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# Numerical convergence for a diffuse limit of hyperbolic systems on bounded domain

Hélène Mathis and Nicolas Therme

**Abstract** This paper deals with the diffusive limit of the scaled Goldstein-Taylor model and its approximation by an Asymptotic Preserving Finite Volume scheme. The problem is set in some bounded interval with nonhomogeneous boundary conditions depending on time. We obtain a uniform estimate in the small parameter  $\varepsilon$  using a relative entropy of the discrete solution with respect to a suitable profile which satisfies the boundary conditions expected to hold as  $\varepsilon$  goes to 0.

**Key words:** Diffusive limit of hyperbolic systems, initial boundary value problem, finite volume approximation, asymptotic preserving scheme

**MSC (2010):** 65M08, 65M12, 35L50

## 1 Introduction

This work is devoted to the numerical analysis of numerical scheme for the initial bounded value hyperbolic problem with diffusive limit. We focus here on the Goldstein-Taylor model which depicts the motion of a Chaplygin gas of density  $v(t, x)$  at velocity  $u(t, x)$

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Hélène Mathis  
LMJL, Université de Nantes,  
2 rue de la Houssinière, 44322 Nantes, France  
e-mail: helene.mathis@univ-nantes.fr

Nicolas Therme  
LMJL, Université de Nantes,  
2 rue de la Houssinière, 44322 Nantes, France  
e-mail: nicolas.therme@univ-nantes.fr

$$\begin{cases} \varepsilon \partial_t v_\varepsilon + \partial_x u_\varepsilon = 0, \\ \varepsilon \partial_t u_\varepsilon + a^2 \partial_x v_\varepsilon = -\frac{\sigma}{\varepsilon} u_\varepsilon, \end{cases} \quad (1)$$

where  $\sigma$  is a positive friction coefficient,  $a$  stands for the speed of sound and  $\varepsilon$  is a positive relaxation parameter. The system is set on the bounded space-time domain  $\Omega := (0, 1) \times (0, T)$  with the following initial and boundary conditions

$$\begin{cases} u(0, x) = u_0(x) \in L^3(0, 1), & v(0, x) = v_0(x) \in L^3(0, 1), \\ (av + u)(t, 0) = \varphi^-(t), & (av - u)(t, 1) = \varphi^+(t), \quad \forall t \in [0, T], \\ \varphi^\pm \in W^{1, \infty}([0, T]). \end{cases} \quad (2)$$

Moreover we assume that  $av_0 + u_0$ ,  $av_0 - u_0$  are nonnegative functions of  $x \in [0, 1]$  and that  $\varphi^\pm$  are nonnegative functions of  $t \in [0, T]$ . According to [4] the relaxation process, as  $\varepsilon \rightarrow 0$ , characterizes limit solutions of (1) as diffusive solutions of

$$\begin{cases} \partial_t v - \frac{a^2}{\sigma} \partial_{xx} v = 0, \\ u = 0, \end{cases} \quad (3)$$

endowed with the following initial and boundary conditions

$$\begin{cases} v(0, x) = v_0(x), & x \in (0, 1), \\ av(t, 0) = \varphi^-(t), & av(t, 1) = \varphi^+(t), \quad t \in (0, T). \end{cases} \quad (4)$$

The diffusive relaxation limit of hyperbolic systems has been the topic of numerous papers, see for instance [2] and included references. In [11] the authors provide a convergence rate in  $\varepsilon^4$  for several hyperbolic systems with their diffusive limits (including the Goldstein-Taylor model) using a relative entropy method. The technique consists in the comparison of the weak entropy solutions of the hyperbolic system toward the regular solutions of the diffusive limit using the entropy function of the hyperbolic system. These papers deal with the initial value problem on the infinite line or with initial boundary value problem with periodic conditions. In [4] the authors establish the diffusion limit for Carleman-type model (including the Goldstein-Taylor model) in bounded domain with nonhomogeneous boundary conditions. Considering boundary conditions of type (2), they obtain an uniform estimate using another relative entropy of the solution  $(u_\varepsilon, v_\varepsilon)$  with respect to a suitable *profile* which satisfies the boundary conditions of the diffusive limit as  $\varepsilon \rightarrow 0$ .

We aim to prove a similar result for the discrete approximation of the Goldstein-Taylor system (1) on a bounded domain using an Asymptotic Preserving (AP) Finite Volume scheme. According to the primary works of Jin [9], a numerical scheme for the system (1) is said to be AP if it is stable and consistent with the solutions of the hyperbolic model (1) for all  $\varepsilon > 0$  and if, at the limit  $\varepsilon \rightarrow 0$ , it converges to a stable and consistent numerical scheme with the solutions of the limit parabolic model (3). Concerning specifically the discretization of hyperbolic systems with source terms in the diffusive limit, the literature is huge. Let us cite the work of Gosse and Toscani

who proposed a well-balanced and AP scheme for the Goldstein-Taylor model in [6]. The same scheme is recover by means of relaxation techniques in [3]. For more general discrete kinetic models, we refer to [7] and [10]. Besides in [1] the authors provide a convergence rate for the semi-discrete in time AP scheme given in [10] for the  $p$ -system with damping. The convergence is proved on infinite domain by adapting the relative entropy method of [11]. We aim at proving the same kind of result on a bounded domain by mimicking the proof of Golse and Salvarani [4] on bounded domain.

To this end we organize this note as follow. In Section 2, we present the numerical scheme given in [3] and state the main convergence theorem. Then in section 3, we introduce the necessary tools to prove the theorem, namely the relative entropy of the system and its production rate. The Section ends with the statement of an inequality satisfied by the entropy and its production rate. This inequality contains remainder terms for which we provide upper bounds in Section 4. The control of these remainders allow to conclude the proof of the main theorem. In order to illustrate the convergence of the AP scheme towards the diffusive limit on bounded domain, we conclude this work in Section 5 by some numerical results.

## 2 AP scheme and main result

The numerical scheme we propose to solve the Goldstein-Taylor system was first introduced in [3]. It is based on relaxation techniques for the construction of well-balanced schemes, following [5] and [8]. The computation domain  $(0, 1)$  is discretized with  $L$  cells of size  $\Delta x$ . The time interval  $[0, T]$  is decomposed in  $N$  time steps  $\Delta t$  submitted to a suitable CFL condition (see below). For the sake of readability we drop the subscript  $\varepsilon$  for the discrete solution of the model (1). For  $n = 0, \dots, N-1$  and  $i = 1, \dots, L-1$ , the numerical scheme reads

$$\begin{cases} u_i^{n+1} = u_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x} \left[ a^2 \frac{v_{i+1}^n - v_{i-1}^n}{2} + \frac{a}{2} (2u_i^n - u_{i+1}^n - u_{i-1}^n) \right] - \frac{\sigma \Delta t}{\varepsilon^2 K_\varepsilon} u_i^{n+1}, \\ v_i^{n+1} = v_i^n - \frac{\Delta t}{\varepsilon K_\varepsilon \Delta x} \left[ \frac{u_{i+1}^n - u_{i-1}^n}{2} + \frac{a}{2} (2v_i^n - v_{i-1}^n - v_{i+1}^n) \right], \\ K_\varepsilon = 1 + \frac{\sigma \Delta x}{2a\varepsilon}. \end{cases} \quad (5)$$

Note that  $\varepsilon K_\varepsilon \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . On the infinite line, it is proved in [3] that the scheme is consistent,  $L^2$ -diminishing under the CFL condition  $\Delta t \leq \frac{\sigma}{2a^2} \Delta x^2$  and convergent towards a consistent discretization of (3). This proof rely on a van Neumann analysis which cannot be used on bounded domains. The key idea is then to adapt tools introduced for the continuous framework in [4]. To do so, we consider initial data  $u_0, v_0 \in L^3(0, 1)$  for (1) discretized as

$$u_i^0 = \frac{1}{\Delta x} \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} u_0(x) dx, \quad v_i^0 = \frac{1}{\Delta x} \int_{x_i - \frac{\Delta x}{2}}^{x_i + \frac{\Delta x}{2}} v_0(x) dx, \quad \forall i = 1, \dots, L-1, \quad (6)$$

and discrete boundary conditions

$$av_0^n + u_0^n = \varphi^-(t^n), \quad av_L^n - u_L^n = \varphi^+(t^n), \quad n = 0, \dots, N. \quad (7)$$

We aim at the following main result

**Theorem 1.** *Let  $(u_{\varepsilon,i}^n, v_{\varepsilon,i}^n)_{i=0,\dots,L, n=0,\dots,N}$  be the solution of the scheme (5) together with boundary and initial conditions (7)–(6). Suppose that the following CFL condition holds,*

$$\Delta t \leq \frac{\sigma}{8a^2} \Delta x^2. \quad (8)$$

*Then for all  $i = 1, \dots, L-1$  and  $n = 0, \dots, N-1$ ,  $u_{\varepsilon,i}^{n+1}$  tends to zero as  $\varepsilon \rightarrow 0$  and  $v_{\varepsilon,i}^{n+1}$  tends towards the solution  $v_{0,i}^{n+1}$  of the consistent discretization of (3) given by*

$$\begin{cases} v_{0,i}^{n+1} = v_{0,i}^n - \frac{a^2 \Delta t}{\sigma \Delta x^2} [2v_{0,i}^n - v_{0,i-1}^n - v_{0,i+1}^n], & \text{if } n \geq 1, \\ v_{0,i}^1 = v_i^0 - \frac{a^2 \Delta t}{\sigma \Delta x^2} [2v_i^0 - v_{i-1}^0 - v_{i+1}^0] - \frac{a \Delta t}{\sigma \Delta x^2} [u_{i+1}^0 - u_{i-1}^0]. \end{cases} \quad (9)$$

The first step is to *diagonalize* the Goldstein-Taylor system to enter the framework of [4]. To this end we set  $\alpha = av + u$  and  $\beta = av - u$  and we get, in the discrete setting, for  $i = 1, \dots, L-1$  and  $n = 0, \dots, N-1$

$$\begin{cases} \varepsilon \bar{\partial}_t \alpha_i^n + \frac{a}{\varepsilon K_\varepsilon} \bar{\partial}_x^- \alpha_i^n = -\frac{\sigma}{2\varepsilon^2 K_\varepsilon} (\alpha_i^{n+1} - \beta_i^{n+1}), \\ \varepsilon \bar{\partial}_t \beta_i^n - \frac{a}{\varepsilon K_\varepsilon} \bar{\partial}_x^+ \beta_i^n = \frac{\sigma}{2\varepsilon^2 K_\varepsilon} (\alpha_i^{n+1} - \beta_i^{n+1}), \end{cases} \quad (10)$$

where,

$$\bar{\partial}_t \alpha_i^n = \frac{\alpha_i^{n+1} - \alpha_i^n}{\Delta t}, \quad \bar{\partial}_x^- \alpha_i^n = \frac{\alpha_i^n - \alpha_{i-1}^n}{\Delta x}, \quad \bar{\partial}_x^+ \beta_i^n = \frac{\beta_{i+1}^n - \beta_i^n}{\Delta x}, \quad (11)$$

which is nothing but the upwind scheme for the state variables  $\alpha$  and  $\beta$ . In the following sections we prove that in the relaxation limit  $\varepsilon \rightarrow 0$ , the sequence defined by  $\alpha_i^n - \beta_i^n = 2u_i^n$ , for  $i = 1, \dots, L-1$  and  $n = 1, \dots, N$ , converges to 0 and that the sequence given by  $(\alpha_i^n + \beta_i^n)/2a = v_i^n$  converges toward a consistent discretization of the diffusive system (3).

### 3 The relative entropy and its production rate

Following the proof of [4], we now introduce a *profile*, consistent with the boundary conditions (7) and with a consistent discretization of the boundary conditions (4) of the limiting diffusion equation. Let define

$$v = \max(\|\varphi^+\|_{W^{1,\infty}(0,T)}, \|\varphi^-\|_{W^{1,\infty}(0,T)}), \quad (12)$$

and a sequence  $(f_i^n)$  as a discrete convex combination of the boundary conditions with

$$\begin{cases} f_0^n = \varphi^-(t^n), & f_L^n = \varphi^+(t^n), \\ f_i^n = (1 - (i-1)\frac{\Delta x}{1-2\Delta x})\varphi^-(t^n) + (i-1)\frac{\Delta x}{1-2\Delta x}\varphi^+(t^n), \end{cases} \quad (13)$$

for all  $i = 1, \dots, L-1$  and  $n = 0, \dots, N$ . Hence  $(f_i^n)$ ,  $\partial_t(f_i^n)$ ,  $\partial_x^\pm(f_i^n)$  are bounded according to  $v$ . Next we define  $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$  a convex function, which acts as an entropy function for both  $\alpha$  and  $\beta$ , namely

$$\phi(x) = \frac{1}{2}(x^2 + (1+v)^2). \quad (14)$$

It satisfies the following property

$$\phi(y) - f_i^n(t, x)y \geq \phi(y) - vy \geq y. \quad (15)$$

We now defined the relative entropy for  $\alpha$  and  $\beta$  with respect to the profile  $f$  for  $n = 0, \dots, N$

$$H^n[\alpha, \beta | f] = \sum_{i=1}^{L-1} \Delta x (\phi(\alpha_i^n) + \phi(\beta_i^n) - 2\phi(f_i^n) - \phi'(f_i^n)(\alpha_i^n + \beta_i^n - 2f_i^n)). \quad (16)$$

By the definition of  $\phi$ , one can notice that

$$H^n[\alpha, \beta | f] = \sum_{i=1}^{L-1} \frac{\Delta x}{2} ((\alpha_i^n - f_i^n)^2 + (\beta_i^n - f_i^n)^2) \geq 0. \quad (17)$$

We also define the entropy production rate,  $\forall n = 0, \dots, N$ , as

$$P^n[\alpha, \beta] = \sum_{i=1}^{L-1} \Delta x (\phi'(\alpha_i^n) - \phi'(\beta_i^n))(\alpha_i^n - \beta_i^n), \quad (18)$$

which boils down to

$$P^n[\alpha, \beta] = \sum_{i=1}^{L-1} \Delta x (\alpha_i^n - \beta_i^n)^2 \geq 0. \quad (19)$$

The proof of the main theorem relies on the following relative entropy inequality.

**Lemma 1.** *The relative entropy and its production rate satisfy*

$$\partial_t H^n[\alpha, \beta | f] + \frac{\sigma}{2\varepsilon^2 K_\varepsilon} P^{n+1}[\alpha, \beta] + R^n \leq Q^n + \sum_{i=1}^{L-1} \Delta x \partial_t (f_i^n)^2, \quad (20)$$

where the remainders  $R^n$  and  $Q^n$  are given by

$$\begin{aligned}
R^n &= \sum_{i=1}^{L-1} \frac{\Delta x}{2\Delta t} [(\alpha_i^{n+1} - \alpha_i^n)^2 + (\beta_i^{n+1} - \beta_i^n)^2] \\
&+ \sum_{i=1}^{L-1} \frac{a}{2\varepsilon K_\varepsilon} [(\alpha_i^n - \alpha_{i-1}^n)^2 + (\beta_{i+1}^n - \beta_i^n)^2] \\
&+ \sum_{i=1}^{L-1} \frac{a}{\varepsilon K_\varepsilon} [(\alpha_i^{n+1} - \alpha_i^n)(\alpha_i^n - \alpha_{i-1}^n) + (\beta_i^{n+1} - \beta_i^n)(\beta_i^n - \beta_{i+1}^n)] \quad (21) \\
&+ \sum_{i=1}^{L-1} \Delta x \Delta t \bar{\partial}_t(f_i^n) \bar{\partial}_t(\alpha_i^n + \beta_i^n) - \frac{a}{\varepsilon K_\varepsilon} \sum_{i=1}^{L-1} \Delta x^2 \bar{\partial}_x^+ f_i^n \bar{\partial}_x^+ \beta_i^n \\
Q^n &= - \sum_{i=1}^{L-1} \Delta x \bar{\partial}_t(f_i^n) (\alpha_i^n + \beta_i^n) - \frac{a}{\varepsilon K_\varepsilon} \sum_{i=1}^{L-1} \Delta x \bar{\partial}_x^+(f_i^n) (\alpha_i^n - \beta_i^n).
\end{aligned}$$

*Proof.* The inequality is obtained by multiplying the first and second equations of the scheme (10) by  $\alpha_i^{n+1} \Delta x$  and  $\beta_i^{n+1} \Delta x$  respectively, summing it over cells  $i = 1, \dots, L-1$  and combining the two relations. Using the definition (14) of  $\varphi$  and basic algebraic manipulations we get

$$\begin{aligned}
&\sum_{i=1}^{L-1} \Delta x \bar{\partial}_t(\phi(\alpha_i^n) + \phi(\beta_i^n) - f_i^n(\alpha_i^n + \beta_i^n)) \\
&+ \frac{a}{\varepsilon K_\varepsilon} \sum_{i=1}^{L-1} \Delta x (\bar{\partial}_x^-(\phi(\alpha_i^n) - f_i^n \alpha_i^n) - \bar{\partial}_x^+(\phi(\beta_i^n) - f_i^n \beta_i^n)) \\
&+ R^n = - \frac{\sigma}{2\varepsilon^2 K_\varepsilon} P^{n+1}[\alpha, \beta] + Q^n. \quad (22)
\end{aligned}$$

The second term in (22) turns to be a nonnegative quantity by convexity of  $\phi$  and the choice of the profile  $f$ , which leads to the desired inequality.

## 4 Control of the remainder terms and proof of the main theorem

### *Control of the remainders $R^n$*

Using a Taylor-Young inequalities on the last three terms of  $R^n$ , one notices that  $R^n \geq -C$ , provided the CFL condition (8) is satisfied, where  $C$  is a positive constant only depending on  $v$ ,  $\Delta x$ ,  $\Delta t$ ,  $a$  and  $\sigma$ .

### *Control of the remainders $Q^n$*

We estimate the remainder  $Q^n$  in term of  $P^n[\alpha, \beta]$  thanks to a Taylor-Young inequality. One gets

$$Q^n \leq - \sum_{i=1}^{L-1} \Delta x \partial_t f_i^n(\alpha_i^n + \beta_i^n) + \frac{\sigma}{4\varepsilon^s K_\varepsilon} P^n[\alpha, \beta] + C',$$

where  $s$  equals 2 if  $n \geq 1$  and 1 otherwise and  $C'$  is a positive constant depending only on  $v$ ,  $\Delta x$ ,  $\Delta t$ ,  $a$  and  $\sigma$ .

Finally, using twice the inequality (15) with  $y = \alpha_i^n$  and  $y = \beta_i^n$  leads to

$$- \sum_{i=1}^{L-1} \Delta x \partial_t f_i^n(\alpha_i^n + \beta_i^n) \leq v H^n[\alpha, \beta | f] + S$$

with  $S$  a positive constant only depending on  $v$ .

### Relative entropy estimate

Using the above estimates, the relative entropy satisfies the following inequality

$$\partial_t H^n[\alpha, \beta | f] + \frac{\sigma}{2\varepsilon^2 K_\varepsilon} P^{n+1}[\alpha, \beta] \leq \frac{\sigma}{4\varepsilon^s K_\varepsilon} P^n[\alpha, \beta] + v H^n[\alpha, \beta | f] + \hat{C}, \quad (23)$$

with  $\hat{C}$  a positive constant only depending on  $v$ ,  $\Delta x$ ,  $\Delta t$ ,  $a$  and  $\sigma$ .

Upon multiplying both side of (23) by  $\Delta t$  and summing it for all time iteration  $n = 0, \dots, N-1$ , one obtains, thanks to a discrete Grönwall inequality,

$$H^N[\alpha, \beta | f] + \frac{\sigma}{4\varepsilon^2 K_\varepsilon} \sum_{n=1}^{N-1} \Delta t P^n[\alpha, \beta] \leq M e^{vT} \quad (24)$$

where  $M = H^0[\alpha, \beta | f] + \frac{a}{2\Delta x} P^0[\alpha, \beta] + \hat{C}T$ . Since  $\varepsilon K_\varepsilon \rightarrow \frac{\sigma \Delta x}{2a}$  as soon as  $\varepsilon \rightarrow 0$ , we can deduce from (24) that  $\alpha_i^n - \beta_i^n = 2u_{\varepsilon,i}^n$  tends to zero for every  $i = 1, \dots, L-1$  and  $n = 1, \dots, N$ . On the other hand the relative entropy satisfies (17) so that, using (24),  $\alpha_i^n + \beta_i^n = 2av_{\varepsilon,i}^n$  is bounded independently of  $\varepsilon$ . Finally thanks to Bolzano–Weierstrass theorem there exists an unlabeled subsequence of  $(v_{\eta,i}^n)_\eta$  converging towards some  $v_{0,i}^n$  for  $i = 1, \dots, L-1$  and  $n = 1, \dots, N$ . Passing to the limit in the second equation of the original scheme (5) leads to the main theorem.

## 5 Numerical illustrations

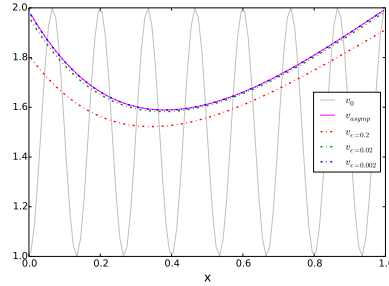
To conclude, we highlight the previous result with some numerical experiments. We consider the following test case:

$$u_0(x) = 0.5, \quad v_0(x) = 1 + \sin\left(\frac{15\pi}{2}x\right)^2, \quad \varphi^-(t) = 1 + \frac{t}{0.05}, \quad \varphi^+(t) = 2.$$

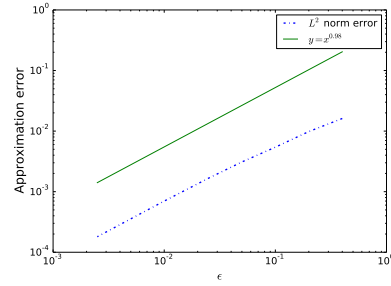


The test is performed on a 100 cells mesh with the parabolic CFL (8). Results obtained with the asymptotic scheme alongside the AP scheme with various epsilon values is plotted below.

A convergence test is then performed to assess the convergence rate. It is numerically close to one.



Solution at time  $T = 0.05$  with  $\Delta x = 0.1$



Convergence test

These results are in a good accordance with the previous theorem. Similar numerical simulations has been performed for the p-system and show similar behavior. Extension of the theorem to the p-system is underway.

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