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Note on a differential inequality

Daniel Barlet.*

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ABSTRACT. We study a second order differential inequality for a function of one real variable which allows to control the behaviour at the origin of a positive \mathcal{C}^2 function F on a punctured ball with center 0 in \mathbb{R}^n ($n \geq 2$) such that $\Delta(F)(x)$ is bounded by $C.\|x\|^{-2}.F(x)$ near $x = 0$.

AMS CLASSIFICATION. 26 A 12 – 34 E 10 – 35 B 09 – 35 B 40.

KEY WORDS. Second order differential inequality– Laplace operator.

Introduction

The motivation of this note is to look for an analogous result to the classical Gronwald lemma which gives some estimate of the growth of a positive function F near the center of a punctured disc D^* of the plane which satisfies the inequality $\Delta F(x) \leq C.\|x\|^{-2}.F(x)$. Using the invariance by rotation of the Laplacian allows to reduce our problem to a special case of the following differential inequality

$$x^2.f''(x) + x.\theta(x).f'(x) \leq \eta(x).f(x) \quad \forall x \in]0, 1[\quad (1)$$

for a positive function $f \in \mathcal{C}^2(]0, 1[)$ where θ is a \mathcal{C}^1 function in $[0, 1[$ and η is a continuous function on $[0, 1[$.

We do not obtain a point-wise estimate for such an f when x goes to 0 but an integral estimate which corresponds precisely to the point-wise estimate we would wait for. But we show that such a point-wise estimate does not hold in general.

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1 The estimate

1.1 The standard situation

We fix the real valued functions $\theta \in \mathcal{C}^1([0, 1[)$ and $\eta \in \mathcal{C}^0([0, 1[)$. Let $a := \eta(0)$. We shall say that we are in the standard situation when the following conditions are fulfilled

1. If $q := \theta(0) \neq 1$ we assume that

$$\limsup_{x \rightarrow 0^+} (\eta(x) - a) \cdot (-\text{Log } x) = \alpha_1 < +\infty.$$

2. If $q = 1$ we assume that

$$\limsup_{x \rightarrow 0^+} (\eta(x) - a) \cdot (-\text{Log } x)^2 = \alpha_2 < +\infty.$$

We want to study the behaviour when $x \rightarrow 0^+$ of \mathcal{C}^2 functions $f :]0, 1[\rightarrow \mathbb{R}^+$ which are solutions of the differential inequality

$$x^2 \cdot f''(x) + x \cdot \theta(x) \cdot f'(x) \leq \eta(x) \cdot f(x) \quad \forall x \in]0, 1[\quad (1)$$

1.2 Reduction to the case $\theta'(0) < 0$

Assume that we are under the standard situation defined above. Then for a given real number u that we shall choose later on, define the function $g :]0, 1[\rightarrow \mathbb{R}^+$ by letting $f(x) = \exp(u \cdot x) \cdot g(x)$. Then

$$\begin{aligned} f'(x) &= u \cdot \exp(u \cdot x) \cdot g(x) + \exp(u \cdot x) \cdot g'(x) \\ f''(x) &= u^2 \cdot \exp(u \cdot x) \cdot g(x) + 2u \cdot \exp(u \cdot x) \cdot g'(x) + \exp(u \cdot x) \cdot g''(x) \end{aligned}$$

so we obtain

$$\begin{aligned} x^2 \cdot f''(x) + \theta(x) \cdot x \cdot f'(x) - \eta(x) \cdot f(x) &= \\ \exp(u \cdot x) \cdot [x^2 \cdot g''(x) + (\theta(x) + 2u \cdot x) \cdot x \cdot g'(x) - (\eta(x) - u \cdot x \cdot \theta(x) - u^2 \cdot x^2) \cdot g(x)] \end{aligned}$$

So the differential inequality (1) for f is equivalent to the following differential inequality for the function g (which depends on the choice of u):

$$x^2 \cdot g''(x) + \theta_u(x) \cdot x \cdot g'(x) \leq \eta_u(x) \cdot g(x) \quad \forall x \in]0, 1[\quad (1_u)$$

with $\theta_u(x) := \theta(x) + 2u \cdot x$ and $\eta_u(x) := \eta(x) - u \cdot x \cdot \theta(x) - u^2 \cdot x^2$. Then we are again in the standard situation with

$$\theta_u(x) = \theta(x) + 2u \cdot x \quad \text{and} \quad \eta_u(x) = \eta(x) + o(1/(-\text{Log } x)^2).$$

So we have $\theta_u(0) = \theta(0)$ and choosing $u \ll 0$ we obtain that $\theta'_u(0) < 0$ so the following condition will be fulfilled

- There exists $\varepsilon > 0$ such that $\theta_u(x) - \theta_u(0) \leq 0$ on $[0, \varepsilon]$. (@)

REMARK. The behaviour when $x \rightarrow 0^+$ of the function f and of the function g are equivalent, so to study the behaviour when $x \rightarrow 0^+$ of a solution f of the differential inequality (1) we may always assume that the condition (@) is satisfied by the function θ .

1.3 The case $\eta(0) = 0$ and $q > 1$

Proposition 1.3.1 *We fix the functions $\theta \in \mathcal{C}^1([0, 1[)$ and $\eta \in \mathcal{C}^0([0, 1[)$ as in the standard situation. We make the hypotheses that $q := \theta(0) > 1$, $a := \eta(0) = 0$ and that the condition (@) is fulfilled. Then we have the following estimate:*

$$\int_0^{1/2} t^{q-2} \cdot (-\text{Log } t)^{-\beta-1} \cdot f(t) \cdot dt < +\infty \quad (2)$$

for all $\beta > 0$ satisfying $\beta \cdot (q - 1) > \alpha_1$

PROOF. First remark that the convergence in (2) is satisfied for $q > 1$ and any β when f is bounded near 0. This is of course the case if f is increasing on some interval $]0, c[$ with $c > 0$. So we may assume that there exists a decreasing sequence $(x_\nu)_{\nu \geq 0}$ in $]0, \varepsilon[$ with limit 0 such that $f'(x_\nu) < 0$ for each $\nu \geq 0$ where ε is the positive number which exists from the condition (@).

Let us compute the following quantity for $x \in]0, 1/2]$ and $\beta \geq 0$ fixed:

$$\begin{aligned} A_\lambda(x) := & \int_x^{1/2} t^\lambda \cdot (-\text{Log } t)^{-\beta} \cdot f''(t) \cdot dt + \int_x^{1/2} t^{\lambda-1} \cdot (-\text{Log } t)^{-\beta} \cdot \theta(t) \cdot f'(t) \cdot dt \leq \\ & \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta} \cdot \eta(t) \cdot f(t) \cdot dt \end{aligned} \quad (1 \text{ bis})$$

Then, integrating by parts the first integral:

$$\begin{aligned} A_\lambda(x) = & 2^{-\lambda} \cdot (\text{Log } 2)^{-\beta} \cdot f'(1/2) - x^\lambda \cdot (-\text{Log } x)^{-\beta} \cdot f'(x) + \\ & - \beta \cdot \int_x^{1/2} t^{\lambda-1} \cdot (-\text{Log } t)^{-\beta-1} \cdot f'(t) \cdot dt + \int_x^{1/2} t^{\lambda-1} \cdot (-\text{Log } t)^{-\beta} \cdot (\theta(t) - \lambda) \cdot f'(t) \cdot dt \end{aligned}$$

and integrating by parts again:

$$\begin{aligned}
A_\lambda(x) &= 2^{-\lambda} \cdot (\text{Log } 2)^{-\beta} \cdot f'(1/2) - x^\lambda \cdot (-\text{Log } x)^{-\beta} \cdot f'(x) - \beta \cdot 2^{1-\lambda} \cdot (\text{Log } 2)^{-\beta-1} f(1/2) + \\
&\quad \beta \cdot x^{\lambda-1} \cdot (-\text{Log } x)^{-\beta-1} \cdot f(x) + \beta \cdot (\lambda - 1) \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta-1} \cdot f(t) \cdot dt + \\
&\quad + \beta \cdot (\beta + 1) \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta-2} \cdot f(t) \cdot dt + [t^{\lambda-1} \cdot (-\text{Log } t)^{-\beta} \cdot (\theta(t) - \lambda) \cdot f(t)]_x^{1/2} + \\
&\quad - (\lambda - 1) \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta} \cdot (\theta(t) - \lambda) \cdot f(t) \cdot dt + \\
&\quad - \beta \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta-1} \cdot (\theta(t) - \lambda) \cdot f(t) \cdot dt + \\
&\quad - \int_x^{1/2} t^{\lambda-1} \cdot (-\text{Log } t)^{-\beta} \cdot \theta'(t) \cdot f(t) \cdot dt \tag{3}
\end{aligned}$$

So we have:

$$\begin{aligned}
A_\lambda(x) &= & (a) \\
2^{-\lambda} \cdot (\text{Log } 2)^{-\beta} \cdot f'(1/2) &+ & (a) \\
- x^\lambda \cdot (-\text{Log } x)^{-\beta} \cdot f'(x) &+ & (b) \\
- \beta \cdot 2^{1-\lambda} \cdot (\text{Log } 2)^{-\beta-1} f(1/2) &+ & (c) \\
\beta \cdot x^{\lambda-1} \cdot (-\text{Log } x)^{-\beta-1} \cdot f(x) &+ & (d) \\
\beta \cdot (\lambda - 1) \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta-1} \cdot f(t) \cdot dt &+ & (e) \\
\beta \cdot (\beta + 1) \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta-2} \cdot f(t) \cdot dt &+ & (f) \\
[t^{\lambda-1} \cdot (-\text{Log } t)^{-\beta} \cdot (\theta(t) - \lambda) \cdot f(t)]_x^{1/2} &+ & (g) \\
- (\lambda - 1) \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta} \cdot (\theta(t) - \lambda) \cdot f(t) \cdot dt &+ & (h) \\
- \beta \cdot \int_x^{1/2} t^{\lambda-2} \cdot (-\text{Log } t)^{-\beta-1} \cdot (\theta(t) - \lambda) \cdot f(t) \cdot dt &+ & (i) \\
- \int_x^{1/2} t^{\lambda-1} \cdot (-\text{Log } t)^{-\beta} \cdot \theta'(t) \cdot f(t) \cdot dt & & (j)
\end{aligned}$$

where the term (g) is the sum of the following two terms:

$$\begin{aligned}
2^{1-\lambda} \cdot (\text{Log } 2)^{-\beta} \cdot (\theta(1/2) - \lambda) \cdot f(1/2) & & (g_1) \\
- x^{\lambda-1} \cdot (-\text{Log } x)^{-\beta} \cdot (\theta(x) - \lambda) \cdot f(x) & & (g_2)
\end{aligned}$$

Now we have

1. (b) is non negative for $x = x_\nu$ as $f'(x_\nu) \leq 0$ by assumption.
2. (d) is non negative as β and f are non negative.
3. (g_2) is non negative for $x = x_\nu, \nu \gg 1$, when we assume that $\theta(x) - \lambda$ is non positive for x near enough to 0.

So, under the hypothesis that the terms (b), (d) and (g_2) are non negative, and this is the case choosing $\lambda = q$ and $x := x_\nu < \varepsilon$, the inequality (1bis) gives

$$\int_{x_\nu}^{1/2} t^{q-2} \cdot (-\text{Log } t)^{-\beta-1} \cdot f(t) \cdot Z(t) \cdot dt \leq -2^{-q} \cdot (\text{Log } 2)^{-\beta} \cdot f'(1/2) + \beta \cdot 2^{1-q} \cdot (\text{Log } 2)^{-\beta-1} f(1/2) - 2^{1-q} (\text{Log } 2)^{-\beta} \cdot (\theta(1/2) - q) \cdot f(1/2) \quad (*)$$

with

$$Z(t) = \beta \cdot (q-1) + \frac{\beta \cdot (\beta+1)}{-\text{Log } t} - (\theta(t) - q) \cdot \left((q-1) \cdot (-\text{Log } t) + \beta \right) - \theta'(t) \cdot t \cdot (-\text{Log } t) - (-\text{Log } t) \cdot \eta(t) \quad (**)$$

Now if $\beta > \sup\{\alpha_1/(q-1), 0\}$ we obtain that $Z(t)$ is bounded below by a fixed positive number when t is small enough, and then (2) holds true. ■

REMARK. Looking to the proof above, the reader may see that the following hypotheses on the function θ are enough to obtain the conclusion of the proposition, but replacing the condition $\beta > \sup\{\alpha_1/(q-1), 0\}$ by the (better) condition¹ $\beta > \sup\{\alpha_1/(q-1) + \gamma, 0\}$.

- i) θ is \mathcal{C}^1 on $]0, 1[$.
- ii) θ extends continuously at the origin and let $\theta(0) := q$. (@@)
- iii) The majoration $\theta(x) \leq q$ is valid for $x \in]0, \varepsilon[$ for some $\varepsilon > 0$.
Then define $\gamma := \limsup_{x \rightarrow 0^+} (\theta(x) - q) \cdot (-\text{Log } x)$.
- iv) $\lim_{x \rightarrow 0^+} \theta'(x) \cdot x \cdot (\text{Log } x) = 0$. □

Note that *iii*) and *iv*) are consequences of the condition (@) which can always be assumed when θ is in $\mathcal{C}^1([0, 1[)$ or may be written $\theta(x) = \theta_0(x) - 2/(-\text{Log } x)$ with $\theta_0 \in \mathcal{C}^1([0, 1[)$.

¹Note that by condition *iii*) $\gamma \leq 0$.

1.4 The case $\eta(0) = 0$ and $q = 1$

We consider now the standard situation with the conditions $q = 1, a = \eta(0) = 0$ and (@). The the computation above in this case gives the inequality:

$$\int_{x_\nu}^{1/2} t^{-1} \cdot (-\text{Log } t)^{-\beta-1} \cdot f(t) \cdot Z(t) \cdot dt \leq -2^{-1} \cdot (\text{Log } 2)^{-\beta} \cdot f'(1/2) + \beta \cdot (\text{Log } 2)^{-\beta-2} f(1/2) - (\text{Log } 2)^{-\beta} \cdot (\theta(1/2) - 1) \cdot f(1/2) \quad (*\text{bis})$$

with

$$Z(t) = \frac{\beta \cdot (\beta + 1)}{-\text{Log } t} - (\theta(t) - 1) \cdot \beta - \theta'(t) \cdot t \cdot (-\text{Log } t) - (-\text{Log } t) \cdot \eta(t) \quad (**\text{bis})$$

Now if $\beta \cdot (\beta + 1) - \alpha_2 > 0$ we have $Z(t) \cdot (-\text{Log } t) \geq (\beta \cdot (\beta + 1) - \alpha_2) / 2$ when t is small enough, and then we obtain:

$$\int_0^{1/2} t^{-1} \cdot (-\text{Log } t)^{-\beta-2} \cdot f(t) \cdot dt < +\infty \quad (2.1)$$

for all $\beta > 0$ such that $\beta \cdot (\beta + 1) > \alpha_2$, so for $\beta > \sup\{-1/2 + \sqrt{1/4 + \alpha_2}, 0\}$ when $\alpha_2 > -1/4$ or for all $\beta > 0$ when $\alpha_2 \leq -1/4$.

Note that, in this case, it would be enough to assume for θ the conditions *i), ii), iii)* and

$$iv)\text{-bis} \quad \lim_{x \rightarrow 0^+} \theta'(x) \cdot x \cdot (\text{Log } x)^2 = 0.$$

in order to obtain (2.1) for $\beta > \sup\{(\gamma - 1)/2 + \sqrt{(\gamma - 1)^2/4 + \alpha_2}, 0\}$ when $(\gamma - 1)^2 + 4\alpha_2 > 0$ or for all $\beta > 0$ when $(\gamma - 1)^2 + 4\alpha_2 \leq 0$.

1.5 The case $\eta(0) = 0$ and $q < 1$

Assume now that $q < 1$ and define $g(x) := x^{-s} \cdot f(x)$ where s is a real number that we shall choose later on. We have

$$\begin{aligned} f'(x) &= s \cdot x^{s-1} \cdot g(x) + x^s \cdot g'(x) \quad \text{and} \\ f''(x) &= s \cdot (s - 1) \cdot x^{s-2} \cdot g(x) + 2s \cdot x^{s-1} \cdot g'(x) + x^s \cdot g''(x) \quad \text{so} \\ x^2 \cdot f''(x) + \theta(x) \cdot x \cdot f'(x) - \eta(x) \cdot f(x) &= \\ x^s \cdot [s(s - 1) \cdot g(x) + 2s \cdot x \cdot g'(x) + x^2 \cdot g''(x) + \theta(x) \cdot (s \cdot g(x) + x \cdot g'(x)) - \eta(x) \cdot g(x)] \end{aligned}$$

So the differential inequation (1) for f is equivalent to the differential inequation

$$x^2 \cdot g''(x) + (\theta(x) + 2s) \cdot x \cdot g'(x) - (\eta(x) - s \cdot \theta(x) - s \cdot (s - 1)) \cdot g(x) \leq 0. \quad (1_s)$$

The differential inequality (1_s) is obtained from (1) by the transformation

$$\theta \mapsto \theta + 2s \quad \text{and} \quad \eta \mapsto \eta - s.\theta - s.(s - 1).$$

Now choose $s = 1 - q > 0$; then we obtain for g the differential inequality (1) with $\tilde{\theta}(x) = \theta(x) + 2 - 2q$, so $\tilde{q} := \tilde{\theta}(0) = 2 - q > 1$ as we assumed $q = \theta(0) < 1$, and with $\tilde{\eta}(x) = \eta(x) - (1 - q).(\theta(x) - q)$. Note that $\tilde{\eta}$ is continuous and vanishes at 0; because θ is \mathcal{C}^1 we have

$$\alpha_1 = \limsup_{x \rightarrow 0^+} \tilde{\eta}(x).(-\text{Log } x) = \limsup_{x \rightarrow 0^+} \eta(x).(-\text{Log } x).$$

Now we apply the case $q > 1$ to g and we conclude that (2) holds for g and this gives:

$$\begin{aligned} \int_0^{1/2} t^{\tilde{q}-2}.(-\text{Log } t)^{-\beta-1}.t^{q-1}.f(t).dt < +\infty \quad \text{and then} \\ \int_0^{1/2} t^{-1}.(-\text{Log } t)^{-\beta-1}.f(t).dt < +\infty \end{aligned} \quad (2.2)$$

for all $\beta > \sup\{\alpha_1/(1 - q), 0\}$.

Note that, in fact, the conditions (@@) on θ are enough to obtain (2.2) for

$$\beta > \sup\{\alpha_1/(1 - q) + \gamma, 0\}.$$

1.6 The case $\eta(0) > 0$ and $q \neq 1$

We consider now the case where a is positive and $q := \theta(0) \neq 1$.

We assume that $\theta :]0, 1[\rightarrow \mathbb{R}$ is still a \mathcal{C}^1 and satisfies² $\theta(x) - \theta(0) \leq 0$ for $x \in]0, \varepsilon[$ and we look at positive solutions of the differential inequality

$$x^2.f''(x) + x.\theta(x).f'(x) \leq \eta(x).f(x) \quad \forall x \in]0, 1[\quad (1)$$

where f is a \mathcal{C}^2 function on $]0, 1[$.

Defining $g(x) = x^{-s}.f(x)$ we have seen that (1) is equivalent to

$$x^2.g''(x) + x.\tilde{\theta}(x).g'(x) \leq \tilde{\eta}(x).g(x) \quad \forall x \in]0, 1[$$

with

$$\tilde{\theta}(x) = \theta(x) + 2s \quad \text{and} \quad \tilde{\eta}(x) = \eta(x) - s.\theta(x) - s.(s - 1),$$

so, choosing for s a solution of the equation $s^2 + (q - 1).s - a = 0$ where $q = \theta(0)$, we obtain

$$\tilde{\eta}(0) = 0, \quad \limsup_{x \rightarrow 0^+} \tilde{\eta}(x).(-\text{Log } x) = \alpha_1 \quad \text{and} \quad \tilde{q} = \tilde{\theta}(0) = q + 2s.$$

²But see the remark at the end of 1.3.

In fact we shall choose the positive root of $s^2 + (q-1)s - a = 0$:

$$s := s_+ = -(q-1)/2 + \sqrt{(q-1)^2/4 + a}.$$

Then $\tilde{q} = q + 2s_+ = 1 + \sqrt{(q-1)^2 + 4a} > 1$ so we may apply to g the proposition 1.3.1. This gives

$$\int_0^{1/2} t^{\tilde{q}-2} \cdot (-\text{Log } t)^{-\beta-1} \cdot g(t) \cdot dt < +\infty$$

for all $\beta > 0$ such that $\beta > \alpha_1/(\tilde{q}-1)$, that is to say

$$\int_0^{1/2} t^{(q-3)/2 + \sqrt{(q-1)^2/4 + a}} \cdot (-\text{Log } t)^{-\beta-1} \cdot f(t) \cdot dt < +\infty$$

for all $\beta > \sup\{\alpha_1/\sqrt{(q-1)^2 + 4a}, 0\}$.

Note that for $a = 0$ this computation gives for $q > 1$

$$(q-3)/2 + \sqrt{(q-1)^2/4 + a} = q-2$$

and for $q < 1$

$$(q-3)/2 + \sqrt{(q-1)^2/4 + a} = -1.$$

1.7 The case $\eta(0) > 0$ and $q = 1$

We assume now that the continuous function η satisfies

$$\limsup_{x \rightarrow 0^+} (\eta(x) - a) \cdot (-\text{Log } x) = 0.$$

where a is a positive real number (so $\alpha_1 = 0$ and $\eta(0) > 0$).

Then let the function g defined by the equality $f(x) = x^s \cdot (-\text{Log } x) \cdot g(x)$. Then we have

$$\begin{aligned} & x^2 \cdot f''(x) + \theta(x) \cdot x \cdot f'(x) - \eta(x) \cdot f(x) = \\ & x^s \cdot (-\text{Log } x) \cdot [x^2 \cdot g''(x) + 2s \cdot x \cdot g'(x) - 2s \cdot g'(x)/(-\text{Log } x) + s \cdot (s-1) \cdot g(x)] + (2s-1) \cdot x^s \cdot g(x) + \\ & \theta(x) \cdot [x^s \cdot (-\text{Log } x) \cdot x \cdot g'(x) + s \cdot x^s \cdot (-\text{Log } x) \cdot g(x) + x^s \cdot g(x)] - \eta(x) \cdot x^s \cdot (-\text{Log } x) \cdot g(x) \end{aligned}$$

So the inequality $x^2 \cdot f''(x) + \theta(x) \cdot x \cdot f'(x) \leq \eta(x) \cdot f(x)$ is equivalent to the inequality

$$x^2 \cdot g''(x) + (\theta(x) + 2s - 2)/(-\text{Log } x) \cdot x \cdot g'(x) \leq [\eta(x) - s \cdot \theta(x) - s \cdot (s-1) - \frac{\theta(x) + 2s - 1}{-\text{Log } x}] \cdot g(x).$$

Choosing $s := -\sqrt{a}$ we obtain as $\tilde{\theta}(x) := \theta(x) + 2s - 2/(-\text{Log } x)$ and, if θ satisfies the conditions (@@), the same will be true for $\tilde{\theta}$ (but changing γ to $\gamma - 2$), then

$$\tilde{q} := \tilde{\theta}(0) = 1 - 2\sqrt{a} \quad \text{and} \quad \tilde{\eta}(x) := \eta(x) - s \cdot \theta(x) - s \cdot (s-1) - \frac{\theta(x) + 2s - 1}{-\text{Log } x}.$$

Then $\tilde{\eta}$ satisfies $\tilde{\eta}(0) = 0$, as $\theta(0) = 1$ and

$$\limsup_{x \rightarrow 0^+} \tilde{\eta}(x) \cdot (-\text{Log } x) = 2\sqrt{a}.$$

So we are under the hypothesis of the case $\tilde{q} < 1$ and $\tilde{\eta}(0) = 0$ for g , as $\tilde{\eta}$ has the suitable estimate at 0^+ . As $\alpha_1/(1 - \tilde{q}) = 1$ we obtain that (here $\gamma = -2$)

$$\int_0^{1/2} t^{\sqrt{a}-1} \cdot (-\text{Log } t)^{-\beta-1} \cdot f(t) \cdot dt < +\infty \quad \forall \beta > 0 \quad (2.3)$$

1.8 Examples

The following computation shows that the result of proposition 1.3.1 is optimal. Let $f(x) := x^{-\lambda} \cdot (-\text{Log } x)^\beta$ and $\eta(x) := \alpha_1/(-\text{Log } x) + \alpha_2/(-\text{Log } x)^2$ for x positive small enough, where α_1 is positive.

Then we have

$$\begin{aligned} f'(x) &= -\lambda \cdot x^{-\lambda-1} \cdot (-\text{Log } x)^\beta - \beta \cdot x^{-\lambda-1} \cdot (-\text{Log } x)^{\beta-1} \\ f''(x) &= (\lambda + 1) \cdot \lambda \cdot x^{-\lambda-2} \cdot (-\text{Log } x)^\beta + \beta \cdot (2\lambda + 1) \cdot x^{-\lambda-2} \cdot (-\text{Log } x)^{\beta-1} + \\ &\quad + (\beta - 1) \cdot \beta \cdot x^{-\lambda-2} \cdot (-\text{Log } x)^{\beta-2} \end{aligned}$$

If we define g by the equality

$$x^2 \cdot f''(x) + q \cdot x \cdot f'(x) = g(x) \cdot f(x)$$

we obtain

$$g(x) := [\lambda \cdot (\lambda + 1 - q) + \beta \cdot (2\lambda + 1 - q) \cdot (-\text{Log } x)^{-1} + (\beta - 1) \cdot \beta \cdot (-\text{Log } x)^{-2}].$$

Assuming that $q > 1$ and choosing

$$\lambda + 1 - q = 0 \quad \text{and} \quad \beta \cdot (2\lambda + 1 - q) = \alpha_1$$

so $\lambda = q - 1$ and $\beta \cdot (q - 1) = \alpha_1$, we obtain, for $q - 1$ small enough in order that $\beta \cdot (\beta - 1) < \alpha_2$ implies $g(x) \leq \eta(x)$ for x small enough,

$$x^2 \cdot f''(x) + q \cdot x \cdot f'(x) \leq \eta(x) \cdot f(x)$$

Remark now that, for the choice $\beta = \alpha_1/(q - 1)$ the convergence of the integral

$$\int_0^{1/2} x^{q-2} \cdot (-\text{Log } x)^{-\delta-1} \cdot f(x) \cdot dx$$

holds if and only if $\beta - \delta - 1 < -1$ that is to say for $\delta > \beta = \alpha_1/(q - 1)$ (compare with proposition 1.3.1).

For $q = 1$, choosing $\eta(x) := \alpha_2 / (\text{Log } x)^2$ where α_2 is positive, $\lambda = 1$ and $\beta := \beta_+$, the positive root of $(\beta - 1) \cdot \beta = \alpha_2$, the integral

$$\int_0^{1/2} x^{-1} \cdot (-\text{Log } x)^{-\delta-2} \cdot f(x) \cdot dx$$

converges if and only if $\beta_+ - \delta - 2 > -1$ so for

$$\delta > \beta_+ - 1 = -1/2 + \sqrt{1/4 + \alpha_2} \quad (\text{compare with the conclusion of 1.4}).$$

The following example shows that for $q = 1$ and $\eta(0) = 0$ the differential inequality (1) does not implies the point-wise bound near 0 of a solution by $C \cdot (-\text{Log } x)$ for any positive constant C :

For $x \in]0, 1/3[$, let

$$f(x) = (-\text{Log } x) \cdot \text{Log}(-\text{Log } x).$$

Then

$$\begin{aligned} f'(x) &= \frac{-1}{x} \cdot \text{Log}(-\text{Log } x) - \frac{1}{x} \\ f''(x) &= \frac{1}{x^2} \cdot \text{Log}(-\text{Log } x) + \frac{1}{x^2 \cdot (-\text{Log } x)} + \frac{1}{x^2} \end{aligned}$$

and so, we obtain

$$x^2 \cdot f''(x) + x \cdot f'(x) = \frac{1}{(-\text{Log } x)} = \eta(x) \cdot f(x)$$

where

$$\eta(x) := \frac{1}{(\text{Log } x)^2 \cdot \text{Log}(-\text{Log } x)}.$$

We have $\lim_{x \rightarrow +\infty} (\text{Log } x)^2 \cdot \eta(x) = 0$ but $f(x) / (-\text{Log } x)$ goes to $+\infty$ when x goes to 0^+ . \square

2 An application

Our application is devoted to a kind of generalization of Gronwald's lemma for the Laplacian in a punctured ball in \mathbb{R}^n . We begin by the case $n \geq 3$.

Corollary 2.0.1 *Let $n \geq 3$ be an integer and consider a \mathcal{C}^2 function $F : B_r^* \rightarrow \mathbb{R}^+$ on the punctured ball B_r^* with center 0 and radius r in \mathbb{R}^n endowed with its natural euclidian norm. Assume that there is a continuous function $\eta : B_r \rightarrow \mathbb{R}$ with*

$$\limsup_{\|x\| \rightarrow 0} \eta(x) \cdot \|x\|^{-2} \cdot (-\text{Log } \|x\|) = \alpha_1 < +\infty$$

such that F satisfies on B_r^*

$$\Delta F(x) \leq \eta(x) \cdot F(x) \quad (6)$$

where Δ is the standard Laplace operator on \mathbb{R}^n .

Then we have, for any $\beta > \sup\{\alpha_1/(n-2), 0\}$

$$\int_{B_{r/2}^*} \|x\|^{-2} \cdot (-\text{Log}(\|x\|))^{-\beta-1} \cdot F(x) \cdot dx < +\infty \quad (7)$$

where dx is the euclidian measure.

PROOF. Define the function $f :]0, r[\rightarrow \mathbb{R}^+$ by $f(t) := \int_{S_t} F(t \cdot \sigma) \cdot d\sigma$ where (t, σ) are polar coordinates on \mathbb{R}^n and where $dx = t^{n-1} \cdot dt \wedge d\sigma$ where $d\sigma$ is the rotation invariant measure on the sphere S_1 with a suitable normalization. Then the function f is \mathcal{C}^2 on $]0, r[$ and we have

$$f''(t) + \frac{n-1}{t} \cdot f'(t) = \int_{S_t} \Delta(F)(t \cdot \sigma) \cdot d\sigma \quad \forall t \in]0, r[\quad (8)$$

thanks to the rotation invariance of the Laplacian.

Then it is easy to apply the proposition 1.3.1 to the function f with $\theta \equiv n-1$. ■

The case $n = 2$ corresponds to the case $\lambda = q = 1$. It gives in a similar way the following corollary.

Corollary 2.0.2 *Let $n = 2$ and consider a \mathcal{C}^2 function $F : B_r^* \rightarrow \mathbb{R}^+$ on the punctured disc D_r^* with center 0 and radius r in \mathbb{R}^2 endowed with its natural euclidian norm. Assume that there is a continuous function $\eta : D_r \rightarrow \mathbb{R}$ with*

$$\limsup_{\|x\| \rightarrow 0} \eta(x) \cdot \|x\|^{-2} \cdot (-\text{Log } \|x\|)^2 = \alpha_2 < +\infty$$

such that F satisfies on D_r^*

$$\Delta F(x) \leq \eta(x) \cdot F(x) \quad (6)$$

where Δ is the standard Laplace operator on \mathbb{R}^n .

Then we have, for any $\beta > 0$ such that $\beta \cdot (\beta + 1) > \alpha_2$:

$$\int_{D_{r/2}}^* \|x\|^{-2} \cdot (-\text{Log}(\|x\|))^{-\beta-2} \cdot F(x) \cdot dx < +\infty \quad (7)$$

where dx is the euclidian measure.