortices.** Let us begin with the equations for compactly supported rotating solutions (1.4)–(1.5) and assume that  $f$  is not vanishing on the boundary. In

the opposite case, the equation (1.5) degenerates and becomes trivial, which implies that we just have one equation to analyze. Thus, (1.5) becomes

$$(2.1) \quad (v(x) - \Omega x^\perp) \cdot \vec{n}(x) = 0, \quad \text{on } \partial D.$$

We will rewrite these equations in the unit disc through the use of the conformal map  $\Phi : \mathbb{D} \rightarrow D$ . Note that from now on and for the sake of simplicity we will identify the Euclidean and the complex planes. Then, we write the velocity field as

$$v(x) = \frac{i}{2\pi} \int_D \frac{(f \circ \Phi^{-1})(y)}{\bar{x} - \bar{y}} dA(y), \quad \forall x \in \mathbb{C},$$

where  $dA$  refers to the planar Lebesgue measure, getting

$$v(\Phi(z)) = \frac{i}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z)} - \overline{\Phi(y)}} |\Phi'(y)|^2 dA(y), \quad \forall z \in \mathbb{D}.$$

Using the conformal parametrization  $\theta \in [0, 2\pi] \mapsto \Phi(e^{i\theta})$  of  $\partial D$ , we find that a normal vector to the boundary is given by  $\vec{n}(\Phi(w)) = w\Phi'(w)$ , with  $w \in \mathbb{T}$ . In order to deal with (1.4), we need to transform carefully the term  $\nabla(f \circ \Phi^{-1})(\Phi(z))$  coming from the density equation. Recall that for any complex function  $\varphi : \mathbb{C} \rightarrow \mathbb{C}$  of class  $\mathcal{C}^1$  seen as a function of  $\mathbb{R}^2$ , we can define

$$\partial_{\bar{z}}\varphi(z) \triangleq \frac{1}{2}(\partial_1\varphi(z) + i\partial_2\varphi(z)) \quad \text{and} \quad \partial_z\varphi(z) \triangleq \frac{1}{2}(\partial_1\varphi(z) - i\partial_2\varphi(z)),$$

which are known in the literature as Wirtinger derivatives. Let us state some of their basic properties:

$$\overline{\partial_z\varphi} = \partial_{\bar{z}}\bar{\varphi}, \quad \overline{\partial_{\bar{z}}\varphi} = \partial_z\bar{\varphi}.$$

Given two complex functions  $\varphi_1, \varphi_2 : \mathbb{C} \rightarrow \mathbb{C}$  of class  $\mathcal{C}^1$  in the Euclidean coordinates, the chain rule comes as follows

$$\begin{aligned} \partial_z(\varphi_1 \circ \varphi_2) &= (\partial_z\varphi_1 \circ \varphi_2) \partial_z\varphi_2 + (\partial_{\bar{z}}\varphi_1 \circ \varphi_2) \partial_z\bar{\varphi}_2, \\ \partial_{\bar{z}}(\varphi_1 \circ \varphi_2) &= (\partial_z\varphi_1 \circ \varphi_2) \partial_{\bar{z}}\varphi_2 + (\partial_{\bar{z}}\varphi_1 \circ \varphi_2) \partial_{\bar{z}}\bar{\varphi}_2. \end{aligned}$$

Moreover, since  $\Phi$  is a conformal map, one has that  $\partial_{\bar{z}}\Phi = 0$ . Identifying the gradient with the operator  $2\partial_{\bar{z}}$  leads to

$$\nabla(f \circ \Phi^{-1}) = 2\partial_{\bar{z}}(f \circ \Phi^{-1}).$$

From straightforward computations using the holomorphic structure of  $\Phi^{-1}$ , combined with the previous properties of the Wirtinger derivatives, we get

$$\begin{aligned} \partial_{\bar{x}}(f \circ \Phi^{-1})(x) &= (\partial_x f)(\Phi^{-1}(x)) \partial_{\bar{x}}\Phi^{-1}(x) + (\partial_{\bar{x}} f)(\Phi^{-1}(x)) \partial_{\bar{x}}\overline{\Phi^{-1}(x)} \\ &= (\partial_{\bar{x}} f)(\Phi^{-1}(x)) \overline{(\Phi^{-1})'(x)}, \end{aligned}$$

where the prime notation  $'$  for  $\Phi^{-1}$  denotes the complex derivative in the holomorphic case. Using that  $(\Phi^{-1} \circ \Phi)(z) = z$  and differentiating it we obtain

$$(\Phi^{-1})'(\Phi(z)) = \frac{\overline{\Phi'(z)}}{|\Phi'(z)|^2},$$

which implies

$$(2.2) \quad \partial_{\bar{z}}(f \circ \Phi^{-1})(\Phi(z)) = \frac{\partial_{\bar{z}} f(z) \Phi'(z)}{|\Phi'(z)|^2}.$$

Putting everything together in (1.4)-(2.1) we find the following equivalent expression

$$(2.3) \quad W(\Omega, f, \Phi) \cdot \nabla f = 0, \quad \text{in } \mathbb{D},$$

$$(2.4) \quad W(\Omega, f, \Phi) \cdot \vec{n} = 0, \quad \text{on } \mathbb{T},$$

where  $W$  is given by

$$(2.5) \quad W(\Omega, f, \Phi)(z) = \left( \frac{i}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z)} - \overline{\Phi(y)}} |\Phi'(y)|^2 dy - i\Omega\Phi(z) \right) \overline{\Phi'(z)}.$$

Then, the vector field  $W(\Omega, f, \Phi)$  is incompressible. This fact is a consequence of the lemma below. Given a vector field  $X : \mathbb{C} \rightarrow \mathbb{C}$  of class  $\mathcal{C}^1$  in the Euclidean variables, let us associate the divergence operator with the Wirtinger derivatives as follows

$$(2.6) \quad \operatorname{div} X(z) = 2\operatorname{Re} [\partial_z X(z)].$$

**Lemma 2.1.** *Given  $X : D_1 \rightarrow \mathbb{C}$  an incompressible vector field,  $\Phi : D_2 \rightarrow D_1$  a conformal map, where  $D_1, D_2 \subset \mathbb{C}$ , then  $(X \circ \Phi)\overline{\Phi}' : D_2 \rightarrow \mathbb{C}$  is incompressible.*

*Proof.* Using (2.6) we have that  $\operatorname{Re} [\partial_x X(x)] = 0$ , for any  $x \in D_1$ .

The properties of the Wirtinger derivatives lead to

$$\begin{aligned} \partial_z \left[ X(\Phi(z))\overline{\Phi'(z)} \right] &= \partial_z [X(\Phi(z))] \overline{\Phi'(z)} + X(\Phi(z))\partial_z \overline{\Phi'(z)} \\ &= (\partial_z X)(\Phi(z))\overline{\Phi'(z)} + (\partial_{\bar{z}} X)(\Phi(z))\partial_z \overline{\Phi'(z)} \\ &\quad + X(\Phi(z))\partial_z \overline{\Phi'(z)} \\ &= (\partial_z X)(\Phi(z))|\Phi'(z)|^2, \quad \forall z \in D_2. \end{aligned}$$

Hence, we have that

$$\operatorname{Re} \left[ \partial_z \left( X(\Phi(z))\overline{\Phi'(z)} \right) \right] = |\Phi'(z)|^2 \operatorname{Re} [(\partial_z X)(\Phi(z))] = 0, \quad \forall z \in D_2,$$

and  $(X \circ \Phi)\overline{\Phi}'$  is incompressible.  $\square$

Let us remark that the equation associated to the V-states in [35] is nothing but the boundary equation (2.4). In [35], V-states close to a trivial solution are obtained by means of a perturbation of the domain via a conformal mapping. Since we are perturbing also the initial density, we must analyze one more equation: the density equation (2.3). In order to apply the Crandall–Rabinowitz Theorem we will not deal with (2.3) because it seems not to be suitable when studying the linearized operator. Hence, we will reformulate this equation in Section 4. Moreover, we will provide an alternative way of writing (2.3) in Section 8 to understand the behavior of the orbits of the dynamical system associated to it.

**2.2. Function spaces.** The right choice of the function spaces will be crucial in order to construct non radial rotating solutions different to the vortex patches.

Before going into further details we introduce the classical Hölder spaces in the unit disc  $\mathbb{D}$ . Let us denote  $\mathcal{C}^{0,\alpha}(\mathbb{D})$  as the set of continuous functions such that

$$\|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \triangleq \|f\|_{L^\infty(\mathbb{D})} + \sup_{z_1 \neq z_2 \in \mathbb{D}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha} < +\infty,$$

for any  $\alpha \in (0, 1)$ . By  $\mathcal{C}^{k,\alpha}(\mathbb{D})$ , with  $k \in \mathbb{N}$ , we denote the  $\mathcal{C}^k$  functions whose  $k$ -order derivative lies in  $\mathcal{C}^{0,\alpha}(\mathbb{D})$ . Recall the Lipschitz space with the semi-norm defined as

$$\|f\|_{\operatorname{Lip}(\mathbb{D})} \triangleq \sup_{z_1 \neq z_2 \in \mathbb{D}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|}.$$

Similarly, we define the Hölder spaces  $\mathcal{C}^{k,\alpha}(\mathbb{T})$  in the unit circle  $\mathbb{T}$ . Let us supplement these spaces with additional symmetry structures:

$$\mathcal{C}_s^{k,\alpha}(\mathbb{D}) \triangleq \left\{ g : \mathbb{D} \rightarrow \mathbb{R} \in \mathcal{C}^{k,\alpha}(\mathbb{D}), \quad g(re^{i\theta}) = \sum_{n \geq 0} g_n(r) \cos(n\theta), \quad g_n \in \mathbb{R}, \quad \forall z = re^{i\theta} \in \mathbb{D} \right\},$$

$$(2.7) \quad \mathcal{C}_a^{k,\alpha}(\mathbb{T}) \triangleq \left\{ \rho : \mathbb{T} \rightarrow \mathbb{R} \in \mathcal{C}^{k,\alpha}(\mathbb{T}), \quad \rho(e^{i\theta}) = \sum_{n \geq 1} \rho_n \sin(n\theta), \quad \forall w = e^{i\theta} \in \mathbb{T} \right\}.$$

These spaces are equipped with the usual norm  $\|\cdot\|_{\mathcal{C}^{k,\alpha}}$ . One can easily check that if the functions  $g \in \mathcal{C}_s^{k,\alpha}(\mathbb{D})$  and  $\rho \in \mathcal{C}_a^{k,\alpha}(\mathbb{T})$ , then they satisfy the following properties

$$(2.8) \quad g(\bar{z}) = g(z), \quad \rho(\bar{w}) = -\rho(w), \quad \forall z \in \mathbb{D}, \forall w \in \mathbb{T}.$$

The space  $\mathcal{C}_s^{k,\alpha}(\mathbb{D})$  will contain the perturbations of the initial radial density. The condition on  $g$  means that this perturbation is invariant by reflexion on the real axis. Let us remark that we introduce also a radial perturbation coming from the frequency  $n = 0$ , this fact will be a key point in the bifurcation argument.

The second kind of function spaces is  $\mathcal{H}\mathcal{C}^{k,\alpha}(\mathbb{D})$ , which is the set of holomorphic functions  $\phi$  in  $\mathbb{D}$  belonging to  $\mathcal{C}^{k,\alpha}(\mathbb{D})$  and satisfying

$$\phi(0) = 0, \quad \phi'(0) = 0 \quad \text{and} \quad \overline{\phi(z)} = \phi(\bar{z}), \quad \forall z \in \bar{\mathbb{D}}.$$

With these properties, the function  $\phi$  admits the following expansion

$$(2.9) \quad \phi(z) = z \sum_{n \geq 1} a_n z^n, \quad a_n \in \mathbb{R}.$$

Thus, we have

$$\mathcal{H}\mathcal{C}^{k,\alpha}(\mathbb{T}) \triangleq \left\{ \phi \in \mathcal{C}^{k,\alpha}(\mathbb{T}), \quad \phi(w) = w \sum_{n \geq 1} a_n w^n, \quad a_n \in \mathbb{R}, \quad \forall w \in \mathbb{T} \right\}.$$

Notice that if  $\Phi \triangleq \text{Id} + \phi$  is conformal then  $\Phi(\mathbb{D})$  is a simply connected domain, symmetric with respect to the real axis and whose boundary is  $\mathcal{C}^{k,\alpha}$ . The space  $\mathcal{H}\mathcal{C}^{k,\alpha}(\mathbb{D})$  is a closed subspace of  $\mathcal{C}^{k,\alpha}(\mathbb{D})$  equipped with the same norm, so it is complete. In the bifurcation argument, we will perturb also the initial domain  $\mathbb{D}$  via a conformal map that will lie in this space.

The last condition on  $\phi$ , given by (2.9), together with the symmetry condition for the density (2.8), means that we are looking for rotating initial data which admit at least one axis of symmetry. For the rotating patch problem this is the minimal requirement that we should impose and up to now we do not know whether such structures without any prescribed symmetry could exist.

Now, we introduce the following trace problem concerning the extension of Cauchy integrals, which is a classical result in complex analysis and potential theory. It is directly linked to [53, Proposition 3.4] and [56, Theorem 2.2]) and for the convenience of the reader we give a proof.

**Lemma 2.2.** *Let  $k \in \mathbb{N}$  and  $\alpha \in (0, 1)$ . Denote by  $\mathcal{C} : \mathcal{H}\mathcal{C}^{k,\alpha}(\mathbb{T}) \rightarrow \mathcal{H}\mathcal{C}^{k,\alpha}(\mathbb{D})$  the linear map defined by*

$$\phi(w) = w \sum_{n \in \mathbb{N}} a_n w^n, \quad \forall w \in \mathbb{T} \implies \mathcal{C}(\phi) = \sum_{n \in \mathbb{N}} a_n z^n, \quad \forall z \in \bar{\mathbb{D}}.$$

*Then,  $\mathcal{C}(\phi)$  is well-defined and continuous.*

*Proof.* First, it is a simple matter to check that the map  $\mathcal{C}$  is well-defined. Thus, it remains to check the continuity. We recall from [56, Theorem 2.2]) the following estimates on the modulus of continuity

$$(2.10) \quad \sup_{\substack{z_1, z_2 \in \mathbb{D} \\ |z_1 - z_2| \leq \delta}} |\mathcal{C}(\phi)(z_1) - \mathcal{C}(\phi)(z_2)| \leq 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq \delta}} |\phi(w_1) - \phi(w_2)|,$$

for any  $\delta < \frac{\pi}{2}$  and for any continuous function  $\mathcal{C}(\phi)$  in  $\bar{\mathbb{D}}$ , analytic in  $\mathbb{D}$  and having trace function  $\phi$  on the unit circle. Therefore, given  $z_1, z_2 \in \mathbb{D}$  with  $|z_1 - z_2| \leq 1$ , we obtain

$$\frac{|\mathcal{C}(\phi)(z_1) - \mathcal{C}(\phi)(z_2)|}{|z_1 - z_2|^\alpha} \leq 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq |z_1 - z_2|}} \frac{|\phi(w_1) - \phi(w_2)|}{|z_1 - z_2|^\alpha}$$

$$\begin{aligned}
&\leq 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} \frac{|\phi(w_1) - \phi(w_2)|}{|w_1 - w_2|^\alpha} \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq |z_1 - z_2|}} \frac{|w_1 - w_2|^\alpha}{|z_1 - z_2|^\alpha} \\
&\leq 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} \frac{|\phi(w_1) - \phi(w_2)|}{|w_1 - w_2|^\alpha}.
\end{aligned}$$

Now, let  $z \in \mathbb{D}$  and  $w \in \mathbb{T}$  such that  $|z - w| \leq 1$ , then we also get from (2.10)

$$|\mathcal{C}(\phi)(z)| \leq |\phi(w)| + 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} |\phi(w_1) - \phi(w_2)| \leq |\phi(w)| + 3 \sup_{\substack{w_1, w_2 \in \mathbb{T} \\ |w_1 - w_2| \leq 1}} \frac{|\phi(w_1) - \phi(w_2)|}{|w_1 - w_2|^\alpha},$$

which implies that

$$\|\mathcal{C}(\phi)\|_{L^\infty(\bar{\mathbb{D}})} \leq 3\|\phi\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})}.$$

Combining the preceding estimates, we deduce that

$$\|\mathcal{C}(\phi)\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq 6\|\phi\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})}.$$

Note that this estimate can be extended to higher derivatives and thus we obtain

$$\|\mathcal{C}(\phi)\|_{\mathcal{C}^{k,\alpha}(\mathbb{D})} \leq 6\|\phi\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})},$$

which completes the proof.  $\square$

### 3. BOUNDARY EQUATION

This section focuses on studying the second equation (2.4) concerning the boundary equation and prove that we can parametrize the solutions in  $(\Omega, f, \Phi)$  close to the initial radial solution  $(f_0, \text{Id})$ , with  $f_0$  being a radial profile, through a mapping  $(\Omega, f) \mapsto \Phi = \mathcal{N}(\Omega, f)$ . We will deal with the unknowns  $g$  and  $\phi$  defined by

$$f = f_0 + g, \quad \Phi = \text{Id} + \phi.$$

Equation (2.4) can be written in the following way

$$(3.1) \quad F(\Omega, g, \phi)(w) \triangleq \text{Im} \left[ \left( \Omega \overline{\Phi(w)} - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y) \right) \Phi'(w) w \right] = 0,$$

for any  $w \in \mathbb{T}$ . Notice that from this formulation we can retrieve the fact that

$$F(\Omega, 0, 0) = 0, \quad \forall \Omega \in \mathbb{R},$$

which is compatible with the fact that any radial initial data leads to a stationary solution of the Euler equations. Indeed, this identity follows from Proposition B.5 which implies that

$$\frac{1}{2\pi} \int_{\mathbb{D}} \frac{f_0(y)}{w - y} dA(y) = \bar{w} \int_0^1 s f_0(s) ds, \quad \forall w \in \mathbb{T}.$$

The idea to solve the nonlinear equation (3.1) is to apply the Implicit Function Theorem. Define the open balls

$$(3.2) \quad \begin{cases} B_{\mathcal{C}_s^{k,\alpha}}(g_0, \varepsilon) &= \{g \in \mathcal{C}_s^{k,\alpha}(\mathbb{D}) \text{ s.t. } \|g - g_0\|_{k,\alpha} < \varepsilon\}, \\ B_{\mathcal{H}^k \mathcal{C}^{k,\alpha}}(\phi_0, \varepsilon) &= \{\phi \in \mathcal{H}^k \mathcal{C}^{k,\alpha}(\mathbb{D}) \text{ s.t. } \|\phi - \phi_0\|_{k,\alpha} < \varepsilon\}, \end{cases}$$

for  $\varepsilon > 0$ ,  $g_0 \in \mathcal{C}_s^{k,\alpha}(\mathbb{D})$  and  $\phi_0 \in \mathcal{C}^{k,\alpha}(\mathbb{D})$ . The first result concerns the well-definition and regularity of the functional  $F$  introduced in (3.1).

**Proposition 3.1.** *Let  $\varepsilon \in (0, 1)$ , then  $F : \mathbb{R} \times B_{\mathcal{C}_s^{1,\alpha}}(0, \varepsilon) \times B_{\mathcal{H}^2 \mathcal{C}^{2,\alpha}}(0, \varepsilon) \rightarrow \mathcal{C}_a^{1,\alpha}(\mathbb{T})$  is well-defined and of class  $\mathcal{C}^1$ .*

**Remark 3.2.** *If  $\phi \in B_{\mathcal{H}^k \mathcal{C}^{k,\alpha}}(0, \varepsilon)$  with  $\varepsilon < 1$ , then  $\Phi = \text{Id} + \phi$  is conformal and bi-Lipschitz.*

*Proof.* Let us show that  $F \in \mathcal{C}^{1,\alpha}(\mathbb{T})$ . Since  $\bar{\Phi}, \Phi' \in \mathcal{C}^{1,\alpha}(\mathbb{T})$  it remains to study the integral term. This is a consequence of Lemma B.2 in Appendix B, which yields  $F \in \mathcal{C}^{1,\alpha}(\mathbb{T})$ .

Let us turn to the persistence of the symmetry. According to (2.8) one has to check that

$$(3.3) \quad F(\Omega, g, \phi)(\bar{w}) = -F(\Omega, g, \phi)(w), \quad \forall w \in \mathbb{T}.$$

Using the symmetry properties of the density and the conformal mapping we write

$$\begin{aligned} F(\Omega, g, \phi)(\bar{w}) &= \operatorname{Im} \left[ \left( \Omega \overline{\Phi(\bar{w})} - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(\bar{w})} - \Phi(y)} |\Phi'(y)|^2 dA(y) \right) \Phi'(\bar{w}) \bar{w} \right] \\ &= \operatorname{Im} \left[ \left( \Omega \Phi(w) - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y) \right) \overline{\Phi'(w) \bar{w}} \right] \\ &= \operatorname{Im} \left[ \left( \Omega \Phi(w) - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y) \right) \overline{\Phi'(w) \bar{w}} \right]. \end{aligned}$$

Therefore, we get (3.3). This concludes that  $F(\Omega, g, \phi)$  is in  $\mathcal{C}_a^{1,\alpha}(\mathbb{T})$ . Notice that the dependance with respect to  $\Omega$  is smooth and we will focus on the Gâteaux derivatives of  $F$  with respect to  $g$  and  $\phi$ . Straightforward computations lead to

$$\begin{aligned} D_g F(\Omega, g, \phi)h(w) &= -\operatorname{Im} \left[ \frac{w\Phi'(w)}{2\pi} \int_{\mathbb{D}} \frac{h(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y) \right], \\ D_\phi F(\Omega, g, \phi)k(w) &= \operatorname{Im} \left[ \Omega \overline{k(w)} \Phi'(w) w + \Omega \overline{\Phi(w)} k'(w) w \right. \\ &\quad - \frac{wk'(w)}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y) \\ &\quad + \frac{w\Phi'(w)}{2\pi} \int_{\mathbb{D}} \frac{k(w) - k(y)}{(\Phi(w) - \Phi(y))^2} f(y) |\Phi'(y)|^2 dA(y) \\ &\quad \left. - \frac{w\Phi'(w)}{\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(w) - \Phi(y)} \operatorname{Re} \left[ \overline{\Phi'(y)} k'(y) \right] dA(y) \right]. \end{aligned} \quad (3.4)$$

Let us use the operator  $\mathcal{F}[\Phi]$  defined in (B.4). Although in Lemma B.2  $\mathcal{F}[\Phi]$  is defined in  $\mathbb{D}$  we can extend it up to the boundary  $\mathbb{D}$  getting the same result. Hence, all the above expressions can be written through this operator as

$$\begin{aligned} D_g F(\Omega, g, \phi)h(w) &= -\operatorname{Im} \left[ \frac{w\Phi'(w)}{2\pi} \mathcal{F}[\Phi](h)(w) \right], \\ D_\phi F(\Omega, g, \phi)k(w) &= \operatorname{Im} \left[ \Omega \overline{k(w)} \Phi'(w) w + \Omega \overline{\Phi(w)} k'(w) w - \frac{wk'(w)}{2\pi} \mathcal{F}[\Phi](f)(w) \right. \\ &\quad + \frac{w\Phi'(w)}{2\pi} \int_{\mathbb{D}} \frac{k(w) - k(y)}{(\Phi(w) - \Phi(y))^2} f(y) |\Phi'(y)|^2 dA(y) \\ &\quad \left. - \frac{w\Phi'(w)}{\pi} \mathcal{F}[\Phi] \left( \frac{\operatorname{Re} \left[ \overline{\Phi'(\cdot)} k'(\cdot) \right]}{|\Phi'(\cdot)|^2} \right) (w) \right]. \end{aligned}$$

Since  $\frac{\operatorname{Re} \left[ \overline{\Phi'(\cdot)} k'(\cdot) \right]}{|\Phi'(\cdot)|^2}$ ,  $\Phi', \bar{\Phi}, k' \in \mathcal{C}^{1,\alpha}(\mathbb{D})$  and are continuous with respect to  $\Phi$ , Lemma B.2 entails that all the terms except the integral one lie in  $\mathcal{C}^{1,\alpha}(\mathbb{D})$  and they are continuous with respect to  $\Phi$ . The continuity with respect to  $f$  comes also from the same result. Note that although our unknowns are  $(g, \phi)$ , studying the continuity with respect to  $(g, \phi)$  is equivalent to doing

it with respect to  $(f, \Phi)$ . We shall now focus our attention on the integral term by splitting it as follows

$$\begin{aligned} \int_{\mathbb{D}} \frac{k(w) - k(y)}{(\Phi(w) - \Phi(y))^2} f(y) |\Phi'(y)|^2 dA(y) &= \int_{\mathbb{D}} \frac{(k(w) - k(y))(f(y) - f(w))}{(\Phi(w) - \Phi(y))^2} |\Phi'(y)|^2 dA(y) \\ &\quad + f(w) \int_{\mathbb{D}} \frac{k(w) - k(y)}{(\Phi(w) - \Phi(y))^2} |\Phi'(y)|^2 dA(y) \\ &\triangleq \mathcal{J}_1[\Phi]f(z) + f(w) \mathcal{J}_2[\Phi]. \end{aligned}$$

First, we deal with  $\mathcal{J}_1[\Phi]$ . Clearly

$$|\mathcal{J}_1[\Phi](w)| \leq C \|f\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})},$$

and we define

$$\begin{aligned} K(w, y) &\triangleq \nabla_w \frac{(k(w) - k(y))(f(y) - f(w))}{(\Phi(w) - \Phi(y))^2} \\ &= -\frac{(k(w) - k(y))}{(\Phi(w) - \Phi(y))^2} \nabla_w f(w) + (f(y) - f(w)) \nabla_w \frac{(k(w) - k(y))}{(\Phi(w) - \Phi(y))^2} \\ &\triangleq -\nabla_w f(w) K_1(w, y) + K_2(w, y). \end{aligned}$$

Using the same argument as in (B.6), we can check that  $K_1$  and  $K_2$  verify both the hypotheses of Lemma B.1. This implies that  $\mathcal{J}_1[\Phi]$  lies in  $\mathcal{C}^{1,\alpha}(\mathbb{D})$ . Taking two conformal maps  $\Phi_1$  and  $\Phi_2$  and estimating  $\mathcal{J}_1[\Phi_1] - \mathcal{J}_1[\Phi_2]$ , we find integrals similar to those treated in Lemma B.1.

Concerning the second integral  $\mathcal{J}_2[\Phi]$ , which seems to be more singular, we use the Cauchy–Pompeiu’s formula (B.3) to find

$$\int_{\mathbb{D}} \frac{k(w) - k(y)}{\Phi(w) - \Phi(y)} |\Phi'(y)|^2 dA(y) = \frac{1}{2i} \int_{\mathbb{T}} \frac{k(w) - k(\xi)}{\Phi(w) - \Phi(\xi)} \overline{\Phi(\xi)} \Phi'(\xi) d\xi.$$

Differentiating it, we deduce

$$\begin{aligned} \int_{\mathbb{D}} \frac{k(w) - k(y)}{(\Phi(w) - \Phi(y))^2} |\Phi'(y)|^2 dA(y) &= \frac{k'(w) \overline{\Phi(w)} \pi}{2\Phi'(w)} + \frac{1}{2i} \int_{\mathbb{T}} \frac{k(\xi) - k(w)}{(\Phi(\xi) - \Phi(w))^2} \overline{\Phi(\xi)} \Phi'(\xi) d\xi \\ &\triangleq \frac{k'(w) \overline{\Phi(w)} \pi}{2\Phi'(w)} + \frac{1}{2i} \mathcal{S}[\Phi](w). \end{aligned}$$

The first term is in  $\mathcal{C}^{1,\alpha}(\mathbb{T})$  and is clearly continuous with respect to  $\Phi$ . Integration by parts in the second term  $\mathcal{S}[\Phi]$  leads to

$$\mathcal{S}[\Phi](w) = -\int_{\mathbb{T}} \frac{k'(\xi) \overline{\Phi(\xi)}}{\Phi(\xi) - \Phi(w)} d\xi + \int_{\mathbb{T}} \frac{k(\xi) - k(w)}{\Phi(\xi) - \Phi(w)} \overline{\xi^2 \Phi'(\xi)} d\xi.$$

Differentiating and integrating it by parts again one obtain the following expression

$$\begin{aligned} \mathcal{S}[\Phi]'(w) &= \Phi'(w) \int_{\mathbb{T}} \frac{\partial_{\xi} \left( \frac{k'(\xi) \overline{\Phi(\xi)}}{\Phi'(\xi)} \right)}{\Phi(\xi) - \Phi(w)} d\xi - \Phi'(w) \int_{\mathbb{T}} \frac{\partial_{\xi} \left( \frac{(k(\xi) - k(w)) \overline{\xi^2 \Phi'(\xi)}}{\Phi'(\xi)} \right)}{\Phi(\xi) - \Phi(w)} d\xi \\ &= \Phi'(w) \int_{\mathbb{T}} \frac{\partial_{\xi} \left( \frac{k'(\xi) \overline{\Phi(\xi)}}{\Phi'(\xi)} \right)}{\Phi(\xi) - \Phi(w)} d\xi - \Phi'(w) \int_{\mathbb{T}} \frac{\frac{k'(\xi) \overline{\xi^2 \Phi'(\xi)}}{\Phi'(\xi)^2}}{\Phi(\xi) - \Phi(w)} \Phi'(\xi) d\xi \\ &\quad - \Phi'(w) \int_{\mathbb{T}} \frac{(k(\xi) - k(w)) \partial_{\xi} \left( \frac{\overline{\xi^2 \Phi'(\xi)}}{\Phi'(\xi)} \right)}{\Phi(\xi) - \Phi(w)} d\xi \end{aligned}$$

$$\begin{aligned}
&= \Phi'(w) \mathcal{S}[\Phi] \left( \frac{\partial \left( \frac{k'(\cdot) \overline{\Phi(\cdot)}}{\Phi'(\cdot)} \right)}{\Phi'(\cdot)} \right) (w) - \Phi'(w) \mathcal{S}[\Phi] \left( \frac{k'(\cdot) \overline{(\cdot)^2 \overline{\Phi'(\cdot)}}}{\Phi'(\cdot)^2} \right) (w) \\
&\quad - \Phi'(w) \mathcal{S}[\Phi] \left( k(\cdot) \partial \left( \frac{\overline{^2 \Phi'(\cdot)}}{\Phi'(\cdot)} \right) \right) (w) + \Phi'(w) k(w) \mathcal{S}[\Phi] \left( \partial \frac{\overline{(\cdot)^2 \overline{\Phi'(\cdot)}}}{\Phi'(\cdot)} \right) (w),
\end{aligned}$$

where  $\mathcal{S}[\Phi]$  is the operator defined in (B.13). Since the functions

$$\frac{\partial \left( \frac{k'(\cdot) \overline{\Phi(\cdot)}}{\Phi'(\cdot)} \right)}{\Phi'(\cdot)}, \quad \frac{k'(\cdot) \overline{(\cdot)^2 \overline{\Phi'(\cdot)}}}{\Phi'(\cdot)^2}, \quad k, \quad \partial \left( \frac{\overline{(\cdot)^2 \overline{\Phi'(\cdot)}}}{\Phi'(\cdot)} \right) \in \mathcal{C}^{0,\alpha}(\mathbb{D}),$$

and are continuous with respect to  $\Phi$ , we can use Lemma B.4. Hence, these terms lie in  $\mathcal{C}^{0,\alpha}(\mathbb{D})$ . Moreover, the same argument gives us the continuity with respect to  $\Phi$ . To conclude, we use the fact that the Gâteaux derivatives are continuous with respect to  $(g, \phi)$  and so they are in fact Fréchet derivatives.  $\square$

The next task is to implement the Implicit Function Theorem in order to solve the boundary equation (3.1) through a two-parameters curve solutions in infinite-dimensional spaces. Given a radial function  $f_0 \in \mathcal{C}^{1,\alpha}(\mathbb{D})$ , we associate to it the singular set

$$(3.5) \quad \mathcal{S}_{\text{sing}} \triangleq \left\{ \widehat{\Omega}_n \triangleq \int_0^1 s f_0(s) ds - \frac{n+1}{n} \int_0^1 s^{2n+1} f_0(s) ds, \quad \forall n \in \mathbb{N}^* \cup \{+\infty\} \right\}.$$

This terminology will be later justified in the proof of the next proposition. Actually, this set corresponds to the location of the points  $\Omega$  where the partial linearized operator  $\partial_\phi F(\Omega, 0, 0)$  is not invertible. Let us establish the following result.

**Proposition 3.3.** *Let  $f_0 : \overline{\mathbb{D}} \rightarrow \mathbb{R}$  be a radial function in  $\mathcal{C}^{1,\alpha}(\mathbb{D})$ . Let  $I$  be an open interval such that  $\overline{I} \subset \mathbb{R} \setminus \mathcal{S}_{\text{sing}}$ . Then, there exists  $\varepsilon > 0$  and a  $\mathcal{C}^1$  function*

$$\mathcal{N} : I \times B_{\mathcal{C}_s^{1,\alpha}}(0, \varepsilon) \longrightarrow B_{\mathcal{H}\mathcal{C}^{2,\alpha}}(0, \varepsilon),$$

with the following property:

$$F(\Omega, g, \phi) = 0 \iff \phi = \mathcal{N}(\Omega, g),$$

for any  $(\Omega, g, \phi) \in I \times B_{\mathcal{C}_s^{1,\alpha}}(0, \varepsilon) \times B_{\mathcal{H}\mathcal{C}^{2,\alpha}}(0, \varepsilon)$ . In addition, we obtain the identity

$$D_g \mathcal{N}(\Omega, 0) h(z) = z \sum_{n \geq 1} A_n z^n,$$

for any  $h \in \mathcal{C}_s^{1,\alpha}(\mathbb{D})$ , with  $h(re^{i\theta}) = \sum_{n \geq 0} h_n(r) \cos(n\theta)$ , and

$$(3.6) \quad A_n = \frac{\int_0^1 s^{n+1} h_n(s) ds}{2n(\widehat{\Omega}_n - \Omega)},$$

for any  $n \geq 1$  where  $\widehat{\Omega}_n$  is defined in (3.5). Moreover, we have

$$(3.7) \quad \|\mathcal{N}(\Omega, 0) h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$

**Remark 3.4.** *From the definition of the function space  $\mathcal{C}_s^{1,\alpha}(\mathbb{D})$  we are adding also a radial perturbation of the initial radial part given by the first mode  $n = 0$ . However, from the expression of  $D_g \mathcal{N}(\Omega, 0) h(z)$  the first frequency disappears and the sum starts at  $n = 1$ . This is an expected fact because  $(\Omega, g, 0)$  is a solution of  $F(\Omega, g, \phi)$  for any radial smooth function  $g$ . This means that  $\mathcal{N}(\Omega, g) = 0$ , and hence  $D_g \mathcal{N}(\Omega, 0) h$  is vanishing when  $h$  is radial.*



*Proof.* Applying the Implicit Function Theorem consists in checking that

$$D_\phi F(\Omega, 0, 0) : \mathcal{H}\mathcal{C}^{2,\alpha}(\mathbb{D}) \longrightarrow \mathcal{C}_a^{1,\alpha}(\mathbb{T}),$$

is an isomorphism. A combination of (3.4) with Proposition B.5 allow us to compute explicitly the differential of  $F(\Omega, g, \phi)$  on the initial solution as follows

$$\begin{aligned} D_\phi F(\Omega, 0, 0)k(w) &= \operatorname{Im} \left[ \overline{\Omega k(w)}w + \Omega \bar{w}k'(w)w - \frac{wk'(w)}{2\pi} \int_{\mathbb{D}} \frac{f_0(y)}{w-y} dA(y) \right. \\ &\quad \left. + \frac{w}{2\pi} \int_{\mathbb{D}} \frac{k(w) - k(y)}{(w-y)^2} f_0(y) dA(y) - \frac{w}{\pi} \int_{\mathbb{D}} \frac{f_0(y)}{w-y} \operatorname{Re}[k'(y)] dA(y) \right] \\ &= \sum_{n \geq 1} a_n \operatorname{Im} \left[ \Omega \bar{w}^n + \Omega(n+1)w^n - (n+1)w^n \int_0^1 s f_0(s) ds \right. \\ &\quad \left. + w^n \int_0^1 s f_0(s) ds - (n+1)\bar{w}^n \int_0^1 s^{2n+1} f_0(s) ds \right] \\ &= \sum_{n \geq 1} a_n n \left\{ \Omega - \int_0^1 s f_0(s) ds + \frac{n+1}{n} \int_0^1 s^{2n+1} f_0(s) ds \right\} \sin(n\theta). \end{aligned}$$

Similarly, we get

$$\begin{aligned} D_g F(\Omega, 0, 0)h(w) &= -\operatorname{Im} \left[ \frac{w}{2\pi} \int_{\mathbb{D}} \frac{h(y)}{w-y} dA(y) \right] = -\frac{\pi}{2\pi} \sum_{n \geq 1} \operatorname{Im} \left[ w \bar{w}^{n+1} \int_0^1 s^{n+1} h_n(s) ds \right] \\ (3.8) \quad &= \frac{1}{2} \sum_{n \geq 1} \int_0^1 s^{n+1} h_n(s) ds \sin(n\theta), \end{aligned}$$

where

$$z \mapsto k(z) = \sum_{n \geq 1} a_n z^{n+1} \in \mathcal{H}\mathcal{C}^{2,\alpha}(\mathbb{D}) \quad \text{and} \quad z \mapsto h(z) = \sum_{n \geq 0} h_n(r) \cos(n\theta) \in \mathcal{C}_s^{1,\alpha}(\mathbb{D}),$$

are given as in (2.9) and (2.7), respectively. Then, we have that  $D_\phi F(\Omega, 0, 0)$  is one-to-one linear mapping and is continuous according to Proposition 3.1, for any  $\Omega \in \bar{\mathbb{I}}$ . Using the Banach Theorem it suffices to check that this mapping is onto. Notice that at the formal level the inverse operator can be easily computed from the expression of  $D_\phi F(\Omega, 0, 0)$  and it is given by

$$(3.9) \quad D_\phi F(\Omega, 0, 0)^{-1} \rho(z) = z \sum_{n \geq 1} \frac{\rho_n}{n(\Omega - \hat{\Omega}_n)} z^n,$$

for any  $\rho(e^{i\theta}) = \sum_{n \geq 1} \rho_n \sin(n\theta)$ . Thus the problem reduces to check that  $D_\phi F(\Omega, 0, 0)^{-1} \rho \in \mathcal{H}\mathcal{C}^{2,\alpha}(\mathbb{D})$ . First, we will prove that this function is holomorphic inside the unit disc  $\mathbb{D}$ . For this purpose we use that

$$(3.10) \quad \rho_n = \frac{1}{\pi} \int_0^{2\pi} \rho(e^{i\theta}) \sin(n\theta) d\theta.$$

Since  $\rho \in L^\infty(\mathbb{T})$ , we obtain that the coefficients sequences  $(\rho_n) \in \ell^\infty$ . Using the facts that  $\lim_{n \rightarrow \infty} \hat{\Omega}_n = \hat{\Omega}_\infty$  and that  $\Omega$  is far away from the singular set, then we deduce that the Fourier coefficients of  $D_\phi F(\Omega, 0, 0)^{-1} \rho$  are bounded. Consequently, this function is holomorphic inside the unit disc. It remains to check that this function belongs to  $\mathcal{C}^{2,\alpha}(\mathbb{D})$ . By virtue of Lemma

2.2 it is enough to check that the restriction on the boundary belongs to  $\mathcal{C}^{1,\alpha}(\mathbb{T})$ . First, we must notice that if  $\rho \in \mathcal{C}^{1,\alpha}(\mathbb{T})$ , then

$$w \mapsto \rho_+(w) \triangleq \sum_{n \geq 1} \rho_n w^n \in \mathcal{C}^{1,\alpha}(\mathbb{T}).$$

For this purpose, let us write  $\rho$  in the form

$$\rho(w) = \frac{1}{2i} \sum_{n \in \mathbb{Z}} \rho_n w^n, \quad \text{with } \rho_{-n} = -\rho_n.$$

Hence,  $\rho_+$  is nothing but the Szegő projection of  $\rho$ , which is continuous on  $\mathcal{C}^{1,\alpha}(\mathbb{T})$ . Note that this latter property is based upon the fact that  $\rho_+$  can be expressed from  $\rho$  through the Cauchy integral operator

$$\rho_+(w) = \frac{1}{\pi} \int_{\mathbb{T}} \frac{\rho(\xi)}{\xi - w} d\xi,$$

and one may use  $T(1)$ -Theorem of Wittmann for Hölder spaces, see for instance [62, Theorem 2.1] and [35, page 10] or Lemma B.4. Secondly, we will prove that  $D_\phi F(\Omega, 0, 0)^{-1} \rho \in \mathcal{C}^{2,\alpha}(\mathbb{T})$ . We define

$$D_\phi F(\Omega, 0, 0)^{-1} \rho(w) \triangleq wq(w).$$

Let us show that  $q$  is bounded. Using (3.10) and integration by parts, we have

$$|\rho_n| \leq 2 \frac{\|\rho'\|_{L^\infty(\mathbb{T})}}{n},$$

which implies that  $q$  is bounded. To prove higher regularity, we write  $q$  as a convolution

$$q = \rho_+ \star K_1,$$

where

$$K_1(w) = \sum_{n \geq 1} \frac{w^n}{n(\Omega - \widehat{\Omega}_n)}.$$

Since  $\rho_+ \in \mathcal{C}^{1,\alpha}$ , we just need to check that  $K_1 \in L^1$ . To do that, we use the Parseval's identity, which provides that  $K_1 \in L^2(\mathbb{T})$ :

$$\|K_1\|_2^2 = \sum_{n \geq 1} \frac{1}{n^2(\Omega - \widehat{\Omega}_n)^2} \leq C \sum_{n \geq 1} \frac{1}{n^2} < +\infty,$$

where  $C$  is a constant connected to the distance between  $\Omega$  and the singular set  $\mathcal{S}_{sing}$  defined in (3.5). To study its derivative, let us write it as

$$q'(w) = \bar{w} \sum_{n \geq 1} \frac{\rho_n}{\Omega - \widehat{\Omega}_n} w^n = \bar{w} \left[ \sum_{n \geq 1} \frac{\rho_n}{\beta} w^n + \sum_{n \geq 1} \rho_n \left( \frac{1}{\beta + u_n} - \frac{1}{\beta} \right) w^n \right] \triangleq \bar{w} \left[ \frac{1}{\beta} \rho_+(w) + S \right],$$

where

$$\beta = \Omega - \widehat{\Omega}_\infty, \quad u_n = \frac{n+1}{n} \int_0^1 s^{2n+1} f_0(s) ds.$$

From the foregoing discussion, we have seen that  $\rho_+ \in \mathcal{C}^{1,\alpha}(\mathbb{T})$ . As to the term  $S$ , it can be written in convolution form

$$S = \rho_+ \star K_2,$$

with

$$K_2(w) = - \sum_{n \geq 1} \frac{u_n}{(\beta + u_n)\beta} w^n.$$

Since  $\rho_+ \in \mathcal{C}^{1,\alpha}(\mathbb{T})$ , then we just need to check that  $K_2 \in L^1(\mathbb{T})$  to conclude. Using Parseval's identity we have that  $K_2 \in L^2$  because

$$\begin{aligned} \|K_2\|_2^2 &= \frac{1}{\beta^2} \sum_{n \geq 1} \frac{u_n^2}{(\beta + u_n)^2} \leq C \sum_{n \geq 1} \frac{(n+1)^2}{n^2} \left( \int_0^1 s^{2n+1} f_0(s) ds \right)^2 \\ &\leq C \sum_{n \geq 1} \frac{(n+1)^2}{n^2(2n+1)^2} < +\infty. \end{aligned}$$

This achieves that  $D_\phi F(\Omega, 0, 0)^{-1} \rho \in \mathcal{H}\mathcal{C}^{2,\alpha}(\mathbb{D})$  and consequently the linearized operator  $D_\phi F(\Omega, 0, 0)$  is an isomorphism. Hence, the Implicit Function Theorem can be used and it ensures the existence of a  $\mathcal{C}^1$ -function  $\mathcal{N}$  such that

$$F(\Omega, g, \phi) = 0 \iff \phi = \mathcal{N}(\Omega, g),$$

for any  $(\Omega, g, \phi) \in I \times B_{\mathcal{C}_s^{1,\alpha}}(0, \varepsilon) \times B_{\mathcal{H}\mathcal{C}^{2,\alpha}}(0, \varepsilon)$ . Differentiating with respect to  $g$ , we obtain

$$D_g F(\Omega, g, \mathcal{N}(\Omega, g)) = \partial_g F(\Omega, g, \phi) + \partial_\phi F(\Omega, g, \phi) \circ \partial_g \mathcal{N}(\Omega, g) = 0,$$

which yields

$$\partial_g \mathcal{N}(\Omega, 0) h(z) = -\partial_\phi F(\Omega, 0, 0)^{-1} \circ \partial_g F(\Omega, 0, 0) h(z).$$

Then, using (3.9) and (3.8), straightforward computations show that

$$D_g \mathcal{N}(\Omega, 0) h(z) = -z \sum_{n \geq 1} \frac{\int_0^1 s^{n+1} h_n(s) ds}{2n(\Omega - \widehat{\Omega}_n)} z^n.$$

This concludes the proof of the announced result.  $\square$

#### 4. DENSITY EQUATION

This section aims at studying the density equation (2.3) in order to get non radial rotating solutions via the Crandall–Rabinowitz Theorem. We will reformulate it in a more convenient way since we are not able to use the original expression (2.3) due to the structural defect on its linearized operator as it has been pointed out previously. We must have in mind that under suitable assumptions, the conformal map is recovered from the angular velocity  $\Omega$  and the density function via Proposition 3.3.

**4.1. Reformulation of the density equation.** Taking an initial data in the form (1.2) and noting that if the density  $f$  is fixed close to  $f_0$  and  $\Omega$  does not lie in the singular set  $\mathcal{S}_{\text{sing}}$ , then the conformal mapping is uniquely determined as a consequence of Proposition 3.3. Now, we turn to the analysis of the first equation of (1.4) that we intend to solve for a restricted class of initial densities. The strategy to implement it is to look for solutions satisfying the specific equation

$$\begin{aligned} \nabla(f \circ \Phi^{-1})(x) &= \mu(\Omega, (f \circ \Phi^{-1})(x)) (v(x) - \Omega x^\perp)^\perp \\ (4.1) \quad &= \mu(\Omega, (f \circ \Phi^{-1})(x)) (v^\perp(x) + \Omega x), \quad \forall x \in D, \end{aligned}$$

for some scalar function  $\mu$ . One can easily check that any solution of (4.1) is a solution of the initial density equation (1.4) but the reversed is not in general true. Remark that from this latter equation we are looking for particular solutions due to the precise dependence of the scalar function  $\mu$  with respect to  $f$ . The scalar function  $\mu$  must be fixed in such a way that the radial profile  $f_0$ , around which we look for non trivial solutions, is also a solution of (4.1). Therefore, for any initial radial profile candidate to be bifurcated, we will obtain a different density equation. Notice also that it is not necessary in general to impose to  $\mu$  to be well-defined on  $\mathbb{R}$  but just on some open interval containing the image of  $\mathbb{D}$  by  $f_0$ .

Now, let us show how to construct concretely the function  $\mu$ . By virtue of Proposition 3.3, the associated conformal map to any radial profile is the identity map. Therefore, it is obvious that a smooth radial profile  $f_0$  is a solution of (4.1) if and only if

$$\begin{aligned} f_0'(r) \frac{z}{r} &= \mu(\Omega, f_0(z)) \left[ -\frac{1}{2\pi} \int_{\mathbb{D}} \frac{z-y}{|z-y|^2} f_0(y) dA(y) + \Omega z \right] \\ &= \mu(\Omega, f_0(z)) \left[ -\frac{1}{2\pi} \int_{\mathbb{D}} \frac{f_0(y)}{z-y} dA(y) + \Omega z \right] \\ &= \mu(\Omega, f_0(z)) \left[ -\frac{1}{r^2} \int_0^r s f_0(s) ds + \Omega \right] z, \quad \forall z \in \mathbb{D}, r = |z|, \end{aligned}$$

where we have used the explicit computations given in Proposition B.5. Thus, we infer that the function  $\mu$  must satisfy the compatibility condition

$$(4.2) \quad \mu(\Omega, f_0(r)) = \frac{1}{r} \frac{f_0'(r)}{\Omega - \frac{1}{r^2} \int_0^r s f_0(s) ds}, \quad \forall r \in (0, 1].$$

We emphasize that not all the radial profiles verify the last equation. In fact, we can violate this equation by working with non monotonic profiles. Taking  $f_0$  verifying (4.2), let us go through the above procedure and see how to reformulate the density equation. Consider the function

$$(4.3) \quad \mathcal{M}_{f_0}(\Omega, \mathfrak{t}) = \int_{t_0}^{\mathfrak{t}} \frac{1}{\mu(\Omega, s)} ds,$$

for some  $t_0 \in \mathbb{R}$ . We use the subscript  $f_0$  in order to stress that the above function depends on the choice of the initial profile  $f_0$ . This rigidity is very relevant in our study and enables us to include the structure of the solution into the formulation. By this way, we expect to remove the pathological behavior of the old formulation and to prepare the problem for the bifurcation arguments. From the expression of the velocity field, it is obvious that

$$v^\perp(x) + \Omega x = -\nabla \left( \frac{1}{2\pi} \int_D \log|x-y|(f \circ \Phi^{-1})(y) dA(y) - \frac{1}{2} \Omega |x|^2 \right).$$

Since  $D$  is a simply connected domain, then integrating (4.1) yields to the equivalent form

$$\mathcal{M}_{f_0}(\Omega, (f \circ \Phi^{-1})(x)) + \frac{1}{2\pi} \int_D \log|x-y|(f \circ \Phi^{-1})(y) dA(y) - \frac{1}{2} \Omega |x|^2 = \lambda, \quad \forall x \in D,$$

for some constant  $\lambda$ . Using a change of variable through the conformal map  $\Phi : \mathbb{D} \rightarrow D$ , we obtain the equivalent formulation in the unit disc

$$(4.4) \quad \mathcal{M}_{f_0}(\Omega, f(z)) + \frac{1}{2\pi} \int_{\mathbb{D}} \log|\Phi(z) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) - \frac{1}{2} \Omega |\Phi(z)|^2 = \lambda, \quad \forall z \in \mathbb{D}.$$

It remains to fix the constant  $\lambda$  by using that the initial radial profile should be a solution of (4.4). Thus the last integral identity in Proposition B.5 entails that

$$(4.5) \quad \lambda = \mathcal{M}_{f_0}(f_0(r)) - \int_r^1 \frac{1}{\tau} \int_0^\tau s f_0(s) ds d\tau - \frac{1}{2} \Omega r^2.$$

Notice that  $\lambda$  does not depend on  $r$  since  $f_0$  verifies (4.2). Then, we finally arrive at the following reformulation for the density equation

$$G_{f_0}(\Omega, g, \phi)(z) \triangleq \mathcal{M}_{f_0}(\Omega, f(z)) + \frac{1}{2\pi} \int_{\mathbb{D}} \log|\Phi(z) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) - \frac{\Omega}{2} |\Phi(z)|^2 - \lambda = 0,$$

for any  $z \in \mathbb{D}$ . The above expression yields

$$G_{f_0}(\Omega, 0, 0) = 0, \quad \forall \Omega \in \mathbb{R}.$$

Thanks to Proposition 3.3, the conformal mapping is parametrized outside the singular set by  $\Omega$  and  $g$  and thus the equation for the density becomes

$$(4.6) \quad \widehat{G}_{f_0}(\Omega, g) \triangleq G_{f_0}(\Omega, g, \mathcal{N}(\Omega, g)) = 0.$$

Next, let us analyze the constraint (4.2), for some particular examples. Since we are looking for smooth solutions, it is convenient to deal with smooth radial profiles. Then, one stands

$$f_0(r) = \widehat{f}_0(r^2),$$

and thus (4.2) becomes

$$(4.7) \quad \mu(\Omega, \widehat{f}_0(r)) = \frac{4r \widehat{f}_0'(r)}{2\Omega r - \int_0^r \widehat{f}_0(s) ds}, \quad \forall r \in (0, 1].$$

At this stage, there are two ways to proceed. The first one is to start with  $\widehat{f}_0$  and reconstruct  $\mu$ , and the second one is to impose  $\mu$  and solve the nonlinear differential equation on  $\widehat{f}_0$ . This last approach is implicit and more delicate to implement. Therefore, let us proceed with the first approach and apply it to some special examples.

4.1.1. *Quadratic profiles.* The first example is the quadratic profile of the type

$$f_0(r) = Ar^2 + B,$$

where  $A, B \in \mathbb{R}$ . In this case,  $\widehat{f}_0(r) = Ar + B$  and thus (4.7) agrees with

$$\mu(\Omega, \widehat{f}_0(r)) = \frac{4Ar}{2\Omega r - \frac{Ar}{2} - Br} = \frac{8A}{4\Omega - B - \widehat{f}_0(r)}, \quad \forall r \in (0, 1].$$

Then, we find

$$(4.8) \quad \mu(\Omega, \mathfrak{t}) = \frac{8A}{4\Omega - B - \mathfrak{t}},$$

which implies from (4.3) that

$$\mathcal{M}_{f_0}(\Omega, \mathfrak{t}) = \frac{4\Omega - B}{8A} \mathfrak{t} - \frac{1}{16A} \mathfrak{t}^2.$$

Thus, using (4.5), we deduce that

$$(4.9) \quad \begin{aligned} \lambda &= \frac{4\Omega - B}{8A} f_0(r) - \frac{1}{16A} f_0(r)^2 - \int_r^1 \frac{1}{\tau} \int_0^\tau s f_0(s) ds d\tau - \frac{\Omega r^2}{2} \\ &= \frac{8\Omega B - 3B^2 - A^2 - 4AB}{16A}. \end{aligned}$$

As we have mentioned before, the conformal mapping is determined by  $\Omega$  and  $g$  and so the last equation takes the form (4.6). The subscript  $f_0$  will be omitted when we refer to this equation with the quadratic profile if there is no confusion.

Let us remark some comparison to the vortex patch problem. The case  $A = 0$  agrees with a vortex patch of the type  $f_0(r) = B$ . It was mentioned before that the boundary equation studied in Section 3 is the one studied in [35] when analyzing the vortex patch problem. Here we have one more equation in  $\mathbb{D}$  given by the density equation. This amounts to look for solutions of the type

$$\omega_0(x) = (B + g)(\Phi^{-1}(x)) \mathbf{1}_{\Phi(\mathbb{D})}(x),$$

which implies that the initial vorticity  $B$  of the vortex patch is perturbed by a function that could not to be constant. However, using (4.1) and evaluating in  $f_0(r) = B$ , for any  $r \in [0, 1]$ , one gets that for this case  $\mu \equiv 0$ . Then if you perturb with  $g$  the equation to be studied is

$$\nabla((f_0 + g) \circ \Phi^{-1})(x) = 0, \quad \forall x \in \Phi(\mathbb{D}).$$

Using the conformal map  $\Phi$  and changing the variables we arrive at

$$\nabla((f_0 + g) \circ \Phi^{-1})(\Phi(z)) = 0, \quad \forall z \in \mathbb{D}.$$

By virtue of (2.2), the above equation leads to

$$\nabla(f_0 + g)(z) = 0,$$

which gives us that  $g$  must be a constant. Hence, using our approach we get that starting with a vortex patch we just can obtain another vortex patch solution.

4.1.2. *Polynomial profiles.* The second example is to consider a general polynomial profile of the type

$$\widehat{f}_0(r) = Ar^m + B, \quad m \in \mathbb{N}^*, \quad A \geq 0, \quad B \in \mathbb{R}.$$

From (4.7) we obtain that

$$\mu(\Omega, \mathfrak{t}) = 4m(m+1)A^{\frac{1}{m}} \frac{(\mathfrak{t} - B)^{\frac{m-1}{m}}}{2\Omega(m+1) - mB - \mathfrak{t}}, \quad \forall \mathfrak{t} \geq B.$$

Consequently, we find that

$$\mathcal{M}_{f_0}(\Omega, \mathfrak{t}) = \frac{1}{4m(m+1)A^{\frac{1}{m}}} \left[ (2\Omega(m+1) - mB) \int_B^{\mathfrak{t}} (s - B)^{\frac{1-m}{m}} ds - \int_B^{\mathfrak{t}} s(s - B)^{\frac{1-m}{m}} ds \right].$$

Remark that for  $m = 1$  we recover the previous quadratic profiles. The discussion developed later about the quadratic profile can be also extended to this polynomial profile as we shall comment in detail in Remark 5.9.

4.1.3. *Gaussian profiles.* Another example which is relevant is given by the Gaussian distribution,

$$\widehat{f}_0(r) = e^{Ar}, \quad A \in \mathbb{R}^*.$$

Inserting  $\widehat{f}_0$  into (4.7) one obtains that

$$\mu(\Omega, \widehat{f}_0(r)) = \frac{4A^2 r}{2\Omega Ar - e^{Ar} + 1},$$

and thus

$$\mu(\Omega, \mathfrak{t}) = \frac{4A \ln \mathfrak{t}}{1 - \mathfrak{t} + 2\Omega \ln \mathfrak{t}}.$$

Then the formula (4.3) allows us to get

$$\mathcal{M}_{f_0}(\Omega, \mathfrak{t}) = \frac{1}{4A} \left[ 2\Omega \mathfrak{t} + \int_1^{\mathfrak{t}} \frac{1-s}{\ln s} ds \right].$$

4.2. **Functional regularity.** In this section, we will be interested in the regularity of the functional  $\widehat{G}$  obtained in (4.6) for a the quadratic profile. Notice that in the case of the quadratic profile (1.7), the singular set (3.5) becomes

$$(4.10) \quad \mathcal{S}_{\text{sing}} = \left\{ \widehat{\Omega}_n \triangleq \frac{A}{4} + \frac{B}{2} - \frac{A(n+1)}{2n(n+2)} - \frac{B}{2n}, \quad \forall n \in \mathbb{N}^* \cup \{+\infty\} \right\}.$$

**Proposition 4.1.** *Let  $f_0$  be the quadratic profile given by (1.7) and  $I$  be an open interval with  $\bar{I} \subset \mathbb{R} \setminus \mathcal{S}_{\text{sing}}$ . Then, there exists  $\varepsilon > 0$  such that*

$$\widehat{G} : I \times B_{\mathcal{C}_s^{1,\alpha}(\mathbb{D})}(0, \varepsilon) \rightarrow \mathcal{C}_s^{1,\alpha}(\mathbb{D}),$$

*is well-defined and of class  $\mathcal{C}^1$ , where  $\widehat{G}$  is defined in (4.6) and  $B_{\mathcal{C}_s^{1,\alpha}(\mathbb{D})}(0, \varepsilon)$  in (3.2).*

*Proof.* Let us show that  $\widehat{G}(\Omega, g) \in \mathcal{C}^{1,\alpha}(\mathbb{D})$ . Clearly,  $\mathcal{M}(\Omega, f)$  is polynomial in  $f$  and by the algebra structure of Hölder spaces we deduce that  $\mathcal{M}(\Omega, f) \in \mathcal{C}^{1,\alpha}(\mathbb{D})$ . Since  $\Phi \in \mathcal{C}^{2,\alpha}(\mathbb{D})$ , then the only term which deserves attention is the integral one. It is clear that

$$\int_{\mathbb{D}} \log |\Phi(\cdot) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) \in \mathcal{C}^0(\mathbb{D}).$$

To estimate its derivative, we note that

$$\nabla_z \log |\Phi(z) - \Phi(y)| = \frac{(\Phi(z) - \Phi(y)) \overline{\Phi'(z)}}{|\Phi(z) - \Phi(y)|^2} = \frac{\overline{\Phi'(z)}}{\Phi(z) - \Phi(y)},$$

which implies that

$$\begin{aligned} \nabla_z \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) &= \overline{\Phi'(z)} \int_{\mathbb{D}} \frac{f(y)}{\Phi(z) - \Phi(y)} |\Phi'(y)| dA(y) \\ &= \overline{\Phi'(z)} \mathcal{F}[\Phi](f)(z), \end{aligned}$$

where the operator  $\mathcal{F}[\Phi]$  is defined in (B.4). Thus, we can use Lemma B.2 obtaining that  $\mathcal{F}[\Phi]$  belongs to  $\mathcal{C}^{1,\alpha}(\mathbb{D})$ . Since  $\Phi' \in \mathcal{C}^{1,\alpha}(\mathbb{D})$  we deduce that the integral term of  $\widehat{G}(\Omega, g)$  lies in the space  $\mathcal{C}^{2,\alpha}(\mathbb{D})$  and is continuous with respect to  $(f, \Phi)$ .

Let us check the symmetry property. Take  $g$  and  $\phi$  satisfying  $g(\bar{z}) = g(z)$  and  $\phi(\bar{z}) = \overline{\phi(z)}$ . It is a simple matter to verify that

$$\mathcal{M}(\Omega, f(\bar{z})) = \mathcal{M}(\Omega, f(z)) \quad \text{and} \quad |\Phi(\bar{z})|^2 = |\Phi(z)|^2, \quad \forall z \in \bar{\mathbb{D}}.$$

For the Newtonian potential, the change of variables  $y \mapsto \bar{y}$  leads to

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(\bar{z}) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) &= \frac{1}{2\pi} \int_{\mathbb{D}} \log |\overline{\Phi(z)} - \Phi(\bar{y})| f(\bar{y}) |\Phi'(\bar{y})|^2 dA(y) \\ &= \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y). \end{aligned}$$

Let us turn to the computations of the Gâteaux derivatives, that can be computed as

$$D_g \widehat{G}(\Omega, g) h(z) = \partial_g G(\Omega, g, \phi) h(z) + \partial_\phi G(\Omega, g, \phi) \circ \partial_g \mathcal{N}(\Omega, g) h(z).$$

By virtue of Proposition 3.3, it is known that  $\mathcal{N}$  is  $\mathcal{C}^1$ , which implies that  $\partial_g \mathcal{N}(\Omega, g)$  is continuous. Gâteaux derivatives are given by

$$\begin{aligned} D_g G(\Omega, g, \phi) h(z) &= D_g \mathcal{M}(\Omega, f(z)) h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| h(y) |\Phi'(y)|^2 dA(y) \\ &= \frac{4\Omega - B}{8A} h(z) - \frac{1}{8A} f(z) h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| h(y) |\Phi'(y)|^2 dA(y), \end{aligned}$$

and

$$\begin{aligned} D_\phi G(\Omega, g, \phi) k(z) &= \frac{\text{Re}}{2\pi} \int_{\mathbb{D}} \frac{k(z) - k(y)}{\Phi(z) - \Phi(y)} f(y) |\Phi'(y)|^2 dA(y) - \Omega \text{Re} [\overline{\Phi(z)} k(z)] \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) \text{Re} [\overline{\Phi'(y)} k'(y)] dA(y). \end{aligned}$$

We focus our attention on the integral terms. The operator  $\mathcal{F}[\Phi]$  in (B.4) allows us to write

$$\int_{\mathbb{D}} \frac{k(\cdot) - k(y)}{\Phi(\cdot) - \Phi(y)} f(y) |\Phi'(y)|^2 dA(y) = k(z) \mathcal{F}[\Phi](f) - \mathcal{F}[\Phi](kf)(z).$$

Thus, Lemma B.2 concludes that this term lies in  $\mathcal{C}^{1,\alpha}(\mathbb{D})$  and is continuous with respect to  $\Phi$ . For the other terms involving the logarithm, we can compute its gradient as before; for instance

$$\nabla_z \int_{\mathbb{D}} \log |\Phi(z) - \Phi(y)| f(y) \text{Re} [\overline{\Phi'(y)} k'(y)] dA(y)$$

$$= \overline{\Phi'(z)} \int_{\mathbb{D}} \frac{f(y)}{\overline{\Phi(z) - \Phi(y)}} \operatorname{Re} \left[ \overline{\Phi'(y)} k'(y) \right] dA(y) = \overline{\Phi'(z)} \mathcal{F}[\overline{\Phi}] \left( \frac{\operatorname{Re} \left[ \overline{\Phi'(\cdot)} k'(\cdot) \right]}{|\Phi'(\cdot)|^2} \right) (z).$$

Since  $\frac{\operatorname{Re} \left[ \overline{\Phi'(\cdot)} k'(\cdot) \right]}{|\Phi'(\cdot)|^2} \in \mathcal{C}^{1,\alpha}(\mathbb{D})$  and is continuous with respect to  $\Phi$ , Lemma B.2 concludes that this term lies in  $\mathcal{C}^{1,\alpha}(\mathbb{D})$  and is continuous with respect to  $\Phi$ . For the other terms involving a logarithmic part, the same procedure can be done. Trivially, both  $D_g G(\Omega, g, \phi)$  and  $D_\phi G(\Omega, g, \phi)$  are continuous with respect to  $g$ . We have obtained that the Gateaux derivatives are continuous with respect to  $(g, \phi)$  and hence they are Fréchet derivatives.  $\square$

**Remark 4.2.** *Although we have done the previous discussion for the quadratic profile, the same argument may be applied for any radial profile  $f_0$ . Note that the only difference with the quadratic profile is that the function  $\mathcal{M}_{f_0}$  and the constant  $\lambda_{f_0}$  will depend on  $f_0$ . Hence, we just have to study the regularity of function  $\mathcal{M}_{f_0}$  in order to give a similar result.*

**4.3. Radial solutions.** The main goal of this sections is the resolution of Equation (4.6) in the class of radial functions but in a small neighborhood of the quadratic profiles (1.7). We establish that except for one singular value for  $\Omega$ , no radial solutions different from  $f_0$  may be found around it. This discussion is essential in order to ensure that with the new reformulation we avoid the main defect of the old one (1.3): the kernel is infinite-dimensional and contains radial solutions. As it was observed before, Proposition 3.3 gives us that the associated conformal mapping of any radial function is the identity map, and therefore (4.6) becomes

$$\widehat{G}(\Omega, f - f_0)(z) = \frac{4\Omega - B}{8A} f(|z|) - \frac{1}{16A} f^2(|z|) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |z - y| f(|y|) dA(y) - \frac{\Omega |z|^2}{2} - \lambda = 0,$$

for any  $z \in \mathbb{D}$  where  $\lambda$  is given by (4.9). Thus the last integral identity of Proposition B.5 gives

$$\frac{4\Omega - B}{8A} f(r) - \frac{1}{16A} f^2(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s f(s) ds d\tau - \frac{1}{2} \Omega r^2 - \lambda = 0, \quad \forall r \in [0, 1].$$

Introduce the function  $G_{\text{rad}} : \mathbb{R} \times \mathcal{C}([0, 1]; \mathbb{R}) \rightarrow \mathcal{C}([0, 1]; \mathbb{R})$ , defined by

$$G_{\text{rad}}(\Omega, f)(r) = \frac{4\Omega - B}{8A} f(r) - \frac{1}{16A} f^2(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s f(s) ds d\tau - \frac{\Omega r^2}{2} - \lambda, \quad \forall r \in [0, 1].$$

It is obvious that  $G_{\text{rad}}$  is well-defined and furthermore it satisfies

$$(4.11) \quad G_{\text{rad}}(\Omega, f_0) = 0, \quad \forall \Omega \in \mathbb{R}.$$

Through this work, it will be more convenient to work with the variable  $x$  instead of  $\Omega$  defined as

$$(4.12) \quad \frac{1}{x} = \frac{4}{A} \left( \Omega - \frac{B}{2} \right).$$

Before stating our result, some properties of the hypergeometric function  $x \in (-1, 1) \mapsto F(1 - \sqrt{2}, 1 + \sqrt{2}; 1; x)$ , are needed. A brief account on some useful properties of Gauss hypergeometric functions will be discussed later in the Appendix C. In view of (C.8) we obtain the identity

$$(4.13) \quad F(1 - \sqrt{2}, 1 + \sqrt{2}; 1; x) = \frac{1}{1 - x} F(-\sqrt{2}, \sqrt{2}; 1; x).$$

According to Appendix C, we have  $F(1 - \sqrt{2}, 1 + \sqrt{2}; 1; 0) = 1$ , and it diverges to  $-\infty$  at 1. This implies that there is at least one root in  $(0, 1)$ . Combined with the fact that its derivative is negative according to (C.4), we may show that this root is unique. Denote this zero by  $x_0 \in (0, 1)$  and set

$$(4.14) \quad \Omega_0 \triangleq \frac{B}{2} + \frac{A}{4x_0}.$$



Setting the ball

$$B(f_0, \varepsilon) = \{f \in \mathcal{C}([0, 1]; \mathbb{R}), \quad \|f - f_0\|_{L^\infty} \leq \varepsilon\},$$

for any  $\varepsilon > 0$ , the first result can be stated as follows.

**Proposition 4.3.** *Let  $f_0$  be the quadratic profile (1.7), with  $A \in \mathbb{R}^*$ ,  $B \in \mathbb{R}$ . Then, there exists  $\varepsilon > 0$  such that*

$$G_{\text{rad}}(\Omega, f) = 0 \iff f = f_0,$$

for any  $(\Omega, f) \in I \times B(f_0, \varepsilon)$  and any bounded interval  $I$ , with  $I \cap ([\frac{B}{2}, \frac{B}{2} + \frac{A}{4}] \cup \{\Omega_0\}) = \emptyset$ .

*Proof.* We remark that  $G_{\text{rad}}$  is a  $\mathcal{C}^1$  function on  $(\Omega, f)$ . The idea is to apply the Implicit Function Theorem to deduce the result. By differentiation with respect to  $f$ , one gets that

$$\begin{aligned} D_f G_{\text{rad}}(\Omega, f_0)h(r) &= \frac{4\Omega - B}{8A}h(r) - \frac{1}{8A}f_0(r)h(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s)dsd\tau \\ (4.15) \qquad \qquad \qquad &= \frac{4\Omega - Ar^2 - 2B}{8A}h(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s)dsd\tau, \end{aligned}$$

for any  $h \in \mathcal{C}([0, 1]; \mathbb{R})$ . Now we shall look for the kernel of this operator, which consists of elements  $h$  solving a Volterra integro-differential equation of the type

$$\frac{4\Omega - Ar^2 - 2B}{8A}h(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s)dsd\tau = 0, \quad \forall r \in [0, 1].$$

The assumption  $\Omega \notin [\frac{B}{2}, \frac{B}{2} + \frac{A}{4}]$  implies that  $r \in [0, 1] \mapsto \frac{4\Omega - Ar^2 - 2B}{8A}$  is not vanishing and smooth. Thus from the regularization of the integral, one can check that any element of the kernel is actually  $\mathcal{C}^\infty$ . Our purpose is to derive a differential equation by differentiating successively this integral equation. With the notation (4.12) the kernel equation can be written in the form

$$\mathcal{L}h(r) \triangleq \left(\frac{1}{x} - r^2\right)h(r) - 8 \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s)dsd\tau = 0, \quad \forall r \in [0, 1].$$

Remark that the assumptions on  $\Omega$  can be translated into  $x$ , as  $x \in (-\infty, 1)$  and  $x \neq 0$ . Differentiating the function  $\mathcal{L}h$  yields

$$(\mathcal{L}h)'(r) = \left(\frac{1}{x} - r^2\right)h'(r) - 2rh(r) + \frac{8}{r} \int_0^r sh(s)ds.$$

Multiplying by  $r$  and differentiating again we deduce that

$$(4.16) \quad \frac{[r(\mathcal{L}h)'(r)]'}{r} = \left(\frac{1}{x} - r^2\right)h''(r) + \left(\frac{1}{x_0} - 5r^2\right)\frac{h'(r)}{r} + 4h(r) = 0, \quad \forall r \in (0, 1).$$

In order to solve the above equation we look for solutions in the form

$$h(r) = \rho(xr^2).$$

This ansatz can be justified *a posteriori* by evoking the uniqueness principle for ODEs. Doing the change of variables  $y = xr^2$ , we transform the preceding equation to

$$y(1-y)\rho''(y) + (1-3y)\rho'(y) + \rho(y) = 0.$$

Appendix C leads to assure that the only bounded solutions close to zero to this hypergeometric equation are given by

$$\rho(y) = \gamma F(1 + \sqrt{2}, 1 - \sqrt{2}; 1; y), \quad \forall \gamma \in \mathbb{R},$$

and thus

$$(4.17) \quad h(r) = \gamma F(1 + \sqrt{2}, 1 - \sqrt{2}; 1; xr^2), \quad \forall \gamma \in \mathbb{R}.$$

It is important to note that from the integral representation (C.2) of hypergeometric functions, we can extend the above solution to  $x \in (-\infty, 1)$ . Coming back to the equation (4.16) and integrating two times, we obtain two real numbers  $\alpha, \beta \in \mathbb{R}$  such that

$$\mathcal{L}h(r) = \alpha \ln r + \beta, \quad \forall r \in (0, 1].$$

Since  $\mathcal{L}h \in \mathcal{C}([0, 1], \mathbb{R})$ , we obtain that  $\alpha = 0$  and thus  $\mathcal{L}h(r) = \beta$ . By definition one has  $\mathcal{L}h(1) = \left(\frac{1}{x} - 1\right)h(1)$ . The fact that  $x \neq 1$  implies that  $\mathcal{L}h = 0$  if and only if  $h(1) = 0$ . According to (4.17), this condition is equivalent to  $\gamma F(1 - \sqrt{2}, 1 + \sqrt{2}; 1; x) = 0$ . It follows that the kernel is trivial ( $\gamma = 0$ ) if and only if  $x \neq x_0$ , with  $x_0$  being the only zero of  $F(1 - \sqrt{2}, 1 + \sqrt{2}; 1; \cdot)$ . However, for  $x = x_0$  the kernel is one-dimensional and is generated by this hypergeometric function. Those claims will be made more rigorous in what follows.

• **Case  $x \neq x_0$ .** As we have mentioned before, the kernel is trivial and it remains to check that  $\mathcal{L}$  is an isomorphism. With this aim, it suffices to prove that  $\mathcal{L}$  is a Fredholm operator of zero index. First, we can split  $\mathcal{L}$  as follows

$$\mathcal{L} \triangleq \mathcal{L}_0 + \mathcal{K}, \quad \mathcal{L}_0 \triangleq \left(\frac{1}{x} - r^2\right) \text{Id} \quad \text{and} \quad \mathcal{K}h(r) \triangleq -8 \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s) ds d\tau.$$

Second, it is obvious that  $\mathcal{L}_0 : \mathcal{C}([0, 1]; \mathbb{R}) \rightarrow \mathcal{C}([0, 1]; \mathbb{R})$  is an isomorphism, it is a Fredholm operator of zero index. Now, since the Fredholm operators with given index are stable by compact perturbation (for more details, see Appendix A), then to check that  $\mathcal{L}$  has zero index it is enough to establish that

$$\mathcal{K} : \mathcal{C}([0, 1]; \mathbb{R}) \rightarrow \mathcal{C}([0, 1]; \mathbb{R}),$$

is compact. One can easily obtain that for  $h \in \mathcal{C}([0, 1]; \mathbb{R})$  the function  $\mathcal{K}h$  belongs to  $\mathcal{C}^1([0, 1]; \mathbb{R})$ . Furthermore, by change of variables

$$(\mathcal{K}h)'(r) = r \int_0^1 sh(rs) ds \quad \text{and} \quad (\mathcal{K}h)'(0) = 0, \quad \forall r \in (0, 1],$$

which implies that

$$\|\mathcal{K}h\|_{\mathcal{C}^1} \leq C \|h\|_{L^\infty}.$$

Since the embedding  $\mathcal{C}^1([0, 1]; \mathbb{R}) \hookrightarrow \mathcal{C}([0, 1]; \mathbb{R})$  is compact, we find that  $\mathcal{K}$  is a compact operator. Finally, we get that  $\mathcal{L}$  is an isomorphism. This ensures that  $D_f G_{\text{rad}}(\Omega, f_0)$  is an isomorphism, and therefore the Implicit Function Theorem together with (4.11) allow us to deduce that the only solutions of  $G(\Omega, f) = 0$  in  $I \times B(f_0, \varepsilon)$  are given by the trivial ones  $\{(\Omega, f_0), \Omega \in I\}$ .

• **Case  $x = x_0$ .** In this special case the kernel of  $\mathcal{L}$  is one-dimensional and is generated by

$$(4.18) \quad \text{Ker} \mathcal{L} = \langle F(1 - \sqrt{2}, 1 + \sqrt{2}; 1; x(\cdot)^2) \rangle.$$

This case will be deeply discussed below in Proposition 4.4. □

Let us focus on the case  $\Omega = \Omega_0$ . From (4.18), the kernel of the linearized operator is one-dimensional, and we will be able to implement the Crandall–Rabinowitz Theorem. Our result reads as follows.

**Proposition 4.4.** *Let  $f_0$  be the quadratic profile (1.7) with  $A \in \mathbb{R}^*$ ,  $B \in \mathbb{R}$  and fix  $\Omega_0$  as in (4.14). Then, there exists an open neighborhood  $U$  of  $(\Omega_0, f_0)$  in  $\mathbb{R} \times \mathcal{C}([0, 1]; \mathbb{R})$  and a continuous curve  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi) \in U$  with  $a > 0$  such that*

$$G_{\text{rad}}(\Omega_\xi, f_\xi) = 0, \quad \forall \xi \in (-a, a).$$

*Proof.* We must check that the hypotheses of the Crandall–Rabinowitz Theorem are achieved. It is clear that

$$G_{\text{rad}}(\Omega, f_0) = 0, \quad \forall \Omega \in \mathbb{R}.$$

It is not difficult to show that the mapping  $(\Omega, f) \mapsto G_{\text{rad}}(\Omega, f)$  is  $\mathcal{C}^1$ . In addition, we have seen in the foregoing discussion that  $D_f G_{\text{rad}}(\Omega_0, f_0)$  is a Fredholm operator with zero index and its kernel is one-dimensional. Therefore to apply the bifurcation arguments it remains just to check the transversality condition in the Crandall-Rabinowitz Theorem. Having this in mind, we should first find a practical characterization for the range of the linearized operator. We note that an element  $d \in \mathcal{C}([0, 1]; \mathbb{R})$  belongs to the range of  $D_f G_{\text{rad}}(\Omega_0, f_0)$  if the equation

$$(4.19) \quad \frac{\frac{1}{x_0} - r^2}{8} h(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s) ds d\tau = d(r), \quad \forall r \in [0, 1],$$

admits a solution  $h$  in  $\mathcal{C}([0, 1]; \mathbb{R})$ , where  $x_0$  is given by (4.12). Consider the auxiliary function  $H$

$$H(r) \triangleq \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s) ds d\tau, \quad \forall r \in [0, 1].$$

Then  $H$  belongs to  $\mathcal{C}^1([0, 1]; \mathbb{R})$  and it satisfies the boundary condition

$$(4.20) \quad H(1) = 0 \quad \text{and} \quad H'(0) = 0.$$

In order to write down an ordinary differential equation for  $H$ , let us define the linear operator

$$\mathcal{L}h(r) \triangleq \int_r^1 \frac{1}{\tau} \int_0^\tau sh(s) ds d\tau, \quad \forall r \in [0, 1],$$

for any  $h \in \mathcal{C}([0, 1]; \mathbb{R})$ . Then, we derive successively,

$$(\mathcal{L}h)'(r) = -\frac{1}{r} \int_0^r sh(s) ds,$$

and

$$r(\mathcal{L}h)''(r) + (\mathcal{L}h)'(r) = -rh(r).$$

Applying this identity to  $H$  one arrives at

$$rH''(r) + H'(r) = -\frac{8rx_0}{1 - x_0r^2} [H(r) + d(r)].$$

Thus,  $H$  solves the second order differential equation

$$(4.21) \quad r(1 - x_0r^2)H''(r) + (1 - x_0r^2)H'(r) + 8rx_0H(r) = -8rx_0d(r),$$

supplemented with boundary conditions (4.20). The argument is to come back to the original equation (4.19) and show that the candidate

$$h(r) \triangleq \frac{8x_0}{1 - x_0r^2} [H(r) + d(r)],$$

is actually a solution to this equation. Then, we need to check that  $\mathcal{L}h = H$ . By setting  $\mathcal{H} \triangleq \mathcal{L}h - H$ , we deduce that

$$r(\mathcal{H})''(r) + (\mathcal{H})'(r) = 0,$$

with the boundary conditions  $\mathcal{H}(1) = 0$  and  $\mathcal{H}'(0) = 0$ , which come from (4.20). The solution of this differential equation is

$$\mathcal{H}(r) = \lambda_0 + \lambda_1 \ln r, \quad \forall \lambda_0, \lambda_1 \in \mathbb{R}$$

Since  $\mathcal{L}h$  and  $H$  belong to  $\mathcal{C}([0, 1]; \mathbb{R})$ , then necessarily  $\lambda_1 = 0$ , and from the boundary condition we find  $\lambda_0 = 0$ . This implies that  $\mathcal{L}h = H$ , and it shows finally that solving (4.19) is equivalent to solving (4.21). Now, let us focus on the resolution of (4.21). For this purpose, we proceed by finding a particular solution for the homogeneous equation and use later the method of variation of constants. Looking for a solution to the homogeneous equation in the form  $\mathcal{H}_0(r) = \rho(x_0r^2)$ , and using the variable  $y = x_0r^2$ , one arrives at

$$(1 - y)y\rho''(y) + (1 - y)\rho'(y) + 2\rho(y) = 0.$$

This is a hypergeometric equation, and one solution is given by  $y \mapsto F(-\sqrt{2}, \sqrt{2}; 1; y)$ . Thus, a particular solution to the homogeneous equation is  $\mathcal{H}_0 : r \in [0, 1] \mapsto F(-\sqrt{2}, \sqrt{2}; 1; x_0 r^2)$ . Then, the general solutions for (4.21) are given through the formula

$$H(r) = \mathcal{H}_0(r) \left[ K_2 + \int_{\delta}^r \frac{1}{\tau \mathcal{H}_0^2(\tau)} \left[ K_1 - 8x_0 \int_0^{\tau} \frac{s \mathcal{H}_0(s)}{1 - x_0 s^2} d(s) ds \right] d\tau \right], \quad \forall r \in [0, 1],$$

where  $K_1, K_2$  are real constants and  $\delta \in (0, 1)$  is any given number. Since  $\mathcal{H}_0$  is smooth on the interval  $[0, 1]$ , with  $\mathcal{H}_0(0) = 1$ , one can check that  $H$  admits a singular term close to zero taking the form  $K_1 \ln r$ . This forces  $K_1$  to vanish because  $H$  is continuous up to the origin. Therefore, we infer that

$$H(r) = \mathcal{H}_0(r) \left[ K_2 - 8x_0 \int_{\delta}^r \frac{1}{\tau \mathcal{H}_0^2(\tau)} \int_0^{\tau} \frac{s \mathcal{H}_0(s)}{1 - x_0 s^2} d(s) ds d\tau \right], \quad \forall r \in [0, 1].$$

The last integral term is convergent at the origin and one may take  $\delta = 0$ . This implies that

$$H(r) = \mathcal{H}_0(r) \left[ K_2 - 8x_0 \int_0^r \frac{1}{\tau \mathcal{H}_0^2(\tau)} \int_0^{\tau} \frac{s \mathcal{H}_0(s)}{1 - x_0 s^2} d(s) ds d\tau \right], \quad \forall r \in [0, 1].$$

From this expression, we deduce the second condition of (4.20). For the first condition,  $H(1) = 0$ , we first note from (4.13) that  $F(-\sqrt{2}, \sqrt{2}; 1; x_0) = 0$ . Then, we can compute the limit at  $r = 1$  via l'Hôpital's rule leading to

$$H(1) = -8x_0 \lim_{r \rightarrow 1^-} \mathcal{H}_0(r) \int_0^r \frac{1}{\tau \mathcal{H}_0^2(\tau)} \int_0^{\tau} \frac{s \mathcal{H}_0(s)}{1 - x_0 s^2} d(s) ds d\tau = 8x_0 \frac{\int_0^1 \frac{s \mathcal{H}_0(s)}{1 - x_0 s^2} d(s) ds}{\mathcal{H}_0'(1)}.$$

From the expression of  $\mathcal{H}_0$  and (C.4) we recover that  $\mathcal{H}_0'(1) = -4x_0 F(1 - \sqrt{2}, 1 + \sqrt{2}; 2; x_0)$ . We point out that this quantity is not vanishing. This can be proved by differentiating the relation (4.13), which implies that

$$F(1 - \sqrt{2}, 1 + \sqrt{2}; 2; x_0) = (x_0 - 1)F(2 - \sqrt{2}, 2 + \sqrt{2}; 2; x_0),$$

and the latter term is not vanishing from the definition of hypergeometric functions. It follows that the condition  $H(1) = 0$  is equivalent to

$$(4.22) \quad \int_0^1 \frac{s \mathcal{H}_0(s)}{1 - x_0 s^2} d(s) ds = 0.$$

This characterizes the elements of the range of the linearized operator. Now, we are in a position to check the transversality condition. According to the expression (4.15), one gets by differentiating with respect to  $\Omega$  that

$$D_{\Omega, f} G_{\text{rad}}(\Omega_0, f_0) h(r) = \frac{1}{2A} h(r).$$

Recall from (4.18) and the relation (4.13) that the kernel is generated by  $r \in [0, 1] \mapsto \frac{\mathcal{H}_0(r)}{1 - x_0 r^2}$ . Hence, from (4.22) the transversality assumption in the Crandall–Rabinowitz Theorem becomes

$$\int_0^1 \frac{s \mathcal{H}_0^2(s)}{(1 - x_0 s^2)^2} ds \neq 0,$$

which is trivially satisfied, and concludes the announced result.  $\square$

## 5. LINEARIZED OPERATOR FOR THE DENSITY EQUATION

This section is devoted to the study of the linearized operator of the density equation (4.6). First, we will compute it with a general  $f_0$  and provide a suitable formula in the case of quadratic profiles. Second, we shall prove that the linearized operator is a Fredholm operator of index zero because it takes the form of a compact perturbation of an invertible operator. More details about Fredholm operators can be found in Appendix A. Later we will focus our attention on the algebraic structure of the kernel and the range and give explicit expressions by using hypergeometric functions. We point out that the kernel description is done through the resolution of a Volterra integro-differential equation.

**5.1. General formula and Fredholm index of the linearized operator.** Let  $f_0$  be an arbitrary smooth radial function satisfying (4.2) and let us compute the linearized operator of the functional  $\widehat{G}_{f_0}$  given by (4.6). First, using Proposition 3.3 one gets

$$(5.1) \quad D_g \mathcal{N}(\Omega, 0)h(z) = z \sum_{n \geq 1} A_n z^n \triangleq k(z),$$

where  $A_n$  is given in (3.6) and  $h \in \mathcal{C}_s^{1,\alpha}(\mathbb{D})$ . Therefore, differentiating with respect to  $g$  yields

$$\begin{aligned} D_g \widehat{G}_{f_0}(\Omega, 0)h &= \partial_g G_{f_0}(\Omega, 0, 0)h + \partial_\phi G_{f_0}(\Omega, 0, 0)\partial_g \mathcal{N}h(\Omega, 0) \\ &= \partial_g G_{f_0}(\Omega, 0, 0)h + \partial_\phi G_{f_0}(\Omega, 0, 0)k. \end{aligned}$$

Using the Fréchet derivatives from Proposition 4.1, we have that

$$\begin{aligned} D_g \widehat{G}_{f_0}(\Omega, 0)h(z) &= D_g \mathcal{M}_{f_0}(\Omega, f_0(z))h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |z - y| h(y) dA(y) - \Omega \operatorname{Re}[\bar{z}k(z)] \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{D}} \operatorname{Re} \left[ \frac{k(z) - k(y)}{z - y} \right] f_0(y) dA(y) \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} \log |z - y| f_0(y) \operatorname{Re}[k'(y)] dA(y). \end{aligned}$$

From the definition of  $\mathcal{M}_{f_0}$  in (4.3) we infer that

$$D_g \mathcal{M}_{f_0}(\Omega, f_0(z))h(z) = \frac{h(z)}{\mu(\Omega, f_0(|z|))}, \quad \forall z \in \mathbb{D}.$$

Putting together the preceding formulas we obtain

$$(5.2) \quad \begin{aligned} D_g \widehat{G}_{f_0}(\Omega, 0)h(z) &= \frac{h(z)}{\mu(\Omega, f_0(r))} + \frac{1}{2\pi} \int_{\mathbb{D}} \log |z - y| h(y) dA(y) - \Omega \operatorname{Re}[\bar{z}k(z)] \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{D}} \operatorname{Re} \left[ \frac{k(z) - k(y)}{z - y} \right] f_0(y) dA(y) \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} \log |z - y| f_0(y) \operatorname{Re}[k'(y)] dA(y), \end{aligned}$$

where  $\mu$  is given by the compatibility condition (4.2). Next, we shall rewrite the linearized operator for the quadratic profile, and we will omit the subscript  $f_0$  for the sake of simplicity. Taking  $\mu$  as in (4.8), we get

$$(5.3) \quad \begin{aligned} D_g \widehat{G}(\Omega, 0)h(z) &= \frac{1}{8} \left( \frac{1}{x} - r^2 \right) h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |z - y| h(y) dA(y) - \Omega \operatorname{Re}[\bar{z}k(z)] \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{D}} \operatorname{Re} \left[ \frac{k(z) - k(y)}{z - y} \right] f_0(y) dA(y) \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} \log |z - y| f_0(y) \operatorname{Re}[k'(y)] dA(y), \end{aligned}$$

where  $x$  is given by (4.12).

In the following result we show that the linearized operator associated to a quadratic profile is a Fredholm operator with index zero. Similar result may be obtained in the general case imposing suitable conditions on the profiles.

**Proposition 5.1.** *Let  $f_0$  be the profile (1.7), with  $A \in \mathbb{R}^*$  and  $B \in \mathbb{R}$ . Assume that  $\Omega \notin [\frac{B}{2}, \frac{B}{2} + \frac{A}{4}] \cup \mathcal{S}_{\text{sing}}$ . Then,  $D_g \widehat{G}(\Omega, 0) : \mathcal{C}_s^{1,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}_s^{1,\alpha}(\mathbb{D})$  is a Fredholm operator with zero index.*

*Proof.* Using (5.3) we have

$$\begin{aligned} D_g \widehat{G}(\Omega, 0)h(z) &= \frac{1}{8} \left( \frac{1}{x} - r^2 \right) h(z) + \frac{1}{2\pi} \int_{\mathbb{D}} \log |z - y| h(y) dA(y) - \Omega \text{Re}[\bar{z}k(z)] \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{D}} \text{Re} \left[ \frac{k(z) - k(y)}{z - y} \right] f_0(y) dA(y) \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} \log |z - y| f_0(y) \text{Re}[k'(y)] dA(y) \\ &\triangleq \left[ \frac{1}{8} \left( \frac{1}{x} - r^2 \right) \text{Id} + \mathcal{K} \right] h(z), \end{aligned}$$

where  $k$  is related to  $h$  through (5.1). The assumption on  $\Omega$  entails that the smooth function  $z \in \bar{\mathbb{D}} \mapsto \frac{1}{x} - |z|^2$  is not vanishing on the closed unit disc. Then the operator

$$\frac{1}{8} \left( \frac{1}{x} - |z|^2 \right) \text{Id} : \mathcal{C}_s^{1,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}_s^{1,\alpha}(\mathbb{D}),$$

is an isomorphism. Hence, it is a Fredholm operator with zero index. To check that  $\mathcal{L}$  is also a Fredholm operator with zero index, it suffices to prove that the operator  $\mathcal{K} : \mathcal{C}_s^{1,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}_s^{1,\alpha}(\mathbb{D})$  is compact. To do that, we will prove that  $\mathcal{K} : \mathcal{C}_s^{1,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}^{1,\gamma}(\mathbb{D})$ , for any  $\gamma \in (0, 1)$ . We split  $\mathcal{K}$  as follows

$$\mathcal{K}h = \sum_{j=1}^4 \mathcal{K}_j h,$$

with

$$\begin{aligned} \mathcal{K}_1 h &= -\Omega \text{Re}[\bar{z}k(z)], \quad \mathcal{K}_2 h(z) = \frac{1}{2\pi} \int_{\mathbb{D}} \log |z - y| h(y) dA(y), \\ \mathcal{K}_3 h(z) &= \frac{1}{\pi} \int_{\mathbb{D}} \log |z - y| f_0(y) \text{Re}[k'(y)] dA(y), \quad \mathcal{K}_4 h = \frac{1}{2\pi} \int_{\mathbb{D}} \text{Re} \left[ \frac{k(z) - k(y)}{z - y} \right] f_0(y) dA(y). \end{aligned}$$

The estimate of the first term  $\mathcal{K}_1 h$  follows from (5.1) and (3.7), leading to

$$(5.4) \quad \|\mathcal{K}_1 h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$

Concerning the term  $\mathcal{K}_2$  we note that

$$\|\mathcal{K}_2 h\|_{L^\infty(\mathbb{D})} \leq C \|h\|_{L^\infty(\mathbb{D})},$$

and differentiating it, we obtain

$$\nabla_z \mathcal{K}_2 h(z) = \frac{1}{2\pi} \int_{\mathbb{D}} \frac{h(y)}{\bar{z} - \bar{y}} dA(y).$$

Lemma B.1 yields

$$\|\nabla_z \mathcal{K}_2 h\|_{\mathcal{C}^{0,\gamma}(\mathbb{D})} \leq C \|h\|_{L^\infty(\mathbb{D})},$$

for any  $\gamma \in (0, 1)$ , and thus

$$(5.5) \quad \|\mathcal{K}_2 h\|_{\mathcal{C}^{1,\gamma}(\mathbb{D})} \leq C \|h\|_{L^\infty(\mathbb{D})}.$$

The estimate of  $\mathcal{K}_3$  is similar to that of  $\mathcal{K}_2$ , and using (5.5) and (3.7) we find that

$$(5.6) \quad \|\mathcal{K}_3 h\|_{\mathcal{C}^{1,\gamma}(\mathbb{D})} \leq C \|f_0\|_{L^\infty(\mathbb{D})} \|k'\|_{L^\infty(\mathbb{D})} \leq C \|f_0\|_{L^\infty(\mathbb{D})} \|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$

Setting

$$K(z, y) = \frac{k(z) - k(y)}{z - y}, \quad \forall z \neq y,$$

it is obvious that  $|K(z, y)| \leq \|k'\|_{L^\infty(\mathbb{D})}$ . Therefore, we have

$$\|\mathcal{K}_4 h\|_{L^\infty} \leq C \|f_0\|_{L^\infty} \|k'\|_{L^\infty(\mathbb{D})}.$$

Moreover, by differentiation we find

$$\nabla_z \mathcal{K}_4 h(z) = \frac{1}{2\pi} \operatorname{Re} \int_{\mathbb{D}} \nabla_z K(z, y) f_0(y) dA(y).$$

Straightforward computations show that

$$\begin{aligned} |\nabla_z K(z, y)| &\leq C \|k'\|_{L^\infty(\mathbb{D})} |z - y|^{-1}, \\ |\nabla_z K(z_1, y) - \nabla_z K(z_2, y)| &\leq C |z_1 - z_2| \left[ \frac{\|k''\|_{L^\infty(\mathbb{D})}}{|z_1 - y|} + \frac{\|k'\|_{L^\infty(\mathbb{D})}}{|z_1 - y| |z_2 - y|} \right]. \end{aligned}$$

Thus, hypotheses (B.2) are satisfied and we can use Lemma B.1 and (3.7) to find

$$\|\nabla_z \mathcal{K}_4 h\|_{\mathcal{C}^{0,\gamma}(\mathbb{D})} \leq C \|f_0\|_{L^\infty} \|k\|_{\mathcal{C}^2(\mathbb{D})} \leq C \|f_0\|_{L^\infty} \|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})},$$

obtaining

$$(5.7) \quad \|\mathcal{K}_4 h\|_{\mathcal{C}^{1,\gamma}(\mathbb{D})} \leq C \|f_0\|_{L^\infty(\mathbb{D})} \|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$

Combining the estimates (5.4), (5.5), (5.6) and (5.7), we deduce

$$\|\mathcal{K}h\|_{\mathcal{C}^{1,\gamma}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})},$$

which concludes the proof.  $\square$

To end this subsection, we give a more explicit form of the linearized operator. Coming back to the general expression in (5.2) and using Proposition B.5 we get that

$$\begin{aligned} D_g \widehat{G}_{f_0}(\Omega, 0) h(z) &= \sum_{n \geq 1} \cos(n\theta) \left[ \frac{h_n(r)}{\mu(\Omega, f_0(r))} - \frac{r}{n} \left( A_n G_n(r) + \frac{1}{2r^{n+1}} H_n(r) \right) \right] \\ &\quad + \frac{h_0(r)}{\mu(\Omega, f_0(r))} - \int_r^1 \frac{1}{\tau} \int_0^\tau s h_0(s) ds d\tau, \end{aligned}$$

for

$$r e^{i\theta} \mapsto h(r e^{i\theta}) = \sum_{n \in \mathbb{N}} h_n(r) \cos(n\theta) \in \mathcal{C}_s^{1,\alpha}(\mathbb{D}),$$

where

$$\begin{aligned} G_n(r) &\triangleq n \Omega r^{n+1} + r^{n-1} \int_0^1 s f_0(s) ds - (n+1) r^{n-1} \int_0^r s f_0(s) ds + \frac{n+1}{r^{n+1}} \int_0^r s^{2n+1} f_0(s) ds, \\ H_n(r) &\triangleq r^{2n} \int_r^1 \frac{1}{s^{n-1}} h_n(s) ds + \int_0^r s^{n+1} h_n(s) ds, \end{aligned}$$

for any  $n \geq 1$ . The value of  $A_n$  is given by (3.6) and recall that it was derived from the expression  $\partial_g \mathcal{N}(\Omega, 0)$  when studying the boundary equation. Moreover, there is another useful expression for  $A_n$  coming from the value of  $G_n(1)$

$$G_n(1) = n \left[ \Omega - \int_0^1 s f_0(s) ds + \frac{n+1}{n} \int_0^1 s^{2n+1} f_0(s) ds \right] = n \left( \Omega - \widehat{\Omega}_n \right).$$

Those preceding identities agrees with

$$A_n = -\frac{H_n(1)}{2G_n(1)}, \quad \forall n \geq 1.$$

In the special case of  $f_0$  being a quadratic profiles of the type (1.7), straightforward computations imply that

$$(5.8) \quad \begin{aligned} D_g \widehat{G}(\Omega, 0)h(z) &= \sum_{n \geq 1} \cos(n\theta) \left[ \frac{\frac{1}{x} - r^2}{8} h_n(r) - \frac{r}{n} \left( A_n G_n(r) + \frac{1}{2r^{n+1}} H_n(r) \right) \right] \\ &+ \frac{\frac{1}{x} - r^2}{8} h_0(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s h_0(s) ds d\tau, \end{aligned}$$

with

$$(5.9) \quad G_n(r) = -\frac{An(n+1)}{4(n+2)} r^{n-1} P_n(r^2),$$

$$(5.10) \quad H_n(r) = r^{2n} \int_r^1 \frac{1}{s^{n-1}} h_n(s) ds + \int_0^r s^{n+1} h_n(s) ds,$$

$$(5.11) \quad P_n(r) = r^2 - \frac{n+2}{n+1} \frac{1}{x} r - \frac{A+2B}{A} \frac{n+2}{n(n+1)},$$

$$(5.12) \quad G_n(1) = -\frac{An(n+1)}{4(n+2)} P_n(1) = n \left[ \frac{A}{4} \left( \frac{1}{x} - 1 \right) + \frac{A(n+1)}{2n(n+2)} + \frac{B}{2n} \right],$$

$$(5.13) \quad A_n = -\frac{H_n(1)}{2G_n(1)} = \frac{H_n(1)}{2n \left\{ \widehat{\Omega}_n - \Omega \right\}}.$$

Remark that  $G_n(1) \neq 0$  since we are assuming that  $\Omega \notin \mathcal{S}_{\text{sing}}$ , the singular set defined in (4.10).

From now on we will work only with the quadratic profiles. Similar study could be implemented with general profiles but the analysis may turn out to be very difficult because the spectral study is intimately related to the distribution of the selected profile.

**5.2. Kernel structure and negative results.** The current objective is to conduct a precise study for the kernel structure of the linearized operator (5.8). We must identify the master equation describing the dispersion relation. As a by-product we connect the dimension of the kernel to the number of roots of the master equation. We shall distinguish in this study between the regular case corresponding to  $x \in (-\infty, 1)$  and the singular case associated to  $x > 1$ . For this latter case we prove that the equation (4.10) has no solution close to the trivial one.

**5.2.1. Regular case.** Let us start with a preliminary result devoted to the explicit resolution of a second order differential equation with polynomial coefficients taking the form

$$(5.14) \quad (1 - xr^2)rF''(r) - (1 - xr^2)(2n - 1)F'(r) + 8rxF(r) = g(r), \quad \forall r \in [0, 1],$$

This will be applied later to the study of the kernel and the range. Before stating our result we need to introduce some functions

$$(5.15) \quad F_n(r) = F(a_n, b_n; c_n; r), \quad a_n = \frac{n - \sqrt{n^2 + 8}}{2}, \quad b_n = \frac{n + \sqrt{n^2 + 8}}{2}, \quad c_n = n + 1.$$

where  $r \in [0, 1) \mapsto F(a, b; c; r)$  denotes the Gauss hypergeometric function defined in (C.1).

**Lemma 5.2.** *Let  $n \geq 1$  be an integer,  $x \in (-\infty, 1)$  and  $g \in \mathcal{C}([0, 1]; \mathbb{R})$ . Then, the general continuous solutions of equation (5.14) supplemented with the initial condition  $F(0) = 0$ , are given by a one-parameter curve*

$$F(r) = r^{2n} F_n(xr^2) \left[ \frac{F(1)}{F_n(x)} - \frac{x^{n-1}}{4} \int_{xr^2}^x \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^\tau \frac{F_n(s)}{1-s} \left( \frac{x}{s} \right)^{\frac{1}{2}} g \left( \left( \frac{s}{x} \right)^{\frac{1}{2}} \right) ds d\tau \right].$$



*Proof.* Consider the auxiliary function  $F(r) = \mathcal{F}(xr^2)$  and set  $y = xr^2$ . Note that  $y \in [0, x]$  when  $x > 0$  and  $y \in [x, 0]$  if  $x < 0$ , then  $r = \left(\frac{y}{x}\right)^{\frac{1}{2}}$  in both cases. Hence, the equation governing this new function is

$$(5.16) \quad (1-y)y\mathcal{F}''(y) - (n-1)(1-y)\mathcal{F}'(y) + 2\mathcal{F}(y) = \frac{1}{4x} \left(\frac{x}{y}\right)^{\frac{1}{2}} g\left(\left(\frac{y}{x}\right)^{\frac{1}{2}}\right),$$

with the boundary condition  $\mathcal{F}(0) = 0$ . The strategy to be followed consists in solving the homogeneous equation and using later the method of variation of constants. The homogeneous problem is given by

$$(1-y)y\mathcal{F}_0''(y) - (n-1)(1-y)\mathcal{F}_0'(y) + 2\mathcal{F}_0(y) = 0.$$

Comparing it with the general differential equation (C.11), we obviously find that  $\mathcal{F}_0$  satisfies a hypergeometric equation with the parameters

$$a = \frac{-n - \sqrt{n^2 + 8}}{2}, \quad b = \frac{-n + \sqrt{n^2 + 8}}{2}, \quad c = 1 - n.$$

The general theory of hypergeometric functions gives us that this differential equation is degenerate because  $c$  is a negative integer, see discussion in Appendix C. However, we still get two independent solutions generating the class of solutions to this differential equation: one is smooth and the second is singular and contains a logarithmic singularity at the origin. The smooth one is given by

$$y \in (-\infty, 1) \mapsto y^{1-c} F(1+a-c, 1+b-c, 2-c, y).$$

With the special parameters (5.15), it becomes

$$y \in (-\infty, 1) \mapsto y^n F(a_n, b_n; c_n; y) = y^n F_n(y).$$

It is important to note that, by Taylor expansion, the hypergeometric function initially defined in the unit disc  $\mathbb{D}$  admits an analytic continuation in the complex plane cut along the real axis from 1 to  $+\infty$ . This comes from the integral representation (C.2).

Next, we use the method of variation of constants with the smooth homogeneous solution and set

$$(5.17) \quad \mathcal{F}_0 : (-\infty, 1) \in \mathbb{D} \mapsto y^n F_n(y).$$

We wish to mention that when using the method of variation of constants with the smooth solution we also find the trace of the singular solution. As we will notice in the next step, this singular part will not contribute for the full inhomogeneous problem due to the required regularity and the boundary condition  $F(0) = 0$ . Now, we solve the equation (5.16) by looking for solutions in the form  $\mathcal{F}(y) = \mathcal{F}_0(y)K(y)$ . By setting  $\mathcal{K} \triangleq K'$ , one has that

$$\mathcal{K}'(y) + \left[ 2 \frac{\mathcal{F}_0'(y)}{\mathcal{F}_0(y)} - \frac{n-1}{y} \right] \mathcal{K}(y) = \frac{1}{4x} \frac{\left(\frac{x}{y}\right)^{\frac{1}{2}} g\left(\left(\frac{y}{x}\right)^{\frac{1}{2}}\right)}{y(1-y)\mathcal{F}_0(y)},$$

which can be integrated in the following way

$$\mathcal{K}(y) = \frac{y^{n-1}}{\mathcal{F}_0^2(y)} \left\{ K_1 + \frac{1}{4x} \int_0^y \frac{\mathcal{F}_0(s)}{s^n(1-s)} \left(\frac{x}{s}\right)^{\frac{1}{2}} g\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds \right\},$$

where  $K_1$  is a constant. Thus integrating successively we find  $K$  and  $\mathcal{F}$  and from the expression of  $\mathcal{F}_0$  we deduce

$$\mathcal{F}(y) = y^n F_n(y) \left[ K_2 - \int_y^{\text{sign } y} \frac{1}{\tau^{n+1} F_n^2(\tau)} \left\{ K_1 + \frac{1}{4x} \int_0^\tau \frac{F_n(s)}{1-s} \left(\frac{x}{s}\right)^{\frac{1}{2}} g\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds \right\} d\tau \right],$$

for any  $y \in (-\infty, 1)$ , with  $K_2$  a constant and where  $sign$  is the sign function. From straightforward computations using integration by parts we get

$$\mathcal{F}(0) = -K_1 \lim_{y \rightarrow 0} y^n F_n(y) \int_y^{\text{sign } y} \frac{1}{\tau^{n+1} F_n^2(\tau)} d\tau = -\frac{K_1}{n}.$$

Combined with the initial condition  $\mathcal{F}(0) = 0$  we obtain  $K_1 = 0$ . Coming back to the original function  $F(r) = \mathcal{F}(xr^2)$ , we obtain

$$F(r) = x^n r^{2n} F_n(xr^2) \left[ K_2 - \frac{1}{4x} \int_{xr^2}^{\text{sign } x} \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^\tau \frac{F_n(s)}{1-s} \left(\frac{x}{s}\right)^{\frac{1}{2}} g\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds d\tau \right].$$

The constant  $K_2$  can be computed by evaluating the preceding expression at  $r = 1$ . We finally get

$$(5.18) \quad F(r) = r^{2n} F_n(xr^2) \left[ \frac{F(1)}{F_n(x)} - \frac{x^{n-1}}{4} \int_{xr^2}^x \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^\tau \frac{F_n(s)}{1-s} \sqrt{x/s} g\left(\sqrt{s/x}\right) ds d\tau \right],$$

for  $r \in [0, 1]$ . Observe from the integral representation (C.2) that the function  $F_n$  does not vanish on  $(-\infty, 1)$  for  $n \geq 1$ . Hence, (5.18) is well-defined and  $F$  is  $C^\infty$  in  $[0, 1]$  when  $x < 1$ .  $\square$

The next goal is to give the kernel structure of the linearized operator  $D_g \widehat{G}(\Omega, 0)$ . We emphasize that according to Proposition 5.1, this is a Fredholm operator of zero index, which implies in particular that its kernel is finite-dimensional. Before that, we introduce the singular set for  $x$  connected to the singular set of  $\Omega$  through the relations (4.12) and (4.10)

$$(5.19) \quad \widehat{\mathcal{S}}_{\text{sing}} = \left\{ \widehat{x}_n = \frac{A}{4 \left( \widehat{\Omega}_n - \frac{B}{2} \right)}, \quad \widehat{\Omega}_n \in \mathcal{S}_{\text{sing}} \right\}.$$

For any  $n \geq 1$ , consider the following sequences of functions

$$(5.20) \quad \zeta_n(x) \triangleq F_n(x) \left[ 1 - x + \frac{A + 2B}{A(n+1)} x \right] + \int_0^1 F_n(\tau x) \tau^n [-1 + 2x\tau] d\tau, \quad \forall x \in (-\infty, 1],$$

where  $F_n$  has been introduced in (5.15). Then we prove the following.

**Proposition 5.3.** *Let  $A \in \mathbb{R}^*$ ,  $B \in \mathbb{R}$  and  $x \in (-\infty, 1) \setminus \{\widehat{\mathcal{S}}_{\text{sing}} \cup \{0, x_0\}\}$ , with (4.12) and (4.14). Define the set*

$$(5.21) \quad \mathcal{A}_x \triangleq \left\{ n \in \mathbb{N}^*, \quad \zeta_n(x) = 0 \right\}.$$

*Then, the kernel of  $D_g \widehat{G}(\Omega, 0)$  is finite-dimensional and generated by the  $\mathcal{C}^\infty$  functions  $\{\mathfrak{h}_n, n \in \mathcal{A}_x\}$ , with  $\mathfrak{h}_n : z \in \overline{\mathbb{D}} \mapsto \text{Re} [\mathcal{G}_n(|z|^2) z^n]$  and*

$$\mathcal{G}_n(\mathbf{t}) = \frac{1}{1-x\mathbf{t}} \left[ -\frac{P_n(\mathbf{t})}{P_n(1)} + \frac{F_n(x\mathbf{t})}{F_n(x)} - \frac{2xF_n(x\mathbf{t})}{P_n(1)} \int_{\mathbf{t}}^1 \frac{1}{\tau^{n+1} F_n^2(x\tau)} \int_0^\tau \frac{s^n F_n(xs)}{1-xs} P_n(s) ds d\tau \right].$$

*As a consequence,  $\dim \text{Ker } D_g \widehat{G}(\Omega, 0) = \text{Card } \mathcal{A}_x$ . The functions  $P_n$  and  $F_n$  are defined in (5.11) and (5.15), respectively.*

**Remark 5.4.** *Notice that the set  $\mathcal{A}_x$  can be empty; in that case the kernel of  $D_g \widehat{G}(\Omega, 0)$  is trivial. Otherwise, the set  $\mathcal{A}_x$  is finite.*

*Proof.* To analyze the kernel structure, we return to (5.8) and solve the equations keeping in mind the relations (5.13). Thus we should solve

$$(5.22) \quad \begin{aligned} h_n(r) + \frac{4r}{n(r^2 - \frac{1}{x})} \left[ -\frac{H_n(1)}{G_n(1)} G_n(r) + \frac{H_n(r)}{r^{n+1}} \right] &= 0, \quad \forall r \in [0, 1], \quad \forall n \in \mathbb{N}^*, \\ \frac{\frac{1}{x} - r^2}{8} h_0(r) - \int_\tau^1 \frac{1}{\tau} \int_0^\tau s h_0(s) ds d\tau &= 0, \quad \forall r \in [0, 1], \end{aligned}$$

where the functions involved in the last expressions are given in (5.9)-(5.13). The term  $(r^2 - \frac{1}{x})$  is not vanishing from the assumptions on  $x$ . Note that the last equation for  $n = 0$  has been already studied in Proposition 4.3, which implies that if  $x \neq x_0$ , then the zero function is the only solution. Hence, let us focus on the case  $n \geq 1$  and solve the associated equation. To deal with this equation we write down a differential equation for  $H_n$  and use Lemma 5.2. Firstly, we define the linear operator

$$(5.23) \quad \mathcal{L}h(r) \triangleq r^{2n} \int_r^1 \frac{1}{s^{n-1}} h(s) ds + \int_0^r s^{n+1} h(s) ds,$$

for any  $h \in \mathcal{C}([0, 1]; \mathbb{R})$ . Then, by differentiation we obtain

$$(\mathcal{L}h)'(r) = 2nr^{2n-1} \int_r^1 \frac{1}{s^{n-1}} h(s) ds = \frac{2n}{r} \left[ \mathcal{L}h(r) - \int_0^r s^{n+1} h(s) ds \right].$$

It is important to precise at this stage that  $\mathcal{L}h$  satisfies the boundary conditions

$$(5.24) \quad (\mathcal{L}h)(0) = (\mathcal{L}h)'(1) = 0.$$

Indeed, the second condition is obvious and to get the first one we use that  $h$  is bounded:

$$|\mathcal{L}h(r)| \leq \|h\|_{L^\infty} \left( \frac{|r^{2n} - r^{2+n}|}{|n-2|} + \frac{r^{n+2}}{n+2} \right),$$

for any  $n \neq 2$ . In the case  $n = 2$  we have

$$|\mathcal{L}h(r)| \leq \|h\|_{L^\infty} \left( r^4 |\ln r| + \frac{r^4}{4} \right),$$

and for  $n = 1$  it is clearly verified. Differentiating again we obtain

$$(5.25) \quad \frac{1}{2n} [r(\mathcal{L}h)'(r)]' - (\mathcal{L}h)'(r) = -r^{n+1} h(r).$$

Since  $H_n = \mathcal{L}h_n$ , one has

$$(5.26) \quad \frac{1}{2n} [rH_n'(r)]' - H_n'(r) = -r^{n+1} h_n(r).$$

Using Equation (5.22), we deduce that  $H_n$  satisfies the following differential equation

$$(5.27) \quad (1 - xr^2)rH_n''(r) - (1 - xr^2)(2n - 1)H_n'(r) + 8rxH_n(r) = 8\frac{H_n(1)x}{G_n(1)}r^{n+2}G_n(r),$$

complemented with the boundary conditions (5.24). Let us show how to recover the full solutions of (5.22) from this equation. Assume that we have constructed all the solutions  $H_n$  of (5.27), with the boundary conditions (5.24). Then, to obtain the solutions of (5.22), we should check the compatibility condition  $\mathcal{L}h_n = H_n$ , by setting

$$(5.28) \quad h_n(r) \triangleq \frac{4r}{n(r^2 - \frac{1}{x})} \left[ H_n(1) \frac{G_n(r)}{G_n(1)} - \frac{H_n(r)}{r^{n+1}} \right].$$

Combining (5.25) and (5.26), we deduce that  $\mathcal{H} \triangleq \mathcal{L}h_n - H_n$  satisfies

$$\frac{1}{2n} [r\mathcal{H}'(r)]' - \mathcal{H}'(r) = 0.$$

By solving this differential equation we obtain the existence of two real constants  $\lambda_0$  and  $\lambda_1$  such that  $\mathcal{H}(r) = \lambda_0 + \lambda_1 r^{2n}$ , for any  $r \in [0, 1]$ . Since both  $\mathcal{L}h_n$  and  $H_n$  satisfy (5.24), then  $\mathcal{H}$  satisfies also these conditions. Hence, we find  $\mathcal{H}(r) = 0$ , for any  $r \in [0, 1]$ , and this concludes that  $h_n$ , given by (5.28), is a solution of (5.22). We emphasize that  $h_n$  satisfies the compatibility condition

$$H_n(1) = \int_0^1 r^{n+1} h_n(r) dr.$$

Indeed, integrating (5.26) from 0 to 1, we obtain

$$\frac{H'_n(1)}{2n} - (H_n(1) - H_n(0)) = - \int_0^1 r^{n+1} h_n(r) dr.$$

Thus, if  $H_n$  satisfies the boundary conditions (5.24), then the compatibility condition is automatically verified. Now, let us come back to the resolution of (5.27). Since  $H_n(0) = 0$ , then one can apply Lemma 5.2 with

$$(5.29) \quad g(r) = 8 \frac{H_n(1)x}{G_n(1)} r^{n+2} G_n(r).$$

Therefore, we obtain after a change of variables that

$$(5.30) \quad \begin{aligned} H_n(r) &= r^{2n} F_n(xr^2) H_n(1) \left[ \frac{1}{F_n(x)} \right. \\ &\quad \left. - \frac{2x^n}{G_n(1)} \int_{xr^2}^x \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^\tau \frac{F_n(s)}{1-s} \left(\frac{s}{x}\right)^{\frac{n+1}{2}} G_n\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds d\tau \right] \\ &= H_n(1) r^{2n} F_n(xr^2) \left[ \frac{1}{F_n(x)} \right. \\ &\quad \left. - \frac{8x}{G_n(1)} \int_r^1 \frac{1}{\tau^{2n+1} F_n^2(x\tau^2)} \int_0^\tau \frac{s^{n+2} F_n(xs^2)}{1-xs^2} G_n(s) ds d\tau \right]. \end{aligned}$$

It remains to check the second initial condition:  $H'_n(1) = 0$ . From straightforward computations using (5.9) and (5.12) we find that

$$(5.31) \quad \begin{aligned} H'_n(1) &= \frac{H_n(1)}{F_n(x)} [2nF_n(x) + 2xF'_n(x)] + \frac{8xH_n(1)}{G_n(1)F_n(x)} \int_0^1 \frac{s^{n+2} F_n(xs^2)}{1-xs^2} G_n(s) ds \\ &= 2 \frac{H_n(1)}{F_n(x)} \left[ \varphi_n(x) - \frac{An(n+1)x}{(n+2)G_n(1)} \int_0^1 \frac{s^{2n+1} F_n(xs^2)}{1-xs^2} P_n(s^2) ds \right] \\ &= \frac{2nH_n(1)}{F_n(x)G_n(1)} \Psi_n(x), \end{aligned}$$

where

$$\Psi_n(x) \triangleq \varphi_n(x) \left[ \frac{A}{4} \left( \frac{1}{x} - 1 \right) + \frac{A}{2} \frac{n+1}{n^2+2n} + \frac{B}{2n} \right] - \frac{A(n+1)x}{n+2} \int_0^1 \frac{s^{2n+1} F_n(xs^2)}{1-xs^2} P_n(s^2) ds,$$

and

$$(5.32) \quad \varphi_n(x) \triangleq nF_n(x) + xF'_n(x).$$

Note that  $G_n(1) \neq 0$  because  $\Omega \notin \mathcal{S}_{\text{sing}}$ . Let us link  $\Psi_n$  to the function  $\zeta_n$ . Recall from (5.16) and (C.11) that

$$x(1-x)F_n'' + (n+1)(1-x)F_n'(x) + 2F_n = 0.$$

Differentiating the function  $\varphi_n(x)$  and using the differential equation for  $F_n$  we realize that

$$\varphi'_n(x) = (n+1)F_n'(x) + xF_n''(x)$$

$$= \frac{1}{1-x} [(1-x)(n+1)F'_n(x) + (1-x)xF''_n(x)] = -\frac{2F_n(x)}{1-x}.$$

The change of variables  $xs^2 \mapsto \tau$  in the integral term yields

$$\int_0^1 \frac{s^{2n+1}F_n(xs^2)}{1-xs^2} P_n(s^2) ds = \frac{1}{2x^{n+1}} \int_0^x \frac{\tau^n F_n(\tau)}{1-\tau} P_n\left(\frac{\tau}{x}\right) d\tau.$$

Therefore we get

$$\Psi_n(x) = \varphi_n(x) \left[ \frac{A}{4} \left( \frac{1}{x} - 1 \right) + \frac{A}{2} \frac{n+1}{n^2+2n} + \frac{B}{2n} \right] + \frac{A(n+1)}{4(n+2)x^n} \int_0^x \varphi'_n(\tau) \tau^n P_n\left(\frac{\tau}{x}\right) d\tau.$$

From Definition (5.11) we find the identity

$$\int_0^x \varphi'_n(\tau) \tau^n P_n\left(\frac{\tau}{x}\right) d\tau = \frac{1}{x^2} \int_0^x \varphi'_n(\tau) \tau^n \left[ \tau^2 - \tau \frac{n+2}{n+1} - \frac{A+2B}{A} \frac{n+2}{n(n+1)} x^2 \right] d\tau.$$

Integrating by parts, we deduce

$$\begin{aligned} \frac{A(n+1)}{4(n+2)x^n} \int_0^x \varphi'_n(\tau) \tau^n P_n\left(\frac{\tau}{x}\right) d\tau &= \frac{A}{4} \varphi_n(x) \left( \frac{n+1}{n+2} - \frac{1}{x} - \frac{A+2B}{An} \right) \\ &\quad - \frac{A(n+1)}{4x^{n+2}} \int_0^x \varphi_n(\tau) \tau^{n-1} \left( \tau^2 - \tau - \frac{A+2B}{A(n+1)} x^2 \right) d\tau. \end{aligned}$$

By virtue of the following identity

$$\frac{A}{4} \left( \frac{1}{x} - 1 \right) + \frac{A}{2} \frac{n+1}{n^2+2n} + \frac{B}{2n} + \frac{A}{4} \left( \frac{n+1}{n+2} - \frac{1}{x} - \frac{A+2B}{An} \right) = 0,$$

the boundary term in the integral is canceled with the first part of  $\Psi_n$ . Thus

$$\Psi_n(x) = -\frac{A(n+1)}{4x^{n+2}} \int_0^x \varphi_n(\tau) \tau^{n-1} \left( \tau^2 - \tau - \frac{A+2B}{A(n+1)} x^2 \right) d\tau,$$

where, after a change of variables in the integral term, we get that

$$\Psi_n(x) = \frac{A(n+1)}{4x} \int_0^1 \varphi_n(\tau x) \tau^{n-1} \left( -x\tau^2 + \tau + \frac{A+2B}{A(n+1)} x \right) d\tau.$$

Setting

$$\zeta_n(x) = \int_0^1 \varphi_n(\tau x) \tau^{n-1} \left( -x\tau^2 + \tau + \frac{A+2B}{A(n+1)} x \right) d\tau,$$

we find the relation

$$\Psi_n(x) = \frac{A(n+1)}{4x} \zeta_n(x).$$

Observe first that the zeroes of  $\Psi_n$  and  $\zeta_n$  are the same. Coming back to (5.32) and integrating by parts we get the equivalent form

$$\zeta_n(x) = F_n(x) \left( 1 - x + \frac{A+2B}{A(n+1)} x \right) + \int_0^1 F_n(\tau x) \tau^n (-1 + 2x\tau) d\tau.$$

This gives (5.20). According to (5.31), the constraint  $H'_n(1) = 0$  is equivalent to  $H_n(1) = 0$  or  $\zeta_n(x) = 0$ . In the first case, we get from (5.30) that  $H_n \equiv 0$  and inserting this into (5.28) we find  $h_n(r) = 0$ , for any  $r \in [0, 1]$ . Thus, for  $n \notin \mathcal{A}_x$ , where  $\mathcal{A}_x$  is given by (5.21), we obtain that there is only one solution for the kernel equation, which is the trivial one. As to the second condition  $\zeta_n(x) = 0$ , which agrees with  $n \in \mathcal{A}_x$ , one gets from (5.30) and (5.28) that the kernel of  $D_g \widehat{G}(\Omega, 0)$  restricted to the level frequency  $n$  is generated by

$$h_n(re^{i\theta}) = h_n^*(r) \cos(n\theta),$$

with

$$h_n^*(r) = \frac{1}{1-xr^2} \left[ -\frac{rG_n(r)}{G_n(1)} + \frac{r^n F_n(xr^2)}{F_n(x)} - 8xr^n F_n(xr^2) \int_r^1 \frac{\int_0^\tau \frac{s^{n+2} F_n(xs^2) G_n(s)}{1-xs^2} \frac{G_n(s)}{G_n(1)} ds}{\tau^{2n+1} F_n^2(x\tau^2)} d\tau \right].$$

The fact that  $h_n = \frac{4x}{n} H_n(1) h_n^*$  together with  $H_n(1) = \int_0^1 s^{n+1} h_n(s) ds$  imply

$$(5.33) \quad \int_0^1 s^{n+1} h_n^*(s) ds = \frac{n}{4x}.$$

Using

$$\frac{rG_n(r)}{G_n(1)} = r^n \frac{P_n(r^2)}{P_n(1)},$$

and a suitable change of variables allows getting the formula,

$$\begin{aligned} \int_r^1 \frac{\int_0^\tau \frac{s^{n+2} F_n(xs^2) G_n(s)}{1-xs^2} \frac{G_n(s)}{G_n(1)} ds}{\tau^{2n+1} F_n^2(x\tau^2)} d\tau &= \frac{1}{P_n(1)} \int_r^1 \frac{\int_0^\tau \frac{s^{2n+1} F_n(xs^2) P_n(s^2)}{1-xs^2} P_n(s^2) ds}{\tau^{2n+1} F_n^2(x\tau^2)} d\tau \\ &= \frac{1}{4P_n(1)} \int_{r^2}^1 \frac{\int_0^\tau \frac{s^n F_n(xs) P_n(s)}{1-xs} P_n(s) ds}{\tau^{n+1} F_n^2(x\tau)} d\tau. \end{aligned}$$

We have

$$h_n^*(r) = \frac{r^n}{1-xr^2} \left[ -\frac{P_n(r^2)}{P_n(1)} + \frac{F_n(xr^2)}{F_n(x)} - \frac{2xF_n(xr^2)}{P_n(1)} \int_{r^2}^1 \frac{1}{\tau^{n+1} F_n^2(x\tau)} \int_0^\tau \frac{s^n F_n(xs)}{1-xs} P_n(s) ds d\tau \right].$$

Setting

$$\mathcal{G}_n(\mathfrak{t}) = \frac{1}{1-x\mathfrak{t}} \left[ -\frac{P_n(\mathfrak{t})}{P_n(1)} + \frac{F_n(x\mathfrak{t})}{F_n(x)} - \frac{2xF_n(x\mathfrak{t})}{P_n(1)} \int_{\mathfrak{t}}^1 \frac{1}{\tau^{n+1} F_n^2(x\tau)} \int_0^\tau \frac{s^n F_n(xs)}{1-xs} P_n(s) ds d\tau \right],$$

we deduce that

$$(5.34) \quad h_n(z) = \mathcal{G}_n(|z|^2) |z|^n \cos(n\theta) = \operatorname{Re} [\mathcal{G}_n(|z|^2) z^n], \quad \forall z \in \overline{\mathbb{D}}.$$

We intend to check that  $h_n$  belongs to  $\mathcal{C}^\infty(\overline{\mathbb{D}}; \mathbb{R})$ . To get this it is enough to verify that  $\mathcal{G}$  belongs to  $\mathcal{C}^\infty([0, 1]; \mathbb{R})$ . Since  $x \in (-\infty, 1)$ , then  $\tau \in [0, 1] \mapsto F_n(x\tau)$  is in  $\mathcal{C}^\infty([0, 1]; \mathbb{R})$ . The change of variables  $s = \tau\theta$  implies

$$\int_{\mathfrak{t}}^1 \frac{\int_0^\tau \frac{s^n F_n(xs)}{1-xs} P_n(s) ds}{\tau^{n+1} F_n^2(x\tau)} d\tau = \int_{\mathfrak{t}}^1 \frac{\int_0^1 \frac{\theta^n F_n(x\tau\theta) P_n(\tau\theta)}{1-x\tau\theta} d\theta}{F_n^2(x\tau)} d\tau.$$

Since  $F_n$  does not vanish on  $(-\infty, 1)$ , then the mapping  $\tau \in [0, 1] \mapsto \frac{1}{F_n(x\tau)}$  belongs to  $\mathcal{C}^\infty([0, 1]; \mathbb{R})$ . It suffices to observe that the integral function is  $\mathcal{C}^\infty$  on  $[0, 1]$ . Then, we have an independent element of the kernel given by  $h_n^*(\cdot)$ , for any  $n \in \mathcal{A}_x$ . This concludes the announced result.  $\square$

**Remark 5.5.** The hypergeometric function  $F_n$  for  $n = 1$  can be computed as  $F_1(r) = F(a_1, b_1; c_1; r) = 1 - r$ . Hence, the function (5.20) becomes

$$\zeta_1(x) = F_1(x) \left[ 1 - x + \frac{A+2B}{2A} x \right] + \int_0^1 F_1(\tau x) \tau [-1 + 2x\tau] d\tau$$

$$= (1-x) \left[ 1-x + \frac{A+2B}{2A}x \right] + \int_0^1 (1-\tau)x \tau [-1+2x\tau] d\tau = (1-x) \left( \frac{B}{A}x + \frac{1}{2} \right).$$

The root  $x = 1$  is not allowed since  $x \notin \widehat{\mathcal{S}}_{\text{sing}}$ . Therefore, the unique root is  $x = -\frac{A}{2B}$ . Coming back to  $\Omega$  using (4.12), one has that  $\Omega = 0$ .

5.2.2. *Singular case.* The singular case  $x \in (1, +\infty)$  is studied in this section. Notice that from (4.12), we obtain

$$(5.35) \quad \frac{B}{2} < \Omega < \frac{B}{2} + \frac{A}{4}.$$

It is worthy to point out that this case is degenerate because the leading terms of the equations of the linearized operator (5.22) vanish inside the unit disc. To understand this operator one should deal with a second differential equation of hypergeometric type with a singularity. Thus, the first difficulty amounts to solving those equations across the singularity and invert the operator. This can be done in a straightforward way getting that the operator is injective with an explicit representation of its formal inverse. However, it is not an isomorphism and undergoes a loss of regularity in the Hölder class. Despite this bad behavior, one would expect at least the persistence of the injectivity for the nonlinear problem. This problem appears in different contexts, for instance in the inverse backscattering problem [58]. The idea to overcome this difficulty is to prove two key ingredients. The first one concerns the coercivity of the linearized operator with a quantified loss in the Hölder class. The second point is to use the Taylor expansion and to establish a soft estimate for the reminder combined with an interpolation argument. This argument leads to the following result.

**Theorem 5.6.** *Let  $0 < \alpha < 1$ ,  $A > 0$  and  $B \in \mathbb{R}$  such that  $\frac{B}{A} \notin [-1, -\frac{1}{2}]$ . Assume that  $\Omega$  satisfies (5.35) and  $\Omega \notin \mathcal{S}_{\text{sing}}$ , where this latter set is defined in (4.10). Then, there exists a small neighborhood  $V$  of the origin in  $\mathcal{C}_s^{2,\alpha}(\mathbb{D})$  such that the nonlinear equation (4.6) has no solution in  $V$ , except the origin. Notice that in the case  $B < -A$ , the condition  $\Omega \notin \mathcal{S}_{\text{sing}}$  follows automatically from (5.35).*

The proof of this theorem will be given at the end of this section. Before we should develop some tools. Let us start with solving the kernel equations, for this reason we introduce some auxiliary functions. Set

$$(5.36) \quad \widehat{F}_n(x) = F(-a_n, b_n; b_n - a_n + 1; x),$$

and define the functions

$$(5.37) \quad \mathcal{F}_{K_1, K_2}(y) = \begin{cases} y^n F_n(y) \left[ \frac{K_1}{F_n(1)} - \int_y^1 \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^{n+1} F_n^2(\tau)} d\tau \right], & y \in [0, 1], \\ y^{a_n} \widehat{F}_n\left(\frac{1}{y}\right) \left[ \frac{K_1}{\widehat{F}_n(1)} + \int_{\frac{1}{y}}^1 \frac{K_2 - \int_\tau^1 \frac{s^{n-a_n-1} \widehat{F}_n(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^{n+1-2a_n} \widehat{F}_n^2(\tau)} d\tau \right], & y \geq 1, \end{cases}$$

and

$$\widehat{\mathcal{F}}_{K_1, K_2}(y) = \begin{cases} y(1-y) \left[ K_1 + \int_0^y \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right], & y \in [0, 1], \\ \frac{1}{y} \widehat{F}_1\left(\frac{1}{y}\right) \left[ \frac{1}{3} \int_0^1 \mathcal{R}\left(\frac{s}{x}\right) ds + \int_{\frac{1}{y}}^1 \frac{K_2 - \int_\tau^1 \frac{s \widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau \right], & y \geq 1, \end{cases}$$

with

$$(5.38) \quad \mathcal{R}(y) = \frac{1}{4x} y^{-\frac{1}{2}} g(y^{\frac{1}{2}}),$$

where  $g$  is the source term in (5.14). Our first result reads as follows.

**Lemma 5.7.** *Let  $n \geq 2$  be an integer,  $x \in (1, +\infty)$  and  $g \in \mathcal{C}([0, 1]; \mathbb{R})$ . Then, the continuous solutions in  $[0, +\infty)$  to the equation (5.14), such that  $F(0) = 0$ , are given by the two-parameters curve*

$$r \in [0, 1] \mapsto F(r) = \mathcal{F}_{K_1, K_2}(xr^2), \quad K_1, K_2 \in \mathbb{R},$$

with the constraint

$$(5.39) \quad \mathcal{R}\left(\frac{1}{x}\right) = 0.$$

Moreover, if  $g \in \mathcal{C}^\mu([0, 1]; \mathbb{R})$ , for some  $\mu > 0$ , then the above solutions are  $\mathcal{C}^1$  on  $[0, 1]$  if and only if the following conditions hold true:

$$(5.40) \quad K_1 = 0 \quad \text{and} \quad K_2 = \frac{\widehat{F}_n(1)}{F_n(1)} \int_0^1 \frac{F_n(s) \mathcal{R}\left(\frac{s}{x}\right)}{1-s} ds.$$

For  $n = 1$ , if  $F(0) = 0$  and  $g \in \mathcal{C}([0, 1]; \mathbb{R})$ , with the condition (5.39), then the continuous solutions are given by the two-parameters curve

$$r \in [0, 1] \mapsto F(r) = \widehat{\mathcal{F}}_{K_1, K_2}(xr^2), \quad K_1, K_2 \in \mathbb{R}.$$

If  $g \in \mathcal{C}^\mu([0, 1]; \mathbb{R})$ , for some  $\mu > 0$ , then this solution is  $\mathcal{C}^1$  if and only if

$$(5.41) \quad \mathcal{R}(0) = \int_0^{\frac{1}{x}} \mathcal{R}(\tau) d\tau = 0,$$

and  $K_1$  and  $K_2$  satisfy

$$(5.42) \quad K_2 = -3 \left( K_1 + \int_0^1 \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right).$$

Moreover, if  $F'(1) = 0$ , we have the additional constraint

$$(5.43) \quad K_2 \left( \frac{1}{x^2} \left[ \widehat{F}_1\left(\frac{1}{x}\right) + x \widehat{F}'_1\left(\frac{1}{x}\right) \right] + \frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)} \right) = \frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)} \int_{\frac{1}{x}}^1 \frac{s \widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds \\ + \frac{1}{x^2} \left[ \widehat{F}_1\left(\frac{1}{x}\right) + \frac{1}{x} \widehat{F}'_1\left(\frac{1}{x}\right) \right] \times \int_{\frac{1}{x}}^1 \frac{\int_\tau^1 \frac{s \widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau.$$

*Proof.* We proceed as in the proof of Lemma 5.2. The resolution of the equation (5.16), in the interval  $[0, 1]$ , is exactly the same and we find

$$\mathcal{F}(y) = \mathcal{F}_{K_1, K_2}(y), \quad \forall y \in [0, 1].$$

In the interval  $[1, x]$ , we first solve the homogeneous equation associated to (5.16). By virtue of Appendix C, one gets two independent solutions, one of them is described by

$$\mathcal{F}_0(y) = y^{an} \widehat{F}_n\left(\frac{1}{y}\right), \quad \forall y \in [1, x].$$

Using the method of variation of constants, we obtain that the general solutions to the equation (5.16) in this interval take the form

$$\mathcal{F}(y) = \mathcal{F}_{K_1, K_2}(y), \quad \forall y \in [1, x].$$

Since  $K_1$  coincides in both sides of (5.37) and the integrals converge at 1 by (5.39), then we deduce that  $\mathcal{F}$  is continuous in  $[0, x]$ . Therefore, we get the first part of the Lemma by using simply that  $F(r) = \mathcal{F}(xr^2)$ , for  $r \in [0, 1]$ .

Let us now select in this class those solutions who are  $\mathcal{C}^1$ . Notice that the solutions  $\mathcal{F}$  are  $\mathcal{C}^1$  in  $[0, x] \setminus \{1\}$ . So it remains to study the derivatives from the left and the right on  $y = 1$ . Since  $F_n$  and  $\widehat{F}_n$  have no derivatives on the left at 1 and verify

$$|F'_n(y)| \sim C \ln(1-y) \quad \text{and} \quad |\widehat{F}'_n(y)| \sim C \ln(1-y),$$



for any  $y \in [0, 1)$ , see (C.6), then the first members of (5.37) have no derivatives at 1. This implies necessary that  $K_1 = 0$ . Moreover, one gets

$$(5.44) \quad \mathcal{F}'_{K_1, K_2}(1^-) = \frac{\int_0^1 \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{F_n(1)}.$$

Since  $F_n(1) > 0$  and  $\mathcal{R}$  is Hölder continuous, then the convergence of this integral is equivalent to the condition  $\mathcal{R}\left(\frac{1}{x}\right) = 0$ . Now, using (5.37), we deduce that the right derivative of  $\mathcal{F}_{K_1, K_2}$  at 1 is given by

$$(5.45) \quad \mathcal{F}'_{K_1, K_2}(1^+) = \frac{K_2}{\widehat{F}_n(1)}.$$

Combining (5.44) with (5.45), we deduce that  $F_n$  admits a derivative at 1 if and only if

$$K_2 = \frac{\widehat{F}_n(1)}{F_n(1)} \int_0^1 \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds.$$

Thus, the  $\mathcal{C}^1$ -solution to (5.37) is given by

$$\begin{aligned} \mathcal{F}(y) &= y^n F_n(y) \int_1^y \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^{n+1} F_n^2(\tau)} d\tau \mathbf{1}_{[0,1]}(y) + \\ & y^{a_n} \widehat{F}_n\left(\frac{1}{y}\right) \int_{\frac{1}{y}}^1 \frac{\frac{\widehat{F}_n(1)}{F_n(1)} \int_0^1 \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds - \int_\tau^1 \frac{s^{n-a_n-1} \widehat{F}_n(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^{n+1-2a_n} \widehat{F}_n^2(\tau)} d\tau \mathbf{1}_{[1,\infty)}(y), \end{aligned}$$

and, therefore, the solution to (5.14) takes the form

$$(5.46) \quad F(r) = \mathcal{F}(xr^2).$$

This implies in particular that there is only one  $\mathcal{C}^1$  solution to (5.14) and satisfies  $F(x^{-\frac{1}{2}}) = 0$ . The case  $n = 1$  is very special since  $F_1(x) = 1 - x$  and hence  $F_1$  vanishes at 1. As in the previous discussion, the solution of (5.16) in  $[0, 1]$  is given by

$$\mathcal{F}(y) = y(1-y) \left[ K_1 + \int_0^y \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right],$$

whereas for  $y \in [1, \infty)$  the solution reads as

$$\mathcal{F}(y) = \frac{1}{y} \widehat{F}_1\left(\frac{1}{y}\right) \left[ K_3 + \int_{\frac{1}{y}}^1 \frac{K_2 - \int_\tau^1 \frac{s \widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau \right],$$

where  $K_1, K_2$  and  $K_3$  are constants. We can check that the continuity of  $\mathcal{F}$  at 1 is satisfied if and only if

$$\begin{aligned} K_3 &= \frac{1}{\widehat{F}_1(1)} \int_0^1 \mathcal{R}\left(\frac{s}{x}\right) ds \\ &= \frac{1}{3} \int_0^1 \mathcal{R}\left(\frac{s}{x}\right) ds, \end{aligned}$$

by using in the last line the explicit expression of  $\widehat{F}_1(1)$  coming from (C.5):

$$\widehat{F}_1(1) = F(1, 2; 4; 1) = \frac{\Gamma(4)\Gamma(1)}{\Gamma(3)\Gamma(2)} = 3.$$

Note that we need (5.39) in order to ensure the convergence of the integral

$$\int_\tau^1 \frac{s \widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds.$$

Let us deal with the derivative, given by

$$\mathcal{F}'(y) = \begin{cases} (1-2y) \left[ K_1 + \int_0^y \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right] + \frac{1}{y(1-y)} \int_0^y \mathcal{R}\left(\frac{s}{x}\right) ds, & y \in [0, 1], \\ -\frac{1}{y^2} \left[ \widehat{F}_1\left(\frac{1}{y}\right) + \frac{1}{y} \widehat{F}'_1\left(\frac{1}{y}\right) \right] \left[ \frac{1}{3} \int_0^1 \mathcal{R}\left(\frac{s}{x}\right) ds \right. \\ \left. + \int_{\frac{1}{y}}^1 \frac{K_2 - \int_\tau^1 \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau \right] + \frac{y}{\widehat{F}_1\left(\frac{1}{y}\right)} \left[ K_2 - \int_{\frac{1}{y}}^1 \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds \right], & y \geq 1. \end{cases}$$

The convergence in 0 and 1 of the first part comes from  $\mathcal{R}(0) = \mathcal{R}\left(\frac{1}{x}\right) = \int_0^1 \mathcal{R}\left(\frac{s}{x}\right) d\tau = 0$ . In which case, one gets

$$\mathcal{F}'(1^-) = -K_1 - \int_0^1 \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau.$$

For the second part of  $\mathcal{F}$ , one needs  $\int_0^1 \mathcal{R}\left(\frac{s}{x}\right) d\tau = 0$  since  $\widehat{F}'$  is singular at 1 as it was mentioned before. Then,

$$\mathcal{F}'(1^+) = \frac{1}{3} K_2.$$

Clearly, we have the constraint

$$(5.47) \quad K_2 = -3 \left( K_1 + \int_0^1 \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right),$$

in order to obtain  $\mathcal{C}^1$  solutions. If, in addition,  $F'(1) = 0$ , which agrees with  $\mathcal{F}'(x) = 0$ , we obtain the following additional equation for  $K_2$ :

$$K_2 \left( \frac{1}{x^2} \left[ \widehat{F}_1\left(\frac{1}{x}\right) + x\widehat{F}'_1\left(\frac{1}{x}\right) \right] + \frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)} \right) = \frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)} \int_{\frac{1}{x}}^1 \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds + \frac{1}{x^2} \left[ \widehat{F}_1\left(\frac{1}{x}\right) + \frac{1}{x} \widehat{F}'_1\left(\frac{1}{x}\right) \right] \int_{\frac{1}{x}}^1 \frac{\int_\tau^1 \frac{s\widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau.$$

Notice that

$$\frac{1}{x^2} \left[ \widehat{F}_1\left(\frac{1}{x}\right) + x\widehat{F}'_1\left(\frac{1}{x}\right) \right] + \frac{x}{\widehat{F}_1\left(\frac{1}{x}\right)} \neq 0,$$

using that  $\widehat{F}_1(z) = F(1, 2; 4; z)$ , for  $z \in (0, 1)$ , which is a positive and increasing function. Then, we can obtain the exact value of  $K_2$  and then  $K_1$  via relation (5.47). Using that  $F(r) = \mathcal{F}(xr^2)$ . The proof is now achieved.  $\square$

**Proposition 5.8.** *Let  $A > 0$  and  $B \in \mathbb{R}$  such that  $\frac{B}{A} \notin [-1, -\frac{1}{2}]$ . Let  $\Omega$  satisfy (5.35) with  $\Omega \notin \mathcal{S}_{\text{sing}}$ , where the last set is defined in (4.10). Then, the following holds true:*

- (1) *The kernel of  $D_g \widehat{G}(\Omega, 0)$  is trivial.*
- (2) *If  $h$  and  $d$  are smooth enough with*

$$D_g \widehat{G}(\Omega, 0)h = d, \quad h(re^{i\theta}) = \sum_{n \in \mathbb{N}} h_n(r) \cos(n\theta), \quad d(re^{i\theta}) = \sum_{n \in \mathbb{N}} d_n(r) \cos(n\theta),$$

*then, there exists an absolute constant  $C > 0$  such that*

$$\|h_n\|_{\mathcal{C}^0([0,1])} \leq C \|d_n\|_{\mathcal{C}^1([0,1])}, \quad \forall n \in \mathbb{N}.$$

- (3) *Coercivity with loss of derivative: for any  $\alpha \in (0, 1)$ , there exists  $C > 0$  such that*

$$\|h\|_{\mathcal{C}^0(\mathbb{D})} \leq C \|D_g \widehat{G}(\Omega, 0)h\|_{\mathcal{C}_s^{2,\alpha}(\mathbb{D})}.$$

*Proof.* (1) First, note that  $x \neq x_0$ , where  $x_0$  is defined in (4.14), because  $x_0 \in (0, 1)$ . Then, Proposition 4.3 implies that the last equation in (5.22) admits only the zero function as a solution. We will check how the condition (5.39) gives us that there are not nontrivial smooth solutions for  $n = 1$ . This can be done easily with the explicit expression of  $g$  given in (5.29) for  $n = 1$ . Since

$$G_1(r) = -2 \left[ r^4 - \frac{3}{2x} r^2 - \frac{3}{2} \left( 1 - \frac{1}{x} \right) \right],$$

one obtains that

$$g(r) = -16H_1(1)xr^3 \left[ r^4 - \frac{3}{2x} r^2 - \frac{3}{2} \left( 1 - \frac{1}{x} \right) \right].$$

Then, condition (5.39) is equivalent to

$$3x^2 - 3x + 1 = 0 \quad \text{or} \quad H_1(1) = 0.$$

Since  $x > 1$ , one has  $H_1(1) = 0$  and  $g \equiv 0$ . Then,  $\mathcal{R} \equiv 0$ . By Lemma 5.7, one has that the solution of (5.27) with  $H_1(0) = 0$  and  $H_1'(1) = 0$  is the trivial one:  $H_1(r) \equiv 0$ . Coming back to (5.28), we deduce that  $h_1 \equiv 0$ . Therefore, the only smooth solution is the zero function, which implies finally that the kernel is trivial.

Let us now deal with  $n \geq 2$ . Applying Lemma 5.7 to the equation (5.27) with (5.38) we get that this equation admits a  $\mathcal{C}^1$  solution if and only if

$$(5.48) \quad H_n(1)G_n(x^{-\frac{1}{2}}) = 0.$$

Using the expression (5.9), we find that

$$G_n(x^{-\frac{1}{2}}) = -\frac{An(n+1)}{4(n+2)} x^{\frac{1-n}{2}} P_n(x^{-1}),$$

and from (5.11) one has

$$(5.49) \quad P_n(x^{-1}) = -\frac{1}{n+1} \left( \frac{1}{x^2} + \frac{A+2B}{A} \frac{n+2}{n} \right).$$

With the assumptions  $\frac{A+2B}{A} \notin [-1, 0]$  and  $x > 1$  one gets

$$P_n(x^{-1}) \neq 0, \quad \forall n \geq 2,$$

obtaining from (5.48) that

$$H_n(1) = 0, \quad \forall n \geq 2.$$

Coming back to (5.27) we find that the source term is vanishing everywhere. Now, from Lemma 5.7 and (5.37) we infer that

$$H_n(r) = 0, \quad \forall n \geq 2, r \in [0, 1].$$

Inserting this into (5.22), we obtain  $h_n \equiv 0$  for any  $n \in \mathbb{N}^*$ . Finally, this implies that the vanishing function is the only element of the kernel.

(2) To get this result we should derive *a priori estimates* for the solutions to the equation

$$D_g \widehat{G}(\Omega, 0)h = d.$$

The pre-image equation is equivalent to solve the infinite-dimensional system

$$(5.50) \quad \begin{aligned} \frac{\frac{1}{x} - r^2}{8} h_n(r) - \frac{r}{n} \left[ A_n G_n(r) + \frac{1}{2r^{n+1}} H_n(r) \right] &= d_n(r), \quad \forall n \geq 1, \\ \frac{\frac{1}{x} - r^2}{8} h_0(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s h_0(s) ds d\tau &= d_0(r), \end{aligned}$$

where we use the notations of (5.9), (5.10) and (5.13). We first analyze the case of large  $n$ , for which we can apply the contraction principle and get the desired estimates. Later, we will deal

with low frequencies, which is more delicate and requires the integral representation (5.46). Let us first work with large values of  $n$ . Observe that the first equation of (5.50) can be transformed into

$$(5.51) \quad h_n(r) = \frac{8}{\frac{1}{x} - r^2} \left[ \frac{A_n}{n} r G_n(r) + \frac{H_n(r)}{2nr^n} + d_n(r) \right].$$

The estimate of  $A_n$ , defined by (5.10) and (5.13), can be done as follows

$$|A_n| \leq \frac{\int_0^1 s^{n+1} |h_n(s)| ds}{2n|\widehat{\Omega}_n - \Omega|}.$$

Keeping in mind the relation (4.12) and the assumption  $x \notin \widehat{\mathcal{S}}_{\text{sing}}$ , we obtain

$$\inf_{n \geq 1} \frac{1}{|\widehat{\Omega}_n - \Omega|} < +\infty,$$

and, on the other hand, it is obvious that

$$\int_0^1 s^{n+1} |h_n(s)| ds \leq \frac{\|h_n\|_{\mathcal{C}^0([0,1])}}{n+2}.$$

Combining the preceding estimates, we find

$$(5.52) \quad |A_n| \leq \frac{C}{n^2} \|h_n\|_{\mathcal{C}^0([0,1])},$$

for some constant  $C$  independent of  $n$ . Let us remark that according to the equation (5.51), one should get the compatibility condition

$$\frac{A_n}{n} x^{-\frac{1}{2}} G_n(x^{-\frac{1}{2}}) + \frac{H_n(x^{-\frac{1}{2}})}{2nx^{-\frac{n}{2}}} + d_n(x^{-\frac{1}{2}}) = 0.$$

Hence, applying the Mean Value Theorem, combined with (5.52), we obtain

$$(5.53) \quad \begin{aligned} \|h_n\|_{\mathcal{C}^0([0,1])} &\leq C \frac{\|h_n\|_{\mathcal{C}^0([0,1])}}{n^3} \|(rG_n(r))'\|_{\mathcal{C}^0([0,1])} \\ &\quad + \frac{1}{2n} \left\| \left( \frac{H_n(r)}{r^n} \right)' \right\|_{\mathcal{C}^0([0,1])} + \|d_n'\|_{\mathcal{C}^0([0,1])}, \end{aligned}$$

where in this inequality  $C$  may depend on  $x$  but not on  $n$ . Now, it is straightforward to check that

$$(5.54) \quad |(rG_n(r))'| \leq Cn^2, \quad \forall r \in [0, 1],$$

and second

$$\left( \frac{H_n(r)}{r^n} \right)' = nr^{n-1} \int_r^1 \frac{h_n(s)}{s^{n-1}} ds - \frac{n}{r^{n+1}} \int_0^r s^{n+1} h_n(s) ds.$$

From this latter identity, we infer that

$$\left| \left( \frac{H_n(r)}{r^n} \right)' \right| \leq \left( \frac{n}{n-2} + \frac{n}{n+2} \right) \|h_n\|_{\mathcal{C}^0([0,1])} \leq C \|h_n\|_{\mathcal{C}^0([0,1])}, \quad \forall r \in [0, 1],$$

for any  $n \geq 3$ . Consequently, we get

$$(5.55) \quad \left\| \left( \frac{H_n(r)}{r^n} \right)' \right\|_{\mathcal{C}^0([0,1])} \leq C \|h_n\|_{\mathcal{C}^0([0,1])}.$$

Plugging (5.55) and (5.54) into (5.53), we find

$$\|h_n\|_{\mathcal{C}^0([0,1])} \leq \frac{C}{n} \|h_n\|_{\mathcal{C}^0([0,1])} + \|d_n'\|_{\mathcal{C}^0([0,1])}.$$

Hence, choosing  $n_0$  large enough we deduce that

$$\|h_n\|_{\mathcal{C}^0([0,1])} \leq \|d'_n\|_{\mathcal{C}^0([0,1])}, \quad \forall n \geq n_0.$$

Next, we deal with the cases  $1 \leq n \leq n_0$ . The preceding argument fails and to invert the operator we recover explicitly the solution  $h_n$  from  $d_n$  according to the integral representation (5.46). For this purpose we will proceed as in the range study in Subsection 5.3. By virtue of (5.69), we find that  $H_n$  satisfies an equation of the type (5.14). Thus, using (5.46), we deduce

$$(5.56) \quad \begin{aligned} H_n(r) &= x^n r^{2n} F_n(xr^2) \int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^{n+1} F_n^2(\tau)} d\tau \mathbf{1}_{[0, x^{-\frac{1}{2}}]}(r) + x^{an} r^{2a_n} \widehat{F}_n\left(\frac{1}{xr^2}\right) \\ &\times \int_{\frac{1}{xr^2}}^1 \frac{\widehat{F}_n(1)}{F_n(1)} \frac{\int_0^1 \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds - \int_\tau^1 \frac{s^{n-an-1} \widehat{F}_n(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^{n+1-2a_n} \widehat{F}_n^2(\tau)} d\tau \mathbf{1}_{(x^{-\frac{1}{2}}, 1]}(r), \end{aligned}$$

for  $n \geq 2$ , and

$$\begin{aligned} H_1(r) &= xr^2(1-xr^2) \left[ K_1 - \int_{xr^2}^0 \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right] \mathbf{1}_{[0, x^{-\frac{1}{2}}]}(r) \\ &+ \frac{1}{xr^2} \widehat{F}_1\left(\frac{1}{xr^2}\right) \left[ \frac{1}{3} \int_0^1 \mathcal{R}\left(\frac{s}{x}\right) ds + \int_{\frac{1}{xr^2}}^1 \frac{K_2 - \int_\tau^1 \frac{s \widehat{F}_1(s)}{1-s} \mathcal{R}\left(\frac{1}{sx}\right) ds}{\tau^4 \widehat{F}_1^2(\tau)} d\tau \right] \mathbf{1}_{(x^{-\frac{1}{2}}, 1]}(r), \end{aligned}$$

where  $K_1$  and  $K_2$  are given in (5.42)-(5.43). From (5.38), we get

$$\mathcal{R}(y) = \frac{1}{4x} \left[ AA_n \frac{n(n+1)}{n+2} y^n P_n(y) - 4xny^{\frac{n}{2}} d_n(y^{\frac{1}{2}}) \right],$$

where  $P_n$  is defined in (5.11).

Let us relate  $A_n$  with  $d_n$ . This can be obtained from the constraint  $\mathcal{R}\left(\frac{1}{x}\right) = 0$ , which implies

$$A_n = \frac{(n+2)x^{\frac{n}{2}+1} d_n(x^{-\frac{1}{2}})}{A(n+1)P_n(x^{-1})}.$$

Let us remark that this relation is different from (5.52), which is not useful for low frequencies. Consequently, using (5.49) we infer that

$$|A_n| \leq C \|d_n\|_{\mathcal{C}^0([0,1])}, \quad \forall n \in [1, n_0].$$

Concerning  $\mathcal{R}$ , we can get successively

$$(5.57) \quad |\mathcal{R}(y)| \leq C \left[ |A_n| y^n + y^{\frac{n}{2}} \left| d_n(y^{\frac{1}{2}}) \right| \right], \quad \forall y \in [0, 1],$$

which implies that

$$\|\mathcal{R}\|_{\mathcal{C}^0([0,1])} \leq C \left[ |A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} \right] \leq C \|d_n\|_{\mathcal{C}^0([0,1])},$$

for any  $n \geq 1$ . In the case  $n \geq 2$ , we also obtain that

$$|\mathcal{R}'(y)| \leq C \left[ |A_n| y^{n-1} + y^{\frac{n}{2}-1} \left( \left| d_n(y^{\frac{1}{2}}) \right| + \left| d'_n(y^{\frac{1}{2}}) \right| \right) \right], \quad \forall y \in [0, 1],$$

which amounts to

$$(5.58) \quad \|\mathcal{R}'\|_{\mathcal{C}^0([0,1])} \leq C \left( \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])} \right), \quad \forall n \geq 2.$$

Note that this last estimate can not be used for  $n = 1$  since  $\mathcal{R}$  is only Hölder continuous. Then, we can find in this case that

$$\|\mathcal{R}\|_{\mathcal{C}^{0,\gamma}([0,1])} \leq C \|d_n\|_{\mathcal{C}^{0,\gamma}([0,1])},$$

for  $\mu = \min\left(\frac{1}{2}, \alpha\right)$ .

Let us begin with  $n \geq 2$ . Using the boundedness property of  $F_n$ , which we shall see later in Lemma 6.4, combined with an integration by parts imply

$$\begin{aligned}
\left| \int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^{n+1} F_n^2(\tau)} d\tau \right| &\leq C \left| \int_1^{xr^2} \frac{\int_0^\tau \frac{|\mathcal{R}(s/x)|}{1-s} ds}{\tau^{n+1}} d\tau \right| \\
&\leq C \int_0^1 \frac{|\mathcal{R}\left(\frac{s}{x}\right)|}{1-s} ds + Cr^{-2n} \int_0^{xr^2} \frac{|\mathcal{R}\left(\frac{s}{x}\right)|}{1-s} ds \\
(5.59) \qquad &+ \left| \int_1^{xr^2} \frac{\tau^{-n} |\mathcal{R}\left(\frac{\tau}{x}\right)|}{1-\tau} d\tau \right|.
\end{aligned}$$

We discuss first the case  $r \in [0, \frac{1}{2}x^{-\frac{1}{2}}]$ . From the compatibility assumption (5.40), we infer that  $\mathcal{R}\left(\frac{1}{x}\right) = 0$ , and therefore we deduce from (5.57) and (5.58) that

$$\begin{aligned}
\int_0^1 \frac{|\mathcal{R}\left(\frac{s}{x}\right)|}{1-s} ds + r^{-2n} \int_0^{xr^2} \frac{|\mathcal{R}\left(\frac{s}{x}\right)|}{1-s} ds &\leq \int_0^1 \frac{|\mathcal{R}\left(\frac{s}{x}\right)|}{1-s} ds + Cr^{-2n} \int_0^{xr^2} \left| \mathcal{R}\left(\frac{s}{x}\right) \right| ds \\
&\leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])} + r^{-n+2} \|d_n\|_{\mathcal{C}^0([0,1])}),
\end{aligned}$$

for  $2 \leq n \leq n_0$ . In a similar way to the last integral term of (5.59), we split it as follows using the estimates (5.57) and (5.58)

$$\begin{aligned}
\int_{xr^2}^1 \frac{\tau^{-n} |\mathcal{R}\left(\frac{\tau}{x}\right)|}{1-\tau} d\tau &= \int_{\frac{1}{2}}^1 \frac{\tau^{-n} |\mathcal{R}\left(\frac{\tau}{x}\right)|}{1-\tau} d\tau + \int_{xr^2}^{\frac{1}{2}} \frac{\tau^{-n} |\mathcal{R}\left(\frac{\tau}{x}\right)|}{1-\tau} d\tau \\
&\leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}) + C \int_{xr^2}^{\frac{1}{2}} \tau^{-n} \left| \mathcal{R}\left(\frac{\tau}{x}\right) \right| d\tau \\
&\leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])} + Cr^{-n+2} \|d_n\|_{\mathcal{C}^0([0,1])}).
\end{aligned}$$

Putting together the preceding estimates we get that

$$\left| \int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^{n+1} F_n^2(\tau)} d\tau \right| \leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])} + Cr^{-n+2} \|d_n\|_{\mathcal{C}^0([0,1])}),$$

for  $r \in [0, \frac{1}{2}x^{-\frac{1}{2}}]$ . Plugging this into (5.56) yields

$$(5.60) \qquad |H_n(r)| \leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}) r^{2n} + Cr^{n+2} \|d_n\|_{\mathcal{C}^0([0,1])},$$

for  $r \in [0, \frac{1}{2}x^{-\frac{1}{2}}]$  and  $2 \leq n \leq n_0$ . Now, we wish to estimate the derivative of  $\frac{H_n(r)}{r^n}$ . Coming back to (5.56), we deduce from elementary computations that

$$H'_n(r) = H_n(r) \left( \frac{2n}{r} + \frac{2xrF'_n(xr^2)}{F_n(xr^2)} \right) + \frac{2}{rF_n(xr^2)} \int_0^{xr^2} \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds,$$

for  $r \in [0, \frac{1}{2}x^{-\frac{1}{2}}]$ . By (5.60), we get

$$\begin{aligned}
\left| H_n(r) \left( \frac{2n}{r} + \frac{2xrF'_n(xr^2)}{F_n(xr^2)} \right) \right| &\leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}) r^{2n-1} \\
&+ Cr^{n+1} \|d_n\|_{\mathcal{C}^0([0,1])}.
\end{aligned}$$

Concerning the integral term, it suffices to apply (5.57) in order to get

$$\frac{2}{rF_n(xr^2)} \int_0^{xr^2} \frac{F_n(s)}{1-s} \left| \mathcal{R}\left(\frac{s}{x}\right) \right| ds \leq \frac{C}{r} \int_0^{xr^2} \left| \mathcal{R}\left(\frac{s}{x}\right) \right| ds \leq C (|A_n| r^{2n+1} + r^{n+1} \|d_n\|_{\mathcal{C}^0([0,1])}).$$

Hence, combining the preceding estimates leads to

$$(5.61) \quad |H'_n(r)| \leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}) r^{2n-1} + Cr^{n+1} \|d_n\|_{\mathcal{C}^0([0,1])}.$$

This estimate together with (5.60) allows getting

$$\left| \left( \frac{H_n(r)}{r^n} \right)' \right| \leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}), \quad \forall r \in \left[ 0, \frac{1}{2}x^{-\frac{1}{2}} \right],$$

for  $2 \leq n \leq n_0$ . The case  $n = 1$  can be done using similar ideas since we only have the singularity at 0 in this interval. Note that  $K_1$  and  $K_2$  can be estimated in terms of  $\mathcal{R}$  having

$$|K_1|, |K_2| \leq \|\mathcal{R}\|_{\mathcal{C}^{0,\gamma}([0,1])}.$$

Let us now move to the intermediate case  $x \in [\frac{1}{2}x^{-\frac{1}{2}}, x^{-\frac{1}{2}}]$ . Then, there is no singularity in this range except for  $r = x^{-\frac{1}{2}}$  due to the logarithmic behavior of  $F_n$  close to this point. This logarithmic divergence can be controlled from the smallness of the integral term in  $H_n$ . Let us show the idea. When we differentiate  $H_n$ , we obtain one term of the type

$$2x^{n+1}r^{2n+1}F'_n(xr^2) \int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^{n+1}F_n^2(\tau)} d\tau,$$

where we notice the logarithmic singularity coming from  $F'_n$  at 1. However, one has

$$\begin{aligned} \left| 2x^{n+1}r^{2n+1}F'_n(xr^2) \int_1^{xr^2} \frac{\int_0^\tau \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^{n+1}F_n^2(\tau)} d\tau \right| &\leq C \left| F'_n(xr^2)(1-xr^2) \frac{\int_0^{xr^2} \frac{F_n(s)}{1-s} \mathcal{R}\left(\frac{s}{x}\right) ds}{(xr^2)^{n+1}F_n^2(xr^2)} \right| \\ &\leq C (\|\mathcal{R}\|_{\mathcal{C}^0([0,1])} + \|\mathcal{R}'\|_{\mathcal{C}^0([0,1])}) \\ &\leq C (\|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}). \end{aligned}$$

Therefore, after straightforward efforts on (5.56) using (5.58), it implies that

$$(5.62) \quad |H'_n(r)| \leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}),$$

for  $2 \leq n \leq n_0$ . Hence, we obtain

$$\left| \left( \frac{H_n(r)}{r^n} \right)' \right| \leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}), \quad \forall r \in \left[ \frac{1}{2}x^{-\frac{1}{2}}, x^{-\frac{1}{2}} \right],$$

for  $2 \leq n \leq n_0$ . In this interval, the case  $n = 1$  is different. This is because we have not singularity coming from the hypergeometric function but we do have it coming from the integral. Hence, some more manipulations are needed. In this case,  $H_1$  reads as

$$H_1(r) = xr^2(1-xr^2) \left[ K_1 + \int_0^{\frac{1}{2}} \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau + \int_{\frac{1}{2}}^{xr^2} \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right].$$

The first integral term can be treated as in the previous computations in the interval  $[0, \frac{1}{2}x^{-\frac{1}{2}}]$ . Let us focus on the singular integral term. By a change of variables, one has

$$\begin{aligned} \left| \int_{\frac{1}{2}}^{xr^2} \frac{\int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^2(1-\tau)^2} d\tau \right| &= \left| \frac{\int_0^{xr^2} \mathcal{R}\left(\frac{s}{x}\right) ds}{x^2r^4(1-xr^2)} - 8 \int_0^{\frac{1}{2}} \mathcal{R}\left(\frac{s}{x}\right) ds - \int_{\frac{1}{2}}^{xr^2} \frac{\tau \mathcal{R}\left(\frac{\tau}{x}\right) - 2 \int_0^\tau \mathcal{R}\left(\frac{s}{x}\right) ds}{\tau^3(1-\tau)} d\tau \right| \\ &\leq C (\|\mathcal{R}\|_{\mathcal{C}^0([0,1])} + \|\mathcal{R}\|_{\mathcal{C}^{0,\gamma}([0,1])}), \end{aligned}$$

where we have used (5.39)-(5.41). Then, we obtain

$$|H_1(r)| \leq C(1-xr^2)\|\mathcal{R}\|_{\mathcal{C}^{0,\gamma}([0,1])}.$$

Similar arguments can be done to find that

$$\left| \left( \frac{H_1(r)}{r} \right)' \right| \leq C \|d_n\|_{\mathcal{C}^{0,\gamma}([0,1])},$$

for any  $r \in [\frac{1}{2}x^{-\frac{1}{2}}, x^{-\frac{1}{2}}]$ .

It remains to establish similar results for the case  $r \in [x^{-\frac{1}{2}}, 1]$ . With this aim we use the second integral in (5.56). Notice that the only singular point is  $r = x^{-\frac{1}{2}}$  due to the logarithmic singularity of the hypergeometric function  $\widehat{F}_n$  defined in (5.36). One can check that this function is strictly increasing, positive and satisfies

$$1 \leq \widehat{F}_n(r) \leq \sup_{n \in \mathbb{N}} \widehat{F}_n(1) < +\infty, \quad \forall r \in [0, 1].$$

As in the previous interval, the smallness of the integral term controls this singularity. The same happens for the case  $n = 1$ . Hence, we get

$$\left| \left( \frac{H_n(r)}{r^n} \right)' \right| \leq C (|A_n| + \|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}), \quad \forall r \in [x^{-\frac{1}{2}}, 1],$$

for  $1 \leq n \leq n_0$ . Therefore, in all the cases we have

$$(5.63) \quad \left| \left( \frac{H_n(r)}{r^n} \right)' \right| \leq C (\|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}), \quad \forall r \in [0, 1],$$

for  $1 \leq n \leq n_0$ . Applying the Mean Value Theorem to (5.51), and using (5.63) and (5.54), allow us to obtain

$$\|h_n\|_{\mathcal{C}^0([0,1])} \leq C (\|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}), \quad \forall n \in [1, n_0].$$

Similar arguments can be done in order to deal with the equation for  $n = 0$ . Note that the resolution of this equation is similar to the work done in Proposition 4.4.

Then, combining all the estimates, it yields

$$\|h_n\|_{\mathcal{C}^0([0,1])} \leq C (\|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])}), \quad \forall n \in \mathbb{N}.$$

This achieves the proof of the announced result.

(3) Let us recall the formula for the Fourier coefficients

$$d_n(r) = \frac{1}{\pi} \int_0^{2\pi} d(r \cos \theta, r \sin \theta) \cos(n\theta) d\theta.$$

We can prove that

$$\|d_n\|_{\mathcal{C}^0([0,1])} \leq \frac{C}{n^{1+\alpha}} \|d\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})},$$

for  $n \geq 1$ . This can be done integrating by parts as

$$d_n(r) = -\frac{r}{n\pi} \int_0^{2\pi} \nabla d(r \cos \theta, r \sin \theta) \cdot (-\sin \theta, \cos \theta) \sin(n\theta) d\theta,$$

and writing it as

$$\begin{aligned} d_n(r) &= \frac{r}{n\pi} \int_0^{2\pi} \nabla d \left( r \cos \left( \theta + \frac{\pi}{n} \right), r \sin \left( \theta + \frac{\pi}{n} \right) \right) \cdot \left( -\sin \left( \theta + \frac{\pi}{n} \right), \cos \left( \theta + \frac{\pi}{n} \right) \right) \sin(n\theta) d\theta \\ &= \frac{r}{2n\pi} \int_0^{2\pi} \left[ \nabla d \left( r \cos \left( \theta + \frac{\pi}{n} \right), r \sin \left( \theta + \frac{\pi}{n} \right) \right) \cdot \left( -\sin \left( \theta + \frac{\pi}{n} \right), \cos \left( \theta + \frac{\pi}{n} \right) \right) \right. \\ &\quad \left. - \nabla d (r \cos \theta, r \sin \theta) \cdot (-\sin \theta, \cos \theta) \right] \sin(n\theta) d\theta. \end{aligned}$$

Consequently,

$$|d_n(r)| \leq \frac{C}{n^{1+\alpha}} \|d\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$



With similar arguments, one achieves that

$$\|d_n\|_{\mathcal{C}^0([0,1])} + \|d'_n\|_{\mathcal{C}^0([0,1])} \leq \frac{C}{n^{1+\alpha}} \|d\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})},$$

and then

$$\|h_n\|_{\mathcal{C}^0([0,1])} \leq \frac{C}{n^{1+\alpha}} \|d\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}.$$

Therefore, we obtain

$$\|h\|_{\mathcal{C}^0(\mathbb{D})} \leq \sum_{n \in \mathbb{N}} \|h_n\|_{\mathcal{C}^0([0,1])} \leq C \|d\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})},$$

which completes the proof.  $\square$

The next target is to provide the proof of Theorem 5.6.

*Proof of Theorem 5.6.* In a small neighborhood of the origin we have the following decomposition through Taylor expansion at the second order

$$\widehat{G}(\Omega, h) = D_g \widehat{G}(\Omega, 0)(h) + \frac{1}{2} D_{g,g}^2 \widehat{G}(\Omega, 0)(h, h) + \mathcal{R}_2(\Omega, h),$$

where  $\mathcal{R}_2(\Omega, h)$  is the remainder term, which verifies

$$\|\mathcal{R}_2(\Omega, h)\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \leq \frac{1}{6} \|D_{g,g,g}^3 \widehat{G}(\Omega, g)(h, h, h)\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})},$$

where  $g = th$  for some  $t \in (0, 1)$ . We intend to show the following,

$$(5.64) \quad \left\| \frac{1}{2} D_{g,g}^2 \widehat{G}(\Omega, 0)(h, h) + \mathcal{R}_2(\Omega, h) \right\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}^{\delta_1} \|h\|_{\mathcal{C}^0(\mathbb{D})}^{\delta_2},$$

for some  $\delta_1 > 0$  and  $\delta_2 \geq 1$ , getting the last bound will be crucial in our argument. First, let us deal with the second derivative of  $\widehat{G}$ . Straightforward computations, similar to what was done in Proposition 4.1, lead to

$$\begin{aligned} D_{g,g}^2 \widehat{G}(\Omega, 0)(h, h)(z) &= -\frac{h(z)^2}{8A} + \frac{\operatorname{Re}}{\pi} \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(z) - \partial_g \phi(\Omega, 0)(h)(y)}{z-y} h(y) dA(y) \\ &\quad + \frac{2}{\pi} \int_{\mathbb{D}} \log |z-y| h(y) \operatorname{Re} [\partial_g \phi(\Omega, 0)(h)'(y)] dA(y) \\ &\quad - \frac{\operatorname{Re}}{2\pi} \int_{\mathbb{D}} \frac{(\partial_g \phi(\Omega, 0)(h)(z) - \partial_g \phi(\Omega, 0)(h)(y))^2}{(z-y)^2} f_0(y) dA(y) \\ &\quad + \frac{2\operatorname{Re}}{\pi} \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(z) - \partial_g \phi(\Omega, 0)(h)(y)}{z-y} f_0(y) \operatorname{Re} [\partial_g \phi(\Omega, 0)(h)'(y)] dA(y) \\ &\quad - \Omega |\partial_g \phi(\Omega, 0)(h)(z)|^2 + \frac{1}{\pi} \int_{\mathbb{D}} \log |z-y| f_0(y) |\partial_g \phi(\Omega, 0)(h)'(y)|^2 dA(y) \\ &\quad - \Omega \operatorname{Re} [\partial_{g,g}^2 \phi(\Omega, 0)(h, h)(z)] \\ &\quad + \frac{\operatorname{Re}}{2\pi} \int_{\mathbb{D}} \frac{\partial_{g,g}^2 \phi(\Omega, 0)(h, h)(z) - \partial_{g,g}^2 \phi(\Omega, 0)(h, h)(y)}{z-y} f_0(y) dA(y) \\ &\quad + \frac{1}{\pi} \int_{\mathbb{D}} \log |z-y| f_0(y) \operatorname{Re} [\partial_{g,g}^2 \phi(\Omega, 0)(h, h)'(y)] dA(y). \end{aligned}$$

Recall the relation between  $\partial_g \phi(\Omega, 0)(h)$  and  $h$

$$\partial_g \phi(\Omega, 0)(h)(z) = z \sum_{n \geq 1} A_n z^n, \quad A_n = \frac{\int_0^1 s^{n+1} h_n(s) ds}{2n(\widehat{\Omega}_n - \Omega)}.$$

By Proposition 3.3, one has that

$$\|\partial_g \phi(\Omega, 0)h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$

We claim that one can reach the limit case,

$$(5.65) \quad \|\partial_g \phi(\Omega, 0)h\|_{\mathcal{C}^1(\mathbb{D})} \leq \|h\|_{\mathcal{C}^0(\mathbb{D})}.$$

The last estimate can be done using Proposition 2.2 in order to work in  $\mathbb{T}$  as follows

$$\begin{aligned} \left| \partial_g \phi(\Omega, 0)h'(e^{i\theta}) \right| &= \left| \sum_{n \geq 1} \int_0^1 s^{n+1} h_n(s) ds e^{in\theta} \right| \\ &= \frac{1}{2} \left| \sum_{n \geq 1} \int_0^{2\pi} \int_0^1 s^{n+1} h(se^{i\theta'}) \cos(n\theta') e^{in\theta} ds d\theta' \right| \\ &= \frac{1}{4} \left| \sum_{n \geq 1} \int_0^{2\pi} \int_0^1 s^{n+1} h(se^{i\theta'}) (e^{in\theta'} + e^{-in\theta'}) e^{in\theta} ds d\theta' \right| \\ &= \frac{1}{4} \left| \sum_{n \geq 1} \int_0^{2\pi} \int_0^1 s^{n+1} h(se^{i\theta'}) (e^{in(\theta+\theta')} + e^{in(\theta-\theta')}) ds d\theta' \right|. \end{aligned}$$

Using Fubini we deduce that

$$\begin{aligned} \left| \partial_g \phi(\Omega, 0)h'(e^{i\theta}) \right| &= \frac{1}{4} \left| \int_0^{2\pi} \int_0^1 s^2 h(se^{i\theta'}) \left( \frac{e^{i(\theta+\theta')}}{1 - se^{i(\theta+\theta')}} + \frac{e^{i(\theta-\theta')}}{1 - se^{i(\theta-\theta')}} \right) ds d\theta' \right| \\ &\leq C \|h\|_{\mathcal{C}^0(\mathbb{D})}. \end{aligned}$$

The latter estimate follows from the convergence of the double integral

$$\int_0^{2\pi} \int_0^1 \frac{ds d\theta}{|1 - se^{i\theta}|} < \infty.$$

The next step is to deal with  $\partial_{g,g}^2 \phi(\Omega, 0)(h, h)$ . By differentiating it, similarly to the proof of in Proposition 3.3, we obtain

$$\begin{aligned} \partial_{g,g}^2 \phi(\Omega, 0)(h, h) &= -\frac{1}{2} \partial_\phi F(\Omega, 0, 0)^{-1} [\partial_{g,g}^2 F(\Omega, 0, 0)(h, h) + 2\partial_{g,\phi}^2 F(\Omega, 0, 0)(h, \partial_g \phi(\Omega, 0)h) \\ &\quad + \partial_{\phi,\phi}^2 F(\Omega, 0, 0)(\partial_g \phi(\Omega, 0)h, \partial_g \phi(\Omega, 0)h)] \\ &= -\frac{1}{2} \sum_{n \geq 1} \frac{\rho_n}{n(\Omega - \widehat{\Omega}_n)} z^{n+1}, \end{aligned}$$

where  $\rho(w) = \sum_{n \geq 1} \rho_n \sin(n\theta)$  and

$$\begin{aligned} \rho(w) &= \partial_{g,g}^2 F(\Omega, 0, 0)(h, h)(w) + 2\partial_{g,\phi}^2 F(\Omega, 0, 0)(h, \partial_g \phi(\Omega, 0)h)(w) \\ &\quad + \partial_{\phi,\phi}^2 F(\Omega, 0, 0)(\partial_g \phi(\Omega, 0)h, \partial_g \phi(\Omega, 0)h)(w) \\ &= \text{Im} \left[ -\frac{w \partial_g \phi(\Omega, 0)(h)'(w)}{\pi} \int_{\mathbb{D}} \frac{h(y)}{w-y} dA(y) \right. \\ &\quad \left. + \frac{w}{\pi} \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(w) - \partial_g \phi(\Omega, 0)(h)(y)}{(w-y)^2} h(y) dA(y) \right. \\ &\quad \left. - \frac{2w}{\pi} \int_{\mathbb{D}} \frac{h(y)}{w-y} \text{Re} [\partial_g \phi(\Omega, 0)(h)'(y)] dA(y) + 2\overline{\partial_g \phi(\Omega, 0)(h)(w)} \partial_g \phi(\Omega, 0)(h)'(w)w \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{w \partial_g \phi(\Omega, 0)(h)'(w)}{\pi} \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(w) - \partial_g \phi(\Omega, 0)(h)(y)}{(w-y)^2} f_0(y) dA(y) \\
& - \frac{2w \partial_g \phi(\Omega, 0)(h)'(w)}{\pi} \int_{\mathbb{D}} \frac{f_0(y)}{w-y} \operatorname{Re} [\partial_g \phi(\Omega, 0)(h)'(y)] dA(y) \\
& - \frac{w}{\pi} \int_{\mathbb{D}} \frac{[\partial_g \phi(\Omega, 0)(h)(w) - \partial_g \phi(\Omega, 0)(h)(y)]^2}{(w-y)^3} f_0(y) dA(y) \\
& + \frac{2w}{\pi} \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(w) - \partial_g \phi(\Omega, 0)(h)(y)}{(w-y)^2} f_0(y) \operatorname{Re} [\partial_g \phi(\Omega, 0)(h)'(y)] dA(y) \\
& - \frac{w}{\pi} \int_{\mathbb{D}} \frac{f_0(y)}{w-y} |\partial_g \phi(\Omega, 0)(h)'(y)|^2 dA(y) \Big].
\end{aligned}$$

By Proposition 3.7 and due to the fact that  $\partial_{g,g}^2 \phi(\Omega, 0)$  can be seen as a convolution operator, one has that

$$\|\partial_{g,g}^2 \phi(\Omega, 0)\|_{\mathcal{C}^{k,\alpha}(\mathbb{D})} \leq C \|\rho\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})}.$$

Moreover, we claim that

$$(5.66) \quad \|\rho\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \leq C \|h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}^{\sigma_1} \|h\|_{\mathcal{C}^0(\mathbb{D})}^{\sigma_2}, \quad k = 0, 1, 2,$$

with  $\sigma_2 \geq 1$ . First, we use the interpolation inequalities for Hölder spaces

$$(5.67) \quad \|h\|_{\mathcal{C}^{k,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^{k_1,\alpha_1}(\mathbb{D})}^{\beta} \|h\|_{\mathcal{C}^{k_2,\alpha_2}(\mathbb{D})}^{1-\beta},$$

for  $k, k_1$  and  $k_2$  non negative integers,  $0 \leq \alpha_1, \alpha_2 \leq 1$  and

$$k + \alpha = \beta(k_1 + \alpha_1) + (1 - \beta)(k_2 + \alpha_2),$$

where  $\beta \in (0, 1)$ . The proof of the interpolation inequality can be found in [31]. In order to get the announced results, we would need to use some classical results in Potential Theory dealing with the Newtonian potential and the Beurling transform, see Appendix B or for instance [26, 40, 44, 47]. Now, let us show the idea behind (5.66). To estimate the first term of  $\rho$ , we combine (5.65) with the law products in Hölder spaces, as follows,

$$\begin{aligned}
& \left\| \partial_g \phi(\Omega, 0)(h)'(\cdot) \int_{\mathbb{D}} \frac{h(y)}{(\cdot) - y} dA(y) \right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \leq C \left\| \partial_g \phi(\Omega, 0)(h)'(\cdot) \right\|_{\mathcal{C}^0(\mathbb{T})} \left\| \int_{\mathbb{D}} \frac{h(y)}{(\cdot) - y} dA(y) \right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \\
& + C \left\| \partial_g \phi(\Omega, 0)(h)'(\cdot) \right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \left\| \int_{\mathbb{D}} \frac{h(y)}{(\cdot) - y} dA(y) \right\|_{\mathcal{C}^0(\mathbb{T})} \\
& \leq C \left( \|h\|_{\mathcal{C}^0(\mathbb{D})} \|h\|_{\mathcal{C}^{k,\alpha}(\mathbb{D})} + \|h\|_{\mathcal{C}^{k,\alpha}(\mathbb{D})} \|h\|_{\mathcal{C}^0(\mathbb{D})} \right).
\end{aligned}$$

Then, (5.66) is satisfied for the first term. Let us deal with the second term of  $\rho$ . For  $k = 0$ , one has

$$\begin{aligned}
& \left\| \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(\cdot) - \partial_g \phi(\Omega, 0)(h)(y)}{(\cdot) - y)^2} h(y) dA(y) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})} \\
& \leq C \left\| \partial_g \phi(\Omega, 0)(h)(\cdot) \text{ p.v. } \int_{\mathbb{D}} \frac{h(y)}{(\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})} \\
& + C \left\| \text{p.v. } \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(y) h(y)}{(\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})} \\
& \leq C \left\| \partial_g \phi(\Omega, 0)(h)(\cdot) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})} \left\| \text{p.v. } \int_{\mathbb{D}} \frac{h(y)}{(\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^0(\mathbb{T})} + C \left\| \partial_g \phi(\Omega, 0)(h) h \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})}
\end{aligned}$$

$$\begin{aligned}
& + C \|\partial_g \phi(\Omega, 0)(h)(\cdot)\|_{\mathcal{C}^0(\mathbb{T})} \left\| \text{p.v.} \int_{\mathbb{D}} \frac{h(y)}{((\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{T})} \\
& \leq C \|h\|_{\mathcal{C}^0(\mathbb{D})} \|h\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})}.
\end{aligned}$$

For  $k = 1, 2$ , we would need the use of the interpolation inequalities:

$$\begin{aligned}
& \left\| \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(\cdot) - \partial_g \phi(\Omega, 0)(h)(y)}{((\cdot) - y)^2} h(y) dA(y) \right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \\
& \leq C \left\| \partial_g \phi(\Omega, 0)(h)(\cdot) \text{p.v.} \int_{\mathbb{D}} \frac{h(y)}{((\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \\
& \quad + C \left\| \text{p.v.} \int_{\mathbb{D}} \frac{\partial_g \phi(\Omega, 0)(h)(y) h(y)}{((\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \\
& \leq C \|\partial_g \phi(\Omega, 0)(h)(\cdot)\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \left\| \text{p.v.} \int_{\mathbb{D}} \frac{h(y)}{((\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^0(\mathbb{T})} \\
& \quad + C \|\partial_g \phi(\Omega, 0)(h)(\cdot)\|_{\mathcal{C}^0(\mathbb{T})} \left\| \text{p.v.} \int_{\mathbb{D}} \frac{h(y)}{((\cdot) - y)^2} dA(y) \right\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} + C \|\partial_g \phi(\Omega, 0)(h) h\|_{\mathcal{C}^{k,\alpha}(\mathbb{T})} \\
& \leq \|h\|_{\mathcal{C}^{k-1,\alpha}(\mathbb{D})} \|h\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} + C \|h\|_{\mathcal{C}^0(\mathbb{D})} \|h\|_{\mathcal{C}^{k,\alpha}(\mathbb{D})}.
\end{aligned}$$

It remains to use the interpolation inequalities in order to conclude. For the case  $k = 1$ , we need to use (5.67) in order to get

$$\|h\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \|h\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^0(\mathbb{D})}^{2 - \frac{2\alpha}{2+\alpha}} \|h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}^{\frac{2\alpha}{2+\alpha}}.$$

We use again (5.67) for  $k = 2$ :

$$\|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} \|h\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^0(\mathbb{D})}^{\frac{3}{2+\alpha}} \|h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}^{\frac{1+2\alpha}{2+\alpha}}.$$

Note that in every case, the exponent of the  $\mathcal{C}^0$ -norm is bigger than 1. As to the remaining terms of  $\rho$ , we develop similar estimates with the same order of difficulties leading to the announced inequality (5.66).

Once we have these preliminaries estimates, we can check that (5.64) holds true. For example, let us illustrate the basic idea to implement through the second term.

$$\left\| \int_{\mathbb{D}} \log |(\cdot) - y| h(y) \text{Re} [\partial_g \phi(\Omega, 0)(h)'(y)] dA(y) \right\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \leq C \|h \partial_g \phi(\Omega, 0)(h)'\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})}.$$

Using the classical law products and (5.65) we find

$$\begin{aligned}
\left\| \int_{\mathbb{D}} \log |(\cdot) - y| h(y) \text{Re} [\partial_g \phi(\Omega, 0)(h)'(y)] dA(y) \right\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} & \leq C \|h\|_{\mathcal{C}^0(\mathbb{D})} \|\partial_g \phi(\Omega, 0)(h)'\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \\
& \quad + C \|h\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \|\partial_g \phi(\Omega, 0)(h)'\|_{\mathcal{C}^0(\mathbb{D})} \\
& \leq C \|h\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \|h\|_{\mathcal{C}^0(\mathbb{D})}.
\end{aligned}$$

The other terms can be estimated in a similar way, achieving (5.64). The same arguments applied to the remainder term lead to

$$\|\mathcal{R}_2(\Omega, h)\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \leq C \|h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}^{\delta_1} \|h\|_{\mathcal{C}^0(\mathbb{D})}^{\delta_2}.$$

The computations are long here but the analysis is straightforward.

Let us see how to achieve the argument. Assume that  $h$  is a zero to  $\widehat{G}$  in a small neighborhood of the origin, then

$$D_g \widehat{G}(\Omega, 0)h = -\frac{1}{2} D_{g,g}^2 \widehat{G}(\Omega, 0)(h, h) - \mathcal{R}_2(\Omega, h).$$

Applying Lemma 5.8–(3) we deduce that

$$\begin{aligned} \|h\|_{\mathcal{C}^0(\mathbb{D})} &\leq C \|D_g \widehat{G}(\Omega, 0)h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \\ &\leq C \left\| \frac{1}{2} D_{g,g}^2 \widehat{G}(\Omega, 0)(h, h) + \mathcal{R}_2(\Omega, h) \right\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})} \\ &\leq C \|h\|_{\mathcal{C}^{2,\alpha}}^{\delta_1} \|h\|_{\mathcal{C}^0(\mathbb{D})}^{\delta_2}, \end{aligned}$$

Consequently, if  $\|h\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}^{\delta_1} < C^{-1}$ , then necessary  $\|h\|_{\mathcal{C}^0(\mathbb{D})} = 0$  since  $\delta_2 \geq 1$ . Therefore, we deduce that there is only the trivial solution in this ball.  $\square$

**Remark 5.9.** *The quadratic profiles are particular cases of the polynomial profiles studied in Section 4.1.2;  $f_0(r) = Ar^{2m} + B$ . Here, we briefly show how to develop this case. Studying the kernel for this case is equivalent to study the equations*

$$\begin{aligned} h_n(r) + \frac{r}{n} \frac{m(2m+2)r^{2m-2}}{\left(r^{2m} - \frac{1}{x_m}\right)} \left[ -\frac{H_n(1)}{G_n(1)} G_n(r) + \frac{H_n(r)}{r^{n+1}} \right] &= 0, \quad \forall r \in [0, 1], \quad \forall n \in \mathbb{N}^*, \\ \frac{\frac{1}{x_m} - r^2}{8} h_0(r) - \int_\tau^1 \frac{1}{\tau} \int_0^\tau s h_0(s) ds d\tau &= 0, \quad \forall r \in [0, 1], \end{aligned}$$

where the functions  $H_n$  and  $G_n$  are defined in (5.9)-(5.10) and

$$\frac{1}{x_m} = \frac{2m+2}{A} \left( \Omega - \frac{B}{2} \right).$$

Thus  $H_n$  verifies the following equation

$$\begin{aligned} r(1 - x_m r^{2m}) H_n''(r) - (2n-1)(1 - x_m r^{2m}) H_n'(r) \\ + 2m(2m+2)r^{2m-1} x_m H_n(r) &= 2m(2m+2)x_m r^{n+2m} \frac{H_n(1)}{G_n(1)} G_n(r). \end{aligned}$$

Using the change of variables  $y = x_m r^{2m}$  and setting  $H_n(r) = \mathcal{F}(x_m r^{2m})$ , one has that

$$\begin{aligned} y(1-y)\mathcal{F}''(y) + \frac{m-n}{m}(1-y)\mathcal{F}'(y) + \frac{m+1}{m}\mathcal{F}(y) \\ = \frac{m+1}{m} \left( \frac{y}{x_m} \right)^{\frac{n+1}{2m}} \frac{H_n(1)}{G_n(1)} G_n \left( \left( \frac{y}{x_m} \right)^{\frac{1}{2m}} \right). \end{aligned}$$

The homogeneous equation of the last differential equation can be solved in terms of hypergeometric functions as it was done in the quadratic profile. Then, similar arguments can be applied to this case.

**5.3. Range structure.** Here, we provide an algebraic description of the range. This will be useful when studying the transversality assumption of the Crandall-Rabinowitz Theorem. Our result reads as follows.

**Proposition 5.10.** *Let  $A \in \mathbb{R}^*$ ,  $B \in \mathbb{R}$  and  $x_0$  be given by (4.14). Let  $x \in (-\infty, 1) \setminus \{\widehat{\mathcal{S}}_{\text{sing}} \cup \{0, x_0\}\}$ , where the set  $\widehat{\mathcal{S}}_{\text{sing}}$  is defined in (5.19). Then*

$$\text{Im } D_g \widehat{G}(\Omega, 0) = \left\{ d \in \mathcal{C}_s^{1,\alpha}(\mathbb{D}) : \int_{\mathbb{D}} d(z) \mathcal{K}_x(z) dz = 0, \quad n \in \mathcal{A}_x \right\},$$

where

$$\mathcal{K}_x(z) = \text{Re} \left[ \frac{F_n(x|z|^2)}{1-x|z|^2} z^n \right], \quad \Omega = \frac{B}{2} + \frac{A}{4x},$$

and the set  $\mathcal{A}_x$  is defined by (5.21).

*Proof.* In order to describe the range of the  $D_g \widehat{G}(\Omega, 0) : \mathcal{C}_s^{1,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}_s^{1,\alpha}(\mathbb{D})$ , we should solve the equation

$$D_g \widehat{G}(\Omega, 0)h = d, \quad h(re^{i\theta}) = \sum_{n \geq 0} h_n(r) \cos(n\theta), \quad d(re^{i\theta}) = \sum_{n \geq 0} d_n(r) \cos(n\theta).$$

From the structure of the linearized operator seen in (5.8), this problem is equivalent to

$$(5.68) \quad \begin{aligned} \frac{\frac{1}{x} - r^2}{8} h_n(r) - \frac{r}{n} \left[ A_n G_n(r) + \frac{1}{2r^{n+1}} H_n(r) \right] &= d_n(r), \quad \forall n \geq 1 \\ \frac{\frac{1}{x} - r^2}{8} h_0(r) - \int_r^1 \frac{1}{\tau} \int_0^\tau s h_0(s) ds d\tau &= d_0(r), \end{aligned}$$

where the functions involved in the last expressions are defined in (5.9)-(5.13). By Proposition 4.3, the case  $n = 0$  can be analyzed through the Inverse Function Theorem getting a unique solution. Let us focus on the case  $n \geq 1$  and proceed as in the preceding study for the kernel. We use the linear operator defined in (5.23),

$$\mathcal{L}h = r^{2n} \int_r^1 \frac{1}{s^{n-1}} h(s) ds + \int_0^r s^{n+1} h(s) ds,$$

for any  $h \in \mathcal{C}([0, 1]; \mathbb{R})$ , which satisfies the boundary conditions in (5.24) and

$$\frac{1}{2n} (r(\mathcal{L}h)'(r))' - (\mathcal{L}h)'(r) = -r^{n+1} h(r).$$

Taking  $H_n \triangleq \mathcal{L}h_n$  and using (5.68) we find that  $H_n$  solves

$$(5.69) \quad \begin{aligned} (1 - xr^2)rH_n''(r) - (1 - xr^2)(2n - 1)H_n'(r) + 8rxH_n(r) \\ = -16A_n x r^{n+2} G_n(r) - 16xnr^{n+1} d_n(r), \end{aligned}$$

complemented with the boundary conditions  $H_n(0) = H_n'(1) = 0$ . This differential equation is equivalent to (5.68). Once we have a solution of the differential equation (5.69) we have to verify that  $\mathcal{L}h_n = H_n$ , where

$$h_n(r) \triangleq \frac{8x}{1 - xr^2} \left[ d_n(r) + \frac{A_n r}{n} G_n(r) + \frac{1}{2nr^n} H_n(r) \right].$$

Denote  $\mathcal{H} \triangleq \mathcal{L}h_n - H_n$ , then it satisfies

$$\frac{1}{2n} [r \mathcal{H}'(r)]' - \mathcal{H}'(r) = 0.$$

From the boundary conditions one obtains that  $\mathcal{H} = 0$  and thus  $\mathcal{L}h_n = H_n$ . Now, since  $H_n(0) = 0$ , Lemma 5.14 can be applied with

$$g(r) = -16A_n x r^{n+2} G_n(r) - 16xnr^{n+1} d_n(r).$$

Thus, the solutions are given by

$$\begin{aligned} H_n(r) &= r^{2n} F_n(xr^2) \left[ \frac{H_n(1)}{F_n(x)} + 4A_n x^n \int_{xr^2}^x \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^\tau \frac{F_n(s)}{1-s} \left(\frac{s}{x}\right)^{\frac{n+1}{2}} G_n\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds d\tau \right. \\ &\quad \left. + 4nx^n \int_{xr^2}^x \frac{1}{\tau^{n+1} F_n^2(\tau)} \int_0^\tau \frac{F_n(s)}{1-s} \left(\frac{s}{x}\right)^{\frac{n}{2}} d_n\left(\left(\frac{s}{x}\right)^{\frac{1}{2}}\right) ds d\tau \right]. \end{aligned}$$

A change of variables combined with (5.13) yield

$$\begin{aligned} H_n(r) &= H_n(1)r^{2n} F_n(xr^2) \left[ \frac{1}{F_n(x)} - \frac{8x}{G_n(1)} \int_r^1 \frac{1}{\tau^{2n+1} F_n^2(x\tau^2)} \int_0^\tau \frac{s^{n+2} F_n(xs^2)}{1 - xs^2} G_n(s) ds d\tau \right] \\ &\quad + 16nxr^{2n} F_n(xr^2) \int_r^1 \frac{1}{\tau^{2n+1} F_n^2(x\tau^2)} \int_0^\tau \frac{s^{n+1} F_n(xs^2)}{1 - xs^2} d_n(s) ds d\tau. \end{aligned}$$

Note that when  $d_n \equiv 0$ , the function  $H_n$  agrees with the one obtained for the kernel. It remains to check the boundary conditions. Clearly  $H_n(0) = 0$ , then we focus on proving  $H'_n(1) = 0$ . Following the computations leading to (5.31), we obtain

$$H'_n(1) = \frac{2nH_n(1)}{F_n(x)G_n(1)}\Psi_n(x) - \frac{16nx}{F_n(x)} \int_0^1 \frac{s^{n+1}F_n(xs^2)}{1-xs^2}d_n(s)ds.$$

We will distinguish two cases. In the first one  $n \notin \mathcal{A}_x$ . By virtue of Proposition 5.3 and Proposition 5.1, we obtain that  $D_g\widehat{G}(\Omega, 0)$  is an isomorphism. Otherwise, we have  $\Psi_n(x) = 0$  and the boundary condition is equivalent to

$$(5.70) \quad \int_0^1 \frac{r^{n+1}F_n(xr^2)}{1-xr^2}d_n(r)dr = 0.$$

Define  $z \in \overline{\mathbb{D}} \mapsto \mathcal{K}_n(z) = \operatorname{Re} \left[ \frac{F_n(x|z|^2)}{1-x|z|^2} z^n \right]$ , and consider the linear form  $T_{\mathcal{K}_n} : \mathcal{C}_s^{1,\alpha}(\mathbb{D}) \mapsto \mathbb{R}$  given by

$$T_{\mathcal{K}_n} d \triangleq \int_{\mathbb{D}} d(y) \mathcal{K}_n(y) dA(y).$$

Condition (5.70) leads to  $T_{\mathcal{K}_n} d = 0$ , which follows from

$$\int_0^{2\pi} d(re^{i\theta}) \cos(n\theta) d\theta = \pi d_n(r).$$

Since  $\mathcal{K}_n$  belongs to  $\mathcal{C}^\infty(\overline{\mathbb{D}}; \mathbb{R})$ , we deduce that  $T_{\mathcal{K}_n}$  is continuous. Thus,  $\operatorname{Ker} T_{\mathcal{K}_n}$  is closed and of co-dimension one. In addition, from the preceding analysis one has that

$$\operatorname{Im} D_g\widehat{G}(\Omega, 0) \subseteq \left\{ d \in \mathcal{C}_s^{1,\alpha}(\mathbb{D}) : \int_{\mathbb{D}} d(z) \mathcal{K}_n(z) dz = 0, n \in \mathcal{A}_x \right\} \subseteq \bigcap_{n \in \mathcal{A}_x} \operatorname{Ker} T_{\mathcal{K}_n}.$$

The elements of the family  $\{\mathcal{K}_n : n \in \mathcal{A}_x\}$  are independent, and thus  $\bigcap_{n \in \mathcal{A}_x} \operatorname{Ker} T_{\mathcal{K}_n}$  is closed and of co-dimension  $\operatorname{card} \mathcal{A}_x$ . As a consequence of Proposition 5.3,  $\operatorname{Ker} D_g\widehat{G}(\Omega, 0)$  is of dimension  $\operatorname{card} \mathcal{A}_x$ . Using Proposition 5.1,  $D_g\widehat{G}(\Omega, 0)$  is a Fredholm operator of index zero. Consequently,  $\operatorname{Im} D_g\widehat{G}(\Omega, 0)$  is of co-dimension  $\operatorname{card} \mathcal{A}_x$ , and thus

$$\operatorname{Im} D_g\widehat{G}(\Omega, 0) = \bigcap_{n \in \mathcal{A}_x} \operatorname{Ker} T_{\mathcal{K}_n}.$$

This achieves the proof of the announced result.  $\square$

## 6. SPECTRAL STUDY

The aim of this section is to study some qualitative properties of the roots of the spectral function (5.20) that will be needed when we apply bifurcation arguments. For instance, to identify the eigenvalues and explore the kernel structure of the linearized operator, we should carefully analyze the existence and uniqueness of roots  $x_n$  of (5.20) at each frequency level  $n$  and study their monotonicity. This part is highly technical and requires cautious manipulations on hypergeometric functions and their asymptotics with respect to  $n$ . Notice that for some special regime in  $A$  and  $B$ , the monotonicity turns to be very intricate and it is only established for higher frequencies through refined expansions of the eigenvalues  $x_n$  with respect to  $n$ . Another problem that one should face is connected to the separation between the eigenvalues set and the singular set associated to (3.5). It seems that the two sequences admit the same leading term and the separation is obtained at the second asymptotics level, which requires much more efforts because the sequence  $\{x_n\}$  converges to 1, which is a singular point for the hypergeometric function involved in (5.20). Recall that  $n$  and  $m$  are non negative integers.

**6.1. Reformulations of the dispersion equation.** In what follows, we intend to write down various formulations for the dispersion equation (5.20) describing the set (1.12). This set is given by the zeroes of (5.20) and the elements of this set are called ‘‘eigenvalues’’. As we shall notice, the study of some qualitative behavior of the zeroes will be much more tractable through the use of different representations connected to some specific algebraic structure of the hypergeometric equations. Recall the use of the notation  $F_n(x) = F(a_n, b_n; c_n; x)$ , where the coefficients are given by (5.15). The Kummer quadratic transformations introduced in Appendix C leads to the following result:

**Lemma 6.1.** *The following identities hold true:*

$$(6.1) \quad \zeta_n(x) = \left[ 1 + x \left( \frac{A + 2B}{A(n+1)} - 1 \right) \right] F(a_n, b_n; n+1; x) \\ + \frac{2x-1}{n+1} F(a_n, b_n; n+2; x) - \frac{2x}{(n+1)(n+2)} F(a_n, b_n; n+3; x),$$

$$(6.2) \quad \zeta_n(x) = \frac{A + 2B}{A(n+1)} x F(a_n, b_n; n+1; x) \\ + \frac{n - (n+1)x}{n+1} F(a_n, b_n; n+2; x) + \frac{2nx}{(n+1)(n+2)} F(a_n, b_n; n+3; x),$$

for any  $n \in \mathbb{N}$  and  $x \in (-\infty, 1)$ , where we have used the notations (5.15).

*Proof.* Let us begin with (6.1). The integral term in (5.20) can be written as follows

$$\int_0^1 F_n(\tau x) \tau^n [-1 + 2x\tau] d\tau = (2x-1) \int_0^1 F_n(\tau x) \tau^n d\tau - 2x \int_0^1 F_n(\tau x) \tau^n (1-\tau) d\tau.$$

This leads to

$$(6.3) \quad \zeta_n(x) = F_n(x) \left[ 1 - x + \frac{A + 2B}{A(n+1)} x \right] + (2x-1) \int_0^1 F_n(\tau x) \tau^n d\tau \\ - 2x \int_0^1 F_n(\tau x) \tau^n (1-\tau) d\tau.$$

We use (C.10) in order to get successively

$$\int_0^1 F(a_n, b_n; n+1; \tau x) \tau^n d\tau = \frac{F(a_n, b_n; n+2; x)}{n+1}$$

and

$$\int_0^1 F(a_n, b_n; n+1; \tau x) \tau^n [1-\tau] d\tau = \frac{F(a_n, b_n; n+3; x)}{(n+1)(n+2)},$$

for any  $x \in (-\infty, 1)$ . Taking into account these identities, we can rewrite (6.3) as (6.1).

In order to obtain (6.2), we use (C.9) with  $a = a_n$ ,  $b = b_n$  and  $c = n+1$ , which yields

$$F(a_n, b_n; n+1)(x-1) = \frac{(n+3)x - (n+1)}{n+1} F(a_n, b_n; n+2; x) \\ + \frac{(a_n - (n+2))((n+2) - b_n)x}{(n+1)(n+2)} F(a_n, b_n; n+3; x) \\ = \frac{(n+3)x - (n+1)}{n+1} F(a_n, b_n; n+2; x) \\ - \frac{(2n+2)x}{(n+1)(n+2)} F(a_n, b_n; n+3; x),$$



where we have taken into account the identities  $a_n + b_n = n$  and  $a_n b_n = -2$ . By virtue of the first assertion of this lemma we obtain

$$\begin{aligned}
\zeta_n(x) &= \frac{A+2B}{A(n+1)} x F(a_n, b_n; n+1; x) \\
&\quad + \frac{(n+1) - (n+3)x}{n+1} F(a_n, b_n; n+2; x) + \frac{(2n+2)x}{(n+1)(n+2)} F(a_n, b_n; n+3; x) \\
&\quad + \frac{2x-1}{n+1} F(a_n, b_n; n+2; x) - \frac{2x}{(n+1)(n+2)} F(a_n, b_n; n+3; x) \\
&= \frac{A+2B}{A(n+1)} x F(a_n, b_n; n+1; x) \\
&\quad + \frac{n - (n+1)x}{n+1} F(a_n, b_n; n+2; x) + \frac{2nx}{(n+1)(n+2)} F(a_n, b_n; n+3; x).
\end{aligned}$$

This achieves the proof of the second identity (6.2).

Let us also remark that using (C.7) we can deduce another useful equivalent expression for  $\zeta_n$

$$\zeta_n(x) = I_n^1(x) F(a_n, b_n; n+1; x) + I_n^2(x) F(a_n+1, b_n; n+2; x) + I_n^3(x) F(a_n, b_n; n+3; x),$$

where

$$\begin{aligned}
I_n^1(x) &\triangleq \frac{n - a_n}{n+1 - a_n} + x \left( \frac{A+2B}{A(n+1)} - \frac{n-1+a_n}{n+1 - a_n} \right), \\
I_n^2(x) &\triangleq -\frac{a_n(2x-1)}{(n+1)(n+1 - a_n)}, \\
I_n^3(x) &\triangleq -\frac{2x}{(n+1)(n+2)}.
\end{aligned}$$

□

**6.2. Qualitative properties of hypergeometric functions.** The main task of this section is to provide suitable properties about the analytic continuation of the mapping  $(n, x) \mapsto F(a_n, b_n; c_n; x)$  and some partial monotonicity behavior. First, applying the integral representation (C.2) with the special coefficients (5.15), we find

$$(6.4) \quad F(a_n, b_n; c_n; x) = \frac{\Gamma(n+1)}{\Gamma(n-a_n)\Gamma(1+a_n)} \int_0^1 \tau^{n-a_n-1} (1-\tau)^{a_n} (1-\tau x)^{-a_n} d\tau,$$

for  $x \in (-\infty, 1]$ . Notice that, due to (C.5) we can evaluate it at 1, obtaining

$$(6.5) \quad F(a_n, b_n; c_n; 1) = \frac{\Gamma(n+1)}{\Gamma(n-a_n+1)\Gamma(1+a_n)},$$

for any  $n \geq 2$ , where we have used the identity  $\Gamma(x+1) = x\Gamma(x)$ . We observe that the representation (6.4) fails for the case  $n=1$  because  $a_1 = -1$ . This does not matter since as we have already mentioned in Remark 5.5, the case  $n=1$  is explicit and the study of  $\zeta_1$  can be done by hand. It is a well-known fact that the Gamma function can be extended analytically to  $\mathbb{C} \setminus \{0, -1, -2, \dots\}$ . Therefore, the map  $n \in \mathbb{N}^* \setminus \{1\} \mapsto F(a_n, b_n; c_n; 1)$  admits a  $\mathcal{C}^\infty$ -extension given by

$$\mathcal{F} : \mathfrak{t} \in ]1, +\infty[ \mapsto \frac{\Gamma(\mathfrak{t}+1)}{\Gamma(\mathfrak{t}-a_{\mathfrak{t}}+1)\Gamma(1+a_{\mathfrak{t}})},$$

with  $a_{\mathfrak{t}} = -\frac{4}{\mathfrak{t} + \sqrt{\mathfrak{t}^2 + 8}}$ .

The first result that we should discuss concerns some useful asymptotic behaviors for  $\mathfrak{t} \mapsto \mathcal{F}(\mathfrak{t})$ .

**Lemma 6.2.** *The following properties are satisfied:*

- (1) Let  $t \geq 1$ , then the function  $x \in (-\infty, 1] \mapsto F(a_t, b_t; c_t; x)$  is positive and strictly decreasing.  
(2) For large  $t \gg 1$ , we have

$$\mathcal{F}(t) = 1 - 2\frac{\ln t}{t} - \frac{2\gamma}{t} + O\left(\frac{\ln^2 t}{t^2}\right)$$

and

$$\mathcal{F}'(t) = 2\frac{\ln t}{t^2} + \frac{2(\gamma - 1)}{t^2} + O\left(\frac{\ln^2 t}{t^3}\right),$$

where  $\gamma$  is the Euler constant. In particular, we have the asymptotics

$$F(a_n, b_n; c_n; 1) = 1 - 2\frac{\ln n}{n} - \frac{2\gamma}{n} + O\left(\frac{\ln^2 n}{n^2}\right)$$

and

$$\frac{d}{dn}F(a_n, b_n; c_n; 1) = 2\frac{\ln n}{n^2} + \frac{2(\gamma - 1)}{n^2} + O\left(\frac{\ln^2 n}{n^3}\right),$$

for large  $n$ .

*Proof.* (1) The case  $t = 1$  follows obviously from the explicit expression given in Remark 5.5. Now let us consider  $t > 1$ . According to (C.4), we can differentiate  $F$  with respect to  $x$ :

$$F'(a_t, b_t; c_t; x) = \frac{a_t b_t}{c_t} F(a_t + 1, b_t + 1; c_t + 1; x), \quad \forall x \in (-\infty, 1).$$

Using the integral representation (C.2) and the positivity of Gamma function, one has that for any  $c > b > 0$

$$(6.6) \quad F(a, b; c, x) > 0, \quad \forall x \in (-\infty, 1).$$

Since  $a_t \in (-1, 0), b_t, c_t > 0$ , then we deduce that  $F(a_t + 1, b_t + 1; c_t + 1; x) > 0$  and thus

$$F'(a_t, b_t; c_t; x) < 0, \quad \forall x \in (-\infty, 1).$$

This implies that  $x \mapsto F(a_n, b_n; c_n; x)$  is strictly decreasing and together with (6.5) we obtain

$$F(a, b; c, x) \geq F(a_t, b_t; c_t, 1) > 0, \quad \forall x \in (-\infty, 1).$$

(2) The following asymptotic behavior

$$\frac{\Gamma(t + \alpha)}{\Gamma(t + \beta)} = \sum_{n \in \mathbb{N}} C_n(\alpha - \beta, \beta) t^{\alpha - \beta - n},$$

holds as  $t \rightarrow +\infty$  by using [59, Identity 12], where the coefficients  $C_n(\alpha - \beta, \beta)$  can be obtained recursively and are polynomials on the variables  $\alpha, \beta$ . In addition, the first coefficients can be calculated explicitly

$$C_0(\alpha - \beta, \beta) = 1 \quad \text{and} \quad C_1(\alpha - \beta, \beta) = \frac{1}{2}(\alpha - \beta)(\alpha + \beta - 1).$$

Taking  $\alpha = 1$  and  $\beta = 1 - a_t$  we deduce

$$\frac{\Gamma(t + 1)}{\Gamma(t + 1 - a_t)} = t^{a_t} + \frac{1}{2}a_t(1 - a_t)t^{a_t - 1} + O\left(\frac{1}{t^2}\right) = t^{a_t} + O\left(\frac{1}{t^2}\right),$$

where we have used that  $a_t \sim -\frac{2}{t}$ . From the following expansion

$$t^{a_t} = e^{a_t \ln t} = 1 - 2\frac{\ln t}{t} + O\left(\frac{\ln^2 t}{t^2}\right),$$

we get

$$(6.7) \quad \frac{\Gamma(\mathfrak{t} + 1)}{\Gamma(\mathfrak{t} + 1 - a_{\mathfrak{t}})} = 1 - 2\frac{\ln \mathfrak{t}}{\mathfrak{t}} + O\left(\frac{\ln^2 \mathfrak{t}}{\mathfrak{t}^2}\right).$$

Using again Taylor expansion, we find  $\Gamma(1 + a_{\mathfrak{t}}) = 1 + a_{\mathfrak{t}}\Gamma'(1) + O(a_{\mathfrak{t}}^2)$ . Therefore, combining this with  $\Gamma'(1) = -\gamma$  and  $a_{\mathfrak{t}} \sim -\frac{2}{\mathfrak{t}}$  yields

$$\Gamma(1 + a_{\mathfrak{t}}) = 1 + \frac{2\gamma}{\mathfrak{t}} + O\left(\frac{1}{\mathfrak{t}^2}\right).$$

Consequently, it follows that  $\mathcal{F}$  admits the following asymptotic behavior at infinity

$$\mathcal{F}(\mathfrak{t}) = \frac{1 - 2\frac{\ln \mathfrak{t}}{\mathfrak{t}} + O(\mathfrak{t}^{-2} \ln^2 \mathfrak{t})}{1 + \frac{2\gamma}{\mathfrak{t}} + O(\mathfrak{t}^{-2})} = 1 - 2\frac{\ln \mathfrak{t}}{\mathfrak{t}} - 2\frac{\gamma}{\mathfrak{t}} + O\left(\frac{\ln^2 \mathfrak{t}}{\mathfrak{t}^2}\right).$$

Since  $\Gamma$  is real analytic, we have that  $\mathcal{F}$  is also real analytic and one may deduce the asymptotics at  $+\infty$  of the derivative  $\mathcal{F}'$  through the differentiation term by term the asymptotics of  $\mathcal{F}$ . Thus, we obtain the second expansion in assertion (2).  $\square$

Our next purpose is to provide some useful estimates for  $F(a_n, b_n; c_n; x)$  and its partial derivatives. More precisely, we state the following result.

**Lemma 6.3.** *With the notations (5.15), the following assertions hold true.*

- (1) *The sequence  $n \in [1, +\infty) \mapsto F(a_n, b_n; c_n; x)$  is strictly increasing, for any  $x \in (0, 1]$ , and strictly decreasing, for any  $x \in (-\infty, 0)$ .*
- (2) *Given  $n \geq 1$  we have*

$$|\partial_n F_n(x)| \leq \frac{-2xF_n(x)}{(n+1)^2},$$

*for any  $x \in (-\infty, 0]$ .*

- (3) *There exists  $C > 0$  such that*

$$|\partial_x F(a_n, b_n; c_n; x)| \leq C + C|\ln(1-x)|,$$

*for any  $x \in [0, 1]$ , and  $n \geq 2$ .*

- (4) *There exists  $C > 0$  such that*

$$|\partial_n F(a_n, b_n; c_n; x)| \leq C\frac{\ln n}{n^2},$$

*for any  $x \in [0, 1]$ , and  $n \geq 2$ .*

- (5) *There exists  $C > 0$  such that*

$$|\partial_{xx} F(a_n, b_n; c_n; x)| \leq \frac{C}{1-x},$$

*for any  $x \in [0, 1]$  and  $n \geq 2$ .*

*Proof.* (1) Recall that  $F_n$  solves the equation

$$x(1-x)F_n''(x) + (n+1)(1-x)F_n'(x) + 2F_n(x) = 0,$$

with  $F_n(0) = 1$  and  $F_n'(0) = \frac{a_n b_n}{c_n} = \frac{-2}{n+1}$ . As we have mentioned in the beginning of this section the dependence with respect to  $n$  is smooth, here we use  $n$  as a continuous parameter instead of  $\mathfrak{t}$ . Then, differentiating with respect to  $n$  we get

$$x(1-x)(\partial_n F_n)''(x) + (n+1)(1-x)(\partial_n F_n)'(x) + 2(\partial_n F_n) = -(1-x)F_n'(x),$$

with  $(\partial_n F_n)(0) = 0$  and  $(\partial_n F_n)'(0) = \frac{2}{(n+1)^2}$ . We can explicitly solve the last differential equation by using the variation of the constant and keeping in mind that  $x \mapsto F_n(x)$  is a homogeneous solution. Thus, we obtain

$$\partial_n F_n(x) = F_n(x) \left[ K_2 - \int_x^1 \frac{1}{F_n^2(\tau)\tau^{n+1}} \left( K_1 - \int_0^\tau F_n'(s)F_n(s)s^n ds \right) d\tau \right],$$

where the constant  $K_1$  must be zero to remove the singularity at 0, in a similar way to the proof of Lemma 5.2. Since  $(\partial_n F_n)(0) = 0$ , we deduce that

$$K_2 = - \int_0^1 \frac{1}{F_n^2(\tau)\tau^{n+1}} \int_0^\tau F_n'(s)F_n(s)s^n ds d\tau,$$

and then

$$(6.8) \quad \partial_n F_n(x) = -F_n(x) \int_0^x \frac{1}{F_n^2(\tau)\tau^{n+1}} \int_0^\tau F_n'(s)F_n(s)s^n ds d\tau.$$

The change of variables  $s = \tau\theta$  leads to

$$(6.9) \quad \partial_n F_n(x) = -F_n(x) \int_0^x \frac{1}{F_n^2(\tau)} \int_0^1 F_n'(\tau\theta)F_n(\tau\theta)\theta^n d\theta d\tau.$$

Hence, it is clear that  $\partial_n F_n(x) > 0$ , for  $x \in [0, 1)$ , using Lemma 6.2-(1). In the case  $x \in (-\infty, 0]$  we similarly get  $\partial_n F_n(x) < 0$ . Let us observe that the compatibility condition  $(\partial_n F_n)'(0) = \frac{2}{(n+1)^2}$  can be directly checked from the preceding representation. Indeed, one has

$$(\partial_n F_n)'(0) = \lim_{x \rightarrow 0} \frac{\partial_n F_n(x)}{x} = - \lim_{x \rightarrow 0^+} \frac{F_n'(0) \int_0^1 \theta^n d\theta}{F_n(0)} = \frac{2}{(n+1)^2}.$$

**(2)** First, notice that

$$F_n'(x) = -\frac{2}{n+1} F(a_n + 1, b_n + 1, c_n + 1, x),$$

and from (6.6) we deduce that

$$|F_n'(x)| \leq \frac{2}{n+1} F(a_n + 1, b_n + 1, c_n + 1, x).$$

Now studying the variation of  $x \in (-\infty, 1) \mapsto F(a_n + 1, b_n + 1, c_n + 1, x)$  by means of the integral representation (C.2), we can show that it is strictly increasing and positive, which implies in turn that

$$0 < F(a_n + 1, b_n + 1, c_n + 1, x) \leq F(a_n + 1, b_n + 1, c_n + 1, 0) = 1, \quad \forall x \in (-\infty, 0].$$

This allows us to get,

$$(6.10) \quad 0 \leq -F_n'(x) \leq \frac{2}{n+1}, \quad \forall x \in (-\infty, 0].$$

Lemma 6.2-(1) implies in particular that

$$F_n(x) \geq 1, \quad \forall x \in (-\infty, 1),$$

and coming back to (6.9) we find

$$|\partial_n F_n(x)| \leq \frac{2F_n(x)}{n+1} \int_x^0 \frac{1}{F_n^2(\tau)} \int_0^1 F_n(\tau\theta)\theta^n d\theta d\tau \leq \frac{2F_n(x)}{(n+1)^2} \int_x^0 \frac{d\tau}{F_n(\tau)} \leq \frac{-2xF_n(x)}{(n+1)^2},$$

for  $x \in (-\infty, 0]$ . This achieves the proof of the announced inequality.

**(3)** From previous computations we have

$$\partial_x F(a_n, b_n; c_n; x) = \frac{-2}{n+1} F(1 + a_n, n + 1 - a_n; n + 2; x),$$

which admits the integral representation

$$\begin{aligned}
\partial_x F(a_n, b_n; c_n; x) &= \frac{-2\Gamma(n+2)}{(n+1)\Gamma(n+1-a_n)\Gamma(1+a_n)} \int_0^1 \tau^{n-a_n} (1-\tau)^{a_n} (1-\tau x)^{-1-a_n} d\tau \\
&= \frac{-2(n+2)}{n+1} \mathcal{F}(n) \int_0^1 \tau^{n-a_n} (1-\tau)^{a_n} (1-\tau x)^{-1-a_n} d\tau \\
(6.11) \quad &= \frac{-2(n+2)}{n+1} \mathcal{F}(n) J_n(x),
\end{aligned}$$

where

$$J_n(x) \triangleq \int_0^1 \tau^{n-a_n} (1-\tau)^{a_n} (1-\tau x)^{-1-a_n} d\tau.$$

Using

$$\sup_{n \geq 1} \frac{2(n+2)}{n+1} \leq 3,$$

and the first assertion of Lemma 6.2, we have  $\mathcal{F}(n) \in [0, 1]$ , and

$$(6.12) \quad \sup_{n \geq 1} \frac{2(n+2)\mathcal{F}(n)}{n+1} \leq 3.$$

Consequently,

$$(6.13) \quad |\partial_x F(a_n, b_n; c_n; x)| \leq 3J_n.$$

To estimate  $J_n$  we simply write

$$J_n \leq \int_0^1 (1-\tau)^{a_n} \left(1 - \frac{\tau}{2}\right)^{-1-a_n} d\tau \leq C \int_0^1 (1-\tau)^{a_n} d\tau \leq \frac{C}{1+a_n} \leq C,$$

for some constant  $C$  independent of  $n \geq 2$ , and for  $x \in [0, \frac{1}{2}]$ . In the case  $x \in [\frac{1}{2}, 1]$ , making the change of variable

$$(6.14) \quad \tau = 1 - \frac{1-x}{x} \tau',$$

and denoting the new variable again by  $\tau$ , we obtain

$$\begin{aligned}
J_n &\leq x^{-a_n-1} \int_0^{\frac{x}{1-x}} \tau^{a_n} (1+\tau)^{-1-a_n} d\tau \\
&\leq x^{-a_n-1} \int_0^1 \tau^{a_n} (1+\tau)^{-1-a_n} d\tau + x^{-a_n-1} \int_1^{\frac{x}{1-x}} \tau^{a_n} (1+\tau)^{-1-a_n} d\tau \\
&\leq C + C \int_1^{\frac{x}{1-x}} \left(\frac{1+\tau}{\tau}\right)^{-a_n} (1+\tau)^{-1} d\tau \leq C + C |\ln(1-x)|,
\end{aligned}$$

which achieves the proof.

(4) According to (6.8) and Lemma 6.2–(1) we may write

$$\begin{aligned}
|\partial_n F_n(x)| &\leq F_n(x) \int_0^x \frac{1}{F_n^2(\tau) \tau^{n+1}} \int_0^\tau |F_n'(s)| F_n(s) s^n ds d\tau \\
(6.15) \quad &\leq \frac{1}{F_n^2(1)} \int_0^1 \tau^{-n-1} \int_0^\tau |F_n'(s)| s^n ds d\tau,
\end{aligned}$$

for  $x \in [0, 1]$  and  $n \geq 2$ . Using (6.13), the definition of  $J_n$  and the fact that  $0 < -a_n < 1$ , we obtain

$$|F_n'(x)| \leq 3 \int_0^1 \tau^{n-a_n} (1-\tau)^{a_n} (1-\tau x)^{-1-a_n} d\tau \leq \frac{3}{1-x} \int_0^1 \tau^{n-a_n} (1-\tau)^{a_n} d\tau.$$

Now, recall the classical result on the Beta function  $B$  defined as follows

$$(6.16) \quad \int_0^1 \tau^{n-a_n} (1-\tau)^{a_n} d\tau = B(n+1-a_n, 1+a_n) = \frac{\Gamma(n+1-a_n)\Gamma(1+a_n)}{\Gamma(n+2)},$$

which implies in view of (6.5) that

$$\int_0^1 \tau^{n-a_n} (1-\tau)^{a_n} d\tau = \frac{1}{(n+1)F_n(1)}.$$

Consequently

$$|F'_n(x)| \leq \frac{3}{(1-x)(n+1)F_n(1)}.$$

Inserting this inequality into (6.15) we deduce that

$$|\partial_n F_n(x)| \leq \frac{1}{(n+1)F_n^3(1)} \int_0^1 \tau^{-n-1} \int_0^\tau \frac{s^n}{1-s} ds d\tau,$$

and integrating by parts we find

$$\begin{aligned} \int_0^1 \tau^{-n-1} \int_0^\tau \frac{s^n}{1-s} ds d\tau &= \left[ \frac{1-\tau^{-n}}{n} \int_0^\tau \frac{s^n}{1-s} ds \right]_0^1 + \frac{1}{n} \int_0^1 \frac{s^n-1}{s-1} ds \\ &= \frac{1}{n} \int_0^1 \sum_{k=0}^{n-1} s^k ds = \frac{1}{n} \sum_{k=1}^n \frac{1}{k}. \end{aligned}$$

Thus, it follows from the classical inequality  $\sum_{k=1}^n \frac{1}{k} \leq 1 + \ln n$  that

$$|\partial_n F_n(x)| \leq \frac{1 + \ln n}{n^2 F_n^3(1)},$$

for  $n \geq 2$ . Since  $F_n(1) > 0$  and converges to 1, as  $n$  goes to  $\infty$ , then one can find an absolute constant  $C > 0$  such that

$$|\partial_n F_n(x)| \leq C \frac{\ln n}{n^2},$$

for any  $n \geq 2$ , which achieves the proof of the estimate.

(5) Differentiating the integral representation (6.11) again with respect to  $x$  we obtain

$$\partial_{xx} F(a_n, b_n; c_n; x) = \frac{(a_n+1)a_n(n-a_n)(n+2)}{n+1} \mathcal{F}(n) \widehat{J}_n(x),$$

with

$$\widehat{J}_n(x) \triangleq \int_0^1 \tau^{n-a_n+1} (1-\tau)^{a_n} (1-\tau x)^{-2-a_n} d\tau.$$

According to (6.12) one finds that

$$(6.17) \quad |\partial_{xx} F(a_n, b_n; c_n; x)| \leq 4\widehat{J}_n(x).$$

The procedure for estimating  $\widehat{J}_n$  matches the one given for  $J_n$  in the previous assertion. Indeed, we have the uniform bound  $\widehat{J}_n(x) \leq C$ , for  $x \in [0, \frac{1}{2}]$ . Now, the change of variables (6.14) leads to

$$\widehat{J}_n(x) \leq \int_0^1 (1-\tau)^{a_n} (1-\tau x)^{-2-a_n} d\tau \leq \frac{x^{-a_n-1}}{1-x} \int_0^{\frac{x}{1-x}} \tau^{a_n} (1+\tau)^{-2-a_n} d\tau \leq \frac{C}{1-x},$$

where we have used the bounds  $0 < -a_n < \frac{2}{1+\sqrt{3}} < 1$ , which are verified for any  $n \geq 2$  and  $x \in [\frac{1}{2}, 1)$ . Inserting this estimate into (6.17) we obtain the announced inequality.  $\square$

Next we shall prove the following.

**Lemma 6.4.** *There exists  $C > 0$  such that*

$$(6.18) \quad |F(a_n, b_n; n+1; x) - 1| \leq C \frac{\ln n}{n},$$

$$(6.19) \quad 1 \leq F(a_n + 1, b_n; n+2; x) \leq Cn,$$

$$(6.20) \quad |F(a_n, b_n; n+3; x) - 1| \leq C \frac{\ln n}{n},$$

for any  $n \geq 2$  and any  $x \in [0, 1]$ .

*Proof.* The estimate (6.18) follows easily from the second assertion of Lemma 6.3, combined with the monotonicity of  $F_n$  and Lemma 6.2. Indeed,

$$|F(a_n, b_n; c_n; 0) - F(a_n, b_n; c_n; x)| \leq F(a_n, b_n; c_n; 0) - F(a_n, b_n; c_n; 1) \leq C \frac{\ln n}{n}.$$

In the case (6.19), applying similar arguments as in the first assertion of Lemma 6.2, we conclude that the function  $x \in [0, 1] \rightarrow F(a_n + 1, b_n; n+2; x)$  is positive and strictly increasing. Hence,

$$1 \leq F(a_n + 1, b_n; n+2; x) \leq F(a_n + 1, b_n; n+2; 1).$$

Combining (C.5) and (6.7) we obtain the estimate:

$$1 \leq F(a_n + 1, b_n; n+2; x) \leq F(a_n + 1, b_n; n+2; 1) \leq \frac{\Gamma(n+2)}{\Gamma(n+1-a_n)\Gamma(2+a_n)} \leq Cn.$$

To check (6.20), we use the first assertion of Lemma 6.2,

$$0 \leq 1 - F(a_n, b_n; n+3; x) \leq 1 - F(a_n, b_n; n+3; 1).$$

Moreover, by virtue of (C.5) one has

$$F(a_n, b_n; n+3; 1) = \frac{\Gamma(n+3)}{\Gamma(n+3-a_n)} \frac{\Gamma(3)}{\Gamma(3+a_n)}.$$

As a consequence of (6.7) and  $a_n \sim -\frac{2}{n}$ , we obtain

$$\frac{\Gamma(n+3)}{\Gamma(n+3-a_n)} = 1 - 2 \frac{\ln n}{n} + O\left(\frac{1}{n}\right) \quad \text{and} \quad \frac{\Gamma(3)}{\Gamma(3+a_n)} = 1 + O\left(\frac{1}{n}\right).$$

Therefore, the following asymptotic expansion

$$F(a_n, b_n; n+3; 1) = 1 - 2 \frac{\ln n}{n} + O\left(\frac{1}{n}\right)$$

holds and the estimate follows easily.  $\square$

Another useful property deals with the behavior of the hypergeometric function with respect to the third variable  $c$ .

**Lemma 6.5.** *Let  $n \geq 1$ , then the mapping  $c \in (b_n, \infty) \mapsto F(a_n, b_n; c; x)$  is strictly increasing for  $x \in (0, 1)$  and strictly decreasing for  $x \in (-\infty, 0)$ .*

*Proof.* First, we check the case  $n = 1$  that comes by

$$F(a_1, b_1; c; x) = 1 - \frac{2}{c}x.$$

Let  $n > 1$  and recall that the hypergeometric function  $F(a_n, b_n; c; x)$  solves the differential equation

$$x(1-x)\partial_{xx}^2 F(a_n, b_n; c; x) + [c - (n+1)x]\partial_x F(a_n, b_n; c; x) + 2F(a_n, b_n; c; x) = 0,$$

with  $F(a_n, b_n; c; 0) = 1$  and  $\partial_x F(a_n, b_n; c; 0) = -\frac{2}{c}$ . Hence by differentiation it is easy to check that  $\partial_c F(a_n, b_n; c; x)$  solves

$$x(1-x)\partial_{xx}^2 (\partial_c F(a_n, b_n; c; x)) + [c - (n+1)x]\partial_x (\partial_c F(a_n, b_n; c; x)) + 2(\partial_c F(a_n, b_n; c; x))$$

$$= -\partial_x F(a_n, b_n; c; x),$$

with initial conditions

$$\partial_c F(a_n, b_n; c; 0) = 0 \quad \text{and} \quad \partial_x(\partial_c F(a_n, b_n; c; 0)) = \frac{2}{c^2}.$$

Note that a homogeneous solution of the last differential equation is  $F(a_n, b_n; c; x)$ . By the variation of constant method one can look for the full solution to the differential equation in the form

$$\partial_c F(a_n, b_n; c; x) = K(x)F(a_n, b_n; c; x),$$

and from straightforward computations we find that  $T = K'$  solves the first order differential equation,

$$T'(x) + \left[ 2 \frac{\partial_x F(a_n, b_n; c; x)}{F(a_n, b_n; c; x)} + \frac{c - (n+1)x}{x(1-x)} \right] T(x) = -\frac{\partial_x F(a_n, b_n; c; x)}{x(1-x)F(a_n, b_n; c; x)}.$$

The general solution to this latter equation is given by

$$T(x) = \frac{(1-x)^{c-(n+1)}}{F(a_n, b_n; c; x)^2 |x|^c} \left[ K_1 - \int_0^x \frac{|s|^c s^{-1} F(a_n, b_n; c; s) \partial_s F(a_n, b_n; c; s)}{(1-s)^{c-n}} ds \right],$$

for  $x \in (-\infty, 1)$  where  $K_1 \in \mathbb{R}$  is a real constant. Thus

$$\partial_c F(a_n, b_n; c; x) = F(a_n, b_n; c; x) \left[ K_2 + \int_{x_0}^x \frac{(1-\tau)^{c-(n+1)}}{F(a_n, b_n; c; \tau)^2 |\tau|^c} \left\{ K_1 - \int_0^\tau \frac{|s|^c s^{-1} F(a_n, b_n; c; s) \partial_s F(a_n, b_n; c; s)}{(1-s)^{c-n}} ds \right\} d\tau \right],$$

where  $K_1, K_2 \in \mathbb{R}$  and  $x_0 \in (-1, 1)$ . Since  $\partial_c F(a_n, b_n; c; x)$  is not singular at  $x = 0$ , we get that  $K_1 = 0$ . Then, changing the constant  $K_2$  one can take  $x_0 = 0$  getting

$$\partial_c F(a_n, b_n; c; x) = F(a_n, b_n; c; x) \left[ K_2 - \int_0^x \frac{(1-\tau)^{c-(n+1)}}{F(a_n, b_n; c; \tau)^2 |\tau|^c} \int_0^\tau \frac{|s|^c s^{-1} F(a_n, b_n; c; s) \partial_s F(a_n, b_n; c; s)}{(1-s)^{c-n}} ds d\tau \right].$$

The initial condition  $\partial_c F(a_n, b_n; c; 0) = 0$  implies that  $K_2 = 0$  and hence

$$\begin{aligned} \partial_c F(a_n, b_n; c; x) &= -F(a_n, b_n; c; x) \times \\ &\int_0^x (1-\tau)^{c-(n+1)} \int_0^\tau \frac{|s|^c s^{-1} F(a_n, b_n; c; s) \partial_s F(a_n, b_n; c; s)}{|\tau|^c F(a_n, b_n; c; \tau)^2 (1-s)^{c-n}} ds d\tau. \end{aligned}$$

Similarly to the proof of Lemma 6.2–(1) one may obtain that

$$F(a_n, b_n; c; x) > 0 \quad \text{and} \quad \partial_x F(a_n, b_n; c; x) < 0,$$

for any  $c > b_n$  and  $x \in (-\infty, 1)$ . This entails that  $\partial_c F(a_n, b_n; c; x)$  is positive for  $x \in (0, 1)$ , and negative when  $x \in (-\infty, 0)$ , which concludes the proof.  $\square$

**6.3. Eigenvalues.** The existence of eigenvalues, that are the elements of the dispersion set defined in (1.12), is connected to the problem of studying the roots of the equation introduced in (5.20). Here, we will develop different cases illustrating strong discrepancy on the structure of the dispersion set. Assuming  $A > 0$  and  $B < -A$ , we find that the dispersion set is infinite. However, for the case  $A > 0$  and  $B \geq -\frac{A}{2}$ , the dispersion set is finite. Notice that the transient regime corresponding to  $-A \leq B \leq -\frac{A}{2}$  is not covered by the current study and turns to be



more complicate due to the complex structure of the spectral function (5.20).  
Let us begin with studying the cases

$$(6.21) \quad A > 0, \quad A + B < 0,$$

$$(6.22) \quad A > 0, \quad A + 2B \leq 0,$$

$$(6.23) \quad A > 0, \quad A + 2B \geq 0.$$

Our first main result reads as follows.

**Proposition 6.6.** *The following assertions hold true:*

- (1) *Given  $A, B$  satisfying (6.21), there exist  $n_0 \in \mathbb{N}^*$ , depending only on  $A$  and  $B$ , and a unique root  $x_n \in (0, 1)$  of (5.20), i.e.  $\zeta_n(x_n) = 0$ , for any  $n \geq n_0$ . In addition,*

$$x_n \in \left(0, 1 + \frac{A+B}{An}\right),$$

*and the sequence  $n \in [n_0, +\infty) \mapsto x_n$  is strictly increasing.*

- (2) *Given  $A, B$  satisfying the weak condition (6.22) and  $n \in \mathbb{N}^*$ , then  $\zeta_n$  has no solution in  $(-\infty, 0]$ .*

- (3) *Given  $A, B$  satisfying (6.23), then  $\zeta_n$  has no solution in  $[0, 1]$ , for  $n \geq 2$ .*

*Proof.* (1) The expression of the spectral equation (6.1) agrees with

$$\zeta_n(x) = I_n^1(x)F(a_n, b_n; n+1; x) + I_n^2(x)F(a_n+1, b_n; n+2; x) + I_n^3(x)F(a_n, b_n; n+3; x),$$

where  $I_n^1, I_n^2$  and  $I_n^3$  are defined in Lemma 6.1. From this expression we get  $\zeta_n(0) = \frac{n}{n+1}$ . To find a solution in  $(0, 1)$  we shall apply the Intermediate Value Theorem, and for this purpose we need to check that  $\zeta_n(1) < 0$ . Applying (C.5) we get

$$\begin{aligned} \zeta_n(1) &= \left[ \frac{1}{n+1-a_n} + \frac{A+2B}{A(n+1)} \right] \frac{\Gamma(n+1)}{\Gamma(n+1-a_n)\Gamma(1+a_n)} \\ &\quad - \frac{a_n\Gamma(n+1)}{\Gamma(n+2-a_n)\Gamma(2+a_n)} - \frac{4\Gamma(n)}{\Gamma(n+3-a_n)\Gamma(3+a_n)}. \end{aligned}$$

Using the following expansion for large  $n \gg 1$

$$\frac{1}{n+1-a_n} = \frac{1}{n+1} + O\left(\frac{1}{n^3}\right),$$

and (6.7), we find

$$\zeta_n(1) = 2\frac{A+B}{n+1} + O\left(\frac{\ln n}{n^2}\right).$$

Thus, under the hypothesis (6.21), there exists  $n_0 \in \mathbb{N}^*$ , depending on  $A, B$ , such that

$$\zeta_n(1) < 0, \quad \forall n \geq n_0.$$

This proves the existence of at least one solution  $x_n \in (0, 1)$  to the equation  $\zeta_n(x_n) = 0$ , for any  $n \geq n_0$ . The next objective is to localize this root and show that  $x_n \in (0, 1 + \frac{A+B}{An})$ . For this goal it suffices to verify that

$$\zeta_n(1-\varepsilon) < 0, \quad \forall \varepsilon \in \left(0, -\frac{A+B}{An}\right).$$

Let us begin with the first term  $I_n^1(x)$  in the expression (6.1) which implies that

$$I_n^1(1-\varepsilon) = \left[ \frac{1-2a_n}{n+1-a_n} + \frac{A+2B}{A(n+1)} \right] - \varepsilon \left[ \frac{A+2B}{A(n+1)} - \frac{n-1+a_n}{n+1-a_n} \right].$$

Now, it is straightforward to check the following asymptotic expansions,

$$\frac{1-2a_n}{n+1-a_n} + \frac{A+2B}{A(n+1)} = 2\frac{A+B}{A(n+1)} - \frac{(2n+1)a_n}{(n+1)(n+1-a_n)} = 2\frac{A+B}{A(n+1)} + O\left(\frac{1}{n^2}\right),$$

and

$$\begin{aligned} -\varepsilon \left[ \frac{A+2B}{A(n+1)} - \frac{n-1+a_n}{n+1-a_n} \right] &= -\varepsilon \left[ -1 + \frac{A+2B}{A(n+1)} + \frac{2-2a_n}{n+1-a_n} \right] \\ &\leq \varepsilon + O\left(\frac{1}{n^2}\right) \leq -\frac{A+B}{An} + O\left(\frac{1}{n^2}\right), \end{aligned}$$

Therefore, we obtain

$$I_n^1(1-\varepsilon) \leq \frac{A+B}{An} + O\left(\frac{1}{n^2}\right).$$

Thanks to (6.18), we deduce

$$|1 - F(a_n, b_n; n+1; 1-\varepsilon)| \leq C \frac{\ln n}{n},$$

which yields in turn

$$(6.24) \quad I_n^1(1-\varepsilon)F(a_n, b_n; n+1; 1-\varepsilon) \leq \frac{A+B}{A(n+1)} + O\left(\frac{\ln n}{n^2}\right).$$

Next, we will deal with the second term  $I_n^2$  of (6.1). Directly from (6.19) we get

$$(6.25) \quad |I_n^2(x)|F(a_n+1, b_n; n+2; x) = \frac{|a_n(2x-1)|}{(n+1)(n+1-a_n)}F(a_n+1, b_n; n+2; x) \leq \frac{C}{n^2}.$$

Similarly, the estimate (6.20) implies that

$$(6.26) \quad |I_n^3(x)|F(a_n, b_n; n+3; x) = \frac{2xF(a_n, b_n; n+3; x)}{(n+1)(n+2)} \leq \frac{C}{n^2}.$$

Inserting (6.24), (6.25) and (6.26) into the expression of  $\zeta_n$  we find

$$\zeta_n(1-\varepsilon) \leq \frac{A+B}{A(n+1)} + O\left(\frac{\ln n}{n^2}\right),$$

for any  $\varepsilon \in (0, -\frac{A+B}{An})$ . From this we deduce the existence of  $n_0$  depending on  $A$  and  $B$  such that

$$\zeta_n(1-\varepsilon) \leq \frac{A+B}{2A(n+1)} < 0,$$

for any  $n \geq n_0$ . Then  $\zeta_n$  has no zero in  $(1 + \frac{A+B}{An}, 1)$  and this achieves the proof of the first result.

Next we shall prove that  $x_n$  is the only zero of  $\zeta_n$  in  $(0, 1)$ . For this purpose it appears to be more convenient to use the expression for  $\zeta_n$  given by (5.20). Let us differentiate  $\zeta_n$  with respect to  $x$  as follows

$$\begin{aligned} \partial_x \zeta_n(x) &= F_n'(x) \left[ 1 - x + \frac{A+2B}{A(n+1)}x \right] + F_n(x) \left[ -1 + \frac{A+2B}{A(n+1)} \right] \\ &\quad + \int_0^1 F_n'(\tau x) \tau^{n+1} [-1 + 2x\tau] d\tau + 2 \int_0^1 F_n(\tau x) \tau^{n+1} d\tau. \end{aligned}$$

From Lemma 6.2–(1), we recall that  $F_n > 0$  and  $F_n' < 0$ . Hence for  $A+2B < 0$  and  $x \in (0, 1)$  we get

$$\partial_x \zeta_n(x) \leq F_n'(x) \frac{A+2B}{A(n+1)}x - F_n(x) + \int_0^1 F_n'(\tau x) \tau^{n+1} [-1 + 2x\tau] d\tau + 2 \int_0^1 F_n(\tau x) \tau^{n+1} d\tau.$$

Applying the third assertion of Lemma 6.3 we find

$$|F_n'(x)| \leq C \ln n, \quad \forall n \geq 2,$$

for any  $x \in (0, 1 + \frac{A+B}{An})$  and with  $C$  a constant depending only on  $A$  and  $B$ . It follows that

$$\partial_x \zeta_n(x) \leq C \frac{|A+2B| \ln n}{A} - F_n(x) + C \frac{\ln n}{n} + \frac{C}{n} \leq -1 + C \frac{2A+2|B| \ln n}{A} + (1 - F_n(x)),$$

for  $x \in [0, 1 + \frac{A+B}{An})$ , which implies according to (6.18) that

$$\partial_x \zeta_n(x) \leq -1 + C \frac{A+|B| \ln n}{A}.$$

Hence, there exists  $n_0$  such that

$$\partial_x \zeta_n(x) \leq -\frac{1}{2}, \quad \forall x \in \left[0, 1 + \frac{A+B}{An}\right), \quad \forall n \geq n_0.$$

Thus, the function  $x \in [0, 1 + \frac{A+B}{An}) \mapsto \zeta_n(x)$  is strictly decreasing and admits only one zero that we have denoted by  $x_n$ .

It remains to show that  $n \in [n_0, +\infty) \mapsto x_n$  is strictly increasing, which implies in particular that

$$(6.27) \quad \zeta_m(x_n) \neq 0, \quad \forall n \neq m \geq n_0.$$

For this aim, it suffices to show that the mapping  $n \in [n_0, +\infty) \mapsto \zeta_n(x)$  is strictly increasing, for any  $x \in (0, 1)$ . Setting

$$(6.28) \quad F_n(x) = 1 + \rho_n(x),$$

we can write

$$(6.29) \quad \begin{aligned} \zeta_n(x) &= \frac{n}{n+1} - \frac{n}{n+2}x + \frac{A+2B}{A(n+1)}x + \rho_n(x) \left[1 - x + \frac{A+2B}{A(n+1)}x\right] \\ &+ \int_0^1 \rho_n(\tau x) \tau^n [-1 + 2x\tau] d\tau \triangleq \frac{n}{n+1} - \frac{n}{n+2}x + \frac{A+2B}{A(n+1)}x + R_n(x). \end{aligned}$$

Since  $F_n$  is analytic with respect to its parameters and we can think in  $n$  as a continuous parameter,  $n \mapsto \zeta_n(x)$  is also analytic. Therefore, differentiating with respect to  $n$ , we deduce that

$$\partial_n \zeta_n(x) = \frac{1}{(n+1)^2} - 2 \frac{x}{(n+2)^2} - \frac{A+2B}{A(n+1)^2}x + \partial_n R_n(x).$$

Consequently,

$$\partial_n \zeta_n(x) \geq \frac{1}{(n+1)^2} \left[1 - 2x - \frac{A+2B}{A}x\right] + \partial_n R_n(x) \geq \frac{1}{(n+1)^2} \left[1 - \frac{3A+2B}{A}x\right] + \partial_n R_n(x).$$

We use the following trivial bound

$$1 - \frac{3A+2B}{A}x \geq \min(1, \kappa), \quad \forall x \in [0, 1],$$

where  $\kappa = -2\frac{A+B}{A}$  is strictly positive due to the assumptions (6.21). Therefore, we can rewrite the bound for  $\partial_n \zeta_n(x)$  as follows

$$(6.30) \quad \partial_n \zeta_n(x) \geq \frac{\min(1, \kappa)}{(n+1)^2} + \partial_n R_n(x).$$

To estimate  $\partial_n R_n(x)$  we shall differentiate (6.29) with respect to  $n$ ,

$$\begin{aligned} \partial_n R_n(x) &= \partial_n F_n(x)(1-x) + \partial_n F_n(x) \frac{A+2B}{A(n+1)}x - \rho_n(x) \frac{A+2B}{A(n+1)^2}x \\ &+ \int_0^1 (\partial_n F_n(\tau x)) \tau^n [-1 + 2x\tau] d\tau + \int_0^1 \rho_n(\tau x) \tau^n \ln \tau [-1 + 2x\tau] d\tau. \end{aligned}$$

From (6.18) and Lemma 6.3, we deduce

$$\begin{aligned}\partial_n R_n(x) &\geq \partial_n F_n(x) \frac{A+2B}{A(n+1)} x - \rho_n(x) \frac{A+2B}{A(n+1)^2} x \\ &\quad + \int_0^1 \partial_n F_n(\tau x) \tau^n [-1+2x\tau] d\tau + \int_0^1 \rho_n(\tau x) \tau^n \ln \tau [-1+2x\tau] d\tau,\end{aligned}$$

and

$$\left| \partial_n F_n(x) \frac{A+2B}{A(n+1)} x - \rho_n(x) \frac{A+2B}{A(n+1)^2} x \right| \leq C \frac{|A+2B| \ln n}{A n^3}.$$

Observe that the first integral can be bounded as follows

$$\left| \int_0^1 \partial_n F_n(\tau x) \tau^n [-1+2x\tau] d\tau \right| \leq C \frac{\ln n}{n^2} \int_0^1 \tau^n | -1+2x\tau | d\tau \leq C \frac{\ln n}{n^3},$$

while for the second one we have

$$\left| \int_0^1 \rho_n(\tau x) \tau^n \ln \tau [-1+2x\tau] d\tau \right| \leq C \frac{\ln n}{n} \int_0^1 \tau^n |\ln \tau| d\tau \leq C \frac{\ln n}{n^3},$$

where we have used (6.28) and Lemma 6.4. Plugging these estimates into (6.30), we find

$$\partial_n \zeta_n(x) \geq \frac{\min(1, \kappa)}{(n+1)^2} - C \frac{|A+2B| \ln(n+1)}{A (n+1)^3}, \quad \forall x \in [0, 1].$$

Then, there exists  $n_0$  depending only on  $A, B$  such that

$$\partial_n \zeta_n(x) \geq \frac{\min(1, \kappa)}{2(n+1)^2},$$

for any  $n \geq n_0$  and any  $x \in [0, 1]$ . This implies that  $n \in [n_0, +\infty[ \mapsto \zeta_n(x)$  is strictly increasing and thus (6.27) holds.

(2) From (6.2) one has

$$\begin{aligned}\zeta_n(x) &= \frac{A+2B}{A(n+1)} x F(a_n, b_n; n+1; x) - \frac{x}{n+1} F(a_n, b_n; n+2; x) \\ &\quad + \frac{n(1-x)}{n+1} F(a_n, b_n; n+2; x) + \frac{2nx}{(n+1)(n+2)} F(a_n, b_n; n+3; x).\end{aligned}$$

Remark that the involved hypergeometric functions are strictly positive, which implies that

$$\zeta_n(x) > \frac{n}{n+1} \left( (1-x) F(a_n, b_n; n+2; x) + \frac{2x}{n+2} F(a_n, b_n; n+3; x) \right).$$

To get the announced result, it is enough to check that

$$(1-x) F(a_n, b_n; n+2; x) + \frac{2x}{(n+2)} F(a_n, b_n; n+3; x) \geq 1, \quad \forall x \in (-\infty, 0),$$

which follows from Lemma 6.5:

$$\begin{aligned}(1-x) F(a_n, b_n; n+2; x) &+ \frac{2x}{(n+2)} F(a_n, b_n; n+3; x) \\ &\geq F(a_n, b_n; n+2; x) \left( 1 - \frac{nx}{n+2} \right) \geq F(a_n, b_n; n+2; x) \geq 1,\end{aligned}$$

for any  $x \leq 0$ . Thus,  $\zeta_n(x) > 0$  for any  $x \in (-\infty, 0)$  and this concludes the proof.

(3) Let us use the expression of  $\zeta_n$  given in (6.2) obtaining

$$\zeta_n(x) = \frac{A+2B}{A(n+1)} x F(a_n, b_n; n+1; x) + \frac{n(1-x)}{n+1} F(a_n, b_n; n+2; x)$$

$$-\frac{x}{n+1}F(a_n, b_n; n+2; x) + \frac{2nx}{(n+1)(n+2)}F(a_n, b_n; n+3; x).$$

Then

$$\zeta_n(x) > \frac{x}{n+1} \left( -F(a_n, b_n; n+2; x) + \frac{2n}{n+2}F(a_n, b_n; n+3; x) \right),$$

for any  $x \in [0, 1)$ . From Lemma 6.5 we deduce

$$-F(a_n, b_n; n+2; x) + \frac{2n}{(n+2)}F(a_n, b_n; n+3; x) \geq \frac{n-2}{n+2}F(a_n, b_n; n+2; x) \geq 0,$$

for any  $n \geq 2$  and  $x \in (0, 1)$ . This implies that  $\zeta_n(x) > 0$ , for  $x \in (0, 1)$  and  $n \geq 2$ . The case  $n = 1$  can be checked directly by the explicit expression stated in Remark 5.5.  $\square$

In the following result, we investigate more the case (6.23). We mention that according to Proposition 6.6–(3) there are no eigenvalues in  $(0, 1)$ . Thus, it remains to explore the region  $(-\infty, 0)$  and study whether one can find eigenvalues there. Our result reads as follows.

**Proposition 6.7.** *Let  $n \geq 2$  and  $A, B \in \mathbb{R}$  satisfying (6.23). Then, the following assertions hold true:*

- (1) *If  $n \leq \frac{B}{A} + \frac{1}{8}$ , there exists a unique  $x_n \in (-1, 0)$  such that  $\zeta_n(x_n) = 0$ .*
- (2) *If  $n \leq \frac{2B}{A}$ , there exists a unique  $x_n \in (-\infty, 0)$  such that  $\zeta_n(x_n) = 0$ , with*

$$(6.31) \quad \frac{1}{1 - \frac{A+2B}{A(n+1)}} < x_n < 0.$$

*In addition, the map  $x \in (-\infty, 0] \mapsto \zeta_n(x)$  is strictly increasing.*

- (3) *If  $n \geq \frac{B}{A} + 1$ , then  $\zeta_n$  has no solution in  $[-1, 0]$ .*
- (4) *If  $n \geq \frac{2B}{A} + 2$ , then  $\zeta_n$  has no solution in  $(-\infty, 0]$ .*

*Proof.* (1) Thanks to (5.20) we have that  $\zeta_n(0) = \frac{n}{n+1} > 0$ . So to apply the Intermediate Value Theorem and prove that  $\zeta_n$  admits a solution in  $[-1, 0]$  it suffices to guarantee that  $\zeta_n(-1) < 0$ . Now coming back to (5.20) and using that  $x \in (-1, 1) \mapsto F_n(x)$  is strictly decreasing we get

$$\begin{aligned} \zeta_n(-1) &= F_n(-1) \left( 2 - \frac{A+2B}{A(n+1)} \right) - \int_0^1 F_n(-\tau) \tau^n (1+2\tau) d\tau \\ &< F_n(-1) \left( 2 - \frac{A+2B}{A(n+1)} \right) - \int_0^1 \tau^n (1+2\tau) d\tau \\ &< F_n(-1) \left( 2 - \frac{A+2B}{A(n+1)} \right) - \frac{3n+4}{(n+1)(n+2)}, \quad \forall x \in [-1, 0). \end{aligned}$$

Consequently, to get  $\zeta_n(-1) < 0$  we impose the condition

$$(6.32) \quad 2 - \frac{A+2B}{A(n+1)} \leq \frac{3n+4}{(n+1)(n+2)F_n(-1)}.$$

Coming back to the integral representation, one gets

$$(6.33) \quad F_n(-1) \leq F_n(0)2^{-a_n} \leq 2,$$

due to the fact  $a_n \in (-1, 0)$ . In addition, it is easy to check that

$$\frac{3n+4}{n+2} \geq \frac{5}{2}, \quad \forall n \geq 2,$$

and the assumption (6.32) is satisfied if

$$2 - \frac{A+2B}{A(n+1)} \leq \frac{5}{4(n+1)}$$

holds, or equivalently, if

$$(6.34) \quad 2 \leq n \leq \frac{B}{A} + \frac{1}{8}.$$

In conclusion, under the assumption (6.34), the function  $\zeta_n$  admits a solution  $x_n \in (-1, 0)$ . Now, we localize this zero. Since  $F_n$  is strictly positive in  $[-1, 1]$ , then the second term in (5.20) is always strictly negative. Let us analyze the sign of the first term

$$F_n(x) \left[ 1 - x + \frac{A + 2B}{A(n + 1)}x \right],$$

which has a unique root

$$(6.35) \quad x_c = \frac{1}{1 - \frac{A+2B}{A(n+1)}}.$$

This root belongs to  $(-\infty, 0)$  if and only if  $n < \frac{2B}{A}$ , which follows automatically from (6.34). Moreover the mapping  $x \mapsto 1 - x + \frac{A+2B}{A(n+1)}x$  will be strictly increasing. Hence, if  $x_c \leq -1$ , then  $x_n > x_c$ . So let us assume that  $x_c \in (-1, 0)$ , then

$$F_n(x) \left[ 1 - x + \frac{A + 2B}{A(n + 1)}x \right] < 0, \quad \forall x \in [-1, x_c],$$

which implies that

$$\zeta_n(x) < 0, \quad \forall x \in [-1, x_c].$$

Therefore, the solution  $x_n$  must belong to  $(x_c, 0)$ , and equivalently

$$\frac{1}{1 - \frac{A+2B}{A(n+1)}} < x_n < 0.$$

The uniqueness of this solutions comes directly from the second assertion.

(2) As in the previous argument we have that  $\zeta_n(0) = \frac{n}{n+1} > 0$  and the idea is to apply also the Intermediate Value Theorem. We intend to find the asymptotic behavior of  $\zeta_n$  for  $x$  going to  $-\infty$ . We first find an asymptotic behavior of  $F_n$ . For this purpose we use the identity (C.3), which implies that

$$F_n(x) = (1 - x)^{-a_n} F \left( a_n, a_n + 1, n + 1, \frac{x}{x - 1} \right), \quad \forall x \leq 0.$$

Setting

$$(6.36) \quad \varphi_n(y) \triangleq F(a_n, a_n + 1, n + 1, y), \quad \forall y \in [0, 1],$$

we obtain

$$\varphi_n'(y) = \frac{a_n(1 + a_n)}{n + 1} F(a_n + 1, a_n + 2, n + 2, y), \quad \forall y \in [0, 1]$$

By a monotonicity argument we deduce that

$$|\varphi_n'(y)| \leq C |F(a_n + 1, a_n + 2, n + 2, y)| \leq C |F(a_n + 1, a_n + 2, n + 2, 1)| \leq C_n, \quad \forall y \in [0, 1],$$

with  $C_n$  a constant depending on  $n$ . However, the dependence with respect to  $n$  does not matter because we are interested in the asymptotics for large negative  $x$  but for a fixed  $n$ . Then, let us drop  $n$  from the subscript of the constant  $C_n$ . Applying the Mean Value Theorem we get

$$|\varphi_n(y) - \varphi_n(1)| \leq C(1 - y), \quad \forall y \in [0, 1].$$

Combining this estimate with (6.36) and  $a_n \in (-1, 0)$ , we obtain

$$|F_n(x) - (1 - x)^{-a_n} \varphi_n(1)| \leq C(1 - x)^{-a_n - 1} \leq C, \quad \forall x \leq 0,$$

which implies in turn that

$$(6.37) \quad |F_n(x) - (1-x)^{-a_n} \varphi_n(1)| \leq C, \quad \forall x \leq -1.$$

Consequently, we deduce that

$$\begin{aligned} \int_0^1 F_n(\tau x) \tau^n (-1 + 2x\tau) d\tau &\sim 2x(-x)^{-a_n} \varphi_n(1) \int_0^1 \tau^{-a_n+n+1} d\tau \\ &\sim \frac{2}{n+2-a_n} x(-x)^{-a_n} \varphi_n(1), \quad \forall -x \gg 1. \end{aligned}$$

Coming back to (5.20) and using once again (6.37) we get the asymptotic behavior

$$\begin{aligned} \zeta_n(x) &\sim \varphi_n(1)(-x)^{-a_n} \left( 1 - x + \frac{A+2B}{A(n+1)}x + \frac{2}{n+2-a_n}x \right) \\ &\sim \varphi_n(1) \left( \frac{A+2B}{A(n+1)} - 1 + \frac{2}{n+2-a_n} \right) (-x)^{-a_n} x, \quad \forall -x \gg 1. \end{aligned}$$

The condition

$$(6.38) \quad n \leq \frac{2B}{A},$$

implies that

$$\frac{A+2B}{A(n+1)} - 1 + \frac{2}{n+2-a_n} > 0.$$

Since  $\varphi_n(1) > 0$ , we deduce that

$$\lim_{x \rightarrow -\infty} \zeta_n(x) = -\infty.$$

Therefore we deduce from the Intermediate Value Theorem that under the assumption (6.38), the function  $\zeta_n$  admits a solution  $x_n \in (-\infty, 0)$ . Moreover, by the previous proof we get (6.31).

It remains to prove the uniqueness of this solution. For this goal we check that the mapping  $x \in (-\infty, 0) \mapsto \zeta_n(x)$  is strictly increasing when  $n \in [1, \frac{2B}{A}]$ . Differentiating  $\zeta_n$  with respect to  $x$  yields

$$\begin{aligned} \zeta'_n(x) &= F'_n(x) \left[ 1 - x + \frac{A+2B}{A(n+1)}x \right] + F_n(x) \left[ \frac{A+2B}{A(n+1)} - 1 \right] \\ &\quad + \int_0^1 F'_n(\tau x) \tau^{n+1} (-1 + 2x\tau) d\tau + 2 \int_0^1 F_n(\tau x) \tau^{n+1} d\tau. \end{aligned}$$

From Lemma 6.2-(1) we infer that  $F'_n(x) < 0$ , for  $x \in (-\infty, 0)$ , and therefore we get

$$\zeta'_n(x) > 0, \quad \forall x \in (-\infty, x_c).$$

Let  $x \in (x_c, 0)$ , then by a monotonicity argument we get

$$0 \leq 1 - x + \frac{A+2B}{A(n+1)}x \leq 1,$$

and thus

$$F'_n(x) \left[ 1 - x + \frac{A+2B}{A(n+1)}x \right] + 2 \int_0^1 F_n(\tau x) \tau^{n+1} d\tau \geq F'_n(0) + \frac{2}{n+2} \geq -\frac{2}{(n+1)(n+2)},$$

by using that  $F'_n(0) = -\frac{2}{n+1}$  and that  $F'_n(x)$  is decreasing and negative in  $(-\infty, 1)$ . From the assumption (6.38) and the positivity of  $F_n$  we get

$$F_n(x) \left( \frac{A+2B}{A(n+1)} - 1 \right) \geq 0, \quad \forall x \in (-\infty, 0].$$

Therefore, putting together the preceding estimates we deduce that

$$\begin{aligned}\zeta'_n(x) &> -\frac{2}{(n+1)(n+2)} + \int_0^1 F'_n(\tau x) \tau^{n+1} (-1 + 2x\tau) d\tau \\ &> -\frac{2}{(n+1)(n+2)} - \int_0^1 F'_n(\tau x) \tau^{n+1} d\tau, \quad \forall x \in (-\infty, 0).\end{aligned}$$

At this stage it suffices to make appeal to (6.10) in order to obtain

$$-\int_0^1 F'_n(\tau x) \tau^{n+1} d\tau \geq \frac{2}{(n+1)(n+2)}, \quad \forall x \in (-\infty, 0],$$

from which it follows

$$\zeta'_n(x) > 0, \quad \forall x \in (-\infty, 0],$$

which implies that  $\zeta_n$  is strictly increasing in  $(-\infty, 0]$ , and thus  $x_n$  is the only solution in this interval.

(3) Using the definition of  $\zeta_n$  in (5.20) and the monotonicity of  $F_n$ , one has

$$(6.39) \quad \zeta_n(x) > F_n(x) \left[ 1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+1} \right].$$

Now it is easy to check that

$$1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+1} \geq \min \left\{ \frac{n}{n+1}, \frac{2n-1}{n+1} - \frac{A+2B}{A(n+1)} \right\},$$

for any  $x \in [-1, 0]$ . This claim can be derived from the fact that the left-hand-side term is polynomial in  $x$  with degree one. Consequently, if we assume

$$n \geq 1 + \frac{B}{A},$$

we get  $\frac{2n-1}{n+1} - \frac{A+2B}{A(n+1)} \geq 0$ , and therefore (6.39) implies

$$\zeta_n(x) > 0, \quad \forall x \in [-1, 0].$$

Then,  $\zeta_n$  has no solution in  $[-1, 0]$ .

(4) Using the expression of  $\zeta_n$  in (5.20) and the monotonicity of  $F_n$ , one has

$$(6.40) \quad \begin{aligned}\zeta_n(x) &> F_n(x) \left[ 1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+1} \right] \\ &\geq F_n(x) \left[ \frac{n}{n+1} - x \left( \frac{n-1}{n+1} - \frac{A+2B}{A(n+1)} \right) \right], \quad \forall x \in (-\infty, 0).\end{aligned}$$

The assumption

$$n \geq 2 + \frac{2B}{A},$$

yields  $\frac{n-1}{n+1} - \frac{A+2B}{A(n+1)} \geq 0$ , and therefore (6.40) implies

$$\zeta_n(x) > 0, \quad \forall x \in (-\infty, 0].$$

Thus,  $\zeta_n$  has no solution in  $(-\infty, 0]$ . □

In the next task we discuss the localization of the zeroes of  $\zeta_n$  and, in particular, we improve the lower bound (6.31). Notice that  $B > 0$  in order to get solutions of  $\zeta_n$  in  $(-\infty, 0]$  in the case  $A > 0$  and  $n \geq 2$ , by using Proposition 6.7-(4). Our result reads as follows.

**Proposition 6.8.** *Let  $A, B > 0$  and  $n \geq 2$ . If  $x_n \in (-\infty, 0)$  is any solution of  $\zeta_n$ , then the following properties are satisfied:*



- (1) We have  $\mathcal{P}_n(x_n) < 0$ , with  $\mathcal{P}_n(x) = \frac{n}{n+1} + x \left[ -\frac{n}{n+2} + \frac{A+2B}{A(n+1)} \right]$ .
- (2) If  $x_n \in (-1, 0)$ , then  $x_* \triangleq -\frac{2n+1}{2(n+1)} \frac{1}{\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2}} < x_n$ .
- (3) We always have  $x_n < -\frac{A}{2B}$ .

*Proof.* (1) Since  $F_n(\tau x) < F_n(x)$ , for any  $\tau \in [0, 1)$  and  $x \in (-\infty, 0)$ , then we deduce from the expression (5.20) that

$$(6.41) \quad \zeta_n(x) > F_n(x) \left[ 1 - x + \frac{A+2B}{A(n+1)}x - \frac{1}{n+1} + \frac{2x}{n+2} \right] = F_n(x) \mathcal{P}_n(x).$$

As  $F_n$  is strictly positive in  $(-\infty, 1)$ , then  $\mathcal{P}_n(x_n) < 0$ , for any root of  $\zeta_n$ .

(2) Recall from Proposition 6.7-(3) that if  $\zeta_n$  admits a solution in  $(-1, 0)$  with  $n \geq 2$  then necessary  $2 \leq n \leq 1 + \frac{B}{A}$ . This implies that  $n \leq \frac{2B}{A}$  and hence the mapping  $x \mapsto 1 - x + \frac{A+2B}{A(n+1)}x$  is increasing. Combined with the definition of (6.35) and (6.31) we deduce that

$$1 - x + \frac{A+2B}{A(n+1)}x \geq 0, \quad \forall x \in (x_c, 0)$$

and  $x_n \in (x_c, 0)$ . Using the monotonicity of  $F_n$  combined with the bound (6.33) we find from (5.20)

$$\zeta_n(x) < 2 \left[ 1 - x + \frac{A+2B}{A(n+1)}x \right] - \frac{1}{n+1} + \frac{2x}{n+2}, \quad \forall x \in (x_c, 0).$$

Evaluating at any root  $x_n$  we obtain

$$-\frac{2n+1}{n+1} < 2x_n \left[ \frac{A+2B}{A(n+1)} - \frac{n+1}{n+2} \right].$$

Keeping in mind that  $n \leq \frac{2B}{A}$ , we get  $\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2} > 0$ , and therefore we find the announced lower bound for  $x_n$ .

(3) In a similar way to the upper bound for  $x_n$ , we turn to (6.41), and evaluate this inequality at  $-\frac{A}{2B}$ . Then we find

$$\zeta_n \left( -\frac{A}{2B} \right) > F_n \left( -\frac{A}{2B} \right) \mathcal{P}_n \left( -\frac{A}{2B} \right) > \mathcal{P}_n \left( -\frac{A}{2B} \right).$$

Explicit computations yield

$$\mathcal{P}_n \left( -\frac{A}{2B} \right) = \frac{n-1}{n+1} + \frac{A}{2B} \left[ \frac{n}{n+2} - \frac{1}{n+1} \right]$$

Since  $\frac{A}{2B} > 0$  and  $n \geq 2$ , then we infer that  $\mathcal{P}_n \left( -\frac{A}{2B} \right) > 0$  and  $\zeta_n \left( -\frac{A}{2B} \right) > 0$ . Now we recall from Proposition 6.7-(2) that  $x \in (-\infty, 0) \mapsto \zeta_n(x)$  is strictly increasing. Thus combined this property with the preceding one we deduce that

$$x_n < -\frac{A}{2B},$$

which achieves the proof.  $\square$

Notice that from Proposition 6.7-(4) when  $B \leq 0$ , the function  $\zeta_n$  has no solution in  $(-\infty, 0]$  for any  $n \geq 2$ . Moreover, in the case that  $0 < B \leq \frac{A}{4}$ , Proposition 6.7-(4) and Proposition 6.8-(1) give us again that  $\zeta_n$  has no solution in  $(-\infty, 0]$  for any  $n \geq 2$ . Combining these facts with Proposition 6.6-(3), we immediately get the following result.

**Corollary 6.9.** *Let  $A > 0$  and  $B$  satisfying*

$$-\frac{A}{2} \leq B \leq \frac{A}{4}.$$

*Then, the function  $\zeta_n$  has no solution in  $(-\infty, 1]$  for any  $n \geq 2$ . However, the function  $\zeta_1$  admits the solution  $x_1 = -\frac{A}{2B}$ . Notice that this latter solution belongs to  $(-\infty, 1)$  if and only if  $B \notin [-\frac{A}{2}, 0]$ .*

In the next result, we study the case when  $x_1 \in (0, \frac{1+\epsilon}{2}]$  for some  $0 < \epsilon < 1$ , showing that there is no intersection with other eigenvalues.

**Proposition 6.10.** *Let  $A > 0$ . There exists  $\epsilon \in (0, 1)$  such that if  $B \leq -\frac{A}{1+\epsilon}$ , then  $\zeta_n(x_1) \neq 0$  for any  $n \geq 2$  and  $x_1 = -\frac{A}{2B}$ , with  $\epsilon \approx 0,0581$ .*

*Proof.* From (5.20), one has

$$\zeta_n(x_1) = F_n(x_1) \frac{n}{n+1} (1-x_1) + \int_0^1 F_n(x_1 \tau) \tau^n [-1 + 2x_1 \tau] d\tau.$$

By the integral representation of  $F_n$  given in (C.2), we obtain that

$$F_n(x) > \frac{\Gamma(n+1)}{\Gamma(n-a_n)\Gamma(a_n+1)} \int_0^1 \mathfrak{t}^{n-a_n-1} (1-\mathfrak{t})^{a_n} d\mathfrak{t} (1-x)^{-a_n} = (1-x)^{-a_n},$$

for any  $x \in (0, 1)$ , using the Beta function (6.16). By the monotonicity of  $F_n(x)$  with respect to  $x$  and the above estimate, we find that

$$(1-x_1)^{-a_n} < F_n(x_1) < 1,$$

for any  $n \geq 2$  and  $x_1 \in (0, 1)$ , which agrees with the hypothesis on  $A$  and  $B$ . Hence, we have that

$$\zeta_n(x_1) > \frac{n}{n+1} (1-x_1)^{1-a_n} + 2x_1 \frac{(1-x_1)^{-a_n}}{n+2} - \frac{1}{n+1}.$$

The above expression is increasing with respect to  $n$ , which implies that

$$\zeta_n(x_1) > \frac{2}{3} (1-x_1)^{1-a_2} + \frac{x_1}{2} (1-x_1)^{-a_2} - \frac{1}{3}.$$

Since  $a_2 = 1 - \sqrt{3}$ , we get

$$\zeta_n(x_1) > \frac{2}{3} (1-x_1)^{\sqrt{3}} + \frac{x_1}{2} (1-x_1)^{\sqrt{3}-1} - \frac{1}{3} \triangleq \mathcal{P}(x_1).$$

The function  $\mathcal{P}$  decreases in  $(0, 1)$  and admits a unique root  $\bar{x}$  whose approximate value is given by  $\bar{x} = 0,52907$ . Hence,  $\mathcal{P}(x_1) \geq 0$  for  $x_1 \in (0, \bar{x}]$ , and consequently we get

$$\zeta_n(x_1) > 0,$$

for  $B \leq -\frac{A}{2\bar{x}}$ , achieving the announced result.  $\square$

We finish this section by the following result concerning the monotonicity of the eigenvalues.

**Proposition 6.11.** *Let  $A > 0$  and  $2B > A$ . Then, the following assertions hold true:*

- (1) *Let  $x \in (-\infty, 0)$ , then  $n \in [1, \frac{2B}{A}] \mapsto \zeta_n(x)$  is strictly increasing. In addition, we have*

$$\left\{ x \in (-\infty, 0], \zeta_n(x) = 0 \right\} \cap \left\{ x \in (-\infty, 0], \zeta_m(x) = 0 \right\} = \emptyset,$$

*for any  $n \neq m \in [1, \frac{2B}{A}]$ , and each set contains at most one element.*

- (2) *The sequence  $n \in [1, \frac{B}{A} + \frac{1}{8}] \mapsto x_n$  is strictly decreasing, where the  $\{x_n\}$  are constructed in Proposition 6.7.*

(3) If  $m \in [1, \frac{2B}{A} - 2]$ , then

$$\zeta_n(x_m) \neq 0, \quad \forall n \in \mathbb{N}^* \setminus \{m\}.$$

*Proof.* (1) We shall prove that the mapping  $n \in [1, \frac{2B}{A}] \mapsto \zeta_n(x)$  is strictly increasing for fixed  $x \in (-\infty, 0]$  and  $2B \geq A$ . Differentiating (5.20) with respect to  $n$  we get

$$\begin{aligned} \partial_n \zeta_n(x) &= \partial_n F_n(x) \left[ 1 - x + \frac{A + 2B}{A(n+1)}x \right] + \int_0^1 \partial_n F_n(\tau x) \tau^n [-1 + 2x\tau] d\tau \\ &\quad - F_n(x) \frac{A + 2B}{A(n+1)^2} x + \int_0^1 F_n(\tau x) \tau^n \ln \tau [-1 + 2x\tau] d\tau. \end{aligned}$$

Using Lemma 6.3-(1) and the positivity of  $F_n$ , we deduce that

$$\int_0^1 \partial_n F_n(\tau x) \tau^n [-1 + 2x\tau] d\tau > 0, \quad \int_0^1 F_n(\tau x) \tau^n \ln \tau [-1 + 2x\tau] d\tau > 0, \quad \forall x \in (-\infty, 0).$$

Due to the assumption  $n \in [1, \frac{2B}{A}]$ , we have that  $x_c < 0$ , where  $x_c$  is defined in (6.35). If  $x \in (-\infty, x_c]$ , we find that

$$1 - x + \frac{A + 2B}{A(n+1)}x \leq 0,$$

which implies

$$\partial_n \zeta_n(x) > 0, \quad \forall x \in (-1, x_c), \forall n \geq 1.$$

We obtain

$$(6.42) \quad 0 < 1 - x + \frac{A + 2B}{A(n+1)}x < 1,$$

for  $x \in (x_c, 0)$ , which yields in view of Lemma 6.3-(2)

$$\begin{aligned} \partial_n \zeta_n(x) &> \partial_n F_n(x) \left[ 1 - x + \frac{A + 2B}{A(n+1)}x \right] - F_n(x) \frac{A + 2B}{A(n+1)^2} x \\ &> \partial_n F_n(x) - F_n(x) \frac{A + 2B}{A(n+1)^2} x > \frac{-2xF_n(x)}{(n+1)^2} \left[ -1 + \frac{A + 2B}{2A} \right]. \end{aligned}$$

Taking into account  $2B \geq A$ , one gets  $\partial_n \zeta_n(x) > 0, \forall x \in (-\infty, 0]$ . It remains to discuss the case  $x_c \leq -1$ . Remark that the estimate (6.42) is satisfied for any  $x \in (-\infty, 0)$ , and then the foregoing inequality holds, and one gets finally

$$\partial_n \zeta_n(x) > 0, \quad \forall x \in [-1, 0], \forall n \in \left[ 1, \frac{2B}{A} \right].$$

Consequently, we deduce that the mapping  $n \in [1, \frac{2B}{A}] \mapsto \zeta_n(x)$  is strictly increasing for any  $x \in (-\infty, 0)$ . This implies in particular that the functions  $\zeta_n$  and  $\zeta_m$  have no common zero in  $(-\infty, 0)$  for  $n \neq m \in [1, \frac{2B}{A}]$ .

(2) This follows by combining that  $x \in (-\infty, 0) \mapsto \zeta_n(x)$  and  $n \in [1, \frac{2B}{A}] \mapsto \zeta_n(x)$  are strictly increasing, proved in Proposition 6.7-(1) and Proposition 6.11-(1).

(3) By the last assertions, this is clear for  $n \leq \frac{2B}{A}$  and it is also true for  $n \geq \frac{2B}{A} + 2$ , since  $\zeta_n$  has not roots in  $(-\infty, 1)$ , by Proposition 6.6 and Proposition 6.7. Then, let us study the case  $n \in (\frac{2B}{A}, \frac{2B}{A} + 2)$ . First, using (6.31), we get that  $x_m$ , which is a solution of  $\zeta_m = 0$  with  $m \leq \frac{2B}{A} - 2$ , verifies

$$(6.43) \quad x_m \geq -\frac{2B - A}{2A}.$$

Now, the strategy is to show that  $\mathcal{P}_n(x_m) > 0$  for  $n \in (\frac{2B}{A}, \frac{2B}{A} + 2)$ , and then Proposition 6.8 will imply that  $\zeta_n(x_m) \neq 0$ . By definition, we have that

$$\mathcal{P}_n(x_m) = \frac{n}{n+1} + x_m \left[ -\frac{n}{n+2} + \frac{A+2B}{A(n+1)} \right].$$

If  $-\frac{n}{n+2} + \frac{A+2B}{A(n+1)} \leq 0$ , then  $\mathcal{P}_n(x_m) > 0$ . Otherwise, we use (6.43) getting

$$\begin{aligned} \mathcal{P}_n(x_m) &> \frac{n}{n+1} - \frac{2B-A}{2A} \left[ -\frac{n}{n+2} + \frac{A+2B}{A(n+1)} \right] \\ &= \frac{1}{2(n+1)(n+2)} \left[ n^2 \left( 1 + \frac{2B}{A} \right) + n \left( 4 + \frac{2B}{A} - 4 \frac{B^2}{A^2} \right) + 2 \left( 1 - 4 \frac{B^2}{A^2} \right) \right]. \end{aligned}$$

Straightforward computations yield that the above parabola is increasing in  $n \in (\frac{2B}{A}, \frac{2B}{A} + 2)$ . Evaluating at  $n = \frac{2B}{A}$  in the parabola, we find

$$\mathcal{P}_n(x_m) > \frac{8\frac{B}{A} + 2}{2(n+1)(n+2)} > 0.$$

□

**6.4. Asymptotic expansion of the eigenvalues.** When solving the boundary equation in Proposition 3.3, one requires that the angular velocity is located outside the singular set (3.5). Consequently, in order to apply the bifurcation argument for the density equation we should check that the eigenvalues  $\{x_n\}$  constructed in Proposition 6.6 do not intersect the singular set. This problem sounds to be very technical and in the case  $A+B < 0$ , where we know that the dispersion set is infinite, we reduce the problem to studying the asymptotic behavior of each sequence. Let us start with a preliminary result.

**Lemma 6.12.** *Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of real numbers in  $(-1, 1)$  such that  $x_n = 1 - \frac{\kappa}{n} + o(\frac{1}{n})$ , for some strictly positive number  $\kappa$ . Then the following asymptotics*

$$F(a_n + 1, b_n; n + 2; x_n) = n(\eta + o(1)),$$

holds, with  $\eta = \kappa \int_0^{+\infty} \frac{\tau e^{-\kappa\tau}}{1+\tau} d\tau$ .

*Proof.* The integral representation of hypergeometric functions (C.2) allows us to write

$$F(a_n + 1, b_n; n + 2; x_n) = \frac{\Gamma(n+2)}{\Gamma(n-a_n)\Gamma(2+a_n)} \ell_n = \frac{n(n+1)\Gamma(n)}{\Gamma(n-a_n)\Gamma(2+a_n)} \ell_n,$$

where

$$\ell_n \triangleq \int_0^1 \tau^{n-a_n-1} (1-\tau)^{1+a_n} (1-\tau x_n)^{-1-a_n} d\tau.$$

Set  $\varepsilon_n = \frac{1-x_n}{x_n}$ , making the change of variables  $\tau = 1 - \varepsilon_n \tau'$ , and keeping the same notation  $\tau$  to the new variable, we obtain

$$\ell_n = x_n^{-a_n-1} \varepsilon_n \int_0^{\frac{1}{\varepsilon_n}} (1 - \varepsilon_n \tau)^{n-a_n-1} \tau^{1+a_n} (1+\tau)^{-a_n-1} d\tau.$$

From the first order expansion of  $x_n$ , one has the pointwise convergence

$$\lim_{n \rightarrow +\infty} (1 - \varepsilon_n \tau)^{n-a_n-1} = e^{-\kappa\tau},$$

for any  $\tau > 0$ . Since the sequence  $n \mapsto (1 - \frac{\kappa}{n})^n$  is increasing, the Lebesgue Theorem leads to

$$\lim_{n \rightarrow +\infty} \int_0^{\frac{1}{\varepsilon_n}} (1 - \varepsilon_n \tau)^{n-a_n-1} \tau^{1+a_n} (1+\tau)^{-a_n-1} d\tau = \int_0^{+\infty} e^{-\kappa\tau} \frac{\tau}{1+\tau} d\tau.$$

Therefore, we obtain the equivalence  $\ell_n \sim \frac{\eta}{n}$ . Combining the previous estimates with (6.7) we find the announced estimate.  $\square$

The next objective is to give the asymptotic expansion of the eigenvalues.

**Proposition 6.13.** *Let  $A$  and  $B$  be such that (6.21) holds. Then, the sequence  $\{x_n, n \geq n_0\}$ , constructed in Proposition 6.6, admits the following asymptotic behavior*

$$x_n = 1 - \frac{\kappa}{n} + \frac{c_\kappa}{n^2} + o\left(\frac{1}{n^2}\right),$$

where

$$\kappa = -2\frac{A+B}{A} \quad \text{and} \quad c_\kappa = \kappa^2 - 2 + 2 \int_0^{+\infty} \frac{e^{-\kappa\tau}}{(1+\tau)^2} d\tau.$$

*Proof.* Let us consider the ansatz

$$x_n = 1 - \frac{\kappa}{n+1} - u_n, \quad u_n = o\left(\frac{1}{n}\right).$$

From (6.1), we know that  $x_n$  satisfies the equation

$$\begin{aligned} \zeta_n(x_n) &= I_n^1(x_n)F(a_n, b_n; n+1; x_n) + I_n^2(x_n)F(a_n+1, b_n; n+2; x_n) \\ &\quad + I_n^3(x_n)F(a_n, b_n; n+3; x_n) = 0, \end{aligned}$$

with

$$\begin{aligned} I_n^1(x_n) &= \frac{n-a_n}{n+1-a_n} - \left(1 - \frac{\kappa}{n+1} - u_n\right) \left(\frac{1+\kappa}{n+1} + \frac{n-1-a_n}{n+1-a_n}\right), \\ I_n^2 &= -\frac{a_n(2x_n-1)}{(n+1)(n+1-a_n)} \quad \text{and} \quad I_n^3 = -\frac{2x_n}{(n+1)(n+2)}. \end{aligned}$$

It is a simple matter to have

$$\begin{aligned} I_n^1 &= -\frac{\kappa(1-\kappa)}{(n+1)^2} + u_n + o(u_n) + O\left(\frac{1}{n^3}\right), \\ I_n^2 &= \frac{2}{(n+1)^3} + o\left(\frac{1}{n^3}\right) \quad \text{and} \quad I_n^3 = \frac{-2}{(n+1)^2} + O\left(\frac{1}{n^3}\right), \end{aligned}$$

where we have used  $a_n \sim -\frac{2}{n}$ . By virtue of the above estimates, Lemma 6.4 and Lemma 6.12, the expansion of  $u_n$  reads as

$$u_n = \frac{\kappa(1-\kappa) + 2 - 2\eta}{(n+1)^2} + o\left(\frac{1}{n^2}\right) = \frac{\kappa - \kappa^2 + 2 - 2\eta}{n^2} + o\left(\frac{1}{n^2}\right).$$

Using  $\frac{1}{n+1} = \frac{1}{n} - \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$ , we find

$$x_n = 1 - \frac{\kappa}{n} + \frac{\kappa^2 - 2 + 2\eta}{n^2} + o\left(\frac{1}{n^2}\right).$$

The final expression holds as a consequence of the following integration by parts

$$\eta = 1 - \kappa \int_0^{+\infty} \frac{e^{-\kappa\tau}}{1+\tau} d\tau = \int_0^{+\infty} \frac{e^{-\kappa\tau}}{(1+\tau)^2} d\tau.$$

$\square$

**6.5. Separation of the singular and dispersion sets.** In Section 2.4 we have established some conditions in order to solve the boundary equation. If we want to apply Proposition 3.3, we must verify that  $\Omega$  does not lie in the singular set  $\mathcal{S}_{\text{sing}}$  given in (4.10). Moreover, from the last analysis, we have checked that the dispersion set  $\mathcal{S}$ , defined in (1.12), contains different sets depending on the assumptions on  $A$  and  $B$ . We will prove the following results.

**Proposition 6.14.** *Let  $A$  and  $B$  such that (6.21) holds. Denote by*

$$\Omega_n \triangleq \frac{B}{2} + \frac{A}{4x_n},$$

where the sequence  $(x_n)_{n \geq n_0}$  has been defined in Proposition 6.6. If  $n_0$  is large enough depending on  $A$  and  $B$ , then

$$\Omega_n \neq \widehat{\Omega}_{np}, \quad \forall n \geq n_0, \forall p \in \mathbb{N}^*,$$

where  $\widehat{\Omega}_{np}$  belongs to the set  $\mathcal{S}_{\text{sing}}$  introduced in (4.10). Moreover, there exists  $\kappa_c > 0$  such that for any  $\kappa > \kappa_c$  we find  $n_0 \in \mathbb{N}$  such that

$$\Omega_n \neq \widehat{\Omega}_m, \quad \forall n, m \geq n_0.$$

The number  $\kappa_c \in (0, 2)$  is the unique solution of the equation

$$\kappa_c - 2 \int_0^{+\infty} \frac{e^{-\kappa_c \tau}}{(1 + \tau)^2} d\tau = 0.$$

*Proof.* It is a simple matter to have

$$\frac{4}{A} \left( \widehat{\Omega}_n - \frac{B}{2} \right) = 1 + \frac{\kappa}{n} + \frac{2}{n^2} + O\left(\frac{1}{n^3}\right).$$

Setting  $\widehat{x}_n = \frac{1}{\frac{4}{A}(\widehat{\Omega}_n - \frac{B}{2})}$ , we obtain

$$(6.44) \quad \widehat{x}_n = 1 - \frac{\kappa}{n} + \frac{\kappa^2 - 2}{n^2} + O\left(\frac{1}{n^3}\right).$$

Thus, condition  $\Omega_n \neq \widehat{\Omega}_m$  is equivalent to  $x_n \neq \widehat{x}_m$ . According to Proposition 6.13, we have

$$x_n = 1 - \frac{\kappa}{n} + \frac{c_\kappa}{n^2} + o\left(\frac{1}{n^2}\right),$$

for  $n \geq n_0$ , where

$$c_\kappa = \kappa^2 - 2 + 2 \int_0^{+\infty} \frac{e^{-\kappa \tau}}{(1 + \tau)^2} d\tau.$$

This implies that  $x_n \neq \widehat{x}_n$  for large  $n$ . Moreover,

$$\widehat{x}_{np} = 1 - \frac{\kappa}{np} + \frac{\kappa^2 - 2}{n^2 p^2} + O\left(\frac{1}{n^3}\right) = 1 - \frac{\kappa}{np} + O\left(\frac{1}{n^2}\right),$$

for any  $p \in \mathbb{N}^*$  and  $O\left(\frac{1}{n^2}\right)$  being uniform on  $p$ . Therefore, we get that  $\widehat{x}_{np} > x_n$ , for any  $k \geq 2$ , with  $n \geq n_0 k$  and  $n_0$  large enough. Consequently, we deduce that  $x_n \neq \widehat{x}_{np}$ , for any  $n \geq n_0$ , and  $p \in \mathbb{N}^*$ . To establish the second assertion, we will use an asymptotic expansion for  $\widehat{x}_{n+1}$ . Thanks to (6.44) we can write

$$\widehat{x}_{n+1} = 1 - \frac{\kappa}{n} + \frac{\kappa^2 + \kappa - 2}{n^2} + O\left(\frac{1}{n^3}\right).$$

Then, since  $(\widehat{x}_n)_{n \geq n_0}$  is strictly increasing, to prove  $\widehat{x}_n \neq x_n$  it suffices just to check that  $\widehat{x}_{n+1} > x_n$ , which leads to

$$g(\kappa) \triangleq \kappa - 2 \int_0^{+\infty} \frac{e^{-\kappa \tau}}{(1 + \tau)^2} d\tau > 0.$$

Since  $g$  is strictly increasing on  $[0, +\infty)$  and satisfies  $g(0) = -2$  and  $g(2) > 0$ , there exists only one solution  $\kappa_c \in (0, 2)$  for the equation  $g(\kappa) = 0$ . This concludes the proof.  $\square$

The next task is to discuss the separation problem when the dispersion set is finite.

**Proposition 6.15.** *Let  $A > 0, B \in \mathbb{R}$  and  $\widehat{\mathcal{S}}_{sing}$  being the set defined in (5.19). Then the following assertions hold true:*

- (1) *If  $B \notin [-\frac{A}{2}, -\frac{A}{4}]$ , then  $x_1 = -\frac{A}{2B} \notin \widehat{\mathcal{S}}_{sing}$ .*
- (2) *If  $B > A$ , then the sequence  $m \in [2, \frac{2B}{A}] \mapsto x_m$  defined in Proposition 6.7 satisfies*

$$x_m \neq \widehat{x}_{nm}, \quad \forall n \geq 1.$$

*Proof.* Recall from the definition of the set  $\widehat{\mathcal{S}}_{sing}$  given in (5.19) that

$$\frac{1}{\widehat{x}_n} = 1 - \frac{2(n+1)}{n(n+2)} - \frac{2B}{An},$$

where we have used (4.10) and (4.12). Notice that when  $A$  and  $B$  are positive then  $n \mapsto \frac{1}{\widehat{x}_n}$  is strictly increasing.

(1) Let us prove that

$$(6.45) \quad -\frac{2B}{A} \neq 1 - \frac{2(n+1)}{n(n+2)} - \frac{2B}{An}, \quad \forall n \geq 1.$$

Note that for  $n = 1$  this constraint is always satisfied since we get

$$-\frac{1}{3} - \frac{2B}{A} \neq -\frac{2B}{A}.$$

Thus (6.45) is equivalent to

$$\frac{n^2 - 2}{n^2 + n - 2} \neq -\frac{2B}{A}, \quad \forall n \geq 2.$$

One can easily check that left part is strictly increasing on  $[2, +\infty)$  and so

$$\frac{1}{2} \leq \frac{n^2 - 2}{n^2 + n - 2} \leq 1, \quad \forall n \geq 2.$$

Consequently, if  $-\frac{2B}{A} \notin [\frac{1}{2}, 1]$  then the condition (6.45) is satisfied and this ensures the first point.

(2) From the expression of  $\mathcal{P}_n$  given in Proposition 6.8, we may write

$$\mathcal{P}_n(x_m) = \frac{n}{n+1} \left( 1 - \frac{x_m}{\widehat{x}_n} \right),$$

and  $\mathcal{P}_m(x_m) < 0$ , which implies  $\frac{1}{\widehat{x}_m} < \frac{1}{x_m}$ . By the monotonicity of  $\frac{1}{\widehat{x}_n}$ , it is enough to prove

$$(6.46) \quad \frac{1}{x_m} < \frac{1}{\widehat{x}_{2m}},$$

in order to conclude. According to (6.31), we have  $\frac{1}{x_m} < 1 - \frac{A+2B}{A(m+1)}$ . To obtain (6.46), it is enough to establish

$$1 - \frac{A+2B}{A(m+1)} \leq \frac{1}{\widehat{x}_{2m}} = 1 - \frac{2m+1}{2m(m+1)} - \frac{B}{Am},$$

which is equivalent to

$$\frac{A+2B}{A(m+1)} \geq \frac{2m+1}{2m(m+1)} + \frac{B}{Am}.$$

This latter one agrees with

$$\frac{B}{A} \geq \frac{1}{2(m-1)},$$

which holds true since  $\frac{B}{A} \geq 1$ . □

**6.6. Transversal property.** This section is devoted to the transversality assumption concerning the fourth hypothesis of the Crandall–Rabinowitz Theorem [A.2](#). We shall reformulate an equivalent tractable statement, where the problem reduces to check the non-vanishing of a suitable integral. However, it is slightly hard to check this property for all the eigenvalues. We give positive results for higher frequencies using the asymptotics, which have been developed in the preceding sections for some special regimes on  $A$  and  $B$ . The first result in this direction is summarized as follows.

**Proposition 6.16.** *Let  $A > 0$  and  $B \in \mathbb{R}$ ,  $x \in (-\infty, 1) \setminus \{\widehat{\mathcal{S}}_{\text{sing}} \cup \{0, x_0\}\}$  and  $n \in \mathcal{A}_x$  where all the elements involved can be found in [\(4.12\)](#)–[\(4.14\)](#)–[\(5.19\)](#) and [\(5.21\)](#). Let*

$$re^{i\theta} \in \overline{\mathbb{D}} \mapsto h_n(re^{i\theta}) = h_n^*(r) \cos(n\theta) \in \text{Ker } D_g \widehat{G}(\Omega, 0),$$

Then

$$D_{\Omega, g} \widehat{G}(\Omega, 0) h_n \notin \text{Im } D_g \widehat{G}(\Omega, 0)$$

if and only if  $h_n^*$  satisfies

$$(6.47) \quad \int_0^1 \frac{s^{n+1} F_n(xs^2)}{1-xs^2} \left[ \frac{h_n^*(s)}{2A} - A_n s^{n+2} + A_n \frac{sG_n(s)}{G_n(1)} \right] ds \neq 0,$$

where

$$A_n = - \frac{\int_0^1 s^{n+1} h_n^*(s) ds}{2G_n(1)},$$

and  $G_n$  is defined by [\(5.9\)](#).

*Proof.* Differentiating the expression of the linearized operator [\(5.8\)](#) with respect to  $\Omega$  we obtain

$$D_{\Omega, g} \widehat{G}(\Omega, 0) h(re^{i\theta}) = \sum_{n \geq 1} \cos(n\theta) \left[ \frac{1}{2A} h_n(r) - \frac{r}{n} \left( (\partial_\Omega A_n) G_n(r) + A_n \partial_\Omega G_n(r) \right) \right] + \frac{1}{2A} h_0(r).$$

Differentiating the identity [\(5.13\)](#) with respect to  $\Omega$  yields

$$\partial_\Omega A_n = \frac{H_n(1)}{2n(\Omega - \widehat{\Omega}_n)^2} = - \frac{nA_n}{G_n(1)}.$$

Similarly, we get from [\(5.9\)](#), [\(5.11\)](#) and the relation [\(4.12\)](#) between  $x$  and  $\Omega$  that

$$\partial_\Omega G_n(r) = \frac{An}{4} r^{n+1} \partial_\Omega \left( \frac{1}{x} \right) = nr^{n+1}.$$

Putting together the preceding identities we find

$$D_{\Omega, g} \widehat{G}(\Omega, 0) h(re^{i\theta}) = \sum_{n \geq 1} \cos(n\theta) \left[ \frac{1}{2A} h_n(r) + A_n \frac{rG_n(r)}{G_n(1)} - A_n r^{n+2} \right] + \frac{1}{2A} h_0(r).$$

Evaluating this formula at  $h_n$  yields

$$D_{\Omega, g} \widehat{G}(\Omega, 0) h_n(re^{i\theta}) = \left[ \frac{h_n^*(r)}{2A} - A_n r^{n+2} + A_n \frac{rG_n(r)}{G_n(1)} \right] \cos(n\theta),$$

where  $A_n$  is related to  $h_n^*$  via [\(3.6\)](#). Now applying Proposition [5.10](#) we obtain that this element does not belong to  $\text{Im } D_g \widehat{G}(\Omega, 0)$  if and only if the function

$$r \in [0, 1] \mapsto d_n^*(r) \triangleq \frac{h_n^*(r)}{2A} - A_n r^{n+2} + A_n \frac{rG_n(r)}{G_n(1)}$$



verifies

$$\int_0^1 \frac{s^{n+1} F_n(xs^2)}{1-xs^2} d_n^*(s) ds \neq 0,$$

which gives the announced result.  $\square$

The next goal is to check the condition (6.47) for large  $n$  in the regime (6.21). We need first to rearrange the function  $d_n^*$  defined above and use the explicit expression of  $h_n$  given in (5.34). From (5.33) we get

$$A_n = -\frac{n}{8xG_n(1)}.$$

Then, multiplying  $d_n^*$  by  $\frac{1}{A_n}$ , we obtain

$$\frac{1}{A_n} \left[ \frac{h_n^*(r)}{2A} - A_n r^{n+2} + A_n \frac{rG_n(r)}{G_n(1)} \right] = \left[ \frac{h_n^*(r)}{2AA_n} - r^{n+2} + \frac{rG_n(r)}{G_n(1)} \right] \triangleq r^n \mathcal{H}(r^2),$$

where the function  $\mathcal{H}$  takes the form

$$\begin{aligned} \mathcal{H}(\mathfrak{t}) = & \frac{4xG_n(1)}{An(1-x\mathfrak{t})} \left[ \frac{P_n(\mathfrak{t})}{P_n(1)} - \frac{F_n(x\mathfrak{t})}{F_n(x)} + \frac{2xF_n(x\mathfrak{t})}{P_n(1)} \int_{\mathfrak{t}}^1 \frac{1}{\tau^{n+1}F_n^2(x\tau)} \int_0^\tau \frac{s^n F_n(xs)}{1-xs} P_n(s) ds d\tau \right] \\ & - \mathfrak{t} + \frac{P_n(\mathfrak{t})}{P_n(1)}. \end{aligned}$$

With the change of variables  $s \rightsquigarrow s^2$  in the integral, condition (6.47) is equivalent to

$$(6.48) \quad \int_0^1 \frac{s^n F_n(xs)}{1-xs} \mathcal{H}(s) ds \neq 0.$$

6.6.1. *Regime  $A > 0$  and  $A + B < 0$ .* Here, we study the transversality assumption in the regime (6.21). Notice that the existence of infinite countable set of eigenvalues has been already established in Proposition 6.6. However, due to the complex structure of the integrand in (6.48) it appears quite difficult to check the non-vanishing of the integral for a given frequency. Thus, we have to overcome this difficulty using an asymptotic behavior of the integral and checking by this way the transversality only for high frequencies. More precisely, we prove the following result.

**Proposition 6.17.** *Let  $A$  and  $B$  satisfying (6.21) and  $\{x_n\}$  the sequence constructed in Proposition 6.6 – (1). Then there exists  $n_0 \in \mathbb{N}$  such that*

$$\int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} \mathcal{H}(s) ds = -\frac{n}{2}(1+o(1)), \quad \forall n \geq n_0.$$

*Proof.* We proceed with studying the asymptotic behavior of the above integral for large frequencies  $n$ . We write  $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$ , with

$$\begin{aligned} \mathcal{H}_1(\mathfrak{t}) &= \frac{4x_n G_n(1)}{An(1-x_n \mathfrak{t})} \left[ \frac{P_n(\mathfrak{t})}{P_n(1)} - \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} \right], \\ \mathcal{H}_2(\mathfrak{t}) &= \frac{4x_n G_n(1)}{An(1-x_n \mathfrak{t})} \frac{2x_n F_n(x_n \mathfrak{t})}{P_n(1)} \int_{\mathfrak{t}}^1 \frac{1}{\tau^{n+1} F_n^2(x_n \tau)} \int_0^\tau \frac{s^n F_n(x_n s)}{1-x_n s} P_n(s) ds d\tau, \\ \mathcal{H}_3(\mathfrak{t}) &= -\mathfrak{t} + \frac{P_n(\mathfrak{t})}{P_n(1)}. \end{aligned}$$

Let us start with the function  $\mathcal{H}_1$ . Proposition 6.13 leads to

$$\frac{1}{x_n} = 1 + \frac{\kappa}{n} - \frac{c_\kappa - \kappa^2}{n^2} + O\left(\frac{1}{n^3}\right),$$

which, together with the expression of  $G_n(1)$  given in (5.12), imply

$$(6.49) \quad G_n(1) \sim -\frac{A}{4n} (c_\kappa - \kappa^2 + 2).$$

Recall that the inequality  $F_n(1) \leq F_n(x_n \mathfrak{t}) \leq 1$  holds for any  $\mathfrak{t} \in [0, 1]$ , and thus Lemma 6.2 gives that

$$\frac{1}{2} \leq F_n(1), \quad \forall n \geq n_0.$$

Hence,  $\mathcal{H}_1$  can be bounded as follows

$$|\mathcal{H}_1(\mathfrak{t})| \leq \frac{C}{n^2(1-x_n \mathfrak{t})} \left[ \frac{|P_n(\mathfrak{t})|}{|P_n(1)|} + 1 \right], \quad \forall \mathfrak{t} \in [0, 1],$$

with  $C$  a constant depending in  $A$  and  $B$ . From the definition of  $P_n$  in (5.11) we may obtain

$$(6.50) \quad |P_n(\mathfrak{t})| \leq C \left[ 1 - x_n \mathfrak{t} + \frac{1}{n} \right], \quad \forall \mathfrak{t} \in [0, 1].$$

Moreover, plugging (6.49) into (5.12) we deduce that

$$(6.51) \quad P_n(1) \sim -\frac{4}{An} G_n(1) \sim \frac{c_\kappa - \kappa^2 + 2}{n^2}.$$

Putting everything together one gets

$$|\mathcal{H}_1(\mathfrak{t})| \leq C \left[ \frac{1}{n(1-x_n \mathfrak{t})} + 1 \right], \quad \forall \mathfrak{t} \in [0, 1],$$

from which we infer that

$$\left| \int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} \mathcal{H}_1(s) ds \right| \leq \frac{C}{n} \int_0^1 \frac{s^n}{(1-x_n s)^2} ds + C \int_0^1 \frac{s^n}{1-x_n s} ds \triangleq C \left( \frac{I_1}{n} + I_2 \right).$$

To estimate the second integral we use the change of variables  $s = 1 - \varepsilon_n \tau$ , with  $\varepsilon_n = \frac{1-x_n}{x_n}$ , leading to the asymptotic behavior

$$(6.52) \quad I_2 = \frac{1}{x_n} \int_0^{\frac{1}{\varepsilon_n}} \frac{(1-\varepsilon_n \tau)^n}{1+\tau} d\tau \sim \int_0^{\frac{n}{\kappa}} \frac{(1-\frac{\kappa \tau}{n})^n}{1+\tau} \sim \int_0^{+\infty} \frac{e^{-\kappa \tau}}{1+\tau} d\tau,$$

where we have used the expansion of  $x_n$  given by Proposition 6.13. As to the first integral we just have to integrate by parts and use the previous computations,

$$I_1 = \frac{1}{x_n} \frac{1}{1-x_n} - \frac{n}{x_n} \int_0^1 \frac{s^{n-1}}{1-x_n s} ds = \left( \frac{1}{\kappa} - \int_0^\infty \frac{e^{-\kappa \tau}}{1+\tau} d\tau \right) n + o(n),$$

and consequently,

$$(6.53) \quad \sup_{n \geq n_0} \left| \int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} \mathcal{H}_1(s) ds \right| < +\infty.$$

The estimate  $\mathcal{H}_2$  it is straightforward. Indeed, from (5.12) we may write

$$(6.54) \quad |\mathcal{H}_2(\mathfrak{t})| \leq \frac{C}{1-x_n \mathfrak{t}} \int_{\mathfrak{t}}^1 \frac{1}{\tau^{n+1}} \int_0^\tau \frac{s^n}{1-x_n s} |P_n(s)| ds d\tau.$$

Using once again (6.50) and Proposition 6.13 we get

$$\begin{aligned} \int_0^\tau \frac{s^n}{1-x_n s} |P_n(s)| ds &\leq \frac{C}{n} \int_0^\tau \frac{s^n}{1-x_n s} ds + C \int_0^\tau s^n ds \\ &\leq \frac{C}{n(1-x_n)} \frac{\tau^{n+1}}{n+1} + C \frac{\tau^{n+1}}{n+1} \leq \frac{C}{n} \tau^{n+1}, \quad \forall \tau \in [0, 1]. \end{aligned}$$

Hence, we deduce that

$$\int_{\mathfrak{t}}^1 \frac{1}{\tau^{n+1}} \int_0^\tau \frac{s^n}{1-x_n s} |P_n(s)| ds \leq C \frac{1-\mathfrak{t}}{n}.$$

Since the function  $\mathfrak{t} \in [0, 1] \mapsto \frac{1-\mathfrak{t}}{1-x_n \mathfrak{t}}$  is strictly decreasing, then we have

$$0 \leq \frac{1-\mathfrak{t}}{1-x_n \mathfrak{t}} \leq 1, \quad \forall \mathfrak{t} \in [0, 1].$$

Consequently, inserting the preceding two estimates into (6.54), we obtain

$$|\mathcal{H}_2(t)| \leq \frac{C}{n}, \quad \forall \mathfrak{t} \in [0, 1].$$

Thus we infer

$$\left| \int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} \mathcal{H}_2(s) ds \right| \leq \frac{C}{n} \int_0^1 \frac{s^n}{1-x_n s} ds,$$

which implies

$$(6.55) \quad \sup_{n \geq n_0} \left| \int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} \mathcal{H}_2(s) ds \right| \leq \frac{C}{n},$$

where we have used (6.52). It remains to estimate the integral term associated with  $\mathcal{H}_3$  which takes the form

$$\int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} \mathcal{H}_3(s) ds = - \int_0^1 \frac{s^{n+1} F_n(x_n s)}{1-x_n s} ds + \frac{1}{P_n(1)} \int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} P_n(s) ds.$$

Similarly to (6.52), one has

$$\sup_{n \geq n_0} \int_0^1 \frac{s^{n+1} F_n(x_n s)}{1-x_n s} ds < +\infty.$$

To finish we just have to deal with the second integral term. Observe from (5.11) that  $P_n$  is a monic polynomial of degree two, and thus from Taylor formula one gets

$$P_n(\mathfrak{t}) = P_n\left(\frac{1}{x_n}\right) + \left(\mathfrak{t} - \frac{1}{x_n}\right) P_n'\left(\frac{1}{x_n}\right) + \left(\mathfrak{t} - \frac{1}{x_n}\right)^2.$$

It is easy to check the following behaviors

$$P_n\left(\frac{1}{x_n}\right) \sim \frac{\kappa}{n}, \quad P_n'\left(\frac{1}{x_n}\right) \sim 1.$$

Hence, we obtain

$$\begin{aligned} \int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} P_n(s) ds &= P\left(\frac{1}{x_n}\right) \int_0^1 \frac{s^n F_n(x_n s)}{1-x_n s} ds - \frac{1}{x_n} P_n'\left(\frac{1}{x_n}\right) \int_0^1 s^n F_n(x_n s) ds \\ &\quad + \frac{1}{x_n^2} \int_0^1 s^n F_n(x_n s) (1-x_n s) ds. \end{aligned}$$

Concerning the last term we use the asymptotics of  $x_n$ , leading to

$$\frac{1}{x_n^2} \int_0^1 s^n F_n(x_n s) (1-x_n s) ds \leq C \int_0^1 s^n (1-x_n s) ds \leq C \left( \frac{1}{n+1} - \frac{x_n}{n+2} \right) \leq \frac{C}{n^2}.$$

For the first and second terms we use the estimate  $|F_n(x_n t) - 1| < C \frac{\ln n}{n}$  coming from Lemma 6.4. Hence, we find

$$-\frac{1}{x_n} P_n'\left(\frac{1}{x_n}\right) \int_0^1 s^n F_n(x_n s) ds = -\frac{1}{n} (1 + o(1)).$$

Again from (6.52) we find

$$\int_0^1 \frac{s^n F_n(x_n s)}{1 - x_n s} ds = \int_0^1 \frac{s^n}{1 - x_n s} ds + o(1) = \int_0^{+\infty} \frac{e^{-\kappa\tau}}{1 + \tau} d\tau + o(1).$$

Putting together the preceding estimates, we obtain

$$\int_0^1 \frac{s^n F_n(x_n s)}{1 - x_n s} P_n(s) ds = \frac{\kappa \int_0^{+\infty} \frac{e^{-\kappa\tau}}{1 + \tau} d\tau - 1}{n} + o\left(\frac{1}{n}\right),$$

and combining this estimate with (6.51), we infer

$$\int_0^1 \frac{s^n F_n(x_n s)}{1 - x_n s} \mathcal{H}_3(s) ds = \frac{\kappa \int_0^{+\infty} \frac{e^{-\kappa\tau}}{1 + \tau} d\tau - 1}{c_\kappa - \kappa^2 + 2} n + o(n).$$

Using the explicit expression of above constants defined in Proposition 6.13, we get that

$$\frac{\kappa \int_0^{+\infty} \frac{e^{-\kappa\tau}}{1 + \tau} d\tau - 1}{c_\kappa - \kappa^2 + 2} = -\frac{1}{2},$$

and therefore

$$\int_0^1 \frac{s^n F_n(x_n s)}{1 - x_n s} \mathcal{H}_3(s) ds = -\frac{n}{2} + o(n).$$

Combining this estimate with (6.55) and (6.53), we deduce that

$$\int_0^1 \frac{s^n F_n(x_n s)}{1 - s x_n} \mathcal{H}(s) ds = -\frac{n}{2} + o(n),$$

which achieves the proof of the announced result.  $\square$

6.6.2. *Regime  $B > A > 0$ .* In this special regime there is only a finite number of eigenvalues that can be indexed by a decreasing sequence, see Proposition 6.11-(1). In what follows we shall prove that the transversality assumption is always satisfied without any additional constraint on the parameters. More precisely, we prove the following result.

**Proposition 6.18.** *Let  $B > A > 0$ . Then, the transversal property (6.47) holds, for every subsequence  $\{x_n; n \in [2, \mathcal{N}_{A,B}]\}$  defined in Proposition 6.7, where*

$$\mathcal{N}_{A,B} \triangleq \max\left(\frac{B}{A} + \frac{1}{8}, \frac{2B}{A} - \frac{9}{2}\right).$$

*Proof.* Let us start with the case  $n \in [2, \frac{B}{A} + \frac{1}{8}]$ . Using the expression of  $P_n$  introduced in (5.11) one has

$$x_n P_n(1) = x_n \left[ 1 - \frac{A + 2B}{A} \frac{n + 2}{n(n + 1)} \right] - \frac{n + 2}{n + 1}.$$

Moreover, from the definition of  $\mathcal{P}$  seen in Proposition 6.8, we get

$$\begin{aligned} \mathcal{P}_n(x_n) &= \frac{n}{n + 1} + x_n \left[ -\frac{n}{n + 2} + \frac{A + 2B}{A(n + 1)} \right] \\ &= \frac{n}{n + 2} \left( \frac{n + 2}{n + 1} + x_n \left[ -1 + \frac{(A + 2B)(n + 2)}{An(n + 1)} \right] \right) = -\frac{n}{n + 2} x_n P_n(1) < 0. \end{aligned}$$

This implies that  $P_n(1) < 0$ . Since  $\mathfrak{t} \in [0, 1] \mapsto P_n(\mathfrak{t})$  is strictly increasing with  $x_n < 0$ , we deduce that

$$(6.56) \quad P_n(\mathfrak{t}) < 0, \quad \forall \mathfrak{t} \in [0, 1].$$

Therefore, we get from (5.12) that  $G_n(1) > 0$ . Let us study every term involved in (6.48) by using the decomposition  $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$  of Proposition 6.17. From the preceding properties, it is clear that  $\mathcal{H}_2(\mathfrak{t}) > 0$ , for  $\mathfrak{t} \in (0, 1)$ . In addition, we also have that  $\mathfrak{t} \in [0, 1] \mapsto -\mathfrak{t} + \frac{P_n(\mathfrak{t})}{P_n(1)}$  is strictly decreasing and thus

$$\mathcal{H}_3(\mathfrak{t}) = -\mathfrak{t} + \frac{P_n(\mathfrak{t})}{P_n(1)} \geq \mathcal{H}_3(1) = 0, \quad \forall \mathfrak{t} \in [0, 1].$$

Concerning  $\mathcal{H}_1$  we first note that the mapping  $\mathfrak{t} \in [0, 1] \mapsto \frac{P_n(\mathfrak{t})}{P_n(1)} - \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)}$  is strictly decreasing which follows from (6.56) and the fact that  $F_n$  is decreasing and  $P_n$  is increasing. Thus,

$$\frac{P_n(\mathfrak{t})}{P_n(1)} - \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} \geq \frac{P_n(1)}{P_n(1)} - \frac{F_n(x_n)}{F_n(x_n)} = 0, \quad \forall \mathfrak{t} \in [0, 1].$$

Combining (6.56) with (5.12), we deduce

$$\mathcal{H}_1(\mathfrak{t}) = \frac{4x_n G_n(1)}{An(1 - x_n \mathfrak{t})} \left[ \frac{P_n(\mathfrak{t})}{P_n(1)} - \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} \right] < 0, \quad \mathfrak{t} \in (0, 1).$$

We continue our analysis assuming that

$$(6.57) \quad \left| \frac{4x_n G_n(1)}{An(1 - x_n \mathfrak{t})} \right| \leq 1, \quad \forall \mathfrak{t} \in [0, 1),$$

holds, we see how to conclude with. Since  $\mathcal{H}_2$  is always positive then  $\mathcal{H}$  will be strictly positive if one can show that  $\mathcal{H}_1(\mathfrak{t}) + \mathcal{H}_3(\mathfrak{t}) > 0$ , for any  $\mathfrak{t} \in (0, 1)$ . With (6.57) in mind, one gets

$$\mathcal{H}_1(\mathfrak{t}) + \mathcal{H}_3(\mathfrak{t}) \geq - \left[ \frac{P_n(\mathfrak{t})}{P_n(1)} - \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} \right] - \mathfrak{t} + \frac{P_n(\mathfrak{t})}{P_n(1)} = \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} - \mathfrak{t}, \quad \forall \mathfrak{t} \in [0, 1].$$

Computing the derivatives of the function in the right-hand side term, we find

$$\partial_{\mathfrak{t}} \left( \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} - \mathfrak{t} \right) = \frac{x_n F'_n(x_n \mathfrak{t})}{F_n(x_n)} - 1, \quad \partial_{\mathfrak{t}\mathfrak{t}}^2 \left( \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} - \mathfrak{t} \right) = \frac{x_n^2 F''_n(x_n \mathfrak{t})}{F_n(x_n)} < 0.$$

The latter fact implies that the first derivative is decreasing, and thus

$$\partial_{\mathfrak{t}} \left( \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} - \mathfrak{t} \right) \leq \frac{x_n F'_n(0)}{F_n(x_n)} - 1 \leq \frac{-2x_n}{(n+1)F_n(x_n)} - 1 \leq \frac{2}{n+1} - 1 < 0, \quad \forall \mathfrak{t} \in [0, 1],$$

where we have used Lemma 6.3–(1). Therefore, we conclude that the mapping  $\mathfrak{t} \in [0, 1] \mapsto \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} - \mathfrak{t}$  decreases and, since it vanishes at  $\mathfrak{t} = 1$ , we get

$$\mathcal{H}_1(\mathfrak{t}) + \mathcal{H}_3(\mathfrak{t}) \geq \frac{F_n(x_n \mathfrak{t})}{F_n(x_n)} - \mathfrak{t} > 0, \quad \forall \mathfrak{t} \in [0, 1).$$

This implies that  $\mathcal{H}(\mathfrak{t}) > 0$  for any  $\mathfrak{t} \in [0, 1)$  and hence the transversality assumption (6.48) is satisfied. Let us now turn to the proof of (6.57) and observe that from (5.12)

$$(6.58) \quad \begin{aligned} \left| \frac{4x_n G_n(1)}{An(1 - x_n \mathfrak{t})} \right| &= \frac{1}{1 - x_n \mathfrak{t}} \frac{n+1}{n+2} x_n P_n(1) \\ &\leq \frac{n+1}{n+2} \left\{ x_n \left[ 1 - \frac{A+2B}{A} \frac{n+2}{n(n+1)} \right] - \frac{n+2}{n+1} \right\}. \end{aligned}$$

Using  $x_\star$ , defined in Proposition 6.8, and the fact that  $\frac{A+2B}{A} \frac{n+2}{n(n+1)} - 1 > 0$ , we obtain

$$\begin{aligned} \left| \frac{4x_n G_n(1)}{An(1 - x_n \mathfrak{t})} \right| &\leq \frac{n+1}{n+2} \left\{ x_\star \left[ 1 - \frac{A+2B}{A} \frac{n+2}{n(n+1)} \right] - \frac{n+2}{n+1} \right\} \\ &\leq \frac{n+1}{n+2} \left\{ \frac{2n+1}{2(n+1)} \frac{\frac{A+2B}{A} \frac{n+2}{n(n+1)} - 1}{\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2}} - \frac{n+2}{n+1} \right\}. \end{aligned}$$

Consequently (6.57) is satisfied provided that

$$\frac{\frac{A+2B}{A} \frac{n+2}{n(n+1)} - 1}{\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2}} \leq 4 \frac{n+2}{2n+1}.$$

Since  $\frac{A+2B}{A(n+1)} - \frac{n+1}{n+2} > 0$ , then the preceding inequality is true if and only if

$$\frac{A+2B}{A(n+1)} \geq \frac{2n^2+3n}{2n^2+3n-2}.$$

It is easy to check that the sequence  $n \geq 2 \mapsto \frac{2n^2+3n}{2n^2+3n-2}$  is decreasing, and then the foregoing inequality is satisfied for any  $n \geq 2$  if

$$\frac{A+2B}{A(n+1)} \geq \frac{7}{6}.$$

From the assumption  $n+1 \leq \frac{B}{A} + \frac{9}{8}$ , the above inequality holds if

$$\frac{B}{A} \geq \frac{5}{16},$$

which follows from the condition  $B > A$ .

Let us now move on the case  $n \in [2, \frac{2B}{A} - \frac{9}{2}]$ . Note that we cannot use  $x_*$  coming from Proposition 6.8 since we do not know if  $x_n \in (-1, 0)$ . In fact Proposition 6.7 gives us that  $x_n < -1$ , for  $n \geq \frac{B}{A} + 1$ . Hence, we should slightly modify the arguments used to (6.57). From (6.58), we deduce that (6.57) is satisfied if

$$x_n \geq \frac{2^{\frac{n+2}{n+1}}}{1 - \frac{A+2B}{A} \frac{n+2}{n(n+1)}}.$$

By (6.31), it is enough to prove that

$$\frac{1}{1 - \frac{A+2B}{A(n+1)}} \geq \frac{2^{\frac{n+2}{n+1}}}{1 - \frac{A+2B}{A} \frac{n+2}{n(n+1)}}.$$

Straightforward computations give that the last inequality is equivalent to

$$\frac{A+2B}{A(n+1)} \geq \frac{(n+3)n}{(n+2)(n-1)}.$$

Therefore, if  $n$  satisfies  $2 \leq n \leq \frac{2B}{A} - \frac{9}{2}$ , then we have

$$\frac{n+1+\frac{9}{2}}{n+1} \leq \frac{A+2B}{A(n+1)}.$$

Hence, one can check that

$$\frac{n+1+\frac{9}{2}}{n+1} \geq \frac{(n+3)n}{(n+2)(n-1)},$$

for any  $n \geq 2$ , and thus (6.57) is verified.  $\square$

**6.6.3. One-fold case.** The main objective is to make a complete study in the case  $n = 1$ . It is very particular because  $F_1$  is explicit according to Remark 5.5 and, therefore, we can get a compact formula for the integral of (6.48). Our main result reads as follows.

**Proposition 6.19.** *Let  $n = 1$  and  $x = -\frac{A}{2B}$ , then we have the formula*

$$\int_0^1 \frac{sF_1(xs)}{1-xs} \mathcal{H}(s) ds = \frac{x-1}{2x}.$$

*In particular the transversal assumption (6.47) is satisfied if and only if  $x \notin \{0, 1\}$ .*

*Proof.* Note that from (5.11)-(5.12) one has

$$P_1(\mathfrak{t}) = \mathfrak{t}^2 - \frac{3}{2x}\mathfrak{t} - \frac{3}{2}\left[1 - \frac{1}{x}\right], \quad P_n(1) = -\frac{1}{2}, \quad G_1(1) = \frac{A}{12}.$$

Moreover, we get  $F_1(\mathfrak{t}) = 1 - \mathfrak{t}$  using Remark 5.5, and thus

$$\begin{aligned} \mathcal{H}(\mathfrak{t}) &= \frac{1}{3} \frac{x}{(1-x\mathfrak{t})} \left[ \frac{P_1(\mathfrak{t})}{P_1(1)} - \frac{F_1(x\mathfrak{t})}{F_1(x)} + \frac{2xF_1(x\mathfrak{t})}{P_1(1)} \int_{\mathfrak{t}}^1 \frac{1}{\tau^2 F_1^2(x\tau)} \int_0^\tau \frac{sF_1(xs)}{1-xs} P_1(s) ds d\tau \right] \\ &\quad - \mathfrak{t} + \frac{P_1(\mathfrak{t})}{P_1(1)} \\ (6.59) \quad &\triangleq \frac{1}{3} \frac{x}{(1-x\mathfrak{t})} \widehat{\mathcal{H}}(\mathfrak{t}) - \mathfrak{t} + \frac{P_1(\mathfrak{t})}{P_1(1)}. \end{aligned}$$

From straightforward computations we deduce

$$\int_{\mathfrak{t}}^1 \frac{1}{\tau^2 F_1^2(x\tau)} \int_0^\tau \frac{sF_1(xs)}{1-xs} P_1(s) ds d\tau = \int_{\mathfrak{t}}^1 \frac{\frac{\tau^2}{4} - \frac{1}{2x}\tau - \frac{3}{4}(1 - \frac{1}{x})}{(1-\tau x)^2} d\tau$$

Denoting by  $\varphi(\tau) = \frac{\tau^2}{4} - \frac{1}{2x}\tau - \frac{3}{4}(1 - \frac{1}{x})$  and integrating by parts, we get

$$\int_{\mathfrak{t}}^1 \frac{\varphi(\tau)}{(1-\tau x)^2} d\tau = \frac{1}{x} \left[ \frac{\varphi(1)}{1-x} - \frac{\varphi(\mathfrak{t})}{1-\mathfrak{t}x} \right] + \frac{1-\mathfrak{t}}{2x^2} = \frac{1}{x} \left[ \frac{1-2x}{4x(1-x)} + \frac{1-\mathfrak{t}}{2x} - \frac{\varphi(\mathfrak{t})}{1-\mathfrak{t}x} \right].$$

Therefore, after standard computations, we get the simplified formula

$$\widehat{\mathcal{H}}(\mathfrak{t}) = \frac{3(1-\mathfrak{t}x)(\mathfrak{t}-1)}{x}.$$

Inserting this into (6.59), we find

$$\mathcal{H}(\mathfrak{t}) = -1 + \frac{P_1(\mathfrak{t})}{P_1(1)} = -2\mathfrak{t}^2 + \frac{3}{x}\mathfrak{t} + 2 - \frac{3}{x}.$$

Plugging it into the integral of (6.48), it yields

$$\int_0^1 s\mathcal{H}(s) ds = \frac{x-1}{2x}.$$

□

## 7. EXISTENCE OF NON-RADIAL TIME-DEPENDENT ROTATING SOLUTIONS

At this stage we are able to give the full statement of our main result, by using the analysis of the previous sections. In order to apply the Crandall–Rabinowitz Theorem, let us introduce the  $m$ -fold symmetries in our spaces.

$$\begin{aligned} \mathcal{C}_{s,m}^{k,\alpha}(\mathbb{D}) &\triangleq \left\{ g \in \mathcal{C}_s^{k,\alpha}(\mathbb{D}) : g(e^{i\frac{2\pi}{m}}z) = g(z), \quad \forall z \in \mathbb{D} \right\}, \\ \mathcal{C}_{a,m}^{k,\alpha}(\mathbb{T}) &\triangleq \left\{ \rho \in \mathcal{C}_a^{k,\alpha}(\mathbb{T}) : \rho(e^{i\frac{2\pi}{m}}w) = \rho(w), \quad \forall w \in \mathbb{T} \right\}, \\ \mathcal{H}\mathcal{C}_m^{k,\alpha}(\mathbb{D}) &\triangleq \left\{ \phi \in \mathcal{H}\mathcal{C}^{k,\alpha}(\mathbb{D}) : \phi(e^{i\frac{2\pi}{m}}z) = e^{i\frac{2\pi}{m}}\phi(z), \quad \forall z \in \mathbb{D} \right\}. \end{aligned}$$

Note that the functions  $g \in \mathcal{C}_{s,m}^{k,\alpha}(\mathbb{D})$ ,  $\rho \in \mathcal{C}_{a,m}^{k,\alpha}(\mathbb{T})$  and  $\phi \in \mathcal{H}\mathcal{C}_m^{k,\alpha}(\mathbb{D})$  admit the following representation:

$$g(re^{i\theta}) = \sum_{n \geq 0} g_{nm}(r) \cos(nm\theta), \quad \rho(e^{i\theta}) = \sum_{n \geq 0} \rho_n \sin(nm\theta) \quad \text{and} \quad \phi(z) = z \sum_{n \geq 1} z^{nm},$$

where  $z \in \mathbb{D}$ ,  $r \in [0, 1]$  and  $\theta \in [0, 2\pi]$ . With these spaces, the functional  $F$  defined in (3.1), concerning the boundary equation, is also well-defined:

**Proposition 7.1.** *Let  $\varepsilon \in (0, 1)$ , then*

$$F : \mathbb{R} \times B_{\mathcal{C}_s^1, \alpha}(0, \varepsilon) \times B_{\mathcal{H}\mathcal{C}_m^{2, \alpha}}(0, \varepsilon) \mapsto \mathcal{C}_{a, m}^{1, \alpha}(\mathbb{T})$$

*is well-defined and of class  $\mathcal{C}^1$ , where the balls were defined in (3.2).*

*Proof.* Thanks to Proposition 3.1, it remains to prove that  $F(\Omega, g, \phi)$  satisfies  $F(\Omega, g, \phi)(e^{i\frac{2\pi}{m}} w) = F(w)$ , with  $w \in \mathbb{T}$ :

$$\begin{aligned} F(\Omega, g, \phi)(e^{i\frac{2\pi}{m}} w) &= \text{Im} \left[ \left( \overline{\Omega \Phi(e^{i\frac{2\pi}{m}} w)} - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y) |\Phi'(y)|^2}{\Phi(e^{i\frac{2\pi}{m}} w) - \Phi(y)} dA(y) \right) \Phi'(e^{i\frac{2\pi}{m}} w) e^{i\frac{2\pi}{m}} w \right] \\ &= \text{Im} \left[ \left( \Omega e^{-i\frac{2\pi}{m}} \overline{\Phi(w)} - \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(e^{i\frac{2\pi}{m}} y) |\Phi'(e^{i\frac{2\pi}{m}} y)|^2}{\Phi(e^{i\frac{2\pi}{m}} w) - \Phi(e^{i\frac{2\pi}{m}} y)} dA(y) \right) \Phi'(w) e^{i\frac{2\pi}{m}} w \right] \\ &= \text{Im} \left[ \left( \Omega e^{-i\frac{2\pi}{m}} \overline{\Phi(w)} - \frac{e^{-i\frac{2\pi}{m}}}{2\pi} \int_{\mathbb{D}} \frac{f(y) |\Phi'(y)|^2}{\Phi(w) - \Phi(y)} dA(y) \right) \Phi'(w) e^{i\frac{2\pi}{m}} w \right] \\ &= F(\Omega, g, \phi)(w), \end{aligned}$$

where we have used that  $\phi(e^{i\frac{2\pi}{m}} z) = e^{i\frac{2\pi}{m}} \phi(z)$ ,  $\phi'(e^{i\frac{2\pi}{m}} z) = \phi'(z)$  and  $g(e^{i\frac{2\pi}{m}} z) = g(z)$ .  $\square$

We must define the singular set (4.10) once we have introduced the symmetry in the spaces. Fixing  $f_0$  as a quadratic profile (1.7), the singular set (4.10) becomes

$$\mathcal{S}_{\text{sing}}^m \triangleq \left\{ \widehat{\Omega}_{mn} \triangleq \frac{A}{4} + \frac{B}{2} - \frac{A(nm+1)}{2nm(nm+2)} - \frac{B}{2nm}, \quad n \in \mathbb{N}^* \cup \{+\infty\} \right\}.$$

For the density equation defined in (4.6) and the new spaces we obtain the following result.

**Proposition 7.2.** *Let  $I$  be an open interval with  $\bar{I} \subset \mathbb{R} \setminus \mathcal{S}_{\text{sing}}^m$ . Then, there exists  $\varepsilon > 0$  such that*

$$\widehat{G} : I \times B_{\mathcal{C}_s^1, \alpha}(\mathbb{D})(0, \varepsilon) \rightarrow \mathcal{C}_{s, m}^{1, \alpha}(\mathbb{D})$$

*is well-defined and of class  $\mathcal{C}^1$ , where  $\widehat{G}$  is defined in (4.6) and  $B_{\mathcal{C}_s^1, \alpha}(\mathbb{D})(0, \varepsilon)$  in (3.2).*

*Proof.* Similarly to the previous result, from Proposition 4.1 we just have to check that  $\widehat{G}(\Omega, g)(e^{i\frac{2\pi}{m}} z) = \widehat{G}(\Omega, g)(z)$ , for  $z \in \mathbb{D}$ . From Proposition 3.3, there exists  $\varepsilon > 0$  such that the conformal map  $\phi$  is given by  $(\Omega, g)$ , and lies in  $B_{\mathcal{H}\mathcal{C}_m^{2, \alpha}}(0, \varepsilon)$ . Then, using  $\widehat{G}(\Omega, g) = G(\Omega, g, \phi(\Omega, g))$ , we have

$$\begin{aligned} G(\Omega, g, \phi)(e^{i\frac{2\pi}{m}} z) &= \frac{4\Omega - B}{8A} f(e^{i\frac{2\pi}{m}} z) - \frac{f(e^{i\frac{2\pi}{m}} z)^2}{16A} \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(e^{i\frac{2\pi}{m}} z) - \Phi(y)| f(y) |\Phi'(y)|^2 dA(y) - \frac{\Omega |\Phi(e^{i\frac{2\pi}{m}} z)|^2}{2} - \lambda \\ &= \frac{4\Omega - B}{8A} f(z) - \frac{f(z)^2}{16A} \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{D}} \log |\Phi(e^{i\frac{2\pi}{m}} z) - \Phi(e^{i\frac{2\pi}{m}} y)| f(e^{i\frac{2\pi}{m}} y) |\Phi'(e^{i\frac{2\pi}{m}} y)|^2 dA(y) \\ &\quad - \frac{\Omega |e^{i\frac{2\pi}{m}} \Phi(z)|^2}{2} - \lambda \\ &= G(\Omega, g, \phi)(z), \end{aligned}$$

where we have used the properties of functions  $g$  and  $\phi$ .  $\square$

We have now all the tools we need to prove the first three points of Theorem 1.1, which can be detailed as follows:



**Theorem 7.3.** *Let  $A > 0$ ,  $B \in \mathbb{R}$  and  $f_0$  a quadratic profile (1.7). Then the following results hold true.*

- (1) *If  $A + B < 0$ , then there is  $m_0 \in \mathbb{N}$  (depending only on  $A$  and  $B$ ) such that for any  $m \geq m_0$  there exists*
- $\Omega_m = \frac{A+2B}{4} + \frac{A\kappa}{4m} + \frac{A}{4} \frac{\kappa^2 - c_\kappa}{m^2} + o\left(\frac{1}{m^2}\right)$ ,
  - $V$  a neighborhood of  $(\Omega_m, 0, 0)$  in  $\mathbb{R} \times \mathcal{C}_{s,m}^{1,\alpha}(\mathbb{D}) \times \mathcal{H}\mathcal{C}_m^{2,\alpha}(\mathbb{D})$ ,
  - a continuous curve  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi) \in V$ ,
- such that (7.2),

$$\omega_0 = (f \circ \Phi^{-1})\mathbf{1}_{\Phi(\mathbb{D})}, \quad f = f_0 + f_\xi, \quad \Phi = \text{Id} + \phi_\xi,$$

defines a curve of non radial solutions of Euler equations that rotates at constant angular velocity  $\Omega_\xi$ . The constants  $\kappa$  and  $c_\kappa$  are defined in (6.13).

- (2) *If  $B > A > 0$ , then for any integer  $m \in [1, \frac{2B}{A} - \frac{9}{2}]$  or  $m \in [1, \frac{B}{A} + \frac{1}{8}]$  there exists*
- $0 \leq \Omega_m < \frac{B}{2}$ ,
  - $V$  a neighborhood of  $(\Omega_m, 0, 0)$  in  $\mathbb{R} \times \mathcal{C}_{s,m}^{1,\alpha}(\mathbb{D}) \times \mathcal{H}\mathcal{C}_m^{2,\alpha}(\mathbb{D})$ ,
  - a continuous curve  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi) \in V$ ,
- such that (7.2) defines a curve of non radial solutions of Euler equations. However, there is no bifurcation with any symmetry  $m \geq \frac{2B}{A} + 2$ .
- (3) *If  $B > 0$  or  $B \leq -\frac{A}{1+\epsilon}$ , for some  $0,0581 < \epsilon < 1$ , then there exists*
- $V$  a neighborhood of  $(0, 0, 0)$  in  $\mathbb{R} \times \mathcal{C}_s^{1,\alpha}(\mathbb{D}) \times \mathcal{H}\mathcal{C}^{2,\alpha}(\mathbb{D})$ ,
  - a continuous curve  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi) \in V$ ,
- such that (7.2) defines a curve of one-fold non radial solutions of Euler equations.
- (4) *If  $-\frac{A}{2} < B < 0$ , then there is no bifurcation with any symmetry  $m \geq 1$ . However, in the case  $0 < B < \frac{A}{4}$ , there is no bifurcation with any symmetry  $m \geq 2$ .*

*Proof.* (1) Let us prove the first assertion in the case  $A + B < 0$ . We will implement the Crandall–Rabinowitz Theorem (A.2) to  $\widehat{G}$ , defined in (4.6). First, we must concrete the domain of  $\Omega$ . From Proposition 6.6, there exist  $n_{0,1}$  and a unique solution  $x_m \in (0, 1)$  of  $\zeta_m(x) = 0$  for any  $m \geq n_{0,1}$ . Then, we fix  $\Omega_m = \frac{A}{4x_m} + \frac{B}{2} > \frac{A}{4} + \frac{B}{2}$ . Note that one can take  $n_{0,1}$  large enough to avoid  $\Omega_0$  defined in (4.14). Moreover, Proposition 6.14 gives us that  $\Omega_m \neq \widehat{\Omega}_{mn}$ , for  $n \in \mathbb{N}$ , with  $\widehat{\Omega}_{mn} \in \mathcal{S}_{\text{sing}}^m$ . Therefore, let  $I$  be an interval with  $\Omega_m \in I$  and

$$\bar{I} \cap \mathcal{S}_{\text{sing}}^m = \emptyset, \quad \Omega_0 \notin \bar{I}.$$

By virtue of Proposition 3.3, we know that there exists  $\varepsilon > 0$  and a  $\mathcal{C}^1$  function  $\mathcal{N} : I \times B_{\mathcal{C}_{s,m}^{1,\alpha}}(0, \varepsilon) \rightarrow B_{\mathcal{H}\mathcal{C}_m^{2,\alpha}}(0, \varepsilon)$ , such that

$$F(\Omega, g, \phi) = 0 \iff \phi = \mathcal{N}(\Omega, g)$$

holds, for any  $(\Omega, g, \phi) \in I \times B_{\mathcal{C}_{s,m}^{1,\alpha}}(0, \varepsilon) \times B_{\mathcal{H}\mathcal{C}_m^{2,\alpha}}(0, \varepsilon)$ . Hence, the conformal map is defined through the density for that  $\varepsilon$ . We define the density equation,

$$\widehat{G} : I \times B_{\mathcal{C}_{s,m}^{1,\alpha}}(0, \varepsilon) \rightarrow \mathcal{C}_{s,m}^{1,\alpha}(\mathbb{D}),$$

with the expression given in (4.6). Thanks to Proposition 7.2, the function  $\widehat{G}$  is well-defined in these spaces and is  $\mathcal{C}^1$  with respect to  $(\Omega, g)$ . It remains to check the spectral properties of the Crandall–Rabinowitz Theorem. Using Proposition 5.3, we know that the dimension of the kernel of the linearized operator  $D_g \widehat{G}(\Omega, 0)$  is given by the number of elements of the set  $\mathcal{A}_x$  defined in (5.21). Note that we have introduced the symmetry  $m$  in our spaces, and consequently in all the functions involved in the linearized operator. Take  $n = 1$ . We know that  $x_m$  is the unique root  $x_m$  of  $\zeta_m(x)$ , and the sequence  $n \in [n_0, +\infty[ \mapsto x_n$  is strictly increasing. Since we fixed  $n = 1$ , we get that  $\mathcal{A}_{x_m} = \{m\}$ . As  $\Omega_m \notin \mathcal{S}_{\text{sing}}^m$ , we have that the kernel is one dimensional.

Since  $D_g \widehat{G}(\Omega, 0)$  is a Fredholm operator of zero index, we have that the codimension of the range is also one. About the transversality condition, note that using Proposition 6.16 and Proposition 6.17 we have that there exists  $n_{0,2}$  such that if  $m > n_{0,2}$  the transversality condition is satisfied. Taking  $n_0 = \max\{n_{0,1}, n_{0,2}\}$ , the Crandall-Rabinowitz Theorem can be applied to  $\widehat{G}$  obtaining  $V$  a neighborhood of  $(\Omega_m, f_0, \text{Id})$  in  $\mathbb{R} \times \mathcal{C}_{s,m}^{1,\alpha}(\mathbb{D}) \times \mathcal{H}\mathcal{C}_m^{2,\alpha}(\mathbb{D})$ , and a continuous curve  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi) \in V$ , solutions of  $\widehat{G}(\Omega, g) = 0$ . The conformal map  $\phi_\xi$  is the one associated to  $f_\xi$  via Proposition 3.3. Moreover, thanks to  $\Omega_\xi \neq \Omega_0$ , Proposition 4.3 gives us that  $f_\xi$  can not be radial because of  $f_\xi \neq f_0$ . Furthermore, by Proposition 6.13 we know the asymptotics of  $x_m$  obtaining

$$\frac{1}{x_m} = 1 + \frac{\kappa}{m} + \frac{\kappa^2 - c_\kappa}{m^2} + o\left(\frac{1}{m^2}\right),$$

where  $\kappa$  and  $c_\kappa$  are defined in (6.13). Using the relation between  $x_m$  and  $\Omega_m$  in (4.12), we get

$$(7.1) \quad \frac{A + 2B}{4} < \Omega_m = \frac{A + 2B}{4} + \frac{A\kappa}{4m} + \frac{A}{4} \frac{\kappa^2 - c_\kappa}{m^2} + o\left(\frac{1}{m^2}\right).$$

Therefore, we obtain that

$$(7.2) \quad \omega_0 = (f \circ \Phi^{-1})\mathbf{1}_{\Phi(\mathbb{D})}, \quad f = f_0 + f_\xi, \quad \Phi = \text{Id} + \phi_\xi,$$

defines a non radial solution of Euler equation, which rotates at constant angular velocity  $\Omega_\xi$ .

(2) Now, we are concerned with the existence of  $m$ -fold non radial solutions of the type (7.2) in the case  $B > A > 0$ , for any integer  $m \in [1, \frac{2B}{A} - \frac{9}{2}]$  or  $m \in [1, \frac{B}{A} + \frac{1}{8}]$ . In this part of the theorem we also prove that there is no bifurcation with the symmetry  $m$ , for any  $m \geq \frac{2B}{A} + 2$ . As in Assertion (1), we check that the Crandall-Rabinowitz Theorem can be applied. From Proposition 6.7 and Proposition 6.6, there is a unique solution  $x_m \in (-\infty, 1)$  of  $\zeta_m(x) = 0$ . In fact,  $x_m < 0$ . Then, we fix  $\Omega_m = \frac{A}{4x_m} + \frac{B}{2}$ . Note that by (6.31) and Proposition 6.8 we get the bounds for  $\Omega_m$ . Moreover, Proposition 6.15 gives that  $\Omega_m \notin \mathcal{S}_{\text{sing}}^m$ . Then, let  $I$  be an interval such that  $\Omega_m \in I$  and  $\bar{I} \cap \mathcal{S}_{\text{sing}}^m = \emptyset$ .

Using again Proposition 3.3 and Proposition 7.2, we get that  $\widehat{G} : I \times B_{\mathcal{C}_{s,m}^{1,\alpha}}(0, \varepsilon) \rightarrow \mathcal{C}_{s,m}^{1,\alpha}(\mathbb{D})$ , is well-defined and  $\mathcal{C}^1$  with respect to  $(\Omega, g)$ .

About the spectral properties, we have stated in the previous proof that the dimension of the kernel of the linearized operator is given by the roots of  $\zeta_{nm}$ . Taking  $n = 1$ , we know that  $x_m$  is the unique root of  $\zeta_m(x)$ . By Corollary 6.11 we get that  $\zeta_{nm}(x_m) \neq 0$ , for any  $n \geq 2$ . Hence  $\mathcal{A}_{x_m} = \{m\}$ . Due to  $\Omega_m \notin \mathcal{S}_{\text{sing}}^m$ , we have that the kernel is one dimensional.  $D_g \widehat{G}(\Omega, 0)$  is a Fredholm operator of zero index, then we have that the codimension of the range is also one. About the transversal condition, note that using Proposition 6.18, we have that the transversal condition is satisfied. Similarly to the previous proof, the Crandall-Rabinowitz Theorem can be applied to  $\widehat{G}$  obtaining a curve  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi)$  solutions of  $\widehat{G}(\Omega, g) = 0$ . Moreover, thanks to  $\Omega_\xi \neq \Omega_0 \in (0, 1)$ , Proposition 4.3 gives that  $f_\xi$  can not be radial since  $f_\xi \neq f_0$ .

First, note that  $\Omega \notin [\frac{B}{2}, \frac{B}{2} + \frac{A}{4}]$  is equivalent to  $x < 1$ . By Proposition 6.6 and Proposition 6.7 we get that  $\zeta_n$  has not solutions in  $(-\infty, 1]$ , for  $m \geq \frac{2B}{A} + 2$ , and then there is no bifurcation with that symmetry. In the opposite case,  $x > 1$ , there is no bifurcation according to Theorem 5.6.

(3) Here, we are concerning with the case  $A, B > 0$  or  $B \leq -\frac{A}{1+\epsilon}$  for some  $\epsilon \in (0, 1)$ , with  $-\frac{A}{2B} \neq x_0$ , where  $x_0$  is defined through (4.14). We work as in (1)-(2) checking the hypothesis of Crandall-Rabinowitz Theorem. Fixing  $\Omega_1 = 0$  agrees with  $x_1 = -\frac{A}{2B}$ , where we use (4.12). Proposition 6.15 allows us to have that  $x_1 \notin \widehat{\mathcal{S}}_{\text{sing}}$ . Then, we can take an interval  $I$  such that

$0 \in I$ , and

$$\bar{I} \cap \mathcal{S}_{\text{sing}} = \emptyset, \quad \Omega_0 \notin \bar{I}.$$

Again, Proposition 3.3 and Proposition 4.1 imply that

$$\widehat{G} : I \times B_{\mathcal{C}_s^{1,\alpha}}(0, \varepsilon) \rightarrow \mathcal{C}_s^{1,\alpha}(\mathbb{D}),$$

is well-defined and is  $\mathcal{C}^1$  in  $(\Omega, g)$ .

We must check the spectral properties. Due to the assumptions on  $A$  and  $B$ , we get that  $x_1 \leq 1$ . By Proposition 6.6, Proposition 6.8 and Proposition 6.10 we have that  $x_1 \neq x_n$  if there exists  $x_n \in (-\infty, 1)$  solution of  $\zeta_n$ . Note that such  $\varepsilon$  comes from the Proposition 6.10. Hence, by Corollary 6.11, we obtain that the kernel of  $D_g \widehat{G}(0, 0)$  is one dimensional, and is generated by (5.34), for  $n = 1$ . Moreover, Proposition 5.1 implies that  $D_g \widehat{G}(0, 0)$  is a Fredholm operator of zero index, and then the codimension of the range is one. The transversal property is verified by virtue of Proposition 6.19. Then, Crandall–Rabinowitz Theorem can be applied obtaining the announced result. Note that the bifurcated solutions are not radial due to Proposition 4.3.

(4) The first assertion concerning the non bifurcation result comes from Proposition 6.6 and Proposition 6.7 due to the fact that  $\zeta_m$  has not solutions in  $(-\infty, 1]$ , for  $m \geq 2$ . Moreover, by Corollary 6.9 and Theorem 5.6 we get that there is no bifurcation for  $m = 1$  since the only possibility agrees with  $\Omega = 0$ , which satisfies (5.35). Finally, the bifurcation with  $x > 1$  is forbidden due again to Theorem 5.6.

The last assertion follows from Corollary 6.9 and Theorem 5.6.  $\square$

## 8. DYNAMICAL SYSTEM AND ORBITAL ANALYSIS

In this section we wish to investigate the particle trajectories inside the support of the rotating vortices that we have constructed in Theorem 1.1. We will show that in the frame of these V-states the trajectories are organized through concentric periodic orbits around the origin. This allows to provide an equivalent reformulation of the density equation (2.3) via the study of the associated dynamical system. It is worth pointing out that some of the material developed in this section about periodic trajectories and the regularity of the period is partially known in the literature and for the convenience of the reader we will provide the complete proofs.

Assuming that (1.2) is a solution of the Euler equations, the level sets of  $\Psi(x) - \Omega \frac{|x|^2}{2}$ , where  $\Psi$  is the stream function associated to (1.2), are given by the collection of the particle trajectories,

$$\begin{aligned} \partial_t \varphi(t, x) &= (v(\varphi(t, x)) - \Omega \varphi(t, x)^\perp) = \nabla^\perp \left( \Psi - \Omega \frac{|\cdot|^2}{2} \right) (\varphi(t, x)), \\ \varphi(0, x) &= x \in \Phi(\bar{\mathbb{D}}). \end{aligned}$$

In the same way we have translated the problem to the unit disc  $\mathbb{D}$  using the conformal map  $\Phi$  via the vector field  $W(\Omega, f, \Phi)$  in (2.5), we analyze the analogue in the level set context. We define the flow associated to  $W$  as

$$(8.1) \quad \partial_t \psi(t, z) = W(\Omega, f, \Phi)(\psi(t, z)), \quad \psi(0, z) = z \in \bar{\mathbb{D}}.$$

Since  $v(x) - \Omega x^\perp$  is divergence free, via Lemma 2.1, we obtain that  $W(\Omega, f, \Phi)$  is incompressible, and then the last system is also Hamiltonian. In the following result, we highlight the relation between  $\varphi$  and  $\psi$ .

**Lemma 8.1.** *The following identity*

$$\varphi(\eta_z(t), \Phi(z)) = \Phi(\psi(t, z)), \quad \forall z \in \bar{\mathbb{D}},$$

holds, where

$$\eta'_z(t) = |\Phi'(\Phi^{-1}(\varphi(\eta_z(t), \Phi(z))))|^2, \quad \eta_z(0) = 0.$$

*Proof.* Let us check that  $Y(t, z) = \Phi^{-1}(\varphi(t, \Phi(z)))$  verifies a similar equation as  $\psi(t, z)$  sets,

$$\begin{aligned}\partial_t Y(t, z) &= (\Phi^{-1})'(\Phi(Y(t, z)))\partial_t \varphi(t, \Phi(z)) = \frac{\overline{\Phi'(Y(t, z))}}{|\Phi'(Y(t, z))|^2}(v(\Phi(Y(t, z))) - \Omega\Phi(Y(t, z))^\perp) \\ &= \frac{W(\Omega, f, \Phi)(Y(t, z))}{|\Phi'(Y(t, z))|^2}.\end{aligned}$$

Now, we rescale the time through the function  $\eta_z$ , and  $Y(\eta_z(t), z)$  satisfies,

$$\partial_t Y(\eta_z(t), z) = \eta'_z(t)(\partial_t Y)(\eta_z(t), z) = \eta'_z(t) \frac{W(\Omega, f, \Phi)(Y(\eta(t), z))}{|\Phi'(Y(\eta(t), z))|^2} = W(\Omega, f, \Phi)(Y(\eta(t), z)).$$

Since  $Y(\eta_z(0), z) = Y(0, z) = \Phi^{-1}(\varphi(0, \Phi(z))) = z$ , we obtain the announced result.  $\square$

The next task is to connect the solutions constructed in Theorem 1.1 with the orbits of the associated dynamical system through the following result:

**Theorem 8.2.** *Let  $m \geq 1$  and  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi)$  be one of the solutions constructed in Theorem 1.1. The flow  $\psi$  associated to  $W(\Omega_\xi, f_\xi, \Phi_\xi)$ , defined in (8.1), verifies the following properties:*

- (1)  $\psi \in \mathcal{C}^1(\mathbb{R}, \mathcal{C}^{1,\alpha}(\mathbb{D}))$ .
- (2) *The trajectory  $t \mapsto \psi(t, z)$  is  $T_z$  periodic, located inside the unit disc and invariant by the dihedral group  $D_m$ . Moreover, if  $m \geq 4$  then the period map  $z \in \mathbb{D} \mapsto T_z$  belongs to  $\mathcal{C}^{1,\alpha}(\mathbb{D})$ .*
- (3) *The family  $(\psi(t))_{t \in \mathbb{R}}$  generates a group of diffeomorphisms of the closed unit disc.*

The proof will be given in Subsection 8.5.

**8.1. Periodic orbits.** Here we explore sufficient conditions for Hamiltonian vector fields defined on the unit disc whose orbits are all periodic. More precisely, we shall establish the following result.

**Proposition 8.3.** *Let  $W : \overline{\mathbb{D}} \rightarrow \mathbb{C}$  be a vector field in  $\mathcal{C}^1(\overline{\mathbb{D}})$  satisfying the following conditions:*

- i) *It has divergence free.*
- ii) *It is tangential to the boundary  $\mathbb{T}$ , i.e.  $\operatorname{Re}(W(z)\bar{z}) = 0, \quad \forall z \in \mathbb{T}$ .*
- iii) *It vanishes only at the origin.*

*Then, we have*

- (1) *All the trajectories are periodic orbits located inside the unit disc, enclosing a simply connected domain containing the origin.*
- (2) *The family  $(\psi(t))_{t \in \mathbb{R}}$  generates a group of diffeomorphisms of the closed unit disc.*
- (3) *If  $W$  is antisymmetric with respect to the real axis, that is,*

$$(8.2) \quad \overline{W(z)} = -W(\bar{z}), \quad \forall z \in \overline{\mathbb{D}}.$$

*then the orbits are symmetric with respect to the real axis.*

- (4) *If  $W$  is invariant by a rotation centered at zero with angle  $\theta_0$ , i.e.  $W(e^{i\theta_0}z) = e^{i\theta_0}W(z), \forall z \in \overline{\mathbb{D}}$ , then all the orbits are invariant by this rotation.*

*Proof.* (1) Let  $\psi$  be the solution associated to the flux  $W$

$$(8.3) \quad \begin{cases} \partial_t \psi(t, z) = W(\psi(t, z)), \\ \psi(0, z) = z \in \overline{\mathbb{D}}. \end{cases}$$

From the Cauchy–Lipschitz Theorem we know that the trajectory  $t \mapsto \psi(t, z)$  is defined in a maximal time interval  $(-T_*, T^*)$ , with  $T_*, T^* > 0$ , for each  $z \in \overline{\mathbb{D}}$ . Note that when  $z$  belongs to the boundary, then the second condition listed above implies necessarily that its trajectory does not leave the boundary. Since the vector field does not vanish anywhere on the boundary

according to the third condition, the trajectory will cover all the unit disc. As the equation is autonomous, this ensures that the unit disc is a periodic orbit.

By condition i) we get that (8.3) is a Hamiltonian system. Let  $H$  be the Hamiltonian function such that  $W = \nabla^\perp H$ . Since  $H$  is  $C^1$  in  $\overline{\mathbb{D}}$  and constant on the boundary  $\mathbb{T}$  according to the assumption ii), then from iii) the origin corresponds to an extremum point.

Now, taking  $|z| < 1$ , the solution is globally well-posed in time, that is,  $T_\star = T^\star = +\infty$ . This follows easily from the fact that different orbits never intersect and consequently we should get

$$|\psi(t, z)| < 1, \quad \forall t \in (-T_\star, T^\star),$$

meaning that the solution is bounded and does not touch the boundary so it is globally defined according to a classical blow-up criterion.

We will check that all the orbits are periodic inside the unit disc. This follows from some straightforward considerations on the level sets of the Hamiltonian  $H$ . Indeed, the  $\omega$ -limit of a point  $z \neq 0$  cannot contain the origin because it is the only critical point and the level sets of  $H$  around this point are periodic orbits. Thus we deduce from Poincaré–Bendixon Theorem that the  $\omega$ -limit of  $z$  will be a periodic orbit. As the level sets cannot be limit cycles then we find that the trajectory of  $z$  coincides with the periodic orbit.

(2) This follows from classical results on autonomous differential equation. In fact, we know that the flow  $\psi : \mathbb{R} \times \overline{\mathbb{D}} \rightarrow \overline{\mathbb{D}}$  is well-defined and  $\mathcal{C}^1$ . For any  $t \in \mathbb{R}$ , it realizes a bijection with  $\psi^{-1}(t, \cdot) = \psi(-t, \cdot)$ , and  $(\psi(t))_{t \in \mathbb{R}}$  generates a group of diffeomorphisms on  $\overline{\mathbb{D}}$ .

(3) The symmetry of the orbits with respect to the real axis is a consequence of the following elementary fact. Given  $z \in \mathbb{D}$  and  $t \mapsto \psi(t, z)$  its trajectory, then it follows that  $t \mapsto \overline{\psi(-t, z)}$  is also a solution of the same Cauchy problem and by uniqueness we find the identity

$$\psi(t, z) = \overline{\psi(-t, z)}, \quad \forall t \in \mathbb{R}.$$

(4) Assume that  $W$  is invariant by the rotation  $R_{\theta_0}$  centered at zero and with angle  $\theta_0$ . Let  $z \in \mathbb{D}$ , then we shall first check the identity

$$(8.4) \quad e^{i\theta_0} \psi(t, z) = \psi(t, e^{i\theta_0} z), \quad \forall t \in \mathbb{R}.$$

To do that, it suffices to verify that both functions satisfy the same differential equation with the same initial data, and thus the identity follows from the uniqueness of the Cauchy problem. Note that (8.4) means that the rotation of a trajectory is also a trajectory. Denote by  $D_{z_0}$  and  $e^{i\theta_0} D_{z_0}$  the domains delimited by the curves  $t \mapsto \psi(t, z_0)$  and  $t \mapsto e^{i\theta_0} \psi(t, z_0)$ , respectively. Then, it is a classical result that those domains are necessary simply connected and they contain the origin according to 1). Since different trajectories never intersect, then we have only two possibilities:  $D_{z_0} \subset e^{i\theta_0} D_{z_0}$  or the converse. Since the rotation is a Lebesgue preserving measure, then  $D_{z_0} = e^{i\theta_0} D_{z_0}$ , which implies that the periodic orbit  $t \mapsto \psi(t, z_0)$  is invariant by the rotation  $R_{\theta_0}$ .  $\square$

**8.2. Reformulation with the trajectory map.** In this section we discuss a new representation of solutions to the equations of the type

$$(8.5) \quad W(z) \cdot \nabla f(z) = 0, \quad \forall z \in \overline{\mathbb{D}},$$

with  $W$  a vector field as in Proposition 8.3 and  $f : \overline{\mathbb{D}} \rightarrow \mathbb{R}$  a  $\mathcal{C}^1$  function.

**Proposition 8.4.** *Let  $W : \overline{\mathbb{D}} \rightarrow \mathbb{C}$  be a vector field satisfying the assumptions i), ii) and iii) of Proposition 8.3. Then, (8.5) is equivalent to the formulation*

$$(8.6) \quad f(z) - \frac{1}{T_z} \int_0^{T_z} f(\psi(\tau, z)) d\tau = 0, \quad \forall z \in \overline{\mathbb{D}},$$

with  $T_z$  being the period of the trajectory  $t \mapsto \psi(t, z)$ .

*Proof.* We first check that (8.5) is equivalent to

$$(8.7) \quad f(\psi(t, z)) = f(z), \quad \forall t \in \mathbb{R}, \forall |z| \leq 1.$$

Although for simplicity we can assume that the equivalence is done pointwise, where we need  $f \in \mathcal{C}^1$ , the equivalence is perfectly valid in a weak sense without nothing more than assuming Hölder regularity on  $f$ . Indeed, if  $f$  is a  $\mathcal{C}^1$  function satisfying (8.7), then by differentiating in time we get

$$(W \cdot \nabla f)(\psi(t, z)) = 0, \quad \forall t \in \mathbb{R}, \forall |z| \leq 1.$$

According to Proposition 8.3, we have that (8.5) is satisfied everywhere in the closed unit disc, for any  $t$ ,  $\psi(t, \overline{\mathbb{D}}) = \overline{\mathbb{D}}$ . Conversely, if  $f$  is a  $\mathcal{C}^1$  solution to (8.5), then differentiating with respect to  $t$  the function  $t \mapsto f(\psi(t, z))$  we get

$$\frac{d}{dt} f(\psi(t, z)) = (W \cdot \nabla f)(\psi(t, z)) = 0.$$

Therefore, we have (8.7). Now, we will verify that (8.6) is in fact equivalent to (8.7). The implication (8.7)  $\implies$  (8.6) is elementary. So it remains to check the converse. From (8.6) one has

$$(8.8) \quad f(\psi(t, z)) - \frac{1}{T_{\psi(t, z)}} \int_0^{T_{\psi(t, z)}} f(\psi(\tau, \psi(t, z))) d\tau = 0.$$

Since the vector field is autonomous, then all the points located at the same orbit generate periodic trajectories with the same period, and of course with the same orbit. Therefore, we have  $T_{\psi(t, z)} = T_z$ . Using  $\psi(\tau, \psi(t, z)) = \psi(t + \tau, z)$ , a change of variables, and the  $T_z$ -periodicity of  $\tau \mapsto f(\psi(\tau, z))$ , then we deduce

$$\frac{1}{T_{\psi(t, z)}} \int_0^{T_{\psi(t, z)}} f(\psi(\tau, \psi(t, z))) d\tau = \frac{1}{T_z} \int_t^{t+T_z} f(\psi(\tau, z)) d\tau = \frac{1}{T_z} \int_0^{T_z} f(\psi(\tau, z)) d\tau = f(z).$$

Combining this with (8.8), we get (8.7). This completes the proof.  $\square$

**8.3. Persistence of the symmetry.** We shall consider a vector field  $W$  satisfying the assumptions of Proposition 8.3 and (8.2) and let  $\psi$  be its associated flow. We define the operator  $f \mapsto Sf$  by

$$Sf(z) = f(z) - \frac{1}{T_z} \int_0^{T_z} f(\psi(\tau, z)) d\tau, \quad \forall z \in \overline{\mathbb{D}}.$$

We shall prove that  $f$  and  $Sf$  share the same planar group of invariance in the following sense.

**Proposition 8.5.** *Let  $f : \overline{\mathbb{D}} \mapsto \mathbb{R}$  be a smooth function. The following assertions hold true:*

- (1) *If  $f$  is invariant by reflection with respect to the real axis, then  $Sf$  is invariant too. This means that*

$$f(\bar{z}) = f(z), \quad \forall z \in \overline{\mathbb{D}} \implies Sf(\bar{z}) = Sf(z), \quad \forall z \in \overline{\mathbb{D}}.$$

- (2) *If  $W$  and  $f$  are invariant by the rotation  $R_{\theta_0}$  centered at zero with angle  $\theta_0 \in \mathbb{R}$ , then  $Sf$  commutes with the same rotation. This means that*

$$f(e^{i\theta_0} z) = f(z), \quad \forall z \in \overline{\mathbb{D}} \implies (Sf)(e^{i\theta_0} z) = Sf(z), \quad \forall z \in \overline{\mathbb{D}}.$$

*Proof.* (1) Let  $z \in \overline{\mathbb{D}}$ , it is a simple matter to check that  $\psi(t, \bar{z}) = \overline{\psi(-t, z)}$ , which implies  $T_{\bar{z}} = T_z$ , and then

$$Sf(\bar{z}) = f(z) - \frac{1}{T_z} \int_0^{T_z} f(\overline{\psi(-\tau, z)}) d\tau = f(z) - \frac{1}{T_z} \int_0^{T_z} f(\psi(-\tau, z)) d\tau = Sf(z).$$

(2) According to Proposition (8.3) and the fact that the vector-field  $W$  is invariant by the rotation  $R_{\theta_0}$ , then we have that the orbits are symmetric with respect to this rotation and

$$T_{e^{i\theta_0}z} = T_z \quad \text{and} \quad \psi(t, e^{i\theta_0}z) = e^{i\theta_0}\psi(t, z),$$

where we have used (8.4), which implies that

$$Sf(e^{i\theta_0}z) = f(z) - \frac{1}{T_z} \int_0^{T_z} f\left(e^{i\theta_0}\psi(\tau, z)\right) d\tau = f(z) - \frac{1}{T_z} \int_0^{T_z} f(\psi(\tau, z)) d\tau = Sf(z).$$

This concludes the proof.  $\square$

**8.4. Analysis of the regularity.** Next, we are interested in studying the regularity of the the flow map (8.3) and the period map. The following result is classical, see for instance [28].

**Proposition 8.6.** *Let  $\alpha \in (0, 1)$ ,  $W : \overline{\mathbb{D}} \mapsto \mathbb{R}^2$  be a vector-field in  $\mathcal{C}^{1,\alpha}(\mathbb{D})$  satisfying the condition ii) of Proposition 8.3 and  $\psi : \mathbb{R} \times \overline{\mathbb{D}} \mapsto \overline{\mathbb{D}}$  its flow map. Then  $\psi \in \mathcal{C}^1(\mathbb{R}, \mathcal{C}^{1,\alpha}(\overline{\mathbb{D}}))$  and there exists  $C > 0$  such that*

$$\|\psi^{\pm 1}(t)\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})} \leq e^{C\|W\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})}|t|} (1 + \|W\|_{\mathcal{C}^{1,\alpha}(\overline{\mathbb{D}})}|t|), \quad \forall t \in \mathbb{R},$$

holds.

Now we intend to study the regularity of the function  $z = (x, y) \in \overline{\mathbb{D}} \mapsto T_z$ . This is a classical subject in dynamical systems and several results are obtained in this direction for smooth Hamiltonians. Notice that in the most studies in the literature the regularity is measured with respect to the energy and not with respect to the positions as we propose to do here.

Let  $z \in \overline{\mathbb{D}}$  be a given non equilibrium point, the orbit  $t \mapsto \psi(t, z)$  is periodic and  $T_z$  is the first strictly positive time such that

$$(8.9) \quad \psi(T_z, z) - z = 0.$$

This is an implicit function equation, from which we expect to deduce some regularity properties. Our result reads as follows.

**Proposition 8.7.** *Let  $W$  be a vector field in  $\mathcal{C}^1(\overline{\mathbb{D}})$ , satisfying the assumptions of Proposition 8.3 and (8.2) and such that*

$$W(z) = izU(z), \quad \forall z \in \overline{\mathbb{D}},$$

with

$$(8.10) \quad \operatorname{Re}\{U(z)\} \neq 0, \quad \forall z \in \overline{\mathbb{D}}.$$

Then the following assertions hold true:

- (1) *The map  $z \in \overline{\mathbb{D}} \mapsto T_z$  is continuous and verifies the upper bound*

$$(8.11) \quad 0 < T_z \leq \frac{2\pi}{\inf_{z \in \overline{\mathbb{D}}} |\operatorname{Re}U(z)|}, \quad \forall z \in \overline{\mathbb{D}}.$$

- (2) *If in addition  $U \in \mathcal{C}^{1,\alpha}(\mathbb{D})$ , then  $z \mapsto T_z$  belongs to  $\mathcal{C}^{1,\alpha}(\mathbb{D})$ .*

**Remark 8.8.** *Since the origin is an equilibrium point for the dynamical system, then its trajectory is periodic with any period. However, as we will see in the proof, there is a minimal strictly positive period denoted by  $T_z$ , for any curves passing through a non vanishing point  $z$ . The mapping  $z \in \overline{\mathbb{D}} \setminus \{0\} \mapsto T_z$  is not only well-defined, but it can be extended continuously to zero. Thus we shall make the following convention*

$$T_0 \equiv \lim_{z \rightarrow 0} T_z.$$

**Remark 8.9.** The upper bound in (8.11) is “almost optimal” for radial profiles, where  $U(z) = U_0(|z|) \in \mathbb{R}$ , and explicit computations yield

$$T_z = \frac{2\pi}{|U_0(|z|)|}.$$

*Proof.* (1) We shall describe the trajectory parametrization using polar coordinates. Firstly, we may write for  $z = re^{i\theta}$

$$W(z) = \left[ W^r(r, \theta) + iW^\theta(r, \theta) \right] e^{i\theta},$$

with

$$W^\theta(r, \theta) = r\operatorname{Re}(U(re^{i\theta})) \quad \text{and} \quad W^r(r, \theta) = -r\operatorname{Im}(U(re^{i\theta})).$$

Given  $0 < |z| \leq 1$ , we look for a polar parametrization of the trajectory passing through  $z$ ,

$$\psi(t, z) = r(t)e^{i\theta(t)}, \quad r(0) = |z|, \quad \theta(0) = \operatorname{Arg}(z).$$

Inserting into (8.3) we obtain the system

$$\begin{aligned} \dot{r}(t) &= -r(t)\operatorname{Im} \left\{ U \left( r(t)e^{i\theta(t)} \right) \right\} \triangleq P(r(t), \theta(t)) \\ \dot{\theta}(t) &= \operatorname{Re} \left\{ U \left( r(t)e^{i\theta(t)} \right) \right\} \triangleq Q(r(t), \theta(t)). \end{aligned}$$

From the assumption (8.2) we find

$$\overline{U(z)} = U(\bar{z}), \quad \forall z \in \overline{\mathbb{D}},$$

which implies in turn that

$$P(r, -\theta) = -P(r, \theta) \quad \text{and} \quad Q(r, -\theta) = Q(r, \theta), \quad \forall r \in [0, 1], \forall \theta \in \mathbb{R}.$$

Thus, we have the Fourier expansions

$$P(r, \theta) = \sum_{n \in \mathbb{N}^*} P_n(r) \sin(n\theta) \quad \text{and} \quad Q(r, \theta) = \sum_{n \in \mathbb{N}} Q_n(r) \cos(n\theta).$$

Denoting by  $T_n$  and  $U_n$  the classical Chebyshev polynomials that satisfy the identities

$$\begin{aligned} \cos(n\theta) &= T_n(\cos \theta) \\ \sin(n\theta) &= \sin(\theta)U_{n-1}(\cos \theta), \end{aligned}$$

we obtain

$$\begin{aligned} P(r, \theta) &= \sin \theta \sum_{n \in \mathbb{N}^*} P_n(r) U_{n-1}(\cos \theta) \equiv \sin \theta F_1(r, \cos \theta). \\ Q(r, \theta) &= \sum_{n \in \mathbb{N}} Q_n(r) T_n(\cos \theta) \equiv F_2(r, \cos \theta). \end{aligned}$$

Coming back to (8.12), we get

$$(8.12) \quad \begin{aligned} \dot{r}(t) &= \sin \theta(t) F_1(r(t), \cos \theta(t)) \\ \dot{\theta}(t) &= F_2(r(t), \cos \theta(t)). \end{aligned}$$

We look for solutions in the form

$$r(t) = f_z(\cos(\theta(t))), \quad \text{with} \quad f_z : [-1, 1] \rightarrow \mathbb{R},$$

and then  $f$  satisfies the differential equation

$$(8.13) \quad f'_z(s) = - \left( \frac{F_1}{F_2} \right) (f_z(s), s), \quad f(\cos \theta) = |z|.$$

Note that the preceding fraction is well-defined since the assumption (8.10) is equivalent to

$$F_2(r, \cos \theta) \neq 0, \quad \forall r \in [0, 1], \theta \in \mathbb{R}.$$



Theorem 8.3-ii) agrees with

$$F_1(0, s) = F_1(1, s) = 0, \quad \forall |s| \leq 1,$$

which implies that the system (8.13) admits a unique solution  $f_z : [-1, 1] \rightarrow \mathbb{R}_+$  such that

$$0 \leq f_z(s) \leq 1, \quad \forall s \in [-1, 1].$$

Hence, integrating the second equation of (8.12) we find after a change of variable

$$\int_{\theta_0}^{\theta(t)} \frac{1}{F_2(f_z(\cos s), \cos s)} ds = t,$$

and, therefore, the following formula for the period is obtained

$$(8.14) \quad T_z = \left| \int_{\theta_0}^{\theta_0+2\pi} \frac{1}{F_2(f_z(\cos s), \cos s)} ds \right| = \int_0^{2\pi} \frac{1}{|F_2(f_z(\cos s), \cos s)|} ds.$$

This gives the bound of the period stated in (8.11). The continuity  $z \mapsto T_z$  follows from the same property of  $z \mapsto f_z$ , which can be derived from the continuous dependence with respect to the initial conditions.

(2) Now, we will study the regularity of the period in  $\mathcal{C}^{1,\alpha}$ . Note that (8.14) involves the function  $f_z$  which is not smooth enough because the initial condition  $z \mapsto f_z(\cos \theta)$  is only Lipschitz. So it seems quite complicate to follow the regularity in  $\mathcal{C}^{1,\alpha}$  from that formula. The alternative way is to study the regularity of the period using the implicit equation (8.9). Thus, differentiating this equation with respect to  $x$  and  $y$  we obtain

$$\begin{aligned} (\partial_x T_z) \partial_t \psi(T_z, x, y) + \partial_x \psi(T_z, x, y) - 1 &= 0, \\ (\partial_y T_z) \partial_t \psi(T_z, x, y) + \partial_y \psi(T_z, x, y) - i &= 0. \end{aligned}$$

From the flow equation and the periodicity condition we get

$$\partial_t \psi(T_z, z) = W(z),$$

which implies

$$(8.15) \quad \begin{cases} (\partial_x T_z) W(z) + \partial_x \psi(T_z, x, y) - 1 = 0, \\ (\partial_y T_z) W(z) + \partial_y \psi(T_z, x, y) - i = 0. \end{cases}$$

Due to the assumption on  $W$ , the flow equation can be written as

$$\partial_t \psi(t, z) = i\psi(t, z)U(\psi(t, z)), \quad \psi(0, z) = z,$$

which can be integrated, obtaining

$$\psi(t, z) = z e^{i \int_0^t U(\psi(\tau, z)) d\tau}.$$

By differentiating this identity with respect to  $x$ , it yields

$$\partial_x \psi(t, z) = e^{i \int_0^t U(\psi(\tau, z)) d\tau} \left[ 1 + iz \int_0^t \partial_x \{U(\psi(\tau, z))\} d\tau \right].$$

Since  $\psi(T_z, z) = z$ , we have

$$e^{i \int_0^{T_z} U(\psi(\tau, z)) d\tau} = 1,$$

and thus

$$\partial_x \psi(T_z, z) = 1 + iz \int_0^{T_z} \partial_x \{U(\psi(\tau, z))\} d\tau.$$

Similarly, we find

$$\partial_y \psi(T_z, z) = i + iz \int_0^{T_z} \partial_y \{U(\psi(\tau, z))\} d\tau.$$

Combining these identities with (8.15), we obtain

$$\begin{cases} (\partial_x T_z)W(z) &= -iz \int_0^{T_z} \partial_x \{U(\psi(\tau, z))\} d\tau, \\ (\partial_y T_z)W(z) &= -iz \int_0^{T_z} \partial_y \{U(\psi(\tau, z))\} d\tau, \end{cases}$$

which, using the structure of  $W$ , reads as

$$\begin{cases} (\partial_x T_z)U(z) &= -i \int_0^{T_z} \partial_x \{U(\psi(\tau, z))\} d\tau, \\ (\partial_y T_z)U(z) &= -i \int_0^{T_z} \partial_y \{U(\psi(\tau, z))\} d\tau. \end{cases}$$

Now, notice that from Theorem 8.3-iv), the vector field  $W$  vanishes only at zero and since  $U(0) \neq 0$ , we find

$$U(z) \neq 0, \quad \forall z \in \overline{\mathbb{D}}.$$

This implies that  $z \in \overline{\mathbb{D}} \mapsto \frac{1}{U(z)}$  is well-defined and belongs to  $\mathcal{C}^{1,\alpha}(\mathbb{D})$ . Therefore, we can write

$$(8.16) \quad \nabla_z T_z = -\frac{i}{U(z)} \int_0^{T_z} \nabla_z \{U(\psi(\tau, z))\} d\tau,$$

where we have used the notation  $\nabla_z = (\partial_x, \partial_y)$ . According to Proposition 8.6 and classical composition laws, we obtain

$$\tau \mapsto \nabla_z \{U(\psi(\tau, \cdot))\} \in \mathcal{C}(\mathbb{R}; \mathcal{C}^{0,\alpha}(\mathbb{D})).$$

Since  $z \mapsto T_z$  is continuous, then we find by composition that

$$\varphi : z \in \overline{\mathbb{D}} \mapsto \int_0^{T_z} \nabla_z \{U(\psi(\tau, z))\} d\tau \in \mathcal{C}(\overline{\mathbb{D}}).$$

Combining this information with (8.16), we deduce that  $z \mapsto T_z \in \mathcal{C}^1(\overline{\mathbb{D}})$ . Hence, we find in turn that  $\varphi \in \mathcal{C}^{0,\alpha}(\mathbb{D})$  by composition. Using (8.16) again, it follows that  $z \mapsto \nabla_z T_z \in \mathcal{C}^{0,\alpha}(\mathbb{D})$ . Thus,  $z \mapsto T_z \in \mathcal{C}^{1,\alpha}(\mathbb{D})$ . This achieves the proof.  $\square$

**8.5. Application to the nonlinear problem.** We intend in this section to prove Theorem 8.2. Let us point out that from Proposition 3.3, the nonlinear vector field  $W(\Omega, f, \Phi)$  is chosen in order to be tangent to the boundary everywhere. We will see that not only this assumption but all the assumptions of Proposition 8.3 are satisfied if  $f$  is chosen close to a suitable radial profile.

**Lemma 8.10.** *Let  $g \in \mathcal{C}_{s,m}^{1,\alpha}(\mathbb{D})$  and  $\phi \in \mathcal{H}\mathcal{C}_m^{2,\alpha}(\mathbb{D})$ , then  $W(\Omega, f, \Phi) \in \mathcal{C}^{1,\alpha}(\mathbb{D})$  and satisfies the symmetry properties (8.2) and*

$$W(\Omega, f, \Phi)(e^{i\frac{2\pi}{m}} z) = e^{i\frac{2\pi}{m}} W(\Omega, f, \Phi)(z).$$

Moreover, if  $m \geq 4$  then

$$(8.17) \quad W(\Omega, f, \Phi)(z) = izU(z),$$

with  $U \in \mathcal{C}^{1,\alpha}(\mathbb{D})$

*Proof.* Using Proposition 3.1 and Proposition 7.1, then the regularity and the symmetry properties of  $W(\Omega, f, \Phi)$  are verified. Let us now check (8.17). Firstly, since  $\Phi(0) = 0$  and  $\phi \in \mathcal{H}\mathcal{C}^{2,\alpha}(\mathbb{D})$ , we have

$$\Phi(z) = z\Phi_1(z), \quad \Phi_1 \in \mathcal{C}^{1,\alpha}(\mathbb{D}).$$

In addition,  $\overline{\Phi'} \in \mathcal{C}^{1,\alpha}(\mathbb{D})$ , and thus we find

$$i\Phi(z)\overline{\Phi'}(z) = izU_1(z) \quad \text{with} \quad U_1 \in \mathcal{C}^{1,\alpha}(\mathbb{D}).$$

Now, to get (8.17) it is enough to check that

$$I(f, \Phi)(z) = zU_2(z), \quad \text{with} \quad U_2 \in \mathcal{C}^{1,\alpha}(\mathbb{D}),$$

where

$$I(f, \Phi)(z) \triangleq \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y)}{\Phi(z) - \Phi(y)} |\Phi'(y)|^2 dA(y).$$

We look for the first order Taylor expansion around the origin of  $I(f, \Phi)(z)$ . Using the  $m$ -fold symmetry of  $f$  and  $\Phi$ , it is clear that

$$I(f, \Phi)(0) = - \int_{\mathbb{D}} \frac{f(y)}{\Phi(y)} |\Phi'(y)|^2 dA(y) = - \int_{\mathbb{D}} \frac{f(e^{i\frac{2\pi}{m}} y)}{\Phi(e^{i\frac{2\pi}{m}} y)} |\Phi'(e^{i\frac{2\pi}{m}} y)|^2 dA(y) = e^{i\frac{2\pi}{m}} I(f, \Phi)(0),$$

which leads to

$$(8.18) \quad I(f, \Phi)(0) = 0,$$

for  $m \geq 2$ . This implies that one can always write  $I(f, \Phi)(z) = zU_2(z)$ , but  $U_2$  is only bounded:

$$\|U_2\|_{L^\infty(\overline{\mathbb{D}})} \leq \|\nabla_z I_2\|_{L^\infty(\mathbb{D})} \leq \|I\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$

We shall see how the extra symmetry helps to get more regularity for  $U_2$ . According to Taylor expansion one gets

$$I(f, \Phi)(z) = az + b\bar{z} + cz^2 + d\bar{z}^2 + e|z|^2 + \text{l.o.t.} \quad \text{with } a, b, c, d, e \in \mathbb{C}.$$

Using the reflection invariance with respect to the real axis of  $f$  and  $\Phi$ , we obtain

$$\overline{I(f, \Phi)(z)} = I(\bar{z}),$$

which implies that  $a, b, c, d, e \in \mathbb{R}$ . Now, the rotation invariance leads to

$$I(f, \Phi)(e^{i\frac{2\pi}{m}} z) = e^{i\frac{2\pi}{m}} I(z).$$

Then, we obtain

$$e = 0, \quad b(e^{i\frac{4\pi}{m}} - 1) = 0, \quad c(e^{i\frac{2\pi}{m}} - 1) = 0, \quad \text{and} \quad d(e^{i\frac{6\pi}{m}} - 1) = 0.$$

This implies that  $b = c = d = 0$ , whenever  $m \geq 4$ . Thus, we have

$$I(f, \Phi)(z) = az + h(z)$$

with  $h \in \mathcal{C}^{2,\alpha}(\mathbb{D})$ , and

$$h(0) = 0, \quad \nabla_z h(0) = 0, \quad \text{and} \quad \nabla_{\bar{z}}^2 h(0) = 0.$$

From this we claim that

$$h(z) = zk(z), \quad \text{with } k \in \mathcal{C}^{1,\alpha}(\mathbb{D}),$$

which concludes the proof.  $\square$

Now we are in a position to prove Theorem 8.2.

*Proof of Theorem 8.2.* The existence of the conformal map  $\Phi$  comes directly from Proposition 3.3, which gives the boundary equation (2.4). Moreover, Lemma 8.10 gives the decomposition (8.17) and provides the necessary properties to apply Proposition 8.3. Furthermore, we can use Proposition 8.4 in order to obtain the equivalence between a rotating solution (1.2) of the Euler equations and the solution of (8.6). Proposition 8.6 gives the regularity of the flow map, and it remains to check (8.10), using Proposition 8.7, in order to get the regularity of the period function. In the case of radial profile, we know that

$$W(\Omega, f_0, \text{Id})(z) = izU_0(z), \quad U_0(z) = -\Omega + \frac{1}{r^2} \int_0^r sf_0(s)ds,$$

which implies

$$\begin{aligned} W(\Omega, f, \Phi)(z) - W(\Omega, f_0, \text{Id})(z) &= -\Omega\phi(z) \left(1 + \overline{g'(z)}\right) - \Omega z \overline{g'(z)} + \overline{g'(z)} I(f, \Phi)(z) \\ &\quad + I(f, \Phi)(z) - I(f_0, \text{Id})(z). \end{aligned}$$

It is easy to see that

$$\left| \frac{\phi(z)}{z} \left( 1 + \overline{g'(z)} \right) \right| \leq \|\phi'\|_{L^\infty} (1 + \|g'\|_{L^\infty(\mathbb{D})}).$$

From (8.18) and Lemma B.2, we have

$$\left| \overline{g'(z)} \frac{I(f, \Phi)(z)}{z} \right| \leq \|g'\|_{L^\infty(\mathbb{D})} \|\nabla_z I(f, \Phi)\|_{L^\infty(\mathbb{D})} \leq C(\Phi) \|f\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} \|g'\|_{L^\infty(\mathbb{D})}.$$

Using again (8.18), we deduce

$$\left| \frac{I(f, \Phi)(z) - I(f_0, \Phi_0)(z)}{z} \right| \leq \left\| \nabla_z [I(f, \Phi) - I(f_0, \Phi_0)] \right\|_{L^\infty(\mathbb{D})}.$$

Straightforward computations imply

$$\begin{aligned} I(f, \Phi)(z) - I(f_0, \Phi_0)(z) &= \frac{1}{2\pi} \int_{\mathbb{D}} \frac{g(y)}{\bar{z} - \bar{y}} dA(y) + \frac{1}{2\pi} \int_{\mathbb{D}} \frac{f(y) [\phi(\bar{y}) - \phi(\bar{z})]}{(\bar{y} - \bar{z}) (\Phi(\bar{y}) - \Phi(\bar{z}))} dA(y) \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{D}} \frac{2\operatorname{Re}\{\phi'(y)\} + |\phi'(y)|^2}{\Phi(\bar{z}) - \Phi(\bar{y})} f(y) dA(y). \end{aligned}$$

From this and using Lemma B.2, we claim that

$$\left\| \nabla_z [I(f, \Phi) - I(f_0, \Phi_0)] \right\|_{L^\infty(\mathbb{D})} \leq C(\Phi) \|g\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}.$$

Combining the preceding estimates we find

$$W(\Omega, f, \Phi)(z) = z \left[ iU_0(z) + \frac{W(\Omega, f, \Phi)(z) - W(\Omega, f_0, \operatorname{Id})(z)}{z} \right] \equiv z \left[ iU_0(z) + \widehat{W}(\Omega, f, \Phi)(z) \right],$$

with

$$\|\widehat{W}(\Omega, f, \Phi)\|_{L^\infty(\mathbb{D})} \leq C(f, \Phi) (\|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} + \|\phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}).$$

Now, we take  $h, \phi$  small enough such that

$$C(f, \Phi) (\|h\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} + \|\phi\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}) \leq \varepsilon,$$

and  $\varepsilon$  verifies

$$0 < 2\varepsilon \leq \inf_{0 < r \leq 1} \left| \Omega - \frac{1}{r^2} \int_0^r s f_0(s) ds \right| = \inf_{z \in \mathbb{D}} |U_0(z)|,$$

in order to have

$$W(\Omega, f, \Phi)(z) = izU(z), \quad \text{with} \quad 2|\operatorname{Re}\{U(z)\}| \geq \inf_{0 < r \leq 1} \left| \Omega - \frac{1}{r^2} \int_0^r s f_0(s) ds \right|.$$

To end the proof let us check that this infimum is strictly positive for the quadratic profile  $f_0(r) = Ar^2 + B$ , where  $A > 0$ . Take  $\xi \in (-a, a) \mapsto (\Omega_\xi, f_\xi, \phi_\xi)$  the bifurcating curve from Theorem 1.1. Then for  $a$  small enough

$$\Omega_\xi = \frac{B}{2} + \frac{A}{4x_\xi}, \quad x_\xi < 1 \quad \text{and} \quad \Omega_\xi \notin \mathcal{S}_{\text{sing}}$$

Thus

$$\Omega_\xi - \frac{1}{r^2} \int_0^r s f_0(s) ds = \frac{A}{4} \left( \frac{1}{x_\xi} - r^2 \right).$$

Consequently

$$\inf_{r \in [0,1]} \left| \Omega_\xi - \frac{1}{r^2} \int_0^r s f_0(s) ds \right| = \inf_{r \in [0,1]} \frac{A}{4} \left| \frac{1}{x_\xi} - r^2 \right| > 0.$$

This achieves the proof.  $\square$

The remaining part of this paper is devoted to some basic materials that we have used in the preceding proofs.

## APPENDIX A. BIFURCATION THEORY AND FREDHOLM OPERATORS

We shall recall Crandall-Rabinowitz Theorem which is a fundamental tool in bifurcation theory. Notice that the main aim of this theory is to explore the topological transitions of the phase portrait through the variation of some parameters. A particular case is to understand this transition in the equilibria set for the stationary problem  $F(\lambda, x) = 0$ , where  $F : \mathbb{R} \times X \rightarrow Y$  is a smooth function and the spaces  $X$  and  $Y$  are Banach spaces. Assuming that one has a particular solution,  $F(\lambda, 0) = 0$  for any  $\lambda \in \mathbb{R}$ , we would like to explore the bifurcation diagram close to this trivial solution, and see whether we can find multiple branches of solutions bifurcating from a given point  $(\lambda_0, 0)$ . When this occurs we say that the pair  $(\lambda_0, 0)$  is a bifurcation point. When the linearized operator around this point generates a Fredholm operator, then one may use Lyapunov-Schmidt reduction in order to reduce the infinite-dimensional problem to a finite-dimensional one. For the latter problem we just formulate suitable assumptions so that the Implicit Function Theorem can be applied. For more discussion in this subject, we refer to see [37, 38].

In what follows we shall recall some basic results on Fredholm operators.

**Definition 1.** *Let  $X$  and  $Y$  be two Banach spaces. A continuous linear mapping  $T : X \rightarrow Y$ , is a Fredholm operator if it fulfills the following properties,*

- (1)  $\dim \text{Ker } T < \infty$ ,
- (2)  $\text{Im } T$  is closed in  $Y$ ,
- (3)  $\text{codim Im } T < \infty$ .

*The integer  $\dim \text{Ker } T - \text{codim Im } T$  is called the Fredholm index of  $T$ .*

Next, we shall discuss the index persistence through compact perturbations, see [37, 38].

**Proposition A.1.** *The index of a Fredholm operator remains unchanged under compact perturbations.*

Now, we recall the classical Crandall-Rabinowitz Theorem whose proof can be found in [17].

**Theorem A.2** (Crandall-Rabinowitz Theorem). *Let  $X, Y$  be two Banach spaces,  $V$  be a neighborhood of 0 in  $X$  and  $F : \mathbb{R} \times V \rightarrow Y$  be a function with the properties,*

- (1)  $F(\lambda, 0) = 0$  for all  $\lambda \in \mathbb{R}$ .
- (2) The partial derivatives  $\partial_\lambda F_\lambda$ ,  $\partial_f F$  and  $\partial_\lambda \partial_f F$  exist and are continuous.
- (3) The operator  $F_f(0, 0)$  is Fredholm of zero index and  $\text{Ker}(F_f(0, 0)) = \langle f_0 \rangle$  is one-dimensional.
- (4) *Transversality assumption:*  $\partial_\lambda \partial_f F(0, 0)f_0 \notin \text{Im}(\partial_f F(0, 0))$ .

*If  $Z$  is any complement of  $\text{Ker}(F_f(0, 0))$  in  $X$ , then there is a neighborhood  $U$  of  $(0, 0)$  in  $\mathbb{R} \times X$ , an interval  $(-a, a)$ , and two continuous functions  $\Phi : (-a, a) \rightarrow \mathbb{R}$ ,  $\beta : (-a, a) \rightarrow Z$  such that  $\Phi(0) = \beta(0) = 0$  and*

$$F^{-1}(0) \cap U = \{(\Phi(\xi), \xi f_0 + \xi \beta(\xi)) : |\xi| < a\} \cup \{(\lambda, 0) : (\lambda, 0) \in U\}.$$

## APPENDIX B. POTENTIAL ESTIMATES

This appendix is devoted to some classical estimates on potential theory that we have used before in Section 3. We shall deal in particular with truncated operators whose kernels are singular along the diagonal. They have the form

$$(B.1) \quad \mathcal{L}f(z) \triangleq \int_{\mathbb{D}} K(z, y)f(y)dA(y), z \in \mathbb{D}$$

The action of such operators over various function spaces and its connection to the singularity of the kernel is widely studied in the literature, see [26, 31, 40, 46, 47, 62]. In the case of Caldéron-Zygmund operator we refer to the recent papers [18, 19, 44] and the references therein.

In what follows we shall establish some useful estimates whose proofs are classical and for the convenience of the reader we decide to provide most the details. The first result reads as follows.

**Lemma B.1.** *Let  $\alpha \in (0, 1)$  and  $K : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$  is smooth off the diagonal and satisfies*

$$(B.2) \quad |K(z_1, y)| \leq \frac{C_0}{|z_1 - y|} \quad \text{and} \quad |K(z_1, y) - K(z_2, y)| \leq C_0 \frac{|z_1 - z_2|}{|z_1 - y||z_2 - y|},$$

for any  $z_1, z_2 \neq y \in \mathbb{D}$ , with  $C_0$  a real positive constant. The operator defined in (B.1)

$$\mathcal{L} : L^\infty(\mathbb{D}) \rightarrow \mathcal{C}^{0,\alpha}(\mathbb{D})$$

is continuous, with the estimate

$$\|\mathcal{L}f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq CC_0\|f\|_{L^\infty(\mathbb{D})}, \quad \forall f \in L^\infty(\mathbb{D}),$$

where  $C$  is a constant depending on  $\alpha$ .

*Proof.* It is easy to see that

$$\|\mathcal{L}f\|_{L^\infty(\mathbb{D})} \leq CC_0\|f\|_{L^\infty(\mathbb{D})}.$$

Using (B.2) combined with an interpolation argument we may write

$$\begin{aligned} |K(z_1, y) - K(z_2, y)| &\leq |K(z_1, y) - K(z_2, y)|^\alpha (|K(z_1, y)|^{1-\alpha} + |K(z_2, y)|^{1-\alpha}) \\ &\leq CC_0|z_1 - z_2|^\alpha \left[ \frac{1}{|z_1 - y|^{1+\alpha}} + \frac{1}{|z_2 - y|^{1+\alpha}} \right], \end{aligned}$$

Thus from the inequality

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} \frac{|f(y)|}{|z - y|^{1+\alpha}} dA(y) \leq CC_0\|f\|_{L^\infty(\mathbb{D})}$$

we deduce the desired result.  $\square$

Before giving the second result, we need to recall Cauchy–Pompeiu’s formula that we shall use later. Let  $D$  be a simply connected domain and  $\varphi : \bar{D} \rightarrow \mathbb{C}$  be a  $\mathcal{C}^1$  complex function, then

$$(B.3) \quad -\frac{1}{\pi} \int_D \frac{\partial_{\bar{y}} \varphi(y)}{w - y} dA(y) = \frac{1}{2\pi i} \int_{\partial D} \frac{\varphi(w) - \varphi(\xi)}{w - \xi} d\xi.$$

The following result deals with a specific type of integrals that we have already encountered in Proposition 3.1. More precisely we shall be concerned with the integral

$$(B.4) \quad \mathcal{F}[\Phi](f)(z) \triangleq \int_{\mathbb{D}} \frac{f(y)}{\Phi(z) - \Phi(y)} |\Phi'(y)|^2 dA(y), \quad z \in \mathbb{D}.$$

**Lemma B.2.** *Let  $\alpha \in (0, 1)$  and  $\Phi : \mathbb{D} \rightarrow \Phi(\mathbb{D}) \subset \mathbb{C}$  be a conformal bi-Lipschitz function of class  $\mathcal{C}^{2,\alpha}(\mathbb{D})$ . Then*

$$\mathcal{F}[\Phi] : \mathcal{C}^{1,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}^{1,\alpha}(\mathbb{D}),$$

is continuous. Moreover, the functional  $\mathcal{F} : \Phi \in \mathcal{U} \mapsto \mathcal{F}[\Phi]$  is continuous, with

$$\mathcal{U} \triangleq \left\{ \Phi \in \mathcal{C}^{2,\alpha}(\mathbb{D}) : \Phi \text{ is bi-Lipschitz and conformal} \right\}.$$

*Proof.* We start with splitting  $\mathcal{F}[\Phi]f$  as follows

$$\begin{aligned} \mathcal{F}[\Phi]f(z) &= \int_{\mathbb{D}} \frac{f(y) - f(z)}{\Phi(z) - \Phi(y)} |\Phi'(y)|^2 dA(y) + f(z) \int_{\mathbb{D}} \frac{|\Phi'(y)|^2}{\Phi(z) - \Phi(y)} dA(y) \\ &\triangleq \mathcal{F}_1[\Phi]f(z) + f(z)\mathcal{F}_2[\Phi]f(z). \end{aligned}$$

Let us estimate the first term  $\mathcal{F}_1[\Phi]f$ . The  $L^\infty(\mathbb{D})$  bound is straightforward and comes from

$$\sup_{z,y \in \mathbb{D}} \frac{|f(y) - f(z)|}{|\Phi(z) - \Phi(y)|} \leq \frac{\|f\|_{\text{Lip}}}{\|\Phi^{-1}\|_{\text{Lip}}}.$$

Setting

$$K[\Phi](z, y) = \nabla_z \left( \frac{f(y) - f(z)}{\Phi(z) - \Phi(y)} \right),$$

then one can easily check that  $K$  satisfies the assumptions

$$|K[\Phi](z, y)| \leq C \|f\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} \frac{1}{|z - y|},$$

$$|K[\Phi](z_1, y) - K[\Phi](z_2, y)| \leq C \|f\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} \max_{i,j \in \{1,2\}} \left[ \frac{|z_i - z_j|^\alpha}{|z_i - y|} + \frac{|z_i - z_j|}{|z_i - y||z_j - y|} \right],$$

where the constant  $C$  depends on  $\Phi$ . Thus, Lemma (B.1) yields

$$(B.5) \quad \|\nabla_{\mathcal{F}_1}[\Phi]f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C \|f\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}$$

and hence we find

$$\|\mathcal{F}_1[\Phi]f\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} \leq C \|f\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})}$$

Let us now check the continuity of the operator  $\mathcal{F}_1 : \Phi \in \mathcal{U} \mapsto \mathcal{F}_1[\Phi]$ . Taking  $\Phi_1, \Phi_2 \in \mathcal{U}$  we may write,

$$\begin{aligned} |\mathcal{F}_1[\Phi_1]f(z) - \mathcal{F}_1[\Phi_2]f(z)| &\leq \int_{\mathbb{D}} \frac{|f(y) - f(z)|}{|\Phi_1(y) - \Phi_1(z)|} \left| |\Phi_1'(y)|^2 - |\Phi_2'(y)|^2 \right| dA(y) \\ &+ \int_{\mathbb{D}} |f(y) - f(z)| |\Phi_2'(y)| \left| \frac{1}{\Phi_1(z) + \Phi_1(y)} - \frac{1}{\Phi_2(z) + \Phi_2(y)} \right| dA(y) \\ &\leq C \|f\|_{\text{Lip}} (\|\Phi_1' - \Phi_2'\|_{L^\infty(\mathbb{D})} + \|\Phi_1 - \Phi_2\|_{L^\infty(\mathbb{D})}). \end{aligned}$$

Similarly we get

$$\begin{aligned} \nabla(\mathcal{F}_1[\Phi_1]f(z) - \mathcal{F}_1[\Phi_2]f(z)) &= \int_D K[\Phi_1](z, y) (|\Phi_1'(y)|^2 - |\Phi_2'(y)|^2) dA(y) \\ &+ \int_{\mathbb{D}} (K[\Phi_1](z, y) - K[\Phi_2](z, y)) |\Phi_2'(y)|^2 dA(y). \end{aligned}$$

Now, performing the same arguments as for (B.5) allows to get,

$$\left\| \int_D K[\Phi_1](\cdot, y) (|\Phi_1'(y)|^2 - |\Phi_2'(y)|^2) dA(y) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C \|\Phi_1' - \Phi_2'\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})}.$$

For the second integral term, we proceed first with splitting  $K[\Phi]$  in the following way

$$\begin{aligned} K[\Phi](z, y) &= -\frac{\nabla_z f(z)}{\Phi(z) - \Phi(y)} + (f(y) - f(z)) \nabla_z \left( \frac{1}{\Phi(z) - \Phi(y)} \right) \\ &\triangleq -\nabla_z f(z) K_1[\Phi](z, y) + K_2[\Phi](z, y). \end{aligned}$$

Let us check that  $K_1[\Phi_1] - K_1[\Phi_2]$  obeys to the assumptions of Lemma B.1. For the first one, it is clear from elementary computations that

$$|K_1[\Phi_1](z, y) - K_1[\Phi_2](z, y)| \leq C \|\Phi_1 - \Phi_2\|_{\text{Lip}(\mathbb{D})} |z - y|^{-1}.$$

Adding and subtracting adequately we obtain

$$\begin{aligned} &\left| (K_1[\Phi_1] - K_1[\Phi_2])(z_1, y) - (K_1[\Phi_1] - K_1[\Phi_2])(z_2, y) \right| \\ &\leq \left| \frac{((\Phi_1 - \Phi_2)(z_1) - (\Phi_1 - \Phi_2)(y))(\Phi_1(z_2) - \Phi_1(y))(\Phi_2(z_2) - \Phi_2(z_1))}{(\Phi_1(z_1) - \Phi_1(y))(\Phi_1(z_2) - \Phi_1(y))(\Phi_2(z_1) - \Phi_2(y))(\Phi_2(z_2) - \Phi_2(y))} \right| \\ &+ \left| \frac{(\Phi_2(z_1) - \Phi_2(y))(\Phi_1(z_2) - \Phi_1(y))((\Phi_1 - \Phi_2)(z_1) - (\Phi_1 - \Phi_2)(z_2))}{(\Phi_1(z_1) - \Phi_1(y))(\Phi_1(z_2) - \Phi_1(y))(\Phi_2(z_1) - \Phi_2(y))(\Phi_2(z_2) - \Phi_2(y))} \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \frac{((\Phi_1 - \Phi_2)(z_2) - (\Phi_1 - \Phi_2)(y))(\Phi_2(z_1) - \Phi_2(y))(\Phi_1(z_1) - \Phi_2(z_2))}{(\Phi_1(z_1) - \Phi_1(y))(\Phi_1(z_2) - \Phi_1(y))(\Phi_2(z_1) - \Phi_2(y))(\Phi_2(z_2) - \Phi_2(y))} \right| \\
& \leq C \|\Phi_1 - \Phi_2\|_{\text{Lip}} \frac{|z_1 - z_2|}{|z_1 - y||z_2 - y|}.
\end{aligned}$$

Therefore, by applying Lemma B.1 we deduce that

$$\left\| \int_{\mathbb{D}} (K_1[\Phi_1](\cdot, y) - K_1[\Phi_2](\cdot, y)) |\Phi_2'(y)|^2 dA(y) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C \|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}.$$

Let us deal with  $K_2$ . Note that  $\frac{1}{\Phi(z) - \Phi(y)}$  is holomorphic, and then we can work with its complex derivative. We write it as

$$(B.6) \quad K_2[\Phi](z, y) = \frac{f(z) - f(y)}{(\Phi(z) - \Phi(y))^2} \Phi'(z).$$

We wish to apply once again Lemma B.1 and for this purpose we should check the suitable estimate for the kernel. From straightforward computations we find that

$$\begin{aligned}
|K_2[\Phi_1](z, y) - K_2[\Phi_2](z, y)| & \leq \left| \frac{f(y) - f(z)}{(\Phi_1(z) - \Phi_1(y))^2} (\Phi_1'(z) - \Phi_2'(z)) \right| \\
& + \left| (f(y) - f(z)) \Phi_2'(z) \left( \frac{1}{(\Phi_1(z) - \Phi_1(y))^2} - \frac{1}{(\Phi_2(z) - \Phi_2(y))^2} \right) \right| \\
& \leq C \|f\|_{\text{Lip}} \|\Phi_1 - \Phi_2\|_{\text{Lip}} |z - y|^{-1}
\end{aligned}$$

while for the second hypothesis we write

$$\begin{aligned}
& |(K_2[\Phi_1] - K_2[\Phi_2])(z_1, y) - (K_2[\Phi_1] - K_2[\Phi_2])(z_2, y)| \\
& \leq \left| \frac{f(y) - f(z_1)}{(\Phi_1(z_1) - \Phi_1(y))^2} (\Phi_1'(z_1) - \Phi_2'(z_1)) - \frac{f(y) - f(z_2)}{(\Phi_1(z_2) - \Phi_1(y))^2} (\Phi_1'(z_2) - \Phi_2'(z_2)) \right| \\
& + \left| (f(y) - f(z_1)) \Phi_2'(z_1) \frac{((\Phi_1 - \Phi_2)(z_1) - (\Phi_1 - \Phi_2)(y))((\Phi_1 + \Phi_2)(z_1) - (\Phi_1 + \Phi_2)(y))}{(\Phi_1(z_1) - \Phi_1(y))^2 (\Phi_2(z_1) - \Phi_2(y))^2} \right. \\
& \left. - (f(y) - f(z_2)) \Phi_2'(z_2) \frac{((\Phi_1 - \Phi_2)(z_2) - (\Phi_1 - \Phi_2)(y))((\Phi_1 + \Phi_2)(z_2) - (\Phi_1 + \Phi_2)(y))}{(\Phi_1(z_2) - \Phi_1(y))^2 (\Phi_2(z_2) - \Phi_2(y))^2} \right| \\
& \leq C \|f\|_{\text{Lip}} (\|\Phi_1 - \Phi_2\|_{\text{Lip}} + \|\Phi_1' - \Phi_2'\|_{\text{Lip}}) \frac{|z_1 - z_2|}{|z_1 - y||z_2 - y|}.
\end{aligned}$$

It follows from Lemma B.1 that

$$\left\| \int_{\mathbb{D}} (K_2[\Phi_1](\cdot, y) - K_2[\Phi_2](\cdot, y)) |\Phi_2'(y)|^2 dA(y) \right\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C \|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})},$$

which concludes the proof of the continuity of  $\mathcal{F}_1$  with respect to  $\Phi$ .

Let us now move to the second term  $\mathcal{F}_2$ . By using a change of variables one may write

$$\mathcal{F}_2[\Phi]f(z) = \int_{\Phi(\mathbb{D})} \frac{1}{\Phi(z) - y} dA(y).$$

First, note that

$$\|\mathcal{F}_2[\Phi]f\|_{L^\infty(\mathbb{D})} \leq C,$$

and

$$\|\mathcal{F}_2[\Phi_1]f - \mathcal{F}_2[\Phi_2]f\|_{L^\infty(\mathbb{D})} \leq C \|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}.$$

By Cauchy–Pompeiu’s formula (B.3) we get

$$\mathcal{F}_2[\Phi]f(z) = \pi \overline{\Phi(z)} - \frac{1}{2i} \int_{\Phi(\mathbb{T})} \frac{\bar{\xi}}{\xi - \Phi(z)} d\xi$$



$$= \pi \overline{\Phi(z)} - \frac{1}{2i} \int_{\mathbb{T}} \frac{\overline{\Phi(\xi)}}{\Phi(\xi) - \Phi(z)} \Phi'(\xi) d\xi = \pi \overline{\Phi(z)} - \frac{1}{2i} \mathcal{C}[\Phi](z).$$

We observe that the mapping  $\Phi \in \mathcal{C}^{2,\alpha}(\mathbb{D}) \mapsto \overline{\Phi} \in \mathcal{C}^{2,\alpha}(\mathbb{D})$  is well-defined and continuous. So it remains to check that  $\mathcal{C}[\Phi] \in \mathcal{C}^{1,\alpha}(\mathbb{D})$  and prove its continuity with respect to  $\Phi$ . Note first that  $\mathcal{C}[\Phi]$  is holomorphic inside the unit disc and its complex derivative is given by

$$[\mathcal{C}[\Phi]]'(z) = \Phi'(z) \int_{\mathbb{T}} \frac{\overline{\Phi(\xi)}}{(\Phi(\xi) - \Phi(z))^2} \Phi'(\xi) d\xi, \quad \forall z \in \mathbb{D}.$$

Using a change of variables, we deduce that

$$[\mathcal{C}[\Phi]]'(z) = -\Phi'(z) \int_{\mathbb{T}} \frac{\overline{\Phi'(\xi)} \xi^2}{\Phi(\xi) - \Phi(z)} d\xi,$$

where we have used the formula

$$\frac{d}{d\xi} \overline{\Phi(\xi)} = -\overline{\xi^2} \Phi'(\overline{\xi}).$$

For this last integral we can use the upcoming Lemma B.3 to obtain that  $[\mathcal{C}[\Phi]]' \in \mathcal{C}^{0,\alpha}(\mathbb{D})$ . Although the last is clear, we show here an alternative procedure useful to check the continuity with respect to  $\Phi$ . According to [55, Lemma 6.4.8], to show that  $[\mathcal{C}[\Phi]]' \in \mathcal{C}^{0,\alpha}(\mathbb{D})$  it suffices to prove that

$$(B.7) \quad \left| [\mathcal{C}[\Phi]]''(z) \right| \leq C(1 - |z|)^{\alpha-1}, \quad \forall z \in \mathbb{D}.$$

Then, by differentiating we get

$$\begin{aligned} [\mathcal{C}[\Phi]]''(z) &= -\Phi''(z) \int_{\mathbb{T}} \frac{\overline{\Phi'(\xi)} \xi^2}{\Phi(\xi) - \Phi(z)} d\xi - (\Phi'(z))^2 \int_{\mathbb{T}} \frac{\overline{\Phi'(\xi)} \xi^2}{(\Phi(\xi) - \Phi(z))^2} d\xi \\ &\triangleq -\Phi''(z) \mathcal{C}_1[\Phi](z) - (\Phi'(z))^2 \mathcal{C}_2[\Phi](z). \end{aligned}$$

For  $\mathcal{C}_1[\Phi]$  we simply write

$$\mathcal{C}_1[\Phi](z) = \int_{\mathbb{T}} \frac{\xi^2 \frac{\overline{\Phi'(\xi)}}{\Phi'(\xi)} - z^2 \frac{\overline{\Phi'(z)}}{\Phi'(z)}}{\Phi(\xi) - \Phi(z)} \Phi'(\xi) d\xi + 2i\pi \frac{\overline{\Phi'(z)}}{\Phi'(z)} z^2.$$

Since  $\xi \in \overline{\mathbb{D}} \mapsto \xi^2 \frac{\overline{\Phi'(\xi)}}{\Phi'(\xi)} \in \mathcal{C}^{0,\alpha}(\overline{\mathbb{D}})$ , then we have

$$|\mathcal{C}_1[\Phi](z)| \leq C \left( \int_{\mathbb{T}} \frac{|z - \xi|^\alpha}{|z - \xi|} |d\xi| + 1 \right) \leq C(1 - |z|)^{\alpha-1}.$$

It remains to estimate  $\mathcal{C}_2[\Phi]$ . Integration by parts implies

$$\mathcal{C}_2[\Phi](z) = \int_{\mathbb{T}} \frac{\psi(\xi)}{\Phi(\xi) - \Phi(z)} \Phi'(\xi) d\xi,$$

with

$$\psi(\xi) = \frac{\left( \frac{\overline{\Phi'(\xi)} \xi^2}{\Phi'(\xi)} \right)' }{\Phi'(\xi)}.$$

Since  $\Phi \in \mathcal{C}^{2,\alpha}(\mathbb{D})$  and is bi-Lipschitz, then  $\psi \in \mathcal{C}^{0,\alpha}(\mathbb{D})$ . Writing

$$\mathcal{C}_2[\Phi](z) = \int_{\mathbb{T}} \frac{\psi(\xi) - \psi(z)}{\Phi(\xi) - \Phi(z)} \Phi'(\xi) d\xi + 2i\pi \psi(z),$$

implies that

$$|\mathcal{C}_2[\Phi]| \leq C \int_{\mathbb{T}} \frac{|\psi(\xi) - \psi(z)|}{|\xi - z|} |d\xi| + 2\pi \|\psi\|_{L^\infty} \leq C \int_{\mathbb{T}} \frac{|\xi - z|^\alpha}{|\xi - z|} |d\xi| + 2\pi \|\psi\|_{L^\infty}.$$

Thus, we have

$$|\mathcal{C}_2[\Phi](z)| \leq C(1 - |z|)^{\alpha-1}.$$

Putting together the previous estimates, we deduce (B.7) which implies  $\mathcal{F}_2[\Phi] \in \mathcal{C}^{1,\alpha}(\mathbb{D})$ . It remains to check the continuity of  $\mathcal{F}_2$  with respect to  $\Phi$ . Splitting  $\mathcal{F}_2[\Phi_1] - \mathcal{F}_2[\Phi_2]$  as

$$\begin{aligned} \mathcal{F}_2[\Phi_1]f(z) - \mathcal{F}_2[\Phi_2]f(z) &= \int_{\mathbb{D}} \frac{|\Phi_1'(y)|^2 - |\Phi_2'(y)|^2}{\Phi_1(z) - \Phi_1(y)} dA(y) \\ &\quad + \int_{\mathbb{D}} \frac{(\Phi_2 - \Phi_1)(z) - (\Phi_2 - \Phi_1)(y)}{(\Phi_1(z) - \Phi_1(y))(\Phi_2(z) - \Phi_2(y))} |\Phi_2'(y)|^2 dA(y), \end{aligned}$$

combined with Lemma B.1 yield

$$\|\mathcal{F}_2[\Phi_1] - \mathcal{F}_2[\Phi_2]\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C\|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}.$$

Now, we need to prove a similar inequality for its derivative,

$$\|\mathcal{F}_2'[\Phi_1] - \mathcal{F}_2'[\Phi_2]\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C\|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}.$$

Since  $\Phi \in \mathcal{C}^{2,\alpha}(\mathbb{D}) \mapsto \Phi' \in \mathcal{C}^{1,\alpha}(\mathbb{D})$  is clearly continuous, we just have to prove that

$$|[\mathcal{C}[\Phi_1]]''(z) - [\mathcal{C}[\Phi_2]]''(z)| \leq C\|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}(1 - |z|)^{\alpha-1}, \quad \forall z \in \mathbb{D}.$$

It is enough to check the above estimate for  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . Let us show how dealing with the first one, and the same arguments can be applied for  $\mathcal{C}_2$ . We have

$$\begin{aligned} |\mathcal{C}_1[\Phi_1](z) - \mathcal{C}_1[\Phi_2](z)| &\leq \left| \int_{\mathbb{T}} \frac{\bar{\xi}^2 \left( \frac{\overline{\Phi_1'(\xi)}}{\Phi_1'(\xi)} - \frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)} \right) - \bar{z}^2 \left( \frac{\overline{\Phi_1'(z)}}{\Phi_1'(z)} - \frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)} \right)}{\Phi_1(\xi) - \Phi_1(z)} \Phi_1'(\xi) d\xi \right| \\ &\quad + \left| \int_{\mathbb{T}} \frac{\bar{z}^2 \frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)} - \bar{\xi}^2 \frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)}}{\Phi_1(\xi) - \Phi_1(z)} (\Phi_1'(\xi) - \Phi_2'(\xi)) d\xi \right| \\ &\quad + \left| \int_{\mathbb{T}} \frac{(\Phi_1 - \Phi_2)(z) - (\Phi_1 - \Phi_2)(\xi)}{(\Phi_1(\xi) - \Phi_1(z))(\Phi_2(\xi) - \Phi_2(z))} \left( \bar{z}^2 \frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)} - \bar{\xi}^2 \frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)} \right) \Phi_2'(\xi) d\xi \right| \\ &\quad + 2\pi \bar{z}^2 \left| \frac{\overline{\Phi_1'(z)} - \overline{\Phi_2'(z)}}{\Phi_1'(z)} + \frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)} \frac{\Phi_2'(z) - \Phi_1'(z)}{\Phi_1'(z)\Phi_2'(z)} \right| \\ &\leq C\|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}(1 - |z|)^{1-\alpha}, \end{aligned}$$

where we have used that

$$\left| \bar{\xi}^2 \left( \frac{\overline{\Phi_1'(\xi)}}{\Phi_1'(\xi)} - \frac{\overline{\Phi_2'(\xi)}}{\Phi_2'(\xi)} \right) - \bar{z}^2 \left( \frac{\overline{\Phi_1'(z)}}{\Phi_1'(z)} - \frac{\overline{\Phi_2'(z)}}{\Phi_2'(z)} \right) \right| \leq C\|\Phi_1 - \Phi_2\|_{\mathcal{C}^{2,\alpha}(\mathbb{D})}|z - \xi|.$$

Then, we obtain the inequality for  $\mathcal{C}_1[\Phi]$ , which concludes the continuity with respect to  $\Phi$ .  $\square$

The following result deals with a Calderon-Zygmund type estimate, which will be necessary in the later development. The techniques used are related to the well-known T(1)-Theorem of Wittmann, see [62]. Let us define

$$(B.8) \quad \mathcal{H}f(z) \triangleq \int_{\mathbb{T}} K(z, \xi)f(\xi)d\xi, \quad \forall z \in \mathbb{D}.$$

**Lemma B.3.** Let  $\alpha \in (0, 1)$  and  $K : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$  smooth outside the diagonal that satisfies

$$(B.9) \quad |K(z_1, y)| \leq C_0 |z_1 - y|^{-1},$$

$$(B.10) \quad |K(z_1, y) - K(z_2, y)| \leq C_0 \frac{|z_1 - z_2|}{|z_1 - y|^2}, \quad \text{if } 2|z_1 - z_2| \leq |z_1 - y|,$$

$$(B.11) \quad \mathcal{K}(\text{Id}) \in \mathcal{C}^{0,\alpha}(\mathbb{D}),$$

$$(B.12) \quad \left| \int_{\partial(\mathbb{D} \cap B_{z_1}(\rho))} K(z_1, \xi) d\xi \right| < C_0,$$

for any  $z_1, z_2 \neq y \in \mathbb{D}$  and  $\rho > 0$ , where  $C_0$  is a positive constant that does not depend on  $z_1, z_2, y$  and  $\rho$ . Then,

$$\mathcal{K} : \mathcal{C}^{0,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}^{0,\alpha}(\mathbb{D})$$

is continuous, with the estimate

$$\|\mathcal{K}f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq CC_0 \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})},$$

where  $C$  is a constant depending only on  $\alpha$ .

*Proof.* From (B.9) and (B.11), we get easily that

$$\begin{aligned} |\mathcal{K}f(z)| &\leq \left| \int_{\mathbb{T}} K(z, \xi)(f(\xi) - f(z))d\xi \right| + \left| f(z) \int_{\mathbb{T}} K(z, \xi)d\xi \right| \\ &\leq C_0 \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \int_{\mathbb{T}} \frac{|d\xi|}{|z - \xi|^{1-\alpha}} + C \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \\ &\leq CC_0 \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})}. \end{aligned}$$

Taking  $z_1, z_2 \in \mathbb{D}$ , we define  $d = |z_1 - z_2|$ . We write

$$\begin{aligned} &\int_{\mathbb{T}} K(z_1, \xi)f(\xi)d\xi - \int_{\mathbb{T}} K(z_2, \xi)f(\xi)d\xi \\ &= \int_{\mathbb{T}} K(z_1, \xi)(f(\xi) - f(z_1))d\xi - \int_{\mathbb{T}} K(z_2, \xi)(f(\xi) - f(z_1))d\xi \\ &\quad + f(z_1) \int_{\mathbb{T}} K(z_1, \xi)d\xi - f(z_1) \int_{\mathbb{T}} K(z_2, \xi)d\xi \\ &= \int_{\mathbb{T} \cap B_{z_1}(3d)} K(z_1, \xi)(f(\xi) - f(z_1))d\xi - \int_{\mathbb{T} \cap B_{z_1}(3d)} K(z_2, \xi)(f(\xi) - f(z_1))d\xi \\ &\quad + \int_{\mathbb{T} \cap B_{z_1}^c(3d)} (K(z_1, \xi) - K(z_2, \xi))(f(\xi) - f(z_1))d\xi \\ &\quad + f(z_1) \int_{\mathbb{T}} K(z_1, \xi)d\xi - f(z_1) \int_{\mathbb{T}} K(z_2, \xi)d\xi \\ &\triangleq I_1 - I_2 + I_3 + f(z_1)(I_4 - I_5). \end{aligned}$$

Using (B.11), we achieve

$$|I_4 - I_5| \leq C|z_1 - z_2|^\alpha.$$

Let us work with  $I_1$  using the Layer Cake Lemma, see [41]. We use that  $|\mathbb{T} \cap B_x(\rho)| \leq C\rho$ , for any  $\rho > 0$  and  $x \in \mathbb{R}^2$ , which means that it is 1-Ahlfors regular curve. In fact, taking any  $z \in \mathbb{D}$  and  $\rho > 0$  ones has that

$$\begin{aligned} \int_{\mathbb{T} \cap B_z(\rho)} \frac{|d\xi|}{|z - \xi|^{1-\alpha}} &= \int_0^\infty \left| \left\{ \xi \in \mathbb{T} \cap B_z(\rho) : \frac{1}{|z - \xi|^{1-\alpha}} \geq \lambda \right\} \right| d\lambda, \\ &= \int_0^\infty \left| \left\{ \xi \in \mathbb{T} \cap B_z(\rho) : |z - \xi| \leq \lambda^{\frac{-1}{1-\alpha}} \right\} \right| d\lambda \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\rho^{\alpha-1}} |\{\xi \in \mathbb{T} : |z - \xi| \leq \rho\}| d\lambda + \int_{\rho^{\alpha-1}}^{+\infty} \left| \{\xi \in \mathbb{T} : |z - \xi| \leq \lambda^{\frac{-1}{1-\alpha}}\} \right| d\lambda \\
&\leq C \left( \rho^{\alpha-1} + \int_{\rho^{\alpha-1}}^{+\infty} \lambda^{\frac{-1}{1-\alpha}} d\lambda \right) \leq C\rho^\alpha,
\end{aligned}$$

where  $|\cdot|$  inside the integral denotes the arch length measure. Applying the last estimate to  $I_1$ , we find

$$|I_1| \leq C_0 \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \int_{\mathbb{T} \cap B_{z_1}(3d)} \frac{|d\xi|}{|z_1 - \xi|^{1-\alpha}} \leq CC_0 \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} |z_1 - z_2|^\alpha.$$

For the term  $I_3$ , we get

$$|I_3| \leq C_0 \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} |z_1 - z_2| \int_{\mathbb{T} \cap B_{z_1}(3d)^c} \frac{|d\xi|}{|z_1 - \xi|^{2-\alpha}},$$

by (B.10). Now, we use again the Layer Cake Lemma, obtaining

$$\begin{aligned}
\int_{\mathbb{T} \cap B_z(\rho)^c} \frac{|d\xi|}{|z - \xi|^{2-\alpha}} &= \int_0^\infty \left| \left\{ \xi \in \mathbb{T} : \frac{1}{|z - \xi|^{2-\alpha}} \geq \lambda, |z - \xi| \geq \rho \right\} \right| d\lambda, \\
&= \int_0^\infty \left| \left\{ \xi \in \mathbb{T} : |z - \xi| \leq \lambda^{\frac{-1}{2-\alpha}}, |z - \xi| \geq \rho \right\} \right| d\lambda \\
&= \int_0^{\rho^{\alpha-2}} \left| \left\{ \xi \in \mathbb{T} : \rho \leq |z - \xi| \leq \lambda^{\frac{-1}{2-\alpha}} \right\} \right| d\lambda \\
&\leq C \int_0^{\rho^{\alpha-2}} \lambda^{\frac{-1}{2-\alpha}} d\lambda \leq C\rho^{\alpha-1}.
\end{aligned}$$

Therefore,

$$|I_3| \leq C_0 \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} |z_1 - z_2|^\alpha.$$

It remains to estimate  $I_2$ . First, let us write it as

$$\begin{aligned}
I_2 &= \int_{\mathbb{T} \cap B_{z_1}(3d)} K(z_2, \xi) (f(\xi) - f(z_2)) d\xi + (f(z_2) - f(z_1)) \int_{\mathbb{T} \cap B_{z_1}(3d)} K(z_2, \xi) d\xi \\
&\triangleq H_1 + (f(z_2) - f(z_1)) H_2.
\end{aligned}$$

$H_1$  can be estimated as  $I_1$  noting that  $B_{z_1}(3d) \subset B_{z_2}(4d)$ . To finish, we just need to check that  $H_2$  is bounded. Decompose it as

$$H_2 = \int_{\mathbb{T} \cap B_{z_2}(2d)} K(z_2, \xi) d\xi + \int_{\mathbb{T} \cap B_{z_1}(3d) \cap B_{z_2}(2d)^c} K(z_2, \xi) d\xi \triangleq J_1 + J_2,$$

since  $B_{z_2}(2d) \subset B_{z_1}(3d)$ . Note that

$$|J_2| \leq C_0 \int_{\mathbb{T} \cap B_{z_1}(3d) \cap B_{z_2}(2d)^c} \frac{|d\xi|}{|z_2 - \xi|} \leq CC_0 d^{-1} |\mathbb{T} \cap B_{z_1}(3d)| \leq CC_0.$$

For the last term, we write

$$J_1 = \int_{\partial(\mathbb{D} \cap B_{z_2}(2d))} K(z_2, \xi) d\xi - \int_{\mathbb{D} \cap \partial B_{z_2}(2d)} K(z_2, \xi) d\xi$$

By using condition (B.12), we get that the first integral is bounded. For the second one, we obtain

$$\left| \int_{\mathbb{D} \cap \partial B_{z_2}(2d)} K(z_2, \xi) d\xi \right| \leq C_0 \int_{\mathbb{D} \cap \partial B_{z_2}(2d)} \frac{|d\xi|}{|z_2 - \xi|} = \frac{1}{2d} |\mathbb{D} \cap \partial B_{z_2}(2d)| \leq C.$$

Combining all the estimates, we achieved the announced result.  $\square$

In the following result, we deal with the Cauchy integral defined as

$$(B.13) \quad \mathcal{S}[\Phi](f)(z) \triangleq \int_{\mathbb{T}} \frac{f(\xi)\Phi'(\xi)}{\Phi(z) - \Phi(\xi)} d\xi.$$

Note that this classical operator is fully studied in [40] in the case that  $\Phi = Id$ , then there we will adapt that proof.

**Lemma B.4.** *Let  $\alpha \in (0, 1)$  and  $\Phi : \mathbb{D} \rightarrow \Phi(\mathbb{D}) \subset \mathbb{C}$  be a conformal bi-Lipschitz function of class  $\mathcal{C}^{2,\alpha}(\mathbb{D})$ . Therefore, we have that*

$$\mathcal{S}[\Phi] : \mathcal{C}^{0,\alpha}(\mathbb{D}) \rightarrow \mathcal{C}^{0,\alpha}(\mathbb{D}),$$

*is continuous. Moreover,  $\mathcal{S} : \Phi \in \mathcal{U} \mapsto \mathcal{S}[\Phi]$  is continuous, where  $\mathcal{U}$  is defined in Lemma B.2.*

*Proof.* Note that

$$\mathcal{S}[\Phi](f)(z) = \int_{\Phi(\mathbb{T})} \frac{(f \circ \Phi^{-1})(\xi)}{\Phi(z) - \xi} d\xi = \mathcal{C}[f \circ \Phi^{-1}](\Phi(z)),$$

where  $\mathcal{C}$  is the Cauchy Integral. Then, it is classical, see [40], that

$$\begin{aligned} \|\mathcal{S}[\Phi](f)\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} &= \|\mathcal{C}[f \circ \Phi^{-1}] \circ \Phi\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \leq C \|f \circ \Phi^{-1}\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \|\Phi\|_{\text{Lip}(\mathbb{D})} \\ &\leq C \|f\|_{\mathcal{C}^{0,\alpha}(\mathbb{D})} \|\Phi^{-1}\|_{\text{Lip}(\mathbb{D})} \|\Phi\|_{\text{Lip}(\mathbb{D})}. \end{aligned}$$

To deal with the continuity with respect to the conformal map, we write

$$\mathcal{S}[\Phi_1]f(z) - \mathcal{S}[\Phi_2]f(z) = \int_{\mathbb{T}} f(\xi) \left( \frac{\Phi_1'(\xi)}{\Phi_1(z) - \Phi_1(\xi)} - \frac{\Phi_2'(\xi)}{\Phi_2(z) - \Phi_2(\xi)} \right) d\xi \triangleq \int_{\mathbb{T}} f(\xi) K(z, \xi) d\xi.$$

We will check that  $K$  verifies (B.9)-(B.12) in order to use Lemma B.3. Straightforward computations yield

$$\begin{aligned} |K(z_1, y)| &\leq C \|\Phi_1 - \Phi_2\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} |z_1 - y|^{-1}, \\ |K(z_1, y) - K(z_2, y)| &\leq C \|\Phi_1 - \Phi_2\|_{\mathcal{C}^{1,\alpha}(\mathbb{D})} \frac{|z_1 - z_2|}{|z_1 - y|^2}, \quad \text{if } 2|z_1 - z_2| \leq |z_1 - y|, \end{aligned}$$

using that  $|z_2 - y| \geq |z_1 - y| - |z_1 - z_2| \geq \frac{1}{2}|z_1 - y|$  in the second property, which concerns (B.9)-(B.10). Moreover,

$$\int_{\mathbb{T}} K(z, \xi) d\xi = \int_{\Phi_1(\mathbb{T})} \frac{d\xi}{\Phi_1(z) - \xi} - \int_{\Phi_2(\mathbb{T})} \frac{d\xi}{\Phi_2(z) - \xi} = 0,$$

which implies (B.11). In fact,

$$\int_{\partial(\mathbb{D} \cap B_z(\rho))} K(z, \xi) d\xi = \int_{\Phi_1(\partial(\mathbb{D} \cap B_z(\rho)))} \frac{d\xi}{\Phi_1(z) - \xi} - \int_{\Phi_2(\partial(\mathbb{D} \cap B_z(\rho)))} \frac{d\xi}{\Phi_2(z) - \xi} = C_0,$$

by applying the Residue Theorem, and where  $C_0$  that does not depend on  $\rho$  neither  $z$ , which agrees with (B.12). Then, we achieve the proof using Lemma B.3.  $\square$

We give the explicit expressions of some integrals which appear in the analysis of the linearized operator.

**Proposition B.5.** *Let  $\alpha \in (0, 1)$ . Given  $h \in \mathcal{C}_s^{1,\alpha}(\mathbb{D})$ ,  $k \in \mathcal{H}\mathcal{C}^{2,\alpha}(\mathbb{D})$  and a radial function  $f_0 \in \mathcal{C}(\mathbb{D})$ , the following identities*

$$\begin{aligned} \int_{\mathbb{D}} \frac{k(z) - k(y)}{(z - y)^2} f_0(y) dA(y) &= 2\pi \sum_{n \geq 1} A_n z^{n-1} \left[ \int_0^{|z|} r f_0(r) dr - n \int_{|z|}^1 r f_0(r) dr \right], \\ \int_{\mathbb{D}} \frac{f_0(y)}{z - y} \text{Re}[k'(y)] dA(y) &= \pi \sum_{n \geq 1} A_n (n + 1) \left[ -z^{n-1} \int_{|z|}^1 r f_0(r) dr + \frac{\bar{z}^{n+1}}{|z|^{2(n+1)}} \int_0^{|z|} r^{2n+1} f_0(r) dr \right], \end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{D}} \frac{f_0(y)}{z-y} dA(y) &= 2\pi \frac{\bar{z}}{|z|^2} \int_0^{|z|} r f_0(r) dr, \\
\int_{\mathbb{D}} \frac{h(y)}{z-y} dA(y) &= \pi \sum_{n \geq 1} \left[ -z^{n-1} \int_{|z|}^1 \frac{1}{r^{n-1}} h_n(r) dr + \frac{\bar{z}^{n+1}}{|z|^{2n+2}} \int_0^{|z|} r^{n+1} h_n(r) dr \right], \\
\int_{\mathbb{D}} \log |z-y| h(y) dA(y) &= -\pi \sum_{n \geq 1} \cos(n\theta) \frac{1}{n} \left[ |z|^n \int_{|z|}^1 \frac{1}{r^{n-1}} h_n(r) dr + \frac{1}{|z|^n} \int_0^{|z|} |z|^{n+1} h_n(r) dr \right], \\
\int_{\mathbb{D}} \frac{k(z) - k(y)}{z-y} f_0(y) dA(y) &= 2\pi \sum_{n \geq 1} A_n z^n \int_0^1 s f_0(s) ds, \\
\int_{\mathbb{D}} \log |z-y| f_0(y) \operatorname{Re}[k'(y)] dA(y) &= -\pi \sum_{n \geq 1} A_n \frac{n+1}{n} \cos(n\theta) \\
&\quad \times \left[ |z|^n \int_{|z|}^1 r f_0(r) dr + \frac{1}{|z|^n} \int_0^{|z|} r^{2n+1} f_0(r) dr \right], \\
\int_{\mathbb{D}} \log |z-y| f_0(y) dA(y) &= 2\pi \left[ \int_0^{|z|} \frac{1}{\tau} \int_0^\tau r f_0(r) dr - \int_0^1 \frac{1}{\tau} \int_0^\tau r f_0(r) dr \right]
\end{aligned}$$

holds for  $z \in \bar{\mathbb{D}}$ .

*Proof.* Note that  $h$  and  $k$  can be given by

$$h(z) = \sum_{n \geq 1} h_n(r) \cos(n\theta), \quad k(z) = z \sum_{n \geq 1} A_n z^n,$$

where  $z = re^{i\theta} \in \mathbb{D}$ .

(1) Using the expression for the function  $k$ , the integral to be computed takes the form

$$\int_{\mathbb{D}} \frac{z^{n+1} - y^{n+1}}{z-y} \frac{f_0(y)}{z-y} dA(y).$$

An expansion of the function inside the integral provides

$$\int_{\mathbb{D}} \frac{z^{n+1} - y^{n+1}}{z-y} \frac{f_0(y)}{z-y} dA(y) = \sum_{k=0}^n z^{n-k} \int_{\mathbb{D}} y^k \frac{f_0(y)}{z-y} dA(y).$$

The use of polar coordinates yields

$$(B.14) \quad \int_{\mathbb{D}} y^k \frac{f_0(y)}{z-y} dA(y) = i \int_0^1 r^k f_0(r) \int_{\mathbb{T}} \frac{\xi^{k-1}}{\xi - \frac{z}{r}} d\xi.$$

We split our study in the cases  $k = 0$  and  $k \geq 1$  by making use of the Residue Theorem. For  $k = 0$ , we obtain

$$\int_{\mathbb{T}} \frac{1}{\xi} \frac{1}{\xi - \frac{z}{r}} d\xi = \begin{cases} 0, & |z| \leq r, \\ -2\pi i \frac{r}{z}, & |z| \geq r, \end{cases}$$

whereas, we find

$$\int_{\mathbb{T}} \frac{\xi^{k-1}}{\xi - \frac{z}{r}} d\xi = \begin{cases} 2\pi i \frac{z^{k-1}}{r^{k-1}}, & |z| \leq r, \\ 0, & |z| \geq r, \end{cases}$$

for any  $k \geq 1$ . This allows us to have the following expression

$$\int_{\mathbb{D}} \frac{z^{n+1} - y^{n+1}}{z-y} \frac{f_0(y)}{z-y} dA(y) = 2\pi z^{n-1} \int_0^{|z|} r f_0(r) dr - 2\pi \sum_{k=1}^n z^{n-k} z^{k-1} \int_{|z|}^1 r f_0(r) dr$$

$$= 2\pi z^{n-1} \left[ \int_0^{|z|} r f_0(r) dr - n \int_{|z|}^1 r f_0(r) dr \right].$$

(2) Note that  $k'(z) = \sum_{n \geq 1} A_n(n+1)z^n$ . Then, the integral to be analyzed is

$$\int_{\mathbb{D}} \frac{f_0(y)}{z-y} \operatorname{Re}[k'(y)] dA(y) = \sum_{n \geq 1} \frac{A_n(n+1)}{2} \int_{\mathbb{D}} \frac{f_0(y)}{z-y} (y^n + \bar{y}^n) dA(y).$$

We study the two terms in the integral by using polar coordinates and the Residue Theorem. For the first one we have

$$\int_{\mathbb{D}} \frac{f_0(y)}{z-y} y^n dA(y) = i \int_0^1 r^n f_0(r) \int_{\mathbb{T}} \frac{\xi^{n-1}}{\xi - \frac{z}{r}} d\xi dr = -2\pi z^{n-1} \int_{|z|}^1 r f_0(r) dr.$$

In the same way, the second one can be written as

$$\begin{aligned} \int_{\mathbb{D}} \frac{f_0(y)}{z-y} \bar{y}^n dA(y) &= i \int_0^1 f_0(r) r^n \int_{\mathbb{T}} \frac{1}{\xi^{n+1}} \frac{1}{\xi - \frac{z}{r}} d\xi dr \\ &= 2\pi \frac{1}{z^{n+1}} \int_0^{|z|} f_0(r) r^{2n+1} dr = \frac{2\pi \bar{z}^{n+1}}{|z|^{2n+2}} \int_0^{|z|} f_0(r) r^{2n+1} dr, \end{aligned}$$

that concludes the proof.

(3) This integral reads as

$$\int_{\mathbb{D}} \frac{f_0(y)}{z-y} dA(y) = i \int_0^1 f_0(r) \int_{\mathbb{T}} \frac{1}{\xi} \frac{1}{\xi - \frac{z}{r}} d\xi dr = \frac{2\pi}{z} \int_0^{|z|} r f_0(r) dr = \frac{2\pi \bar{z}}{|z|^2} \int_0^{|z|} r f_0(r) dr.$$

(4) We use the expression of  $h(z)$  to deduce that

$$\int_{\mathbb{D}} \frac{h(y)}{z-y} dA(y) = \frac{1}{2} \sum_{n \geq 1} \int_{\mathbb{D}} \frac{h_n(r)(e^{in\theta} + e^{-in\theta})}{z-y} dA(y).$$

The two terms involved in the integral can be computed as follows

$$\begin{aligned} \int_{\mathbb{D}} \frac{h_n(r)e^{in\theta}}{z-y} dA(y) &= i \int_0^1 h_n(r) \int_{\mathbb{T}} \frac{\xi^{n-1}}{\xi - \frac{z}{r}} d\xi dr = -2\pi z^{n-1} \int_{|z|}^1 \frac{1}{r^{n-1}} h_n(r) dr, \\ \int_{\mathbb{D}} \frac{h_n(r)e^{-in\theta}}{z-y} dA(y) &= i \int_0^1 h_n(r) \int_{\mathbb{T}} \frac{1}{\xi^{n+1}} \frac{1}{\xi - \frac{z}{r}} d\xi dr \\ &= \frac{2\pi}{z^{n+1}} \int_0^{|z|} r^{n+1} h_n(r) dr = \frac{2\pi \bar{z}^{n+1}}{|z|^{2(n+1)}} \int_0^{|z|} r^{n+1} h_n(r) dr. \end{aligned}$$

(5) Let us differentiate with respect to  $r$  having that

$$\begin{aligned} \partial_r \int_{\mathbb{D}} \log |re^{i\theta} - y| h(y) dA(y) &= \operatorname{Re} \left[ \frac{z}{r} \int_{\mathbb{D}} \frac{h(y)}{z-y} dA(y) \right] \\ &= \operatorname{Re} \left[ \frac{z}{r} \sum_{n \geq 1} \pi \left[ -z^{n-1} \int_r^1 \frac{h_n(s)}{s^{n-1}} ds + \frac{1}{z^{n+1}} \int_0^r s^{n+1} h_n(s) ds \right] \right] \\ &= \pi \sum_{n \geq 1} \cos(n\theta) \left[ -r^{n-1} \int_r^1 \frac{h_n(s)}{s^{n-1}} ds + \frac{1}{r^{n+1}} \int_0^r s^{n+1} h_n(s) ds \right]. \end{aligned}$$

This last integral was computed before by the Residue Theorem. Now, we realize that

$$-\partial_r \frac{1}{n} \left[ r^n \int_r^1 \frac{1}{s^{n-1}} h_n(s) ds + \frac{1}{r^n} \int_0^r s^{n+1} h_n(s) ds \right]$$

$$= -r^{n-1} \int_r^1 \frac{1}{s^{n-1}} h_n(s) ds + \frac{1}{r^{n+1}} \int_0^r s^{n+1} h_n(s) ds.$$

Then, we obtain

$$\int_{\mathbb{D}} \log |z - y| h(y) dA(y) = -\pi \sum_{n \geq 1} \frac{1}{n} \left[ r^n \int_r^1 \frac{h_n(s)}{s^{n-1}} ds + \frac{1}{r^n} \int_0^r s^{n+1} h_n(s) ds \right] \cos(n\theta) + H(\theta),$$

where  $H$  is a function that only depends on  $\theta$ . Taking  $r = 0$  we have that

$$H(\theta) = \int_{\mathbb{D}} \log(|y|) h(y) dA(y) = 0.$$

The last is equal to zero due to the form of the function  $h$ :  $h(re^{i\theta}) = \sum_{n \geq 1} h_n(r) \cos(n\theta)$ .

(6) This integral can be done by splitting it as follows

$$\begin{aligned} \int_{\mathbb{D}} \frac{k(z) - k(y)}{z - y} f_0(y) dA(y) &= k(z) \int_{\mathbb{D}} \frac{f_0(y)}{z - y} dA(y) - \int_{\mathbb{D}} \frac{k(y) f_0(y)}{z - y} dA(y) \\ &= \sum_{n \geq 1} A_n \left[ z^{n+1} \int_{\mathbb{D}} \frac{f_0(y)}{z - y} dA(y) - \int_{\mathbb{D}} \frac{y^{n+1} f_0(y)}{z - y} dA(y) \right]. \end{aligned}$$

Note that these integrals have been done before. Hence, we conclude using Integral (3) for the first one and (B.14) for the second one.

(7) Similarly to Integral (5), we differentiate with respect to  $r$

$$\begin{aligned} \partial_r \int_{\mathbb{D}} \log |re^{i\theta} - y| f_0(y) \operatorname{Re} [k'(y)] dA(y) &= \operatorname{Re} \left[ \frac{z}{r} \int_{\mathbb{D}} \frac{f_0(y) \operatorname{Re} [k'(y)]}{z - y} dA(y) \right] \\ &= \pi \sum_{n \geq 1} A_n (n+1) \operatorname{Re} \left[ \frac{z}{r} \left[ -z^{n-1} \int_r^1 s f_0(s) ds + \frac{\bar{z}^{n+1}}{r^{2(n+1)}} \int_0^r s^{2n+1} f_0(s) ds \right] \right], \\ &= \pi \sum_{n \geq 1} A_n (n+1) \cos(n\theta) \left[ -r^{n-1} \int_r^1 s f_0(s) ds + \frac{1}{r^{n+1}} \int_0^r s^{2n+1} f_0(s) ds \right], \end{aligned}$$

where we use Integral (2). With the same argument than in Integral (5) we realize that

$$\begin{aligned} -\partial_r \frac{1}{n} \left[ r^n \int_r^1 s f_0(s) ds + \frac{1}{r^n} \int_0^r s^{2n+1} f_0(s) ds \right] \\ = -r^{n-1} \int_r^1 s f_0(s) ds + \frac{1}{r^{n+1}} \int_0^r s^{2n+1} f_0(s) ds, \end{aligned}$$

and hence

$$\begin{aligned} \int_{\mathbb{D}} \log |re^{i\theta} - y| f_0(y) \operatorname{Re} [k'(y)] dA(y) \\ = -\pi \sum_{n \geq 1} A_n \frac{n+1}{n} \cos(n\theta) \left[ r^n \int_r^1 s f_0(s) ds + \frac{1}{r^n} \int_0^r s^{2n+1} f_0(s) ds \right] + H(\theta), \end{aligned}$$

where  $H$  is a function that only depends on  $\theta$ . Evaluating in  $r = 0$  as in Integral (5), we get that  $H \equiv 0$ , obtaining the announced identity.

(8) As in Integral (5) and (7) we differentiate with respect to  $r$  having

$$\partial_r \int_{\mathbb{D}} \log |re^{i\theta} - y| f_0(y) dA(y) = \operatorname{Re} \left[ \frac{z}{r} \int_{\mathbb{D}} \frac{f_0(y)}{z - y} dA(y) \right] = 2\pi \frac{1}{r} \int_0^r s f_0(s) ds,$$



where the last integral is done in Integral (3). Hence,

$$\int_{\mathbb{D}} \log |re^{i\theta} - y| f_0(y) dA(y) = 2\pi \int_0^r \frac{1}{\tau} \int_0^\tau s f_0(s) ds d\tau + H(\theta),$$

where  $H$  is a function that only depends on  $\theta$ . Evaluating in  $r = 0$  we get that

$$H(\theta) = \int_0^1 \int_0^{2\pi} s \log s f_0(s) ds d\theta = -2\pi \int_0^1 \frac{1}{\tau} \int_0^\tau s f_0(s) ds d\tau,$$

concluding the proof.  $\square$

### APPENDIX C. GAUSS HYPERGEOMETRIC FUNCTION

We give a short introduction to the Gauss hypergeometric functions and discuss some of their basic properties. The formulae listed below were crucial in the computations of the linearized operator associated to the V-states equation and the analysis of its spectral study. Recall that for any real numbers  $a, b \in \mathbb{R}$ ,  $c \in \mathbb{R} \setminus (-\mathbb{N})$  the hypergeometric function  $z \mapsto F(a, b; c; z)$  is defined on the open unit disc  $\mathbb{D}$  by the power series

$$(C.1) \quad F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad \forall z \in \mathbb{D}.$$

The Pochhammer symbol  $(x)_n$  is defined by

$$(x)_n = \begin{cases} 1, & n = 0, \\ x(x+1) \cdots (x+n-1), & n \geq 1, \end{cases}$$

and verifies

$$(x)_n = x(1+x)_{n-1}, \quad (x)_{n+1} = (x+n)(x)_n.$$

The series converges absolutely for all values of  $|z| < 1$ . For  $|z| = 1$  we have that it converges absolutely if  $\operatorname{Re}(a+b-c) < 0$  and it diverges if  $1 \leq \operatorname{Re}(a+b-c)$ . See [4] for more details.

We recall the integral representation of the hypergeometric function, see for instance [54, p. 47]. Assume that  $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$ , then we have

$$(C.2) \quad F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a} dx, \quad \forall z \in \mathbb{C} \setminus [1, +\infty).$$

Notice that this representation shows that the hypergeometric function initially defined in the unit disc admits an analytic continuation to the complex plane cut along  $[1, +\infty)$ . Another useful identity is the following:

$$(C.3) \quad F(a, b; c; z) = (1-z)^{-a} F\left(a, c-b; c; \frac{z}{z-1}\right), \quad \forall |\arg(1-z)| < \pi,$$

for  $\operatorname{Re} c > \operatorname{Re} b > 0$ .

The function  $\Gamma : \mathbb{C} \setminus \{-\mathbb{N}\} \rightarrow \mathbb{C}$  refers to the gamma function, which is the analytic continuation to the negative half plane of the usual gamma function defined on the positive half-plane  $\{\operatorname{Re} z > 0\}$ . It is defined by the integral representation

$$\Gamma(z) = \int_0^{+\infty} \tau^{z-1} e^{-\tau} d\tau,$$

and satisfies the relation  $\Gamma(z+1) = z\Gamma(z)$ ,  $\forall z \in \mathbb{C} \setminus (-\mathbb{N})$ . From this we deduce the identities

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad (x)_n = (-1)^n \frac{\Gamma(1-x)}{\Gamma(1-x-n)},$$

provided that all the quantities in the right terms are well-defined. We can differentiate the hypergeometric function obtaining

$$(C.4) \quad \frac{d^k F(a, b; c; z)}{dz^k} = \frac{(a)_k (b)_k}{(c)_k} F(a + k, b + k; c + k; z),$$

for  $k \in \mathbb{N}$ . Depending on the parameters, the hypergeometric function behaves differently at 1. When  $\operatorname{Re} c > \operatorname{Re} b > 0$  and  $\operatorname{Re}(c - a - b) > 0$ , it can be shown that it is absolutely convergent on the closed unit disc and one finds the expression

$$(C.5) \quad F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)},$$

whose proof can be found in [54, page 49]. However, in the case  $a + b = c$ , the hypergeometric function exhibits a logarithmic singularity as follows

$$(C.6) \quad \lim_{z \rightarrow 1^-} \frac{F(a, b; c; z)}{-\ln(1 - z)} = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)},$$

see for instance [1] for more details. Next we recall some Kummer's quadratic transformations of the hypergeometric series, see [54],

$$(C.7) \quad c F(a, b; c; z) - (c - a) F(a, b; c + 1; z) - a F(a + 1, b; c + 1; z) = 0,$$

$$(C.8) \quad (b - c) F(a, b - 1; c; z) + (c - a - b) F(a, b; c; z) - a(z - 1) F(a + 1, b; c; z) = 0,$$

$$(C.9) \quad \frac{(2c - a - b + 1)z - c}{c} F(a, b; c + 1; z) + \frac{(a - c - 1)(c - b + 1)z}{c(c + 1)} F(a, b; c + 2; z) = F(a, b; c; z)(z - 1).$$

Other formulas which have been used in the preceding sections are

$$(C.10) \quad \begin{cases} \int_0^1 F(a, b; c; \tau z) \tau^{c-1} d\tau = \frac{1}{c} F(a, b; c + 1; z), \\ \int_0^1 F(a, b; c; \tau z) \tau^{c-1} (1 - \tau) d\tau = \frac{1}{c(c + 1)} F(a, b; c + 2; z). \end{cases}$$

The last point that we wish to recall concerns the differential equation governing the hypergeometric equation, which is given by,

$$(C.11) \quad z(1 - z)F(z)'' + (c - (a + b + 1)z)F(z)' - abF(z) = 0,$$

with  $a, b, \in \mathbb{R}$  and  $c \in \mathbb{R} \setminus (-\mathbb{N})$  given. One of the two independent solutions of the last differential equation around  $z = 0$  is the hypergeometric function:  $F(a, b; c; z)$ . It remains to identify the second independent solution. If none of  $c, c - a - b, a - b$  is an integer, then the other independent solution around the singularity  $z = 0$  is

$$(C.12) \quad z^{1-c} F(a - c + 1, b - c + 1; 2 - c; z).$$

We will be interested in the critical case when  $c$  is a negative integer. In this case, the hypergeometric differential equation has as a smooth solution given by (C.12). However, the second independent solution is singular and contains a logarithmic singularity, see [54, p. 55] for more details. Real solutions around  $+\infty$  may be also obtained as it is done in [1]. In fact, the two independent solutions are given by

$$z^{-a} F\left(a, a + 1 - c; a + 1 - b; \frac{1}{z}\right) \quad \text{and} \quad z^{-b} F\left(b, b + 1 - c; b + 1 - a; \frac{1}{z}\right).$$

## REFERENCES

- [1] G. R. Andrews, R. Askey, R. Roy, *Special Functions*. Cambridge University Press, 1999.
- [2] V. I. Arnold, *Hamiltonian nature of the Euler equations in the dynamics of a rigid body and of an ideal fluid*. In: Givental A., Khesin B., Varchenko A., Vassiliev V., Viro O. (eds) Vladimir I. Arnold - Collected Works. Vladimir I. Arnold - Collected Works, vol 2. Springer, Berlin, Heidelberg, 1969.
- [3] H. Bahouri, J.-Y. Chemin, *Équations de transport relatives à des champs de vecteurs non-lipschitziens et mécanique des fluides*. Arch. Ration. Mech. Anal. **127** (2) (1994), 159–181.
- [4] H. Bateman, *Higher Transcendental Functions Vol. I–III*. McGraw-Hill Book Company, New York, 1953.
- [5] J. Bedrossian, M. Coti Zelati, V. Vicol, *Vortex axisymmetrization, inviscid damping, and vorticity depletion in the linearized 2D Euler equations*, arXiv:1711.03668, 2017.
- [6] A. L. Bertozzi, P. Constantin, *Global regularity for vortex patches*. Comm. Math. Phys. **152**(1) (1993), 19–28.
- [7] J. Burbea, *Motions of vortex patches*. Lett. Math. Phys. **6** (1982), 1–16.
- [8] G. R. Burton, *Steady symmetric vortex pairs and rearrangements*. Proc. Roy. Soc. Edinburgh Sect. A. **108** (1988) 269–290.
- [9] J. A. Carrillo, J. Soler, *On the evolution of an angle in a vortex patch*. J. Nonlinear Sci. **10** (2000), 23–47.
- [10] A. Castro, D. Córdoba, J. Gómez-Serrano, *Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations*. Duke Math. J. **165**(5) (2016), 935–984.
- [11] A. Castro, D. Córdoba, J. Gómez-Serrano, *Uniformly rotating analytic global patch solutions for active scalars*. J. Ann. PDE **2**(1) (2016), Art. 1, 34.
- [12] A. Castro, D. Córdoba, J. Gómez-Serrano, *Uniformly rotating smooth solutions for the incompressible 2D Euler equations*, arXiv:1612.08964, 2016.
- [13] A. Castro, D. Córdoba, J. Gómez-Serrano, *Global smooth solutions for the inviscid SQG equation*, arXiv:1603.03325, 2016.
- [14] J.-Y. Chemin, *Persistence de structures géométriques dans les fluides incompressibles bidimensionnels*. Ann. Sci. Ec. Norm. Sup. **26** (1993), 1–26.
- [15] A. Choffrut, V. Šverák, *Local structure of the set of steady-state solutions to the 2D incompressible Euler equations*. Geom. Funct. Anal. **22**(1) (2012), 136–201.
- [16] A. Choffrut, L. Székelyhidi Jr., *Weak solutions to the stationary incompressible Euler equations*. SIAM J. Math. Anal. **46**(6) (2014), 4060–4074.
- [17] M. G. Crandall, P. H. Rabinowitz, *Bifurcation from simple eigenvalues*. J. Funct. Anal. **8** (1971), 321–340.
- [18] V. Cruz, J. Mateu, J. Orobitg, *Beltrami equation with coefficient in Sobolev and Besov spaces*. Canad. J. Math. **65**(1) (2013), 1217–1235.
- [19] V. Cruz, X. Tolsa, *Smoothness of the Beurling transform in Lipschitz domains*. J. Funct. Anal. **262**(10) (2012), 4423–4457.
- [20] G. S. Deem, N. J. Zabusky, *Vortex waves: Stationary “V-states”, Interactions, Recurrence, and Breaking*. Phys. Rev. Lett. **40** (1978), 859–862.
- [21] D. G. Dritschel, T. Hmidi, C. Renault, *Imperfect bifurcation for the quasi-geostrophic shallow-water equations*, arXiv:1801.02092, 2018, to appear in Arch. Ration. Mech. Anal.
- [22] F. De la Hoz, T. Hmidi, J. Mateu, J. Verdera, *Doubly connected V-states for the planar Euler equations*. SIAM J. Math. Anal. **48** (2016), 1892–1928.
- [23] F. De la Hoz, Z. Hassainia, T. Hmidi, J. Mateu, *An analytical and numerical study of steady patches in the disc*. Anal. PDE **9**(7) (2016), 1609–1670.
- [24] C. De Lellis, L. Székelyhidi Jr, *High dimensionality and h-principle in PDE*. Bull. Amer. Math. Soc. **54** (2017), 247–282.
- [25] T. M. Elgindi, I. J. Jeong, *Symmetries and Critical Phenomena in Fluids*, arXiv:1610.09701, 2016.
- [26] A. Enciso, D. Poyato, J. Soler, *Stability Results, Almost Global Generalized Beltrami Fields and Applications to Vortex Structures in the Euler Equations*. Commun. Math. Phys. **360** (2018), 197–269.
- [27] P. Gravejat, D. Smets, *Smooth traveling-wave solutions to the inviscid surface quasi-geostrophic equation*. International Mathematics Research Notices, page rnx177, 2017.
- [28] P. Hartman, *Ordinary Differential Equations*. John Wiley & Sons, Inc., New York, London, Sydney, 1964.
- [29] Z. Hassainia, T. Hmidi, *On the V-states for the generalized quasi-geostrophic equations*. Comm. Math. Phys. **337**(1) (2015), 321–377.
- [30] Z. Hassainia, N. Masmoudi, M. H. Wheeler, *Global bifurcation of rotating vortex patches*, arXiv:1712.03085, 2017.
- [31] L. L. Helms, *Potential theory*, Springer-Verlag London, 2009.
- [32] T. Hmidi, F. De la Hoz, J. Mateu, J. Verdera, *Doubly connected V-states for the planar Euler equations*. SIAM J. Math. Anal. **48**(3), 1892–1928.

- [33] T. Hmidi, J. Mateu, *Bifurcation of rotating patches from Kirchhoff vortices*. Discret. Contin. Dyn. Syst. **36** (2016), 5401–5422.
- [34] T. Hmidi, J. Mateu, *Existence of corotating and counter-rotating vortex pairs for active scalar equations*. Comm. Math. Phys. **350**(2) (2017), 699–747.
- [35] T. Hmidi, J. Mateu, J. Verdera, *Boundary regularity of rotating vortex patches*. Arch. Ration. Mech. Anal. **209** (2013), 171–208.
- [36] T. Hmidi, C. Renault, *Existence of small loops in a bifurcation diagram near degenerate eigenvalues*. Non-linearity **30**(10) (2017), 3821–3852.
- [37] T. Kato, *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin-Heidelberg-New York, 1995.
- [38] H. Kielhöfer, *Bifurcation Theory: An Introduction with Applications to PDEs*. Springer-Verlag, Berlin-Heidelberg-New York, 2004.
- [39] A. Kiselev, V. Šverák, *Small scale creation for solutions of the incompressible two-dimensional Euler equation*. Ann. of Math. **180**(3) (2014), 1205–1220.
- [40] R. Kress, *Linear Integral Equations*. Springer–New York–Heidelberg Dordrecht–London, 2014.
- [41] E. Lieb, M. Loss, *Analysis*. American Mathematical Society, 1997.
- [42] X. Luo, R. Shvydkoy, *2D homogeneous solutions to the Euler equation*. Comm. Part. Differ. Equat. **40**(9) (2015), 1666–1687.
- [43] A. Majda, A. Bertozzi, *Vorticity and Incompressible Flow*. Cambridge University Press, 2002.
- [44] J. Mateu, J. Orobitg, J. Verdera, *Extra cancellation of even Calderón–Zygmund operators and quasiconformal mappings*. J. Math. Pures Appl. **91**(4) (2009), 402–431.
- [45] M. V. Melander, N. J. Zabusky, A. S. Styczek, *A moment model for vortex interactions of the two-dimensional Euler equations. Part 1. Computational validation of a Hamiltonian elliptical representation*. J. Fluid Mech. **167** (1986), 95–115.
- [46] C. Miranda, *Partial Differential Equations of Elliptic Type*. Springer, Berlin, 1970.
- [47] C. Miranda, *Sulle proprietà di regolarità di certe trasformazioni integrali*. Mem. Acc. Lincei **7** (1965), 303–336.
- [48] P. J. Morrison, *Hamiltonian description of the ideal fluid*. Rev. Mod. Phys. **70** (1998), 467–521.
- [49] N. Nadirashvili, *On stationary solutions of two-dimensional Euler equation*. Arch. Ration. Mech. Anal. **209**(3) (2013), 729–745.
- [50] P. J. Olver, *A Nonlinear Hamiltonian Structure for the Euler Equations*. J. Math. Anal. and Applications **89** (1982), 233–250.
- [51] E.A. Overman II, *Steady-state solutions of the Euler equations in two dimensions. II. Local analysis of limiting V-states*. SIAM J. Appl. Math. **46**(5) (1986), 765–800.
- [52] M. M. Peloso, *Classical spaces of holomorphic functions*. Technical report, Università di Milano, 2011.
- [53] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*. Springer, Berlin, 1992.
- [54] E. D. Rainville, *Special Functions*. The Macmillan Co., 1973.
- [55] W. Rudin, *Real and Complex Analysis, 3rd ed.* McGraw-Hill, 1987.
- [56] L. A. Rubel, A. L. Shields, B. A. Taylor, *Mergelyan Sets and the Modulus of Continuity of Analytic Functions*. J. Approx. Theory **15** (1975), 23–40.
- [57] P. Serfati, *Une preuve directe d’existence globale des vortex patches 2D*. C. R. Acad. Sci. Paris Sér. I Math. **318**(6) (1994), 515–518.
- [58] P. Stefanov, G. Uhlmann, *Linearizing non-linear inverse problems and an application to inverse backscattering*. J. Funct. Anal. **256** (2009) 2842–2866.
- [59] F. G. Tricomi, A. Erdélyi, *The asymptotic expansion of a ratio of gamma functions*. Pacific J. Math. **1** (1951), 133–142.
- [60] H.T. Yau, *Derivation of the Euler Equation from Hamiltonian Systems with Negligible Random Noise*. In: Schmüdgen K. (eds) Mathematical Physics X. Springer, Berlin, Heidelberg, 1992.
- [61] Y. Yudovich, *Nonstationary flow of an ideal incompressible liquid*. Zh. Vych. Mat. **3** (1963), 1032–1066.
- [62] R. Wittmann, *Application of a Theorem of M.G. Krein to singular integrals*. Trans. Amer. Math. Soc. **299**(2) (1987), 581–599.

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