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## THIN LAYERS FOR THE LANDAU-LIFSCHITZ EQUATION

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**ABSTRACT.** In this paper we study the behaviour of the magnetization in a thin layer of ferromagnetic material when the exchange coefficient is small. We explain the interaction between the boundary layer phenomenon and the thin layer phenomenon.

**AMS (MOS) Subject Classification.** 35Q60, 78M35, 34E20.

### 1. INTRODUCTION

We consider here ferromagnetic media which are spontaneously magnetized. They are used for example in computers and in aeronautics for the skins of the planes. In the quasi-stationary case, the behaviour of the magnetization, denoted by  $u$ , is described by the Landau-Lifschitz equation:

$$(1) \quad \begin{cases} \frac{\partial u}{\partial t} = u \wedge H_e - u \wedge (u \wedge H_e) & \text{in } \mathbb{R}_t^+ \times \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \end{cases}$$

where  $\Omega$  is a domain filled with ferromagnetic material and immersed in the vacuum,  $n$  is the outward unitary normal to  $\partial\Omega$  and  $H_e$  is the effective magnetic field given by:

$$(2) \quad H_e = \varepsilon^2 \Delta u + \mathcal{H}(u).$$

Here,  $\varepsilon^2$  is a small parameter called the exchange coefficient,  $\mathcal{H}(u)$  is called the demagnetizing field given by the magnetostatic equations (we assume that we always are at the electromagnetic equilibrium):

$$(3) \quad \begin{cases} \mathcal{H}(u) \in \mathbb{L}^2(\mathbb{R}^3), \\ \operatorname{div} (\mathcal{H}(u) + \bar{u}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ \operatorname{curl} \mathcal{H}(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \end{cases}$$

where  $\bar{u}$  is the extension of  $u$  by 0 outside  $\Omega$ .

When the exchange coefficient  $\varepsilon$  is fixed, G. Carbou and P. Fabrie have proved in [6] the existence and uniqueness of local regular solutions for the equations (1)-(3). When the exchange coefficient  $\varepsilon$  goes to zero, G. Carbou, P. Fabrie and O. Guès have proved in [8] the following theorem on  $u^\varepsilon$ , solution of the previous equations:

**Theorem 1.1** (G. Carbou, P. Fabrie, O. Guès). *Let  $u_0 \in \mathbb{H}^5(\Omega)$ ,  $|u_0| = 1$ ,  $\left(\frac{\partial u_0}{\partial n}\right)_{|\partial\Omega} = 0$ , there exists a function  $U^0 \in \mathcal{C}([0, +\infty[; \mathbb{H}^5(\Omega))$  solution of*

$$(4) \quad \begin{cases} \frac{\partial U^0}{\partial t} = U^0 \wedge \mathcal{H}(U^0) - U^0 \wedge (U^0 \wedge \mathcal{H}(U^0)) & \text{in } [0, +\infty[ \times \Omega, \\ U^0(0, x) = u_0(x) & \text{on } \Omega, \end{cases}$$

which, moreover, fulfills  $|U^0(t, x)| = 1$  for all  $(t, x) \in [0, +\infty[ \times \Omega$ . There also exists a function  $\tilde{U}^1 \in \mathbb{L}_{loc}^\infty(\mathbb{R}_t^+; \mathbb{H}^4(\Omega) \otimes \mathbb{H}^4(\mathbb{R}^+))$  and a time  $T^\varepsilon > 0$  such that  $\lim_{\varepsilon \rightarrow 0} T^\varepsilon = +\infty$ , so that the solution  $u^\varepsilon$  of the equations (1)-(3) can be written in the form

$$u^\varepsilon(t, x) = U^0(t, x) + \varepsilon \tilde{U}^1\left(t, x, \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon v^\varepsilon(t, x), \quad t \in [0, T],$$

$\forall T < T^\varepsilon$  and with  $\varphi(x) = \text{dist}(x, \partial\Omega)$ . Moreover  $v^\varepsilon$  is bounded in  $\mathbb{L}^\infty(0, T; \mathbb{H}^1(\Omega))$ , and  $\varepsilon v^\varepsilon$  is bounded in  $\mathbb{L}^\infty(0, T; \mathbb{H}^2(\Omega)) \cap \mathbb{L}^2(0, T; \mathbb{H}^3(\Omega))$ .

So the solution  $u^\varepsilon$  of the equations (1)-(3) admits an asymptotic expansion when  $\varepsilon$  goes to zero on a time interval  $[0, T^\varepsilon[$  with  $T^\varepsilon \rightarrow +\infty$ , whose first order term is the solution of the Landau-Lifschitz equation with  $\varepsilon = 0$  and whose second order term is a perturbation which lives in a neighbourhood of the boundary  $\partial\Omega$  and whose characteristic thickness is  $\varepsilon$ .

In [16] we study the case where the demagnetizing field is replaced by an anisotropic field  $\Psi(x, u)$  in  $H_e$ .

**Remark 1.2.** *We can study these equations with both the demagnetizing field and the anisotropic field but we ignore the latter for simplification.*

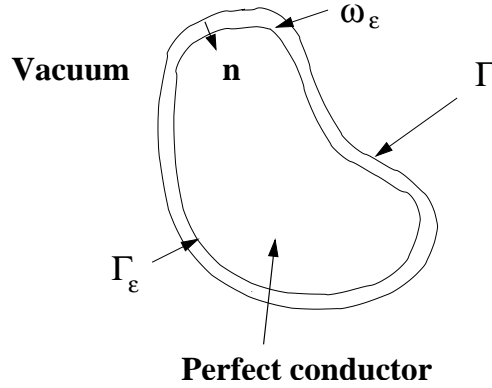
According to the previous result, the perturbation occurs in a  $\varepsilon$ -thick neighbourhood of  $\partial\Omega$ . If we now consider a domain with thickness  $\varepsilon$ , there will be a competition between the behaviours due to the small thickness and the boundary layer that we will describe in the following: in a previous work (see [17]) we considered the case of a  $\varepsilon$ -thick periodic sheet of ferromagnetic material immersed in the vacuum. We will present here the case of a thin layer of ferromagnetic material spread on a non-flat perfect conductor. We then take into account the geometry of the domain (see [18] for the whole proof).

2. THIN LAYERS

Let us precise the notations. Let  $\Omega$  be a regular bounded domain of  $\mathbb{R}^3$ . We denote  $\Gamma = \partial\Omega$ . For  $x \in \Omega$  we introduce  $\varphi(x) = \text{dist}(x, \Gamma)$  and  $P_\Gamma(x)$  the orthogonal projection of  $x$  onto  $\Gamma$ . We remark that since  $\Gamma$  is a regular surface of  $\mathbb{R}^3$ ,  $\varphi$  and  $P_\Gamma$  are regular in a neighbourhood of  $\Gamma$ .

We set

$$\omega_\varepsilon = \{x \in \Omega, 0 < \varphi(x) < \varepsilon\}.$$



We denote  $\mathcal{U}^\varepsilon = \Omega \setminus \omega_\varepsilon$  the domain where lies the perfect conductor.

The equations we will consider are:

$$(5) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} = u^\varepsilon \wedge (\varepsilon^2 \Delta u^\varepsilon + \mathcal{H}(u^\varepsilon)) - u^\varepsilon \wedge (u^\varepsilon \wedge (\varepsilon^2 \Delta u^\varepsilon + \mathcal{H}(u^\varepsilon))) & \text{in } \omega_\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \partial\omega_\varepsilon, \\ u^\varepsilon(0, \cdot) = u_0 & \text{in } \omega_\varepsilon, \end{cases}$$

where the operator  $\mathcal{H}$  fulfills now

$$(6) \quad \begin{cases} \mathcal{H}(u^\varepsilon) \in \mathbb{L}^2(\mathbb{R}^3 \setminus \mathcal{U}^\varepsilon), \\ \text{curl } \mathcal{H}(u^\varepsilon) = 0 & \text{in } \mathbb{R}^3 \setminus \mathcal{U}^\varepsilon, \\ \text{div } (\mathcal{H}(u^\varepsilon) + \overline{u^\varepsilon}) = 0 & \text{in } \mathbb{R}^3 \setminus \mathcal{U}^\varepsilon, \\ [\mathcal{H}(u^\varepsilon) \wedge n] = 0 & \text{on } \Gamma, \\ \mathcal{H}(u^\varepsilon) \wedge n = 0 & \text{on } \Gamma_\varepsilon, \text{ (Boundary condition for a perfect conductor)} \\ [(\mathcal{H}(u^\varepsilon) + \overline{u^\varepsilon}) \cdot n] = 0 & \text{on } \Gamma, \\ (\mathcal{H}(u^\varepsilon) + \overline{u^\varepsilon}) \cdot n = 0 & \text{on } \Gamma_\varepsilon, \text{ (Boundary condition for a perfect conductor)} \end{cases}$$

where  $n$  denotes the outward unitary normal to  $\partial\omega_\varepsilon$  and  $[v]$  denotes the jump of  $v$  at the interface  $\partial\omega_\varepsilon$ .

In order to perform an asymptotic expansion and get the profiles we deal with the rescaled equation. We assume the following conditions are fulfilled:

(H1) The initial data  $u_0^\varepsilon$  satisfies  $\frac{\partial u_0^\varepsilon}{\partial n} = 0$  on  $\partial\omega_\varepsilon$  and can be written in the form:

$$u_0^\varepsilon(x) = U_0^0(P(x)) + \varepsilon U_0^1\left(P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^2 U_0^2\left(P(x), \frac{\varphi(x)}{\varepsilon}\right) \\ + \varepsilon^3 U_0^3\left(P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^3 r_0^\varepsilon(x), \quad \forall x \in \omega_\varepsilon,$$

(H2)  $|u_0^\varepsilon(x)| = 1$  for all  $x \in \omega_\varepsilon$ ,

(H3)  $U_0^0 \in \mathbb{W}^{7,\infty}(\Gamma)$  such that  $|U_0^0| \equiv 1$ ,

(H4)  $U_0^1 \in \mathbb{H}^6(\Gamma) \otimes \mathbb{H}^5(0,1)$ ,

(H5)  $U_0^2 \in \mathbb{H}^5(\Gamma) \otimes \mathbb{H}^5(0,1)$ ,

(H6)  $U_0^3 \in \mathbb{H}^4(\Gamma) \otimes \mathbb{H}^5(0,1)$ ,

(H7)  $r_0^\varepsilon \in \mathbb{H}^2(\omega_\varepsilon)$  is the initial data for the remainder term  $r^\varepsilon$  and fulfills the additional condition:

$$\|r_0^\varepsilon\|_{\mathbb{L}^2(\omega_\varepsilon)}^2 + \|\nabla_{\Gamma_z} r_0^\varepsilon\|_{\mathbb{L}^2(\omega_\varepsilon)}^2 + \|\varepsilon \partial_z r_0^\varepsilon\|_{\mathbb{L}^2(\omega_\varepsilon)}^2 + \varepsilon^2 \left( \|\Delta_{\Gamma_z} r_0^\varepsilon\|_{\mathbb{L}^2(\omega_\varepsilon)}^2 + \|\partial_z^2 r_0^\varepsilon\|_{\mathbb{L}^2(\omega_\varepsilon)}^2 \right) \leq C,$$

where  $C$  is an  $\varepsilon$ -independent constant,  $\nabla_{\Gamma_z}$  and  $\Delta_{\Gamma_z}$  are the tangential parts of the gradient and Laplace operators.

**Remark 2.1.** We work with Sobolev spaces adapted to the anisotropy of the problem:

$$\mathbb{L}^p(\omega_\varepsilon) = \left\{ u \in \mathcal{D}(\omega_\varepsilon), \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} |u|^p dx \right\}, \quad \mathbb{H}^0(\omega_\varepsilon) = \mathbb{L}^2(\omega_\varepsilon), \\ \mathbb{H}^{p+1}(\omega_\varepsilon) = \left\{ u \in \mathbb{L}^2(\omega_\varepsilon), \nabla_{\Gamma_z} u \in \mathbb{H}^p(\omega_\varepsilon) \text{ and } \varepsilon \partial_z u \in \mathbb{H}^p(\omega_\varepsilon) \right\},$$

and with the natural norms on these spaces.

Our main result is the following:

**Theorem 2.2.** Under the assumptions (H1)-(H7), if  $T^\varepsilon$  is the maximum time of existence of the regular solution  $u^\varepsilon$  of the equation (5) then  $\lim_{\varepsilon \rightarrow 0} T^\varepsilon = +\infty$  and there exists some profiles  $U^0, U^1, U^2, U^3$  defined on  $\mathbb{R}_t^+ \times \Gamma \times (0,1)$  such that for all  $T < T^\varepsilon$ ,  $x \in \omega_\varepsilon$  and  $t < T$ ,

$$u^\varepsilon(t, x) = U^0(t, P(x)) + \varepsilon U^1\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^2 U^2\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) \\ + \varepsilon^3 U^3\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^3 r^\varepsilon(t, x),$$

where for all  $T > 0$ ,

- $U^0 \in \mathcal{C}^\infty(\mathbb{R}_t^+; \mathbb{W}^{7,\infty}(\Gamma))$  is solution of the equation (5) with  $\varepsilon = 0$ , i.e.

$$\begin{cases} \frac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) & \text{in } \Gamma, \\ U^0(0, \cdot) = U_0^0 & \text{in } \Gamma, \end{cases}$$

where  $H^0 = -U_3^0 e^3$  becomes a local operator,

- $U^1 \in \bigcap_{k=0}^2 [\mathbb{W}^{k,\infty}(0, T; \mathbb{H}^6(\Gamma) \otimes \mathbb{H}^{5-2k}(0, 1)) \cap \mathbb{H}^k(0, T; \mathbb{H}^6(\Gamma) \otimes \mathbb{H}^{6-2k}(0, 1))],$
- $U^2 \in \bigcap_{k=0}^2 [\mathbb{W}^{k,\infty}(0, T; \mathbb{H}^5(\Gamma) \otimes \mathbb{H}^{5-2k}(0, 1)) \cap \mathbb{H}^k(0, T; \mathbb{H}^5(\Gamma) \otimes \mathbb{H}^{6-2k}(0, 1))],$
- $U^3 \in \bigcap_{k=0}^2 [\mathbb{W}^{k,\infty}(0, T; \mathbb{H}^4(\Gamma) \otimes \mathbb{H}^{5-2k}(0, 1)) \cap \mathbb{H}^k(0, T; \mathbb{H}^4(\Gamma) \otimes \mathbb{H}^{6-2k}(0, 1))],$  *item*  
 for all  $T < T^\varepsilon$  the function  $r^\varepsilon$  is bounded in  $\mathbb{L}^\infty(0, T; \mathbb{H}^1(\omega_\varepsilon))$  and the function  $\varepsilon r^\varepsilon$  is bounded in  $\mathbb{L}^\infty(0, T; \mathbb{H}^2(\omega_\varepsilon))$ .

We have then obtained a first order asymptotic expansion valid on the time interval  $(0, T^\varepsilon)$  with  $T^\varepsilon \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$  and whose first order term is solution of the previous equations taken with  $\varepsilon = 0$ , *i.e.* solution of the 2D Landau-Lifschitz equation with a null exchange coefficient. Also the magnetic field in the 2D Landau-Lifschitz equation is no longer a solution of equation (6), it is now a local operator orthogonal to the surface  $\Gamma$  (in accordance with the physics). Moreover the second order perturbation lies in the whole domain contrary to the boundary layer case.

*Sketch of the proof.*

- Since we can prove that the norm of  $u$  is conserved, the Landau-Lifschitz equation is equivalent to the following equation (see [6]):

$$(7) \quad \begin{cases} \frac{\partial u^\varepsilon}{\partial t} - \varepsilon^2 \Delta u^\varepsilon = \varepsilon^2 |\nabla u^\varepsilon|^2 u^\varepsilon + \varepsilon^2 u^\varepsilon \wedge \Delta u^\varepsilon + u^\varepsilon \wedge \mathcal{H}(u^\varepsilon) \\ \qquad \qquad \qquad - u^\varepsilon \wedge (u^\varepsilon \wedge \mathcal{H}(u^\varepsilon)) \quad \text{in } \mathbb{R}_t^+ \times \omega_\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} = 0 \quad \text{on } \mathbb{R}_t^+ \times (\Gamma \cup \Gamma_\varepsilon), \\ u^\varepsilon(0, \cdot) = u_0^\varepsilon \quad \text{in } \omega_\varepsilon, \end{cases}$$

since  $u \cdot \Delta u = -|\nabla u|^2$  when  $|u| \equiv 1$ .

- We look for solutions in the form of an asymptotic expansion in  $\varepsilon$  where the profiles are defined on  $\mathbb{R}_t^+$  and on the new coordinates  $\Gamma \times (0, 1)$ ,

$$\begin{aligned} u^\varepsilon(t, x) &= U^0 \left( t, P_\Gamma(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon U^1 \left( t, P_\Gamma(x), \frac{\varphi(x)}{\varepsilon} \right) + \dots \\ H^\varepsilon(t, x) &= H^0 \left( t, P_\Gamma(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon H^1 \left( t, P_\Gamma(x), \frac{\varphi(x)}{\varepsilon} \right) + \dots \end{aligned}$$

where  $P_\Gamma(x)$  is the orthogonal projection of  $x$  onto  $\Gamma$  and  $\varphi(x) = \text{dist}(x, \Gamma)$ .

In the vacuum outside  $\omega$  we can prove that there is no boundary layer, so for all  $x \in \mathbb{R}^3 \setminus \omega$ ,

$$H^\varepsilon(t, x) = H^0(t, x) + \varepsilon H^1(t, x) + \dots$$

We substitute to  $u$  and  $H$  their asymptotic expansion in (6)-(7) and we obtain

- transmission conditions for the profiles  $H^i$  deduced from the transmission condition written in equation (6),

- the Neumann conditions written in the new coordinates for the profiles,
- the equations fulfilled by the magnetic field outside  $\omega_\varepsilon$  and which can be solved by Lax-Milgram theorem,
- the equations fulfilled by the profiles of  $u$  and  $H$  in the thin layer. Thanks to the introduction of the new coordinates, the classical differential operators split into two parts, a tangential operator and a normal operator (see [5] and [18]) which allows us to perform the calculus in a simpler way. For example:

If  $u : \omega_\varepsilon \rightarrow \mathbb{R}$  or  $\mathbb{R}^3$ , we have, setting  $\tilde{u}(\sigma, z) = u(x)$  with  $\sigma = P_\Gamma(x)$ ,  $z = \varphi(x)$ :

$$\nabla u(x) = \frac{\partial \tilde{u}}{\partial z}(\sigma, z)n(\sigma) + (\nabla_{\Gamma_z} \tilde{u})(\sigma, z),$$

where  $\Gamma_z = \{x \in \Omega, \text{dist}(x, \Gamma) = z\}$ ,  $\nabla_{\Gamma_z}$  is the tangential gradient on  $\Gamma_z$  when  $\Gamma_z$  is parameterized by  $\Gamma$ .

With this formalism we obtain the equations fulfilled by  $H^0$ :

$$\begin{cases} \frac{\partial}{\partial z} [(H^0 + U^0) \cdot n] = 0, \\ n \wedge \frac{\partial}{\partial z} H^0 = 0. \end{cases}$$

We can solve them thanks to the transmission conditions and the solution we obtained outside  $\omega_\varepsilon$ . We obtain  $H^0 = -(U^0 \cdot n)n$ , *i.e.*  $\mathcal{H}$  behaves itself as a local operator at order  $\varepsilon^0$ .

Next  $U^0$  fulfilled the following equation:

$$\frac{\partial U^0}{\partial t} - U_{zz}^0 = |U_z^0|^2 U^0 + U^0 \wedge U_{zz}^0 + U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) \quad \text{in } \Gamma \times (0, 1).$$

According to the assumption (H3) we can prove that  $U^0$  does not depend on  $z$ , so  $U^0$  fulfills

$$\frac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) \quad \text{in } \Gamma \times (0, 1).$$

By the same way we get the equations fulfilled by  $H^i, U^i$ . We obtain for  $i = 1, 2, 3$ ,

$$H^i = -(U^i \cdot n)n + a(U^1, \dots, U^{i-1}),$$

where  $a$  is a linear operator, and  $U^i$  fulfills

$$\frac{\partial U^i}{\partial t} - U_{zz}^i = U^0 \wedge U_{zz}^1 + F_i(U^0, \dots, U^{i-1}, U^i) + G_i(U^0, \dots, U^{i-1}),$$

where  $F_i$  is a linear operator in its last variable  $U^i$ . We can then obtain the existence of the profiles and the properties we announced above.

- the main difficulty lies in the estimates on the remainder term. Except the technical part of these estimates, there are two important points: we first need to estimate the

remainder term  $Q^\varepsilon$  of  $H^\varepsilon$  with the one of  $u^\varepsilon$  ( $r^\varepsilon$ ) in the same spaces:  $Q^\varepsilon$  solves

$$\begin{cases} Q^\varepsilon \in \mathbb{L}^2(\omega_\varepsilon), \\ \operatorname{curl} Q^\varepsilon + A(H^0, \bar{U}^0, H^1, \bar{U}^1, H^2, \bar{U}^2, H^3, \bar{U}^3) = 0 \text{ in } \omega_\varepsilon, \\ \operatorname{div}(Q^\varepsilon + \bar{r}^\varepsilon) + B(H^0, \bar{U}^0, H^1, \bar{U}^1, H^2, \bar{U}^2, H^3, \bar{U}^3) = 0 \text{ in } \omega_\varepsilon, \end{cases}$$

where  $A$  and  $B$  are linear functions of their arguments, with the following transmission conditions

$$\begin{cases} [Q^\varepsilon \wedge n] = 0 \text{ on } \Gamma, \\ [(Q^\varepsilon + \bar{r}^\varepsilon) \cdot n] = 0 \text{ on } \Gamma, \\ Q^\varepsilon \wedge n = 0 \text{ on } \Gamma_\varepsilon, \\ (Q^\varepsilon + \bar{r}^\varepsilon) \cdot n = 0 \text{ on } \Gamma_\varepsilon. \end{cases}$$

**Lemma 2.3.** *If  $r^\varepsilon \in \mathbb{H}^p$ , there exists an unique  $Q^\varepsilon = R^\varepsilon + S^\varepsilon$  and a constant  $c_0$  such that  $S^\varepsilon$  is regular,  $R^\varepsilon$  is linear in  $r^\varepsilon$  and :*

$$\|R^\varepsilon\|_{\mathbb{H}^p(\omega_\varepsilon)} \leq c_0 \|r^\varepsilon\|_{\mathbb{H}^p(\omega_\varepsilon)}.$$

Secondly we need  $\varepsilon$ -independent estimates on the remainder term of  $u$ . As we use Sobolev embeddings we need to precise the dependence on  $\varepsilon$  of the Sobolev constants. Following the work of R. Temam and M. Ziane [19], we obtain some other estimates (See [17], [18]). For example we have in the anisotropic spaces (see Remark 2.1):

**Lemma 2.4.** *There exists  $c_0$  independent of  $\varepsilon$  such that  $\forall u \in \mathbb{H}^1(\omega_\varepsilon)$  and  $\forall 2 \leq p \leq 6$ ,*

$$\|u\|_{\mathbb{L}^p(\omega_\varepsilon)} \leq c_0 \|u\|_{\mathbb{L}^2(\omega_\varepsilon)}^{\frac{3}{p}-\frac{1}{2}} \|u\|_{\mathbb{H}^1(\omega_\varepsilon)}^{\frac{3}{2}-\frac{3}{p}}$$

and  $\forall u \in \mathbb{H}^2(\omega_\varepsilon)$ ,  $\forall 2 \leq p \leq +\infty$ ,

$$\|u\|_{\mathbb{L}^p(\omega_\varepsilon)} \leq c_0 \|u\|_{\mathbb{L}^2(\omega_\varepsilon)}^{\frac{1}{4}+\frac{3}{2p}} \|u\|_{\mathbb{H}^2(\omega_\varepsilon)}^{\frac{3}{4}-\frac{3}{2p}}.$$

□

**Remark 2.5.** *We have studied here the behaviour of Landau-Lifschitz equations with an exchange coefficient  $\varepsilon^2$  in a thin layer of thickness  $\varepsilon$ . That is the case where the thickness of the boundary layer (if there were any, cf [8]) and the domain's thickness are of the same size. If we instead consider a thin layer with a thickness  $\varepsilon^\alpha$  ( $\alpha > 0$ ,  $\alpha = \frac{p}{q} \in \mathbb{Q}$ ), we perform now an asymptotic expansion at the scale  $\varepsilon^{\frac{1}{q}}$  and we see two different behaviours. If  $\alpha > 1$  we get the same type of behaviour than that of the case  $\alpha = 1$ , i.e. the behaviour of a thin layer and the localization of  $H$  at the low orders. If  $\alpha < 1$ , the layer's thickness is bigger than the characteristic thickness of the boundary layer. We then obtain a boundary layer in the thin layer but with a smaller thickness ( $\varepsilon^{1-\alpha}$ ) and we also get at the low orders the localization of  $H$ .*



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