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## Thin layer for the Landau-Lifschitz equation

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**Abstract.** In this paper we study the behaviour of the Landau-Lifschitz equation in a thin layer spread on a perfect conductor. As the thickness of the domain and the exchange coefficient of the equation simultaneously tend to zero we take into account the geometry of the domain and perform an asymptotic expansion to precise the solution for well prepared initial data.

**Résumé.** Nous étudions dans cet article le comportement de l'équation de Landau-Lifschitz dans une couche mince de matériau ferromagnétique étendue à la surface d'un conducteur parfait. L'épaisseur de la couche mince et le coefficient d'échange dans l'équation tendant simultanément vers zéro, nous prenons en compte la géométrie du domaine et effectuons un développement asymptotique de la solution pour une donnée initiale bien préparée.

## 1 Introduction

We consider a thin layer  $\omega_\varepsilon$  with a thickness  $\varepsilon$  of ferromagnetic material covering a perfectly conductive body  $\mathcal{U}^\varepsilon$ . The ferromagnetic material is characterized by a spontaneous magnetization modeled by a magnetic moment  $u$  defined on the domain  $\omega_\varepsilon \subset \mathbb{R}^3$  in which the material is confined. This moment satisfies  $|u| \equiv 1$  and links the magnetic field  $H$  and the magnetic induction  $B$  by the relation  $B = H + \bar{u}$ , where  $\bar{u}$  is the extension of  $u$  by zero outside  $\omega_\varepsilon$ . The evolution of the magnetic moment  $u$  is governed by the Landau-Lifschitz equation:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} = u \wedge H_{eff} - u \wedge (u \wedge H_{eff}) \text{ in } \omega_\varepsilon, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\omega_\varepsilon, \\ u(0, \cdot) = u_0 \text{ on } \omega_\varepsilon, \end{array} \right. \quad (1.1)$$

where  $n$  denotes the outward unitary normal on  $\partial\omega_\varepsilon$  and  $H_{eff}$  is the effective magnetic field in the ferromagnetic domain:

$$H_{eff} = \varepsilon^2 \Delta u + H.$$

The first term in the expression of  $H_{eff}$  is an exchange term whose coefficient  $\varepsilon^2$ , called the exchange coefficient, is intended to tend to zero; the second one is the magnetic field. We assume that we always are at the electromagnetic equilibrium. The magnetic field then fulfills

the magnetostatic equations:

$$\begin{cases} H(u) \in \mathbb{L}^2(\mathbb{R}^3), \\ \operatorname{div}(H(u) + \bar{u}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ \operatorname{curl} H(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ H(u) = 0 \text{ in } \mathcal{U}^\varepsilon \text{ where the perfect conductor lies.} \end{cases} \quad (1.2)$$

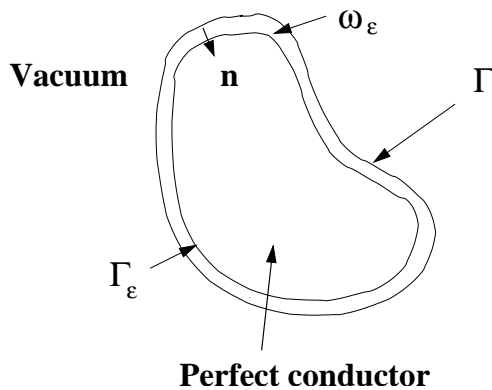
When the exchange coefficient is fixed and the domain  $\omega_\varepsilon$ , an open set placed in the vacuum, does not depend on  $\varepsilon$ , Carbou and Fabrie prove in [9] the local existence of a regular solution  $u$  of the equation (1.1). But when the exchange coefficient  $\varepsilon^2$  goes to zero (and the domain still does not depend on  $\varepsilon$ ), the time interval given in [9] also tends to zero. In [11], Carbou, Fabrie and Guès prove by showing the formation of a boundary layer of characteristic thickness  $\varepsilon$  that the time of existence of  $u$  goes to infinity when  $\varepsilon$  goes to zero for an initial data in  $\mathbb{H}^5(\Omega)$ . Moreover they get that the solution  $u$  tends to the solution of the hyperbolic system formally obtained by taking  $\varepsilon = 0$  in Eq. (1.1).

Since the ferromagnetic materials are often used in thin domains (coating of a plane, reading head of hard disks) the behaviour of such materials and of the Landau-Lifschitz equation in that kind of domain naturally arises.

In [6], Carbou has studied the limit of the magnetic moment  $u$  in a thin domain when the thickness  $\varepsilon$  of the domain goes to 0 and the exchange coefficient is fixed. In a previous paper [24] we study the behaviour of  $u$  when the exchange coefficient is  $\varepsilon^2$  and in an  $\varepsilon$ -thick sheet  $\Omega_\varepsilon = (0, 1)^2 \times (-\varepsilon, \varepsilon)$  which is periodic on  $\mathbb{R}^2$ . Through an asymptotic expansion we then describe the interaction between the boundary layer phenomenon and the thin layer. We want here to enhance this result by considering a thin layer of ferromagnetic material spread on a perfect conductor delimited by a surface. In this case we have to investigate the influence of the geometry on the differential operators to obtain a decomposition suitable for the thin layer.

Let us precise the notations. Let  $\Omega$  be a regular bounded domain of  $\mathbb{R}^3$ . We denote  $\Gamma = \partial\Omega$ . For  $x \in \Omega$  we introduce  $\varphi(x) = \operatorname{dist}(x, \Gamma)$  and  $P(x)$  the orthogonal projection of  $x$  onto  $\Gamma$ . We remark that since  $\Gamma$  is a regular surface of  $\mathbb{R}^3$ ,  $\varphi$  and  $P$  are regular in a neighbourhood of  $\Gamma$ . We set

$$\omega_\varepsilon = \{x \in \Omega, 0 < \varphi(x) < \varepsilon\}.$$



We denote  $\mathcal{U}^\varepsilon = \Omega \setminus \omega_\varepsilon$ .

The equations we will consider are:

$$\begin{cases} \frac{\partial u^\varepsilon}{\partial t} = u^\varepsilon \wedge (\varepsilon^2 \Delta u^\varepsilon + H(u^\varepsilon)) - u^\varepsilon \wedge (u^\varepsilon \wedge (\varepsilon^2 \Delta u^\varepsilon + H(u^\varepsilon))) & \text{in } \omega_\varepsilon, \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \partial\omega_\varepsilon, \\ u^\varepsilon(0, \cdot) = u_0^\varepsilon & \text{in } \omega_\varepsilon, \end{cases} \quad (1.3)$$

and

$$\begin{cases} H(u^\varepsilon) = 0 & \text{in } \mathcal{U}^\varepsilon, \\ H(u^\varepsilon) \in \mathbb{L}^2(\mathbb{R}^3 \setminus \mathcal{U}^\varepsilon), \\ \text{curl } H(u^\varepsilon) = 0 & \text{in } \mathbb{R}^3 \setminus \mathcal{U}^\varepsilon, \\ \text{div } (H(u^\varepsilon) + \overline{u^\varepsilon}) = 0 & \text{in } \mathbb{R}^3 \setminus \mathcal{U}^\varepsilon, \\ [H(u^\varepsilon) \wedge n] = 0 & \text{on } \partial\omega_\varepsilon, \\ [(H(u^\varepsilon) + \overline{u^\varepsilon}) \cdot n] = 0 & \text{on } \partial\omega_\varepsilon, \end{cases} \quad (1.4)$$

where  $n$  denotes the outward unitary normal to  $\partial\omega_\varepsilon$  and  $[v]$  denotes the jump of  $v$  at the interface  $\partial\omega_\varepsilon$  (the transmission conditions are deduced from (1.2)).

In order to perform the asymptotic expansion and get the profiles we work on the rescaled equation. We get the following result:

**Theorem 1.1.** *Let  $U_0^0 \in \mathbb{W}^{7,\infty}(\Gamma)$  an  $\varepsilon$ -independent function satisfying  $|U_0^0| \equiv 1$  and let the initial data  $u_0^\varepsilon$  fulfill*

$$u_0^\varepsilon(x) = U_0^0(P(x)) \quad \forall x \in \omega_\varepsilon.$$

*If  $T^\varepsilon$  is the maximum existence time of the regular solution  $u^\varepsilon$  of Eq. (1.3) then  $\lim_{\varepsilon \rightarrow 0} T^\varepsilon = +\infty$  and for all  $T < T^\varepsilon$  there exists some profiles  $U^0$  and  $U^1$  that fulfill*

- $U^0 \in \mathcal{C}^\infty(\mathbb{R}_t^+; \mathbb{W}^{7,\infty}(\Gamma))$  is solution of Eq. (1.3) with  $\varepsilon = 0$  i.e.

$$\begin{cases} \frac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) & \text{in } \Gamma, \\ U^0(0, \cdot) = U_0^0 & \text{in } \Gamma, \end{cases}$$

where  $H^0 = -(U^0 \cdot n)n$  is a local operator,

- for all  $0 < T < T^\varepsilon$ ,

$$U^1 \in \bigcap_{k=0}^2 \left[ \mathbb{W}^{k,\infty} \left( 0, T; \mathbb{H}^6(\Gamma) \otimes \mathbb{H}^{5-2k}(I_1) \right) \cap \mathbb{H}^k \left( 0, T; \mathbb{H}^7(\Gamma) \otimes \mathbb{H}^{6-2k}(I_1) \right) \right]$$

such that  $u^\varepsilon$  satisfies:

$$u^\varepsilon(t, x) = U^0(t, P(x)) + \varepsilon U^1 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^2 r^\varepsilon(t, x),$$

where  $r^\varepsilon \in \mathbb{L}^\infty(0, T; \mathbb{L}^p(\omega_\varepsilon))$  for all  $2 \leq p \leq +\infty$  and  $0 < T < T^\varepsilon$ , and fulfills:

$$\sup_{t \in [0, T]} \|r^\varepsilon\|_{\mathbb{L}^p(\omega_\varepsilon)} = \mathcal{O}(\varepsilon^{1/p}).$$

**Remark 1.1.** *The limit model for a thin layer of ferromagnetic material we obtain here corresponds to the physical observations and expectations: the magnetic moment follows the 2D-Landau-Lifschitz equation without exchange term and the magnetic moment is orthogonal to the surface  $\Gamma$  of the body.*

**Remark 1.2.** *For the sake of clarity we have chosen a simple initial data. We can without restrictions assume that the initial data admits a regular asymptotic expansion if the homogeneous Neumann condition  $\frac{\partial u_0^\varepsilon}{\partial n} = 0$  is fulfilled on  $\partial\omega_\varepsilon$ .*

*The proof of the theorem requires an asymptotic expansion up to third order. We can then enhance the theorem and give a second order asymptotic expansion of the solution instead of a first order one.*

The geometry of the domain we consider here raises some problems: the main difficulty consists in identifying the geometrical effects of the thin layer. The development of geometrical tools gives the correct framework to obtain the asymptotic expansion and allows us to introduce tools suitable to the geometry (Sobolev spaces, inequalities,...).

This paper is organized in the following way. The second part consists in the reminder and proof of some geometrical tools. In the third part we formally build the asymptotic expansion of  $u^\varepsilon$  and we prove its existence for all time and the regularity of its terms. We then introduce in the fourth part anisotropic Sobolev spaces as well as the corresponding embeddings and inequalities to take into account the small thickness of the domain. We study the remainder term of  $H^\varepsilon$  in the fifth part where we also explain the equation satisfied by the remainder of  $u^\varepsilon$  and we perform the energy estimates which conclude the proof of Theorem 1.1 in the sixth part.

## 2 Geometrical tools

We have here a thin non-flat domain that we consider with non-cartesian coordinates. We then need to express the differential operators in a way which will take into account this geometry (See Carbou [7]).

We will use the following notations :

- $(p \cdot q)$  is the scalar product in  $\mathbb{R}^3$ .
- $\Gamma = \partial\Omega$ ,
- for  $\sigma \in \Gamma$ ,  $n(\sigma)$  is the unitary normal to  $\Gamma$  at the point  $\sigma$ , entering in  $\Omega$ ,
- for  $\sigma \in \Gamma$ ,  $T_\sigma\Gamma$  is the tangent plane of  $\Gamma$  at the point  $\sigma$  :

$$T_\sigma\Gamma = (n(\sigma))^\perp$$

- $\varphi(x) = \text{dist}(x, \Gamma)$  for  $x \in \Omega$ ,
- $P(x)$  the orthogonal projection of  $x$  onto  $\Gamma$ , for  $x \in \Omega$ ,

- $g$  is the matrix of the scalar product in the coordinates  $(\sigma, z)$ .
- $\omega_\varepsilon = \left\{ x \in \Omega, \quad 0 < \varphi(x) < \varepsilon \right\}$ ,
- for  $s > 0$ ,  $\Gamma_s = \left\{ x \in \Omega, \quad \varphi(x) = s \right\}$ .

For  $\eta_0 > 0$  little enough, we define a parametrization of  $\omega_{\eta_0}$  by :

$$\begin{aligned} \Psi : \Gamma \times ]0, \eta_0[ &\longrightarrow \omega_{\eta_0} \\ (\sigma, z) &\longmapsto \sigma + zn(\sigma) \end{aligned}$$

Since  $\Gamma = \partial\Omega$  is a regular compact surface of  $\mathbb{R}^3$  without boundary, there exists  $\eta_0 > 0$  such that  $\Psi$  is a  $\mathcal{C}^\infty$ -diffeomorphism from  $\Gamma \times ]0, \eta_0[$  onto  $\omega_{\eta_0}$ . We remark that for  $\varepsilon < \eta_0$  the restriction of  $\Psi$  to  $\Gamma \times ]0, \varepsilon[$  is a  $\mathcal{C}^\infty$ -parametrization of  $\omega_\varepsilon$ .

Furthermore  $\varphi$  and  $P$  are regular on  $\omega_{\eta_0}$  and

$$\forall x \in \omega_{\eta_0}, \quad \nabla\varphi(x) = n(P(x)).$$

We now precise the expression of the differential operators in the new coordinates  $(\sigma, z)$ .

On the submanifold  $\Gamma$  we classically define the integrale and the differential operators  $\nabla_\Gamma$ ,  $\text{div}_\Gamma$  and  $\Delta_\Gamma$ . Furthermore,  $n$  is a map defined from  $\Gamma$  with values in the unit sphere  $\mathbb{S}^2$ . For  $\sigma \in \Gamma$ , the differential  $dn(\sigma)$  is a linear map from  $T_\sigma\Gamma$  into  $T_{n(\sigma)}\mathbb{S}^2$  and since  $T_{n(\sigma)}\mathbb{S}^2 = T_\sigma\Gamma$ , we can consider  $dn(\sigma)$  as an endomorphism of  $T_\sigma\Gamma$ .

**Integration :** we now set for  $s \in [0, \eta_0[$  and  $\sigma \in \Gamma$  :

$$\gamma_s(\sigma) = \det(Id + s dn(\sigma)).$$

We remark that if  $u : \omega_{\eta_0} \longrightarrow \mathbb{R}$ , denoting  $\tilde{u} = u \circ \Psi$ , we have :

$$\int_{\omega_{\eta_0}} u = \int_0^{\eta_0} \int_\Gamma \tilde{u}(\sigma, s) \gamma_s(\sigma) d\sigma ds.$$

**Gradient :** for  $v : \Gamma \longrightarrow \mathbb{R}$ , we define :

$$\nabla_{\Gamma_s} v(\sigma) = (Id + s dn(\sigma))^{-1} (\nabla_\Gamma v(\sigma))$$

and if  $u : \omega_{\eta_0} \longrightarrow \mathbb{R}$ , denoting  $\tilde{u} = u \circ \Psi$ , we have :

$$\nabla u(x) = \frac{\partial \tilde{u}}{\partial z}(P(x), \varphi(x)) n(P(x)) + \left( \nabla_{\Gamma_{\varphi(x)}} \tilde{u} \right) (P(x), \varphi(x)). \quad (2.1)$$

**Divergence Operator :** let  $Y : \Gamma \longrightarrow T\Gamma$  be a tangent vector field defined on  $\Gamma$ . We define :

$$\text{div}_{\Gamma_s} Y(\sigma) = \frac{1}{\gamma_s(\sigma)} \text{div}_\Gamma [\gamma_s (Id + s dn)^{-1} Y] (\sigma)$$

and if  $Z : \omega_{\eta_0} \longrightarrow \mathbb{R}^3$ , denoting  $\tilde{Z} = Z \circ \Psi$ , we have

$$\text{div} Z(x) = \frac{\partial \tilde{Z}_N}{\partial z}(P(x), \varphi(x)) + G_{\varphi(x)}(P(x)) \tilde{Z}_N(P(x), \varphi(x)) + \left( \text{div}_{\Gamma_{\varphi(x)}} \tilde{Z}_T \right) (P(x), \varphi(x)), \quad (2.2)$$

where

$$\tilde{Z}_N(\sigma, z) = \tilde{Z}(\sigma, z) \cdot n(\sigma), \quad \tilde{Z}_T(\sigma, z) = \tilde{Z}(\sigma, z) - \tilde{Z}_N(\sigma, z)n(\sigma).$$

and where :  $G_s(\sigma) = \frac{1}{\gamma_s(\sigma)} \frac{\partial \gamma_s}{\partial s}(\sigma)$ .

**Laplace operator :** for  $v : \Gamma \longrightarrow \mathbb{R}$  we define

$$\Delta_{\Gamma_s} v = \operatorname{div}_{\Gamma_s} \nabla_{\Gamma_s} v$$

and if  $u : \omega_{\eta_0} \longrightarrow \mathbb{R}$ , denoting  $\tilde{u} = u \circ \Psi$ , we have :

$$\Delta u(x) = \frac{\partial^2 \tilde{u}}{\partial z^2}(P(x), \varphi(x)) + G_{\varphi(x)}(P(x)) \frac{\partial \tilde{u}}{\partial z}(P(x), \varphi(x)) + \left( \Delta_{\Gamma_{\varphi(x)}} \tilde{u} \right) (P(x), \varphi(x)).$$

**Curl operator:** Let  $Y : \Gamma \longrightarrow T\Gamma$  be a tangent vector field on defined on  $\Gamma$ . We define:

$$\operatorname{curl}_{\Gamma_s} Y(\sigma) = \frac{1}{\gamma_s(\sigma)} \operatorname{curl}_{\Gamma} [(Id + s \, dn)Y] (\sigma),$$

where  $\operatorname{curl}_{\Gamma} = -\operatorname{div}_{\Gamma}(n(\sigma) \wedge \cdot)$ , and if  $Z : \omega_{\eta_0} \longrightarrow \mathbb{R}^3$ , denoting  $\tilde{Z} = Z \circ \Psi$ , we have:

$$\begin{aligned} \operatorname{curl} Z = & n(P(x)) \wedge \frac{\partial \tilde{Z}}{\partial z}(P(x), \varphi(x)) - n(P(x)) \wedge \nabla_{\Gamma_{\varphi(x)}} V_N(P(x), \varphi(x)) \\ & + n(P(x)) \wedge [(I + z \, dn)^{-1} dn V_T] (P(x), \varphi(x)) \\ & + \left[ \operatorname{curl}_{\Gamma_{\varphi(x)}} V_T(P(x), \varphi(x)) \right] n(P(x)). \end{aligned} \quad (2.3)$$

**Proof:** with the cartesian coordinates we have:

$$\operatorname{curl} Z = \begin{pmatrix} \partial_{x_2} Z_3 - \partial_{x_3} Z_2 \\ \partial_{x_3} Z_1 - \partial_{x_1} Z_3 \\ \partial_{x_1} Z_2 - \partial_{x_2} Z_1 \end{pmatrix}$$

We substitute to  $\partial_{x_i} Z_j$  their expression in the new coordinates obtained thanks to the one of the gradient (2.1) and we obtain:

$$\operatorname{curl} V = \begin{pmatrix} n_2 \partial_z \tilde{V}_3 - n_3 \partial_z \tilde{V}_2 + (e_2 \cdot \nabla_{\Gamma_z} \tilde{V}_3) - (e_3 \cdot \nabla_{\Gamma_z} \tilde{V}_2) \\ n_3 \partial_z \tilde{V}_1 - n_1 \partial_z \tilde{V}_3 + (e_3 \cdot \nabla_{\Gamma_z} \tilde{V}_1) - (e_1 \cdot \nabla_{\Gamma_z} \tilde{V}_3) \\ n_1 \partial_z \tilde{V}_2 - n_2 \partial_z \tilde{V}_1 + (e_1 \cdot \nabla_{\Gamma_z} \tilde{V}_2) - (e_2 \cdot \nabla_{\Gamma_z} \tilde{V}_1) \end{pmatrix},$$

where  $(e_i)_{i=1,2,3}$  is the usual basis in the cartesian coordinates and  $n_i = (n \cdot e_i)$ . This also writes:

$$\operatorname{curl} V = n \wedge \partial_z \tilde{V}_T + A(\tilde{V}_N) + B(\tilde{V}_T), \quad (2.4)$$

where  $A$  and  $B$  are linear differential operators in which the  $z$ -derivative does not appear. To obtain their expression we use the equalities “ $\operatorname{div} \operatorname{curl} = 0$ ” and “ $\operatorname{curl} \nabla = 0$ ” with the equation (2.4) as well as the expressions of the gradient (2.1) and of the divergence operator (2.2):

$$\begin{aligned} \operatorname{curl} (\nabla u) &= n \wedge \partial_z \nabla_{\Gamma_z} \tilde{u} + A(\partial_z \tilde{u}) + B(\nabla_{\Gamma_z} \tilde{u}) \\ &= n \wedge \nabla_{\Gamma_z} (\partial_z \tilde{u}) - n \wedge [(I + z \, dn)^{-1} dn \nabla_{\Gamma_z} \tilde{u}] + A(\partial_z \tilde{u}) + B(\nabla_{\Gamma_z} \tilde{u}) \\ &= 0. \end{aligned}$$

Since there is no  $z$ -derivative in  $A$  and  $B$ , we deduce:

$$\begin{aligned} A_T(\widetilde{V}_N) &= -n \wedge \nabla_{\Gamma_z} \widetilde{V}_N, \\ B_T(\widetilde{V}_T) &= n \wedge (I + z \, dn)^{-1} \, dn \, \widetilde{V}_T, \\ A_N(\partial_z \widetilde{u}) &= 0, \\ B_N(\nabla_{\Gamma_z} u) &= 0. \end{aligned}$$

We then use “ $\operatorname{div} \operatorname{curl} = 0$ ” and we remark that  $\gamma_z(I + z \, dn)^{-1} n \wedge ((I + z \, \partial n)^{-1} W_T) = W_T$  to get:

$$\begin{aligned} \operatorname{div} \operatorname{curl} V &= \frac{1}{\gamma_z} \partial_z \left( \gamma_z (A_N(\widetilde{V}_N) + B_N(\widetilde{V}_T)) \right) \\ &\quad + \operatorname{div}_{\Gamma_z} \left( n \wedge \partial_z \widetilde{V}_T - n \wedge \nabla_{\Gamma_z} \widetilde{V}_N + n \wedge (I + z \, dn)^{-1} \, dn \, \widetilde{V}_T \right) \\ &= \frac{1}{\gamma_z} \partial_z \left( \gamma_z A_N(\widetilde{V}_N) \right) + G_z B_N(\widetilde{V}_T) + B_N(\partial_z \widetilde{V}_T) + (\partial_z B_N)(\widetilde{V}_T) \\ &\quad + \frac{1}{\gamma_z} \operatorname{div}_{\Gamma} \left[ n \wedge \left( (I + z \, dn) \partial_z \widetilde{V}_T \right) \right] - \frac{1}{\gamma_z} \operatorname{div}_{\Gamma} \left( n \wedge \nabla_{\Gamma} \widetilde{V}_N \right) \\ &\quad + \frac{1}{\gamma_z} \operatorname{div}_{\Gamma} \left( n \wedge (dn \, \widetilde{V}_T) \right) \\ &= \frac{1}{\gamma_z} \partial_z \left( \gamma_z A_N(\widetilde{V}_N) \right) + G_z B_N(\widetilde{V}_T) + B_N(\partial_z \widetilde{V}_T) + (\partial_z B_N)(\widetilde{V}_T) \\ &\quad + \frac{1}{\gamma_z} \operatorname{div}_{\Gamma} \left[ n \wedge \left( (I + z \, dn) \partial_z \widetilde{V}_T \right) \right] + \frac{1}{\gamma_z} \operatorname{div}_{\Gamma} \left( n \wedge (dn \, \widetilde{V}_T) \right) \\ &= 0. \end{aligned}$$

We obtain:

$$\begin{aligned} A_N(\widetilde{V}_N) &= 0, \\ B_N(\widetilde{V}_T) &= \frac{1}{\gamma_z} \operatorname{curl}_{\Gamma} \left( (I + z \, dn) \widetilde{V}_T \right), \end{aligned}$$

where  $\operatorname{curl}_{\Gamma} = -\operatorname{div}_{\Gamma}(n \wedge \cdot)$ .  $\square$

### 3 Formal asymptotic expansion

We now look for an asymptotic expansion of  $u^\varepsilon$  and  $H^\varepsilon$  in the form :

- in  $\omega_\varepsilon$ ,

$$\begin{aligned} u^\varepsilon(t, x) &= U^0 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon U^1 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \dots \\ H^\varepsilon(t, x) &= H^0 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon H^1 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \dots \end{aligned} \tag{3.1}$$

where  $x \in \omega_\varepsilon$ ,

- in  $\Omega' = \mathbb{R}^3 \setminus \Omega$ ,

$$H^\varepsilon(t, y) = H^0(t, y) + \varepsilon H^1(t, y) + \dots \tag{3.2}$$

where  $y \in \Omega'$ .

As we consider a non-flat domain we also need to perform an asymptotic expansion of the differential operators since the geometry also depends on  $\varepsilon$ .



If  $u(x) = \tilde{u}(\sigma, z) \in \mathbb{R}$  and  $V(x) = \tilde{V}(\sigma, z) \in \mathbb{R}^3$ , with  $\sigma = P(x)$  and  $z = \frac{\varphi(x)}{\varepsilon}$ , we get :

$$\nabla u(x) = \frac{1}{\varepsilon} \frac{\partial \tilde{u}}{\partial z}(\sigma, z) n(\sigma) + \sum_{i=0}^{\dots} \varepsilon^i \nabla_{\Gamma_0, i} \tilde{u}(\sigma, z),$$

$$\Delta V(x) = \frac{1}{\varepsilon^2} \frac{\partial^2 \tilde{V}}{\partial z^2}(\sigma, z) + \frac{1}{\varepsilon} \left( \sum_{i=0}^{\dots} \varepsilon^i a_i(z) \right) \frac{\partial \tilde{V}}{\partial z}(\sigma, z) + \sum_{i=0}^{\dots} \varepsilon^i \Delta_{\Gamma_0, i} \tilde{V}(\sigma, z),$$

where  $a_i(z) = \frac{z^i}{i!} (\partial_z^i G)|_{z=0}$ ,  $\nabla_{\Gamma_0, i} = \frac{z^i}{i!} \left( \frac{\partial^i}{\partial s^i} \nabla_{\Gamma_s} \right)_{|s=0}$  and  $\Delta_{\Gamma_0, i} = \frac{z^i}{i!} \left( \frac{\partial^i}{\partial s^i} \Delta_{\Gamma_s} \right)_{|s=0}$ .

$$\operatorname{div} V(x) = \frac{1}{\varepsilon} \frac{\partial \tilde{V}}{\partial z}(\sigma, z) \cdot n(\sigma) + \sum_{i=0}^{\dots} \varepsilon^i a_i(z) \tilde{V}(\sigma, z) \cdot n(\sigma) + \sum_{i=0}^{\dots} \varepsilon^i \Theta^i(\sigma, z) (\tilde{V}_T)(\sigma, z),$$

$$\operatorname{curl} V(x) = \frac{1}{\varepsilon} n(\sigma) \wedge \frac{\partial \tilde{V}}{\partial z}(\sigma, z) + \sum_{i=0}^{\dots} \varepsilon^i \Lambda^i(\sigma, z) (\tilde{V})(\sigma, z),$$

where

$$\begin{cases} \Theta^i(\sigma, z)(V) = \frac{z^i}{i!} (\partial_s^i \operatorname{div} \Gamma_s)_{|s=0} V_T, \\ \Lambda^i(\sigma, z)(V) = \frac{z^i}{i!} \left[ -n \wedge (\partial_s^i \nabla_{\Gamma_s})_{|s=0} V_N + n \wedge (\partial_s^i (I + s \operatorname{dn})^{-1})_{|s=0} \operatorname{dn} V_T \right. \\ \left. + (\partial_s^i \operatorname{curl} \Gamma_s)_{|s=0} V_T n \right]. \end{cases}$$

### 3.1 Equations of the profiles

As it is proved in [9], the equation (1.3) is equivalent for regular enough solutions to :

$$\frac{\partial u^\varepsilon}{\partial t} - \varepsilon^2 \Delta u^\varepsilon = \varepsilon^2 |\nabla u^\varepsilon|^2 u^\varepsilon + \varepsilon^2 u^\varepsilon \wedge \Delta u^\varepsilon + u^\varepsilon \wedge H^\varepsilon - u^\varepsilon \wedge (u^\varepsilon \wedge H^\varepsilon). \quad (3.3)$$

Since this equation isolates the dissipative term we will use it in the following.

We substitute to  $u^\varepsilon$ ,  $H^\varepsilon$  and the differential operators their asymptotic expansions (3.1) and (3.2) in (3.3)-(1.4) in order to get the expression fulfilled by the profiles for  $(x, z) \in \mathbb{S}^2 \times (0, 1)$  or  $y \in \Omega'$ .

As the equations (1.4) are linear, each  $H^i$  fulfills the magnetostatic equations (1.4) in  $\Omega'$ . The transmission conditions (1.4) become for all  $i \in \mathbb{N}$ :

$$\begin{cases} [H^i \wedge n] = 0 \quad \text{on } \Gamma, \\ [(H^i + \bar{U}^i) \cdot n] = 0 \quad \text{on } \Gamma \cup \Gamma_\varepsilon. \end{cases}$$

The Neumann condition also becomes  $\left( \frac{\partial U^i}{\partial z} \right)_{z=0, \varepsilon} = 0$  for all  $i \in \mathbb{N}$ .

In the following  $V_z$  denotes  $\frac{\partial V}{\partial z}$  and we let  $I_1 = (0, 1)$ . Back in (3.3)-(1.4) the terms of the equations at the different orders in  $\varepsilon$  vanish:

**Terms in  $\varepsilon^{-1}$ :** in  $\Gamma \times I_1$ ,

$$\begin{cases} \frac{\partial}{\partial z} [(H^0 + U^0) \cdot n] = 0, \\ n \wedge \frac{\partial}{\partial z} H^0 = 0. \end{cases}$$

We have then  $(H^0 + U^0) \cdot n = 0$  in  $\Gamma \times I_1$ . By solving the magnetostatic equations in the exterior domain  $\Omega'$  with the boundary condition  $H^0 \cdot n = 0$  on  $\Gamma$  we obtain that  $H^0 = 0$  in  $\Omega'$  and  $H^0 \wedge n = 0$  on  $\Gamma \times \{0\}$ . Since  $n$  does not depend on  $z$ , the second equation gives that  $H^0 \wedge n = 0$  in  $\Gamma \times I_1$ . Finally

$$H^0 = - (U^0 \cdot n) n.$$

**Remark 3.1.** *Although  $H$  fulfills a global equation in  $\mathbb{R}^3$ , it becomes a local operator at the 0-order.*

**Terms in  $\varepsilon^0$ :** in  $\Gamma \times I_1$ ,

$$\frac{\partial U^0}{\partial t} - U_{zz}^0 = |U_z^0|^2 U^0 + U^0 \wedge U_{zz}^0 + U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0),$$

Since we consider an initial data which does not vary in the thickness of  $\omega_\varepsilon$ , we take for  $U^0$  a  $z$ -independent solution of the previous equation.  $U^0$  fulfills then:

$$\begin{cases} \frac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) & \text{on } \Gamma, \\ U^0(t=0) = U_0^0 & \text{on } \Gamma, \end{cases}$$

where  $H^0$  is the local operator previously defined.

$$\begin{cases} \partial_z [(H^1 + U^1) \cdot n] + \Theta^0(H^0 + \overline{U}^0) + G_0 n \cdot (H^0 + \overline{U}^0) = 0, \\ n \wedge \partial_z H^1 + \Lambda^0(H^0) = 0, \end{cases}$$

**Terms in  $\varepsilon$ :** in  $\Gamma \times I_1$ ,

$$\begin{aligned} \frac{\partial U^1}{\partial t} - U_{zz}^1 = & |U_z^0|^2 U^1 + 2(U_z^0 \cdot U_z^1) U^0 + U^0 \wedge (U_{zz}^1) + U^1 \wedge (U_{zz}^0) \\ & + U^1 \wedge H^0 + U^0 \wedge H^1 - U^0 \wedge (U^0 \wedge H^1) - U^0 \wedge (U^1 \wedge H^0) \\ & - U^1 \wedge (U^0 \wedge H^0) + G_0 (U_z^0 + U^0 \wedge U_z^0), \end{aligned}$$

$$\begin{cases} \partial_z [(H^2 + U^2) \cdot n] + \Theta^0(H^1 + \overline{U}^1) + G_0 n \cdot (H^1 + \overline{U}^1) \\ \quad + \Theta^1(H^0 + \overline{U}^0) + a_1 n \cdot (H^0 + \overline{U}^0) = 0, \\ n \wedge \partial_z H^2 + \Lambda^0(H^1) + \Lambda^1(H^0) = 0, \end{cases}$$

**Terms in  $\varepsilon^2$ :** in  $\Gamma \times I_1$ ,

$$\begin{aligned} \frac{\partial U^2}{\partial t} - \partial_z^2 U^2 = & |\partial_z U^0|^2 U^2 + 2(\partial_z U^0 \cdot \partial_z U^2) U^0 + U^0 \wedge \partial_z^2 U^2 + U^2 \wedge \partial_z^2 U^0 + U^2 \wedge H^0 \\ & + U^0 \wedge H^2 - U^0 \wedge (U^0 \wedge H^2) - U^0 \wedge (U^2 \wedge H^0) - U^2 \wedge (U^0 \wedge H^0) \\ & + a_1 (\partial_z U^0 + U^0 \wedge \partial_z U^0) + G_0 (\partial_z U^1 + U^0 \wedge \partial_z U^1 + U^1 \wedge \partial_z U^0) \\ & + \Delta_{\Gamma_0} U^0 + |\partial_z U^1|^2 U^0 + 2(\partial_z U^0 \cdot \partial_z U^1) U^1 + |\nabla_{\Gamma_0} U^0|^2 U^0 + U^0 \wedge \Delta_{\Gamma_0} U^0 \\ & + U^1 \wedge \partial_z^2 U^1 + U^1 \wedge H^1 - U^0 \wedge (U^1 \wedge H^1) - U^1 \wedge (U^0 \wedge H^1) \\ & - U^1 \wedge (U^1 \wedge H^0). \end{aligned}$$

$$\begin{cases} \partial_z [(H^3 + U^3) \cdot n] + \Theta^0(H^2 + \bar{U}^2) + G_0 n \cdot (H^2 + \bar{U}^2) \\ \quad + \Theta^1(H^1 + \bar{U}^1) + a_1 n \cdot (H^1 + \bar{U}^1) \\ \quad + \Theta^2(H^0 + \bar{U}^0) + a_2 n \cdot (H^0 + \bar{U}^0) = 0, \\ n \wedge \partial_z H^3 + \Lambda^0(H^2) + \Lambda^1(H^1) + \Lambda^2(H^0) = 0, \end{cases}$$

**Terms in  $\varepsilon^3$ :** in  $\Gamma \times I_1$ ,

$$\begin{aligned} \frac{\partial U^3}{\partial t} - \partial_z^2 U^3 = & |\partial_z U^0|^2 U^3 + 2(\partial_z U^0 \cdot \partial_z U^3) U^0 + U^0 \wedge \partial_z^2 U^3 + U^3 \wedge \partial_z^2 U^0 + U^3 \wedge H^0 \\ & + U^0 \wedge H^3 - U^0 \wedge (U^0 \wedge H^3) - U^0 \wedge (U^3 \wedge H^0) - U^3 \wedge (U^0 \wedge H^0) \\ & + a_2 (\partial_z U^0 U^0 \wedge \partial_z U^0) + a_1 (\partial_z U^1 + U^0 \wedge \partial_z U^1 + U^1 \wedge \partial_z U^0) \\ & + G_0 (\partial_z U^2 + U^0 \wedge \partial_z U^2 + U^2 \wedge \partial_z U^0 + U^1 \wedge \partial_z U^1) + \Delta_{\Gamma_0} U^1 + \Delta_{\Gamma_0,1} U^0 \\ & + |\partial_z U^1|^2 U^1 + 2(\partial_z U^0 \cdot \partial_z U^1) U^2 + 2(\partial_z U^0 \cdot \partial_z U^2) U^1 + 2(\partial_z U^1 \cdot \partial_z U^2) U^0 \\ & + |\nabla_{\Gamma_0} U^0|^2 U^1 + 2(\nabla_{\Gamma_0} U^0 \cdot \nabla_{\Gamma_0} U^1) U^0 + 2(\nabla_{\Gamma_0} U^0 \cdot \nabla_{\Gamma_0,1} U^0) U^0 \\ & + U^0 \wedge \Delta_{\Gamma_0} U^1 + U^1 \wedge \Delta_{\Gamma_0} U^0 + U^0 \wedge \Delta_{\Gamma_0,1} U^0 + U^2 \wedge \partial_z^2 U^1 + U^1 \wedge \partial_z^2 U^2 \\ & + U^1 \wedge H^2 + U^2 \wedge H^1 - U^1 \wedge (U^1 \wedge H^1) - U^0 \wedge (U^1 \wedge H^2) \\ & - U^0 \wedge (U^2 \wedge H^1) - U^1 \wedge (U^0 \wedge H^2) - U^1 \wedge (U^2 \wedge H^0) \\ & - U^2 \wedge (U^0 \wedge H^1) - U^2 \wedge (U^1 \wedge H^0) \end{aligned}$$

## 3.2 Existence of the asymptotic expansion

### 3.2.1 Behaviour of $H$ outside $\Omega$

We are interested in the behaviour of  $H$  outside  $\Omega$ .  $H$  fulfills :

$$\begin{cases} H \in \mathbb{L}^2(\Omega'), \\ \text{curl } H = 0 \text{ in } \Omega', \\ \text{div } H = 0 \text{ in } \Omega', \\ H \cdot n = f \text{ on } \Gamma, \end{cases} \quad (3.4)$$

where  $f \in \mathbb{H}^{l-\frac{1}{2}}(\Gamma)$  and  $l \in \mathbb{N}$ .

We want to get a relationship between  $H \wedge n$  and  $H \cdot n$  on  $\Gamma$ . We need the following regularity theorem:

**Theorem 3.1.** (cf [13])

Let  $m \in \mathbb{N}^*$  and  $\mathcal{O}$  a bounded open set of  $\mathbb{R}^3$  with a regular enough boundary, then

$$\mathbb{H}^m(\mathcal{O}) = \left\{ v \in (\mathbb{L}^2(\mathcal{O}))^3, \text{curl } v \in (\mathbb{H}^{m-1}(\mathcal{O}))^3, \text{div } v \in \mathbb{H}^{m-1}(\mathcal{O}), v \cdot n \in \mathbb{H}^{m-1/2}(\partial\mathcal{O}) \right\},$$

and there exists a constant  $C(m, \mathcal{O})$  such that

$$\|v\|_{\mathbb{H}^m(\mathcal{O})} \leq c \left( \|v\|_{(\mathbb{L}^2(\mathcal{O}))^3}^2 + \|\text{curl } v\|_{(\mathbb{H}^{m-1}(\mathcal{O}))^3}^2 + \|\text{div } v\|_{(\mathbb{H}^{m-1}(\mathcal{O}))^3}^2 + \|v \cdot n\|_{\mathbb{H}^{m-1/2}(\partial\mathcal{O})} \right).$$

**Remark 3.2.** This result stands whether  $\mathcal{O}$  is bounded or not, and with  $v \wedge n$  instead of  $v \cdot n$ .

We get:

**Lemma 3.2.** *There exists an unique solution  $H \in \mathbb{H}^l(\Omega')$  to (3.4) and so an unique application  $\psi$  from  $\mathbb{H}^{l-\frac{1}{2}}(\Gamma)$  in itself which maps  $H \cdot n$  on  $H \wedge n$ .*

**Proof:** The problem (3.4) has classically an unique solution  $H \in \mathbb{L}^2(\Omega')$  (cf [12]). As the boundary condition is regular we can apply theorem 3.1 and we get the announced regularity as well as the existence and uniqueness of  $\psi$ .  $\square$

### 3.2.2 Existence of $U^0$

**Theorem 3.3.** *Given  $U_0^0$  such that  $|U_0^0| \equiv 1$  and  $U_0^0 \in \mathbb{W}^{7,\infty}(\Gamma)$ , there exists an unique solution to (3.5):*

$$\begin{cases} \frac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge (U^0 \wedge H^0) & \text{in } \Gamma, \\ U^0(0, \sigma) = U_0^0(\sigma), \quad \forall \sigma \in \Gamma, \\ H^0 = -(U^0 \cdot n) n \end{cases} \quad (3.5)$$

such that  $|U^0| = 1$  and  $U^0 \in \mathcal{C}^\infty(\mathbb{R}^+; \mathbb{W}^{7,\infty}(\Gamma))$ .

**Remark 3.3.** *If we choose an initial data  $U_0^0$  which depends on  $z$  and if we want to get the existence of the solution on  $\mathbb{R}_t^+$  we need the  $z$ -dependent part of  $U_0^0$  i.e.*

$\widetilde{U}_0^0 = U_0^0 - \overline{U}_0^0 = U_0^0 - \int_{-1}^1 U_0^0(\sigma, z) dz$  to be smaller than  $\overline{U}_0^0$ . We then obtain at first order the 1D-Landau-Lifschitz equation and have to take into account the dependence on  $z$  in the following. Since the  $z$ -independence is conserved in this equation, we only consider a  $z$ -independent initial data for  $U_0^0$ .

**Proof:** As  $\mathbb{W}^{7,\infty}(\Gamma; \mathbb{R}^3)$  is an algebra we can solve easily the previous ordinary differential equation and we remark that  $|U^0(t)| = 1$  which proves the result.  $\square$

### 3.2.3 Existence of $U^1$

**Theorem 3.4.** *Under the assumptions of theorem 3.3 and with the additional assumption that  $U_0^1 \in \mathbb{H}^6(\Gamma) \otimes \mathbb{H}^5(I_1)$  there exists an unique solution to*

$$\begin{cases} \frac{\partial U^1}{\partial t} - U_{zz}^1 = U^0 \wedge U_{zz}^1 + U^1 \wedge H^0 + U^0 \wedge H^1 - U^0 \wedge (U^0 \wedge H^1) \\ \quad - U^0 \wedge (U^1 \wedge H^0) - U^1 \wedge (U^0 \wedge H^0) & \text{in } \Gamma \times I_1, \\ U_z^1 = 0 & \text{on } \Gamma \times \{0, 1\}, \\ U^1(0, \cdot) = U_0^1(\cdot) & \text{in } \Gamma \times I_1, \end{cases} \quad (3.6)$$

where  $H^1$  fulfills

$$\begin{cases} \partial_z [(H^1 + U^1) \cdot n] + \Theta^0(H^0 + \overline{U}^0) + G_0 n \cdot (H^0 + \overline{U}^0) = 0, \\ n \wedge \partial_z H^1 + \Lambda^0(H^0) = 0, \end{cases} \quad (3.7)$$

such that for all  $T > 0$ ,

$$U^1 \in \bigcap_{k=0}^2 \left[ \mathbb{W}^{k,\infty} \left( 0, T; \mathbb{H}^6(\Gamma) \otimes \mathbb{H}^{5-2k}(I_1) \right) \cap \mathbb{H}^k \left( 0, T; \mathbb{H}^6(\Gamma) \otimes \mathbb{H}^{6-2k}(I_1) \right) \right].$$

**Proof:** The equations (3.7), the transmission conditions (1.4) and the lemma 3.2 give:

$$\begin{cases} \left( H^1 + \overline{U^1} \right) \cdot n = (1-z) \left[ \Theta^0 \left( H^0 + \overline{U^0} \right) + G_0 n \cdot \left( H^0 + \overline{U^0} \right) \right], \\ H^1 \wedge n = \psi \left( \left( (H^1 + \overline{U^1}) \cdot n \right) \Big|_{z=0} \right) + z \Lambda^0(H^0). \end{cases}$$

and then  $H^1 = - (U^1 \cdot n) n + q$ , with

$$\begin{aligned} q = & (1-z) \left[ \Theta^0 \left( H^0 + \overline{U^0} \right) + G_0 n \cdot \left( H^0 + \overline{U^0} \right) \right] n \\ & + n \wedge \psi \left( \Theta^0 \left( H^0 + \overline{U^0} \right) + G_0 n \cdot \left( H^0 + \overline{U^0} \right) \right) + n \wedge (z \Lambda^0(H^0)), \end{aligned}$$

and  $q \in \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{H}^6(\Gamma) \otimes \mathcal{C}^\infty(I_1))$ .

The proof follows then the one given in [24]. We just perform the estimates on  $\Gamma$  instead of  $(0,1)^2$ : we consider  $\sigma \in \Gamma$  as a parameter and we use Galerkin's method on the  $z$ -dependent equation. We then perform some energy estimates on the equation and its derivative in time to get regularity in  $z$  and  $t$ . The regularity in time allows then us to get more regularity in space and the existence in the announced spaces.  $\square$

### 3.2.4 Existence of $U^2$

**Theorem 3.5.** *Under the assumptions of theorem 3.3 and 3.4 and with the extra assumption that  $U_0^2 \in \mathbb{H}^5(\Gamma) \otimes \mathbb{H}^5(I_1)$  there exists a unique solution to*

$$\begin{cases} \frac{\partial U^2}{\partial t} - U_{zz}^2 = U^0 \wedge U_{zz}^2 + U^2 \wedge H^0 + U^0 \wedge H^2 - U^0 \wedge (U^0 \wedge H^2) \\ \quad - U^0 \wedge (U^2 \wedge H^0) - U^2 \wedge (U^0 \wedge H^0) + F \text{ in } \Gamma \times I_1, \\ U_z^2 = 0 \text{ on } \Gamma \times \{0,1\}, \\ U^2(0, \cdot) = U_0^2(\cdot) \text{ in } \Gamma \times I_1, \end{cases} \quad (3.8)$$

where

$$\begin{cases} \partial_z \left[ (H^2 + U^2) \cdot n \right] + \Theta^0(H^1 + \overline{U^1}) + G_0 n \cdot (H^1 + \overline{U^1}) \\ \quad + \Theta^1(H^0 + \overline{U^0}) + a_1 n \cdot (H^0 + \overline{U^0}) = 0, \\ n \wedge \partial_z H^2 + \Lambda^0(H^1) + \Lambda^1(H^0) = 0, \end{cases} \quad (3.9)$$

and

$$\begin{aligned} F = & G_0 \left( \partial_z U^1 + U^0 \wedge \partial_z U^1 \right) + \Delta_{\Gamma_0} U^0 + |\partial_z U^1|^2 U^0 + |\nabla_{\Gamma_0} U^0|^2 U^0 + U^0 \wedge \Delta_{\Gamma_0} U^0 \\ & + U^1 \wedge \partial_z^2 U^1 + U^1 \wedge H^1 - U^0 \wedge (U^1 \wedge H^1) - U^1 \wedge (U^0 \wedge H^1) - U^1 \wedge (U^1 \wedge H^0). \end{aligned}$$

such that for all  $T > 0$ ,

$$U^2 \in \bigcap_{k=0}^2 \left[ \mathbb{W}^{k,\infty} \left( 0, T; \mathbb{H}^5(\Gamma) \otimes \mathbb{H}^{5-2k}(I_1) \right) \cap \mathbb{H}^k \left( 0, T; \mathbb{H}^5(\Gamma) \otimes \mathbb{H}^{6-2k}(I_1) \right) \right].$$

**Proof:** The proof follows the same scheme as the previous one. See also [24].  $\square$

### 3.2.5 Existence of $U^3$

**Theorem 3.6.** *Under the assumptions of theorems 3.3, 3.4 and 3.5 and with the additional assumption  $U_0^3 \in \mathbb{H}^4(\Gamma) \otimes \mathbb{H}^5(I_1)$ , there exists a unique solution to:*

$$\left\{ \begin{array}{l} \frac{\partial U^3}{\partial t} - \partial_z^2 U^3 = U^0 \wedge \partial_z^2 U^3 + U^3 \wedge H^0 + U^0 \wedge H^3 - U^0 \wedge (U^0 \wedge H^3) \\ \quad - U^0 \wedge (U^3 \wedge H^0) - U^3 \wedge (U^0 \wedge H^0) + F \text{ in } \Gamma \times I_1, \\ \partial_z U^3 = 0 \text{ on } \Gamma \times \{0, 1\}, \\ U^3(0, \cdot) = U_0^3(\cdot) \text{ in } \Gamma \times I_1, \end{array} \right. \quad (3.10)$$

where

$$\left\{ \begin{array}{l} \partial_z \left[ (H^3 + \bar{U}^3) \cdot n \right] + \Theta^0 (H^2 + \bar{U}^2) + G_0 n \cdot (H^2 + \bar{U}^2) \\ \quad + \Theta^1 (H^1 + \bar{U}^1) + a_1 n \cdot (H^1 + \bar{U}^1) \\ \quad + \Theta^2 (H^0 + \bar{U}^0) + a_2 n \cdot (H^0 + \bar{U}^0) = 0, \\ n \wedge \partial_z H^3 + \Lambda^0(H^2) + \Lambda^1(H^1) + \Lambda^2(H^0) = 0, \end{array} \right. \quad (3.11)$$

and

$$\begin{aligned} F = & a_1 (\partial_z U^1 + U^0 \wedge \partial_z U^1) + G_0 (\partial_z U^2 + U^0 \wedge \partial_z U^2 + U^1 \wedge \partial_z U^1) \\ & + \Delta_{\Gamma_0} U^1 + \Delta_{\Gamma_{0,1}} U^0 + |\partial_z U^1|^2 U^1 + 2 (\partial_z U^1 \cdot \partial_z U^2) U^0 \\ & + |\nabla_{\Gamma_0} U^0|^2 U^1 + 2 (\nabla_{\Gamma_0} U^0 \cdot \nabla_{\Gamma_0} U^1) U^0 + 2 (\nabla_{\Gamma_0} U^0 \cdot \nabla_{\Gamma_{0,1}} U^0) U^0 \\ & + U^0 \wedge \Delta_{\Gamma_0} U^1 + U^1 \wedge \Delta_{\Gamma_0} U^0 + U^0 \wedge \Delta_{\Gamma_{0,1}} U^0 + U^2 \wedge \partial_z^2 U^1 + U^1 \wedge \partial_z^2 U^2 \\ & + U^1 \wedge H^2 + U^2 \wedge H^1 - U^1 \wedge (U^1 \wedge H^1) - U^0 \wedge (U^1 \wedge H^2) \\ & - U^0 \wedge (U^2 \wedge H^1) - U^1 \wedge (U^0 \wedge H^2) - U^1 \wedge (U^2 \wedge H^0) \\ & - U^2 \wedge (U^0 \wedge H^1) - U^2 \wedge (U^1 \wedge H^0) \end{aligned}$$

such that for all  $T > 0$ ,

$$U^3 \in \bigcap_{k=0}^2 \left[ \mathbb{W}^{k,\infty} \left( 0, T; \mathbb{H}^4(\Gamma) \otimes \mathbb{H}^{5-2k}(I_1) \right) \cap \mathbb{H}^k \left( 0, T; \mathbb{H}^4(\Gamma) \otimes \mathbb{H}^{6-2k}(I_1) \right) \right].$$

**Proof:** The proof follows the same scheme as the previous one. See also [24].  $\square$

## 4 Sobolev embeddings and energy estimate in the space $\omega_\varepsilon$

In the following we are lead to perform energy estimates on the remainder terms of  $u^\varepsilon$  and  $H^\varepsilon$ . These terms will be defined in  $\omega_\varepsilon$  and we need to track the dependence on the domain and particularly on  $\varepsilon$  of the Sobolev embeddings and other classical inequalities. [24, 25]. Thanks to a rescaling (to get rid of the  $\varepsilon$  in the definition of the domain), we introduce anisotropic Sobolev spaces on the domain  $\omega_\varepsilon$ . We let  $v(\sigma, z) = u(\sigma, \varepsilon z)$  for  $u$  defined on  $\omega_\varepsilon$ .

**Definition 4.1.** *For all  $1 \leq p < +\infty$  we define the normalized Lebesgue spaces  $\mathbb{L}_\varepsilon^p$  in the following way:*

$$\mathbb{L}_\varepsilon^p = \left\{ u \in \mathcal{D}'(\omega_\varepsilon), \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} |u(x)|^p dx < +\infty \right\},$$

$$|u|_{\varepsilon,p} = \left( \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} |u(x)|^p dx \right)^{1/p}.$$

We let  $\mathbb{W}_\varepsilon^{0,p} = \mathbb{L}_\varepsilon^p$  and we define the anisotropic Sobolev spaces  $\mathbb{W}_\varepsilon^{m,p}$ ,  $m \in \mathbb{N}$ , and their norms as follows:

$$\begin{aligned} \mathbb{W}_\varepsilon^{m+1,p} &= \left\{ u \in \mathbb{L}_\varepsilon^p, \nabla_{\Gamma_{\varphi(x)}} u \in \mathbb{W}_\varepsilon^{m,p} \text{ and } (\varepsilon \partial_z) u \in \mathbb{W}_\varepsilon^{m,p} \right\}, \\ \|u\|_{\mathbb{W}_\varepsilon^{m+1,p}} &= \left( |u|_{\varepsilon,p}^p + \|\nabla_{\Gamma_{\varphi(x)}} u\|_{\mathbb{W}_\varepsilon^{m,p}}^p + \|\varepsilon \partial_z u\|_{\mathbb{W}_\varepsilon^{m,p}}^p \right)^{1/p}. \end{aligned}$$

When  $p = 2$  we let  $\mathbb{H}_\varepsilon^m = \mathbb{W}_\varepsilon^{m,2}$ , and  $\|u\|_{\varepsilon,m} = \|u\|_{\mathbb{W}_\varepsilon^{m,2}}$ .

When  $p = +\infty$ , we let:

$$\begin{aligned} \mathbb{L}_\varepsilon^\infty &= \left\{ u \in \mathcal{D}'(\omega_\varepsilon), \sup_{x \in \omega_\varepsilon} |u(x)| < +\infty \right\}, \\ |u|_{\varepsilon,\infty} &= \sup_{x \in \omega_\varepsilon} |u(x)|. \end{aligned}$$

With  $\mathbb{W}_\varepsilon^{0,\infty} = \mathbb{L}_\varepsilon^\infty$ , we define the anisotropic Sobolev spaces  $\mathbb{W}_\varepsilon^{m,\infty}$ ,  $m \in \mathbb{N}$ , and their norms as follows:

$$\begin{aligned} \mathbb{W}_\varepsilon^{m+1,\infty} &= \left\{ u \in \mathbb{L}_\varepsilon^\infty, \nabla_{\Gamma_{\varphi(x)}} u \in \mathbb{W}_\varepsilon^{m,\infty} \text{ and } (\varepsilon \partial_z) u \in \mathbb{W}_\varepsilon^{m,\infty} \right\}, \\ \|u\|_{\mathbb{W}_\varepsilon^{m+1,\infty}} &= \left( |u|_{\varepsilon,\infty} + \|\nabla_{\Gamma_{\varphi(x)}} u\|_{\mathbb{W}_\varepsilon^{m,\infty}} + \|\varepsilon \partial_z u\|_{\mathbb{W}_\varepsilon^{m,\infty}} \right). \end{aligned}$$

In a first time we remind some anisotropic Sobolev inequalities, the first two being proved in [25], the last one in [24] :

**Theorem 4.1 (Anisotropic Agmon's inequality).** *Let  $\Omega_0 = ]0, 1[^3$ . There exists an absolute constant  $C$  such that for all  $u \in \mathbb{H}^2(\Omega_0)$ ,*

$$\|u\|_{\mathbb{L}^\infty(\Omega_0)} \leq C \|u\|_{\mathbb{L}^2(\Omega_0)}^{\frac{1}{4}} \prod_{i=1}^3 \left( \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{\mathbb{L}^2(\Omega_0)} + \left\| \frac{\partial u}{\partial x_i} \right\|_{\mathbb{L}^2(\Omega_0)} + \|u\|_{\mathbb{L}^2(\Omega_0)} \right)^{\frac{1}{4}}. \quad (4.1)$$

**Theorem 4.2 (Anisotropic Ladyzhenskaya's inequality).** *Let  $\Omega_0 = \prod_{i=1}^3 ]0, \lambda_i[$  with  $\lambda_i > 0$  for  $i = 1, 2, 3$ . There exists an absolute constant  $C$  such that for all  $u \in \mathbb{H}^1(\Omega_0)$ ,*

$$\|u\|_{\mathbb{L}^6(\Omega)} \leq C \prod_{i=1}^3 \left( \left\| \frac{\partial u}{\partial x_i} \right\|_{\mathbb{L}^2(\Omega)} + \frac{1}{\lambda_i} \|u\|_{\mathbb{L}^2(\Omega)} \right)^{\frac{1}{3}}. \quad (4.2)$$

**Theorem 4.3.** *Let  $\Omega_0 = ]0, 1[^3$ . There exists an absolute constant  $C$  such that for all  $u \in \mathbb{H}^2(\Omega_0)$ ,*

$$\|u\|_{\mathbb{L}^{12}(\Omega_0)} \leq C \prod_{i=1}^3 \left( \sum_{j=0}^1 \left\| \frac{\partial^j u}{\partial x_i^j} \right\|_{\mathbb{L}^2(\Omega_0)} \right)^{\frac{1}{4}} \left( \sum_{j=0}^2 \left\| \frac{\partial^j u}{\partial x_i^j} \right\|_{\mathbb{L}^2(\Omega_0)} \right)^{\frac{1}{12}}. \quad (4.3)$$

We now use the previous rescaling to deduce anisotropic versions of the Sobolev embeddings:

**Lemma 4.4.** *Let  $u \in \mathbb{L}_\varepsilon^p$  and  $v(\sigma, x) = u(\Psi(\sigma, x))$  for all  $(\sigma, x) \in \Gamma \times \varepsilon I_1$ .  $v \in \mathbb{L}^2(\Gamma \times \varepsilon I_1)$  and*

$$\frac{1}{M_\varepsilon^{1/p}} |u|_{\varepsilon,p} \leq \int_{\Gamma \times \varepsilon I_1} |v|^p \, d\sigma \, dx \leq M_\varepsilon^{1/p} |u|_{\varepsilon,p},$$

where  $M_\varepsilon = \frac{\sup_{\sigma, x \in \Gamma \times \varepsilon I_1} |\text{Jac} \Psi(\sigma, x)|}{\inf_{\sigma, x \in \Gamma \times \varepsilon I_1} |\text{Jac} \Psi(\sigma, x)|}$ . Moreover there exists some  $\varepsilon$ -independent constants  $\mu, \nu$  such that for  $\varepsilon$  small enough,  $0 < \nu \leq M_\varepsilon \leq \mu$ .

**Proof:** The application  $\Psi$  is a regular diffeomorphism for all  $\varepsilon \in [0, \eta_0]$  so  $|\text{Jac}\Psi|$  is continuous and take non-null values on  $\Gamma \times \varepsilon I_1$  and for  $\varepsilon \in [0, \eta_0]$ .  $\square$

**Corollary 4.5.** *There exists an  $\varepsilon$ -independent constant  $C$  such that:*

- for all  $u \in \mathbb{H}_\varepsilon^1$  and  $1 \leq p \leq 6$ ,

$$|u|_{\varepsilon,p} \leq C |u|_{\varepsilon,2}^{\frac{3}{p}-\frac{1}{2}} \|u\|_{\varepsilon,1}^{\frac{3}{2}-\frac{3}{p}}. \quad (4.4)$$

- for all  $u \in \mathbb{H}_\varepsilon^2$  and  $2 \leq p \leq +\infty$ ,

$$|u|_{\varepsilon,p} \leq C |u|_{\varepsilon,2}^{\frac{1}{4}+\frac{3}{2p}} \|u\|_{\varepsilon,2}^{\frac{3}{4}-\frac{3}{2p}}. \quad (4.5)$$

- for all  $u \in \mathbb{H}_\varepsilon^2$ ,

$$|u|_{\varepsilon,12} \leq C \|u\|_{\varepsilon,1}^{\frac{3}{4}} \|u\|_{\varepsilon,2}^{\frac{1}{4}}. \quad (4.6)$$

**Proof:** Thanks to Lemma (4.4), an atlas covering  $\Gamma$  and a partition of unity we only have to prove the result in  $]0, 1[ \times I_1$ .

We obtain immediatly the first estimate in the case  $p = 6$  thanks to Th. 4.2. An interpolation between  $\mathbb{L}_\varepsilon^2$  and  $\mathbb{L}_\varepsilon^6$  give the result.

With the rescaling strategy we use to introduce the anisotropic Sobolev spaces we deduce from Th. 4.1 and 4.3 the two inequalities.  $\square$

#### 4.1 Trace and lifting

Let us remind a definition of the spaces  $\mathbb{H}^s(\Omega)$ ,  $s = k + \theta > 0$ ,  $0 < s < 1$  (cf [21, 20]):

$$\mathbb{H}^s(\Omega) = \left\{ u \in \mathbb{H}^k(\Omega), \int_{\Omega} \int_{\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2s}} dx dy < +\infty, \forall |\alpha| = k \right\},$$

$$\|u\|_{\mathbb{H}^s(\Omega)} = \left( \|u\|_{\mathbb{H}^k(\Omega)}^2 + \sum_{|\alpha|=k} \int_{\Omega} \int_{\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x - y|^{n+2s}} dx dy \right)^{1/2}.$$

With the same type of rescaling we used for the embedding theorem we define anisotropic spaces on the boundary  $\partial\omega_\varepsilon = \Psi(\Gamma \times \{0, \varepsilon\})$  ( $\Gamma$  has no boundary):

$$\mathbb{H}_\varepsilon^s(\partial\omega_\varepsilon) = \left\{ u \in \mathbb{H}_\varepsilon^k(\partial\omega_\varepsilon), \frac{1}{|\partial\omega_\varepsilon|^2} \int_{\partial\omega_\varepsilon} \int_{\partial\omega_\varepsilon} \frac{|D_\varepsilon^\alpha u(y) - D_\varepsilon^\alpha u(y')|^2}{(|\sigma - \sigma'|^2 + \varepsilon^{-2}|z - z'|^2)^{n/2+s}} dy dy' < +\infty, \right. \\ \left. \forall |\alpha| = k \right\},$$

$$\|u\|_{\mathbb{H}_\varepsilon^s(\partial\omega_\varepsilon)} = \left( \|u\|_{\mathbb{H}_\varepsilon^k(\partial\omega_\varepsilon)}^2 + \sum_{|\alpha|=k} \frac{1}{|\partial\omega_\varepsilon|^2} \int_{\partial\omega_\varepsilon} \int_{\partial\omega_\varepsilon} \frac{|D_\varepsilon^\alpha u(y) - D_\varepsilon^\alpha u(y')|^2}{(|\sigma - \sigma'|^2 + \varepsilon^{-2}|z - z'|^2)^{n/2+s}} dy dy' \right)^{1/2},$$

where  $y = \Psi(\sigma, z)$ ,  $y' = \Psi(\sigma', z')$ ,  $D_\varepsilon = (\nabla_{\Gamma_{\varphi(x)}}^t, \varepsilon \partial_z)^t$  and  $\alpha \in \mathbb{N}^k$ .



**Lemma 4.6.** *There exists an  $\varepsilon$ -independent constant  $C$  such that for all  $\varepsilon > 0$  small enough and  $u \in \mathbb{H}_\varepsilon^1$ ,*

$$\|u\|_{\mathbb{H}_\varepsilon^{\frac{1}{2}}(\partial\omega_\varepsilon)} \leq C \|u\|_{\varepsilon,1}.$$

*There exists an  $\varepsilon$ -independent constant  $C$  such that for all  $0 < \varepsilon$  small enough and  $f \in \mathbb{H}_\varepsilon^{\frac{1}{2}}(\partial\omega_\varepsilon)$  there exists a lifting  $\psi \in \mathbb{H}_\varepsilon^1$  of  $f$  such that:*

$$\|\psi\|_{\varepsilon,1} \leq C \|f\|_{\mathbb{H}_\varepsilon^{\frac{1}{2}}(\partial\omega_\varepsilon)}.$$

**Proof:**

**Anisotropic trace theorem:** for all  $(\sigma, z) \in \Gamma \times I_1$ , we let  $v(\sigma, z) = (u \circ \Psi)(\sigma, \varepsilon z)$  and we apply the classical trace theorem to  $v$ :

$$\|v\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma \times \{0,1\})} \leq c \|v\|_{\mathbb{H}^1(\Gamma \times I_1)},$$

where  $c$  does not depend on  $\varepsilon$ . We now perform some estimations on  $v$ : thanks to an atlas covering  $\Gamma$  and a partition of unity we are lead back to functions with a compact support in  $\Psi(U \times \varepsilon I_1)$  where  $U$  is an open subset of  $\Gamma$ . As in lemmas 4.4 and 4.5 there exists some  $\varepsilon$ -independent constants  $C$  and  $C'$  such that:

$$C \|u\|_{\varepsilon,1} \leq \|v\|_{\mathbb{H}^1(\Gamma \times I_1)} \leq C' \|u\|_{\varepsilon,1}.$$

Since the spaces  $\mathbb{H}_\varepsilon^{\frac{1}{2}}(\partial\omega_\varepsilon)$  take into account the anisotropy of the problem (as well as the Sobolev spaces) we obtain the following estimates on the norme  $\mathbb{H}_\varepsilon^{\frac{1}{2}}$ :

$$C \|u\|_{\mathbb{H}_\varepsilon^{\frac{1}{2}}(\partial\omega_\varepsilon)} \leq \|v\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma \times \{0,1\})} \leq C' \|u\|_{\mathbb{H}_\varepsilon^{\frac{1}{2}}(\partial\omega_\varepsilon)},$$

which gives the result.

**Anisotropic lifting theorem:** By the same way we apply the classical lifting theorem to  $g(\sigma, z) = f \circ \Psi(\sigma, \varepsilon z)$ ,  $(\sigma, z) \in \Gamma \times \{0, 1\}$  to obtain a lifting  $v \in \mathbb{H}^1(\Gamma \times I_1)$ . Since the application  $\Psi$  is a diffeomorphism on  $\omega_{\eta_0}$  we let  $u(\Psi(\sigma, \varepsilon z)) = v(\sigma, z)$ . Then  $u$  is a lifting of  $f \in \mathbb{H}_\varepsilon^{\frac{1}{2}}(\partial\omega_\varepsilon)$  and the previous estimates give the result.  $\square$

## 4.2 Some inequalities

In the following we also use the following inequalities:

**Lemma 4.7.** *For all  $u \in \mathbb{H}_\varepsilon^2$  such that  $\frac{\partial u}{\partial n} = 0$  on  $\partial\omega_\varepsilon$  there exists an  $\varepsilon$ -independent constant  $C$  such that:*

$$\|u\|_{\varepsilon,2} \leq C \left( |u|_{\varepsilon,2}^2 + \left| \Delta_{\Gamma_{\varphi(x)}} u \right|_{\varepsilon,2}^2 + \left| \varepsilon^2 \frac{1}{\gamma_z} \partial_z (\gamma_z \partial_z u) \right|_{\varepsilon,2}^2 \right)^{1/2}, \quad (4.7)$$

*For all  $u \in \mathbb{H}_\varepsilon^m$ ,  $m \in \mathbb{N}^*$  there exists an  $\varepsilon$ -independent constant  $C$  such that:*

$$\begin{aligned} \|u\|_{\varepsilon,m} \leq & C \left( |u|_{\varepsilon,2} + \left\| \operatorname{div}_{\Gamma_{\varphi(x)}} u \right\|_{\varepsilon,m-1} + \left\| \varepsilon \frac{1}{\gamma_z} \partial_z (\gamma_z (u \cdot n)) \right\|_{\varepsilon,m-1} \right. \\ & \left. + \left\| \nabla_{\Gamma_{\varphi(x)}} \wedge u + \varepsilon (n \partial_z) \wedge u \right\|_{\varepsilon,m-1} + \|u \cdot n\|_{\mathbb{H}_\varepsilon^{m-\frac{1}{2}}(\partial\omega_\varepsilon)} \right). \end{aligned} \quad (4.8)$$

**Proof:** We let  $v(\sigma, z) = u(\Psi(\sigma, \varepsilon z))$  to only consider the domain  $\Gamma \times I_1$  and we apply to  $v$  the classical inequality

$$\|v\|_{\mathbb{H}^2(\Gamma \times I_1)} \leq C \left( \|v\|_{\mathbb{L}^2(\Gamma \times I_1)}^2 + \|\Delta v\|_{\mathbb{L}^2(\Gamma \times I_1)}^2 \right)^{1/2}$$

and Lemma 3.1. We then deduce the result thanks to Lemma 4.4 and 4.6.  $\square$

**Corollary 4.8.** *There exists an  $\varepsilon$ -independent constant  $C$  such that for all  $u \in \mathbb{H}_\varepsilon^2$  satisfying  $\frac{\partial u}{\partial n} = 0$  on  $\partial\omega_\varepsilon$  we have:*

$$\left\| (\nabla_{\Gamma_{\varphi(x)}} + \varepsilon(n\partial_z))u \right\|_{\varepsilon,1} \leq C \left( \left| \nabla_{\Gamma_{\varphi(x)}} u \right|_{\varepsilon,2} + |\varepsilon\partial_z u|_{\varepsilon,2} + \left| \Delta_{\Gamma_{\varphi(x)}} u \right|_{\varepsilon,2} + \left| \varepsilon^2 \frac{1}{\gamma_z} \partial_z (\gamma_z \partial_z u) \right|_{\varepsilon,2} \right). \quad (4.9)$$

Moreover if  $u \in \mathbb{H}_\varepsilon^3$  there exists an  $\varepsilon$ -independent constant  $C$  such that:

$$\left\| (\nabla_{\Gamma_{\varphi(x)}} + \varepsilon(n\partial_z))u \right\|_{\varepsilon,2} \leq C \left( \left| \nabla_{\Gamma_{\varphi(x)}} u \right|_{\varepsilon,2} + |\varepsilon\partial_z u|_{\varepsilon,2} + \left\| \Delta_{\Gamma_{\varphi(x)}} u \right\|_{\varepsilon,1} + \left\| \varepsilon^2 \frac{1}{\gamma_z} \partial_z (\gamma_z \partial_z u) \right\|_{\varepsilon,1} \right). \quad (4.10)$$

## 5 Equations satisfied by the remainder term

- In  $\omega_\varepsilon$  we let

$$\left\{ \begin{array}{l} u^\varepsilon(t, x) = U^0(t, P(x)) + \varepsilon U^1 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^2 U^2 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) \\ \quad + \varepsilon^3 U^3 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^3 r^\varepsilon(t, x), \\ H^\varepsilon(t, x) = H^0(t, P(x)) + \varepsilon H^1 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^2 H^2 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) \\ \quad + \varepsilon^3 H^3 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^3 Q^\varepsilon(t, x), \end{array} \right.$$

- In  $\Omega'$ ,

$$H^\varepsilon(t, x) = H^0(t, x) + \varepsilon H^1(t, x) + \varepsilon^2 H^2(t, x) + \varepsilon^3 H^3(t, x) + \varepsilon^3 Q^\varepsilon(t, x),$$

where  $r^\varepsilon$  and  $Q^\varepsilon$  are the remainder terms of  $u$  and  $H$ .

**Remark 5.1.** *All the terms which depend on  $U^0, H^0, U^1, \dots$  naturally live on  $\Gamma \times (0, 1)$ . From now on, when we consider that this kind of terms in the equations on the remainder are defined on the space  $\omega_\varepsilon$ , we assume that these terms are valued on  $\Psi^{-1}(x)$  where  $x \in \omega_\varepsilon$ .*

We also need to precise the expressions of the operator we used. We let

$$a_r = \frac{1}{\varepsilon^2} \int_0^{\varepsilon z} \frac{(\varepsilon z - s)^2}{2} \partial_z^3 G(\sigma, s) ds$$

and we get:

- $\nabla u = (\partial_z u)n + \nabla_{\Gamma_0} u + \varepsilon \nabla_{\Gamma_0,1} + \varepsilon \nabla^r u$ , where  $\nabla^r u = \frac{1}{\varepsilon} \left[ \int_0^{\varepsilon z} (\varepsilon z - s) (\partial_z^2 \nabla_{\Gamma_z}) (\sigma, s) ds \right] (u)$ .

- $\operatorname{div} V = \frac{1}{\varepsilon} \partial_z (V \cdot n) + \sum_{i=0}^2 \varepsilon^i a_i n \cdot V + \varepsilon^2 a_r n \cdot V + \sum_{i=0}^2 \varepsilon^i \Theta^i(V) + \varepsilon^2 \Theta^r(V)$ ,

where  $\Theta^r(V) = \frac{1}{\varepsilon^2} \left[ \int_0^{\varepsilon z} \frac{(\varepsilon z - s)^2}{2} (\partial_z^3 \operatorname{div}_{\Gamma_z}) (\sigma, s) ds \right] (V_T)$ .

- $\operatorname{curl} V = \frac{1}{\varepsilon} n \wedge \partial_z V + \sum_{i=0}^2 \varepsilon^i \Lambda^i(V) + \varepsilon^2 \Lambda^r(V)$ ,

where

$$\begin{aligned} \Lambda^r(V) = & -n \wedge \nabla^r V_N + \frac{1}{\varepsilon^2} \left[ \int_0^{\varepsilon z} \frac{(\varepsilon z - s)^2}{2} n \wedge \partial_z^3 ((I + z dn)^{-1}) dn ds \right] (V_T) \\ & + \frac{1}{\varepsilon^2} \left[ \int_0^{\varepsilon z} \frac{(\varepsilon z - s)^2}{2} \partial_z^3 \operatorname{curl}_{\Gamma_z} \right] (V_T) n. \end{aligned}$$

- $\Delta u = \frac{1}{\varepsilon^2} \frac{\partial^2 u}{\partial z^2} + \frac{1}{\varepsilon} \sum_{i=0}^2 \varepsilon^i a_i \frac{\partial u}{\partial z} + \varepsilon^2 a_r \frac{\partial u}{\partial z} + \Delta_{\Gamma_0} u + \varepsilon \Delta_{\Gamma_0,1} u + \varepsilon \Delta^r u$ ,

where  $\Delta^r u = \frac{1}{\varepsilon} \left( \int_0^{\varepsilon z} (\varepsilon z - s) (\partial_z^2 \Delta_{\Gamma_z}) (\sigma, s) ds \right) u$ .

For all of the previous remainder term we have

$$\|\mathcal{A}^r u\|_{\mathbb{H}^p(\Gamma \times I_1)} \leq C\varepsilon \|u\|_{\mathbb{H}^{p+d}(\Gamma \times I_1)},$$

where  $d$  is the degree of the operator  $\mathcal{A}$ .

## 5.1 Remainder term of $H$

Back in the equations (1.4) we simplify by using the equations satisfied by the profiles. We get:

$$\left\{ \begin{array}{ll} Q^\varepsilon \in \mathbb{L}^2(\Omega' \cup \omega_\varepsilon), & \\ \operatorname{curl} Q^\varepsilon + f_1 = 0 & \text{in } \Omega' \cup \omega_\varepsilon, \\ \operatorname{div} (Q^\varepsilon + \bar{r}^\varepsilon) + f_2 = 0 & \text{in } \Omega' \cup \omega_\varepsilon, \\ [Q^\varepsilon \wedge n] = 0 & \text{on } \Gamma, \\ [(Q^\varepsilon + \bar{r}^\varepsilon) \cdot n] = 0 & \text{on } \Gamma \cup \Gamma_\varepsilon. \end{array} \right. \quad (5.1)$$

where

$$f_1 = \Lambda^0(H^3) + \Lambda^1(H^2 + \varepsilon H^2) + \Lambda^2(H^1 + \varepsilon H^2 + \varepsilon^2 H^3) + \frac{1}{\varepsilon} \Lambda^r(H^0 + \varepsilon H^1 + \varepsilon^2 H^2 + \varepsilon^3 H^3)$$

$$\begin{aligned} f_2 = & G_0(H^3 + \bar{U}^3) \cdot n + a^1(H^2 + \bar{U}^2 + \varepsilon H^3 + \varepsilon \bar{U}^3) \cdot n \\ & + a^2(H^1 + \bar{U}^1 + \varepsilon H^2 + \varepsilon \bar{U}^2 + \varepsilon^2 H^3 + \varepsilon^2 \bar{U}^3) \cdot n \\ & + \frac{1}{\varepsilon} a^r(H^0 + \varepsilon H^1 + \varepsilon^2 H^2 + \varepsilon^3 H^3 + \bar{U}^0 + \varepsilon \bar{U}^1 + \varepsilon^2 \bar{U}^2 + \varepsilon^3 \bar{U}^3) \cdot n \\ & + \Theta^0(H^3 + \bar{U}^3) + \Theta^1(H^2 + \bar{U}^2 + \varepsilon H^3 + \varepsilon \bar{U}^3) \\ & + \Theta^2(H^1 + \bar{U}^1 + \varepsilon H^2 + \varepsilon \bar{U}^2 + \varepsilon^2 H^3 + \varepsilon^2 \bar{U}^3) \\ & + \frac{1}{\varepsilon} \Theta^r(H^0 + \varepsilon H^1 + \varepsilon^2 H^2 + \bar{U}^0 + \varepsilon \bar{U}^1 + \varepsilon^2 \bar{U}^2) \end{aligned}$$

We then consider the remainder  $Q^\varepsilon$  as a function of  $r^\varepsilon$  to get:

**Lemma 5.1.** *Let  $R^\varepsilon$  and  $S^\varepsilon$  the solution of*

$$\left\{ \begin{array}{ll} \operatorname{curl} R^\varepsilon = 0 & \text{in } \Omega' \cup \omega_\varepsilon, \\ \operatorname{div} (R^\varepsilon + \bar{r}^\varepsilon) = 0 & \text{in } \Omega' \cup \omega_\varepsilon, \\ (R^\varepsilon + \bar{r}^\varepsilon) \cdot n = 0 & \text{on } \Gamma_\varepsilon. \end{array} \right. \quad \left\{ \begin{array}{ll} \operatorname{curl} S^\varepsilon = f_1 & \text{in } \Omega' \cup \omega_\varepsilon, \\ \operatorname{div} S^\varepsilon = f_2 & \text{in } \Omega' \cup \omega_\varepsilon, \\ S^\varepsilon \cdot n = 0 & \text{on } \Gamma_\varepsilon. \end{array} \right.$$

Under the assumptions of Th. 3.3, 3.4, 3.5 and 3.6, and if  $r^\varepsilon$  belongs to  $\mathbb{H}_\varepsilon^p$ , there exists an unique  $Q^\varepsilon = R^\varepsilon + S^\varepsilon$  and some  $\varepsilon$ -independent constants  $c_0$  such that  $R^\varepsilon$  is a linear operator in  $r^\varepsilon$  and for all  $i \in \{0, \dots, 3\}$ ,

$$\|R^\varepsilon\|_{\varepsilon,p} \leq c_0 \|r^\varepsilon\|_{\varepsilon,p},$$

$$\|S^\varepsilon\|_{\varepsilon,i} \leq c_0 \sum_{i=0}^3 \left( \|U^i\|_{\varepsilon,i} + \|H^i\|_{\varepsilon,i} \right)$$

**Proof:** We consider at first  $R^\varepsilon$ : there exists  $\Phi^\varepsilon \in W^1(\Omega' \cup \omega_\varepsilon)$  such that  $R^\varepsilon = -\nabla \Phi^\varepsilon$  and  $\Phi^\varepsilon$  is solution of:

$$\left\{ \begin{array}{ll} -\Delta \Phi^\varepsilon = -\operatorname{div} r^\varepsilon & \text{in } \omega_\varepsilon, \\ -\Delta \Phi^\varepsilon = 0 & \text{in } \Omega', \\ \frac{\partial \Phi^\varepsilon}{\partial n} = \bar{r}^\varepsilon \cdot n & \text{on } \Gamma_\varepsilon, \\ [\Phi^\varepsilon] = 0, \left[ \frac{\partial \Phi^\varepsilon}{\partial n} \right] = -r^\varepsilon \cdot n & \text{on } \Gamma. \end{array} \right.$$

For all  $\Phi, \Psi \in W^1(\Omega' \cup \omega_\varepsilon)$  we let  $\alpha(\Phi, \Psi) = \int_{\Omega' \cup \omega_\varepsilon} \nabla \Phi \cdot \nabla \Psi \, dx$  and

$$\beta(\Psi) = - \int_{\Omega' \cup \omega_\varepsilon} \operatorname{div} \bar{r}^\varepsilon \Psi \, dx + \int_{\Gamma_\varepsilon} (\bar{r}^\varepsilon \cdot n) \Psi \, d\gamma.$$

$\alpha$  is continuous and coercive in  $W^1(\Omega' \cup \omega_\varepsilon)$  and

$$\begin{aligned} |\beta(\Psi)| &= \int_{\Omega' \cup \omega_\varepsilon} \bar{r}^\varepsilon \cdot \nabla \Psi \, dx \\ &\leq \|\bar{r}^\varepsilon\|_{\mathbb{L}^2(\Omega' \cup \omega_\varepsilon)} \|\nabla \Psi\|_{\mathbb{L}^2(\Omega' \cup \omega_\varepsilon)}. \end{aligned}$$

We then apply the Lax-Milgram theorem to obtain the existence of  $\Phi^\varepsilon \in W^1(\Omega' \cup \omega_\varepsilon)$ . We have  $\|R^\varepsilon\|_{\mathbb{L}^2(\Omega' \cup \omega_\varepsilon)} = \|\Phi^\varepsilon\|_{W^1(\Omega' \cup \omega_\varepsilon)} \leq \|r^\varepsilon\|_{\mathbb{L}^2(\omega_\varepsilon)}$  and then  $|R^\varepsilon|_{\varepsilon,2} \leq |r^\varepsilon|_{\varepsilon,2}$ . To get more regularity for the solution we proceed as in [9, 11, 19]: we seek the equations fulfilled by the derivatives of  $\Phi^\varepsilon$ .

To obtain them we introduce vector fields  $X$  which are normal or tangential to  $\Gamma$  on an open subset of  $\omega_\varepsilon \cup \{x \in \Omega', \varphi(x) < \eta_0\}$  and we prolongate them by zero to the whole domain  $\omega_\varepsilon$  and an exterior neighbourhood of  $\Gamma$  thanks to a bump function. We then define tangential and normal derivatives by  $(X \cdot \nabla)$ , this definition remaining valid in  $\Omega' \cup \omega_\varepsilon$ .

We obtain the equations for the tangential derivatives of  $\Phi^\varepsilon$  by differentiating tangentially the equations on  $\Phi^\varepsilon$ . When we differentiate the equations in the normal direction we multiply by  $\varepsilon$  to take into account the anisotropy of the domain and in the Sobolev spaces and we deduce the boundary conditions for the equation on  $\Phi^\varepsilon$  (if we have enough regularity in the tangential directions). Finally we get  $\|R^\varepsilon\|_{\varepsilon,p} \leq C \|r^\varepsilon\|_{\varepsilon,p}$  where  $C$  is an  $\varepsilon$ -independent constant.

Back to  $S^\varepsilon$  the functions  $f_1$  and  $f_2$  both belong to  $\mathbb{L}_\varepsilon^2$ . There exists then an application  $v \in W^1(\Omega' \cup \omega_\varepsilon)$  such that:

$$\operatorname{div} v = 0, \quad \operatorname{curl} v = f_1 \quad \text{in } \Omega' \cup \omega_\varepsilon \quad \text{and} \quad v \wedge n = 0 \quad \text{on } \Gamma_\varepsilon.$$

Using again the ideas taken from [9, 11, 19] we prove that  $\|v\|_{\varepsilon,p} \leq C \|f_1\|_{\varepsilon,p}$ . We now let  $w = S^\varepsilon - v$ .  $w$  fulfills:

$$\begin{cases} \operatorname{curl} w = 0 & \text{in } \Omega' \cup \omega_\varepsilon, \\ \operatorname{div} w = f_2 & \text{in } \Omega' \cup \omega_\varepsilon, \\ w \cdot n = -v \cdot n & \text{on } \Gamma_\varepsilon. \end{cases}$$

Since  $\operatorname{curl} w = 0$  there exists  $\Phi \in W^1(\Omega' \cup \omega_\varepsilon)$  such that  $w = -\nabla\Phi$  and we have to consider the problem:

$$\begin{cases} -\Delta\Phi = f_2 & \text{in } \Omega' \cup \omega_\varepsilon, \\ \frac{\partial\Phi}{\partial n} = v \cdot n & \text{on } \Gamma_\varepsilon. \end{cases}$$

Let  $\alpha$  and  $\beta$  be the applications:

$$\begin{aligned} \alpha(\Phi, \Psi) &= \int_{\Omega' \cup \omega_\varepsilon} \nabla\Phi \cdot \nabla\Psi \, dx, \\ \beta(\Psi) &= \int_{\Omega' \cup \omega_\varepsilon} f_2 \cdot \Psi \, dx + \int_{\Gamma_\varepsilon} (v \cdot n) \Psi \, d\gamma. \end{aligned}$$

The application  $\alpha$  is continuous and coercive in  $W^1(\Omega' \cup \omega_\varepsilon)$  and

$$\begin{aligned} |\beta(\Psi)| &\leq \|f_2\|_{\mathbb{L}^2(\omega_\varepsilon)} \|\Psi\|_{\mathbb{L}^2(\omega_\varepsilon)} + C \|v \cdot n\|_{\mathbb{H}^{-1/2}(\Gamma_\varepsilon)} \|\Psi\|_{\mathbb{H}^{1/2}(\Gamma_\varepsilon)} \\ &\leq C \|f_2\|_{\mathbb{L}^2(\omega_\varepsilon)} \|\Psi\|_{W^1(\omega_\varepsilon)} + C (\|v\|_{\mathbb{L}^2(\omega_\varepsilon)} + \|\operatorname{div} v\|_{\mathbb{L}^2(\omega_\varepsilon)}) \|\Psi\|_{W^1(\Omega' \cup \omega_\varepsilon)} \\ &\leq C (\|f_2\|_{\mathbb{L}^2(\omega_\varepsilon)} + \|v\|_{\mathbb{L}^2(\omega_\varepsilon)}) \|\Psi\|_{W^1(\Omega' \cup \omega_\varepsilon)}. \end{aligned}$$

We then apply the Lax-Milgram theorem in  $\mathbb{W}^1(\Omega' \cup \omega_\varepsilon)$  and get the existence and uniqueness of  $\Phi$  in  $\mathbb{W}^1(\Omega' \cup \omega_\varepsilon)$ . Moreover  $\Phi$  fulfills:

$$\|\Phi\|_{\mathbb{W}^1(\Omega' \cup \omega_\varepsilon)} = \|w\|_{\mathbb{L}^2(\Omega' \cup \omega_\varepsilon)} \leq C (\|f_2\|_{\mathbb{L}^2(\omega_\varepsilon)} + \|v\|_{\mathbb{L}^2(\omega_\varepsilon)}),$$

so  $|w|_{\varepsilon,2} \leq C(|f_2|_{\varepsilon,2} + |v|_{\varepsilon,2})$ . Thanks to the strategy given in [9, 11, 19] that we use to improve the regularity of  $R^\varepsilon$  we obtain:

$$\|w\|_{\varepsilon,p} \leq C(\|f_2\|_{\varepsilon,p} + \|v\|_{\varepsilon,p})$$

Back to the definition of  $w$  we have:

$$\begin{aligned} \|S^\varepsilon\|_{\varepsilon,p} &\leq C(\|f_2\|_{\varepsilon,p} + \|f_1\|_{\varepsilon,p}) \\ &\leq C \sum_{i=0}^3 (\|U^i\|_{\varepsilon,p} + \|H^i\|_{\varepsilon,p}). \quad \square \end{aligned}$$

## 5.2 Remainder term of $u$

According to lemma 5.1 we let in  $\omega_\varepsilon$

$$a^\varepsilon(t, x) = U^0(t, P(x)) + \varepsilon U^1\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^2 U^2\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^3 U^3\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right),$$

$Q^\varepsilon = R^\varepsilon + S^\varepsilon$  where  $R^\varepsilon$  is linear in  $r^\varepsilon$ ,

$$K^\varepsilon(t, x) = H^0(t, P(x)) + \varepsilon H^1\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^2 H^2\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^3 H^3\left(t, P(x), \frac{\varphi(x)}{\varepsilon}\right) + \varepsilon^3 S^\varepsilon.$$

Back in the equations 3.3 we simplify by using the equations of the profiles. We get:

$$\left\{ \begin{array}{l} \frac{\partial r^\varepsilon}{\partial t} - \varepsilon^2 \Delta r^\varepsilon = T_1 + \dots + T_8 + F^\varepsilon \quad \text{in } \omega_\varepsilon, \\ \frac{\partial r^\varepsilon}{\partial n} = 0 \quad \text{on } \Gamma \cup \Gamma_\varepsilon, \\ r^\varepsilon(0, x) = r_0^\varepsilon(x) \quad \text{in } \omega_\varepsilon, \end{array} \right. \quad (5.2)$$

where

$$\left\{ \begin{array}{l} T_1 = \varepsilon^8 |\nabla r^\varepsilon|^2 r^\varepsilon, \\ T_2 = \varepsilon^5 (|\nabla r^\varepsilon|^2 a^\varepsilon + 2(\nabla r^\varepsilon \cdot \nabla a^\varepsilon) r^\varepsilon), \\ T_3 = \varepsilon^2 (|\nabla a^\varepsilon|^2 r^\varepsilon + 2(\nabla r^\varepsilon \cdot \nabla a^\varepsilon) a^\varepsilon), \\ T_4 = \varepsilon^2 (r^\varepsilon \wedge \Delta a^\varepsilon + a^\varepsilon \wedge \Delta r^\varepsilon + \varepsilon^3 r^\varepsilon \wedge \Delta r^\varepsilon), \\ T_5 = a^\varepsilon \wedge R^\varepsilon + r^\varepsilon \wedge K^\varepsilon + \varepsilon^3 r^\varepsilon \wedge R^\varepsilon, \\ T_6 = -[a^\varepsilon \wedge (a^\varepsilon \wedge R^\varepsilon) + a^\varepsilon \wedge (r^\varepsilon \wedge K^\varepsilon) + r^\varepsilon \wedge (a^\varepsilon \wedge K^\varepsilon)], \\ T_7 = -\varepsilon^3 [r^\varepsilon \wedge (r^\varepsilon \wedge K^\varepsilon) + r^\varepsilon \wedge (a^\varepsilon \wedge R^\varepsilon) + a^\varepsilon \wedge (r^\varepsilon \wedge R^\varepsilon)], \\ T_8 = -\varepsilon^6 r^\varepsilon \wedge (r^\varepsilon \wedge R^\varepsilon), \end{array} \right.$$

and  $F^\varepsilon = \varepsilon A + a^\varepsilon \wedge S^\varepsilon - a^\varepsilon \wedge (a^\varepsilon \wedge S^\varepsilon)$ , where  $A$  is composed of sums and products of profiles as well as their derivatives up to second order and the asymptotic expansion of the differential operators, each one valued in  $\left(P(x), \frac{\varphi(x)}{\varepsilon}\right)$ . The full expression of  $F^\varepsilon$  is given in Annex A.

According to the regularity results we get on the profiles we have the following proposition:

**Proposition 5.2.** *Let  $U^0 \in \mathbb{W}^{7,\infty}(\Gamma)$ . For all  $p$ ,  $1 \leq p \leq +\infty$  and for all  $T > 0$ , there exists some  $\varepsilon$ -independent constants  $C_p$  such that for all  $\varepsilon > 0$ ,  $t \in [0, T]$  and  $i = 0, 1, 2$ ,*

$$\|a^\varepsilon(t, \cdot)\|_{\mathbb{W}_\varepsilon^{i,p}} \leq C_p.$$

*For all  $p$ ,  $2 \leq p < +\infty$  and  $T > 0$  there exists some  $\varepsilon$ -independent constants  $C_p$  such that for all  $\varepsilon > 0$  and  $t \in [0, T]$ ,*

$$\|a^\varepsilon(t, \cdot)\|_{\mathbb{W}_\varepsilon^{3,p}} \leq C_p,$$

*For all  $p$ ,  $1 \leq p \leq +\infty$  and  $T > 0$  there exists some  $\varepsilon$ -independent constants  $C_p$  such that for all  $\varepsilon > 0$  and  $t \in [0, T]$ ,*

$$\|K^\varepsilon(t, \cdot)\|_{\mathbb{W}_\varepsilon^{1,p}} \leq C_p, \quad \|F^\varepsilon(t, \cdot)\|_{\varepsilon,1} \leq C_2.$$

**Proof :** Since  $U_0^0 \in \mathbb{W}^{7,\infty}(\Gamma)$ , the regularity results on the profiles (Th. 3.3, 3.4, 3.5, 3.6) and the classical Sobolev embeddings give for all  $i = 0, 1, 2, 3$ ,

$$\|U^i\|_{\mathbb{W}^{2,\infty}(\Gamma \times I_1)} \leq C,$$

$$\|U^i\|_{\mathbb{W}^{3,p}(\Gamma \times I_1)} \leq C, \quad 2 \leq p < +\infty.$$

Thanks to a rescaling we define the functions on  $\omega_\varepsilon$  and prove the result.

By the same way we prove estimates in  $\mathbb{W}^{1,\infty}(\Gamma \times I_1)$  for  $H^i$  and Lemma 5.1 and 4.5 give the estimates on  $S^\varepsilon$ .

We then easily deduce the estimates on  $F^\varepsilon$  from the previous results.  $\square$

**Remark 5.2.** *We could obtain more precise results since  $U^0$  does not depend on  $z$ :*

$$\begin{aligned} |\nabla a^\varepsilon|_{\varepsilon,p} &\leq C_p, \\ |D^2 a^\varepsilon|_{\varepsilon,p} &\leq C_p \varepsilon^{-1}, \\ |D^3 a^\varepsilon|_{\varepsilon,p} &\leq C_p \varepsilon^{-2}, \\ |\nabla K^\varepsilon|_{\varepsilon,p} &\leq C_p, \end{aligned}$$

but we do not need such estimates in the following. The estimations that follow remain then valid even if  $U_0^0$  depend on  $z$ .

## 6 Estimate on the remainder term

We will now give an estimate on the remainder term  $r^\varepsilon$ . We will then obtain a new existence proof of the magnetic moment  $u^\varepsilon$  but for an initial data with regularity  $\mathbb{W}^{7,\infty}(\Gamma)$  instead of  $\mathbb{H}^2$ . This loss in the regularity is typical from the asymptotic expansion method we use (See also [16, 5, 11, 23, 24]).

Thanks to the formalism given to the differential operators and thanks to the anisotropic lemmas and regularity results (we get the same loss of exponent  $\varepsilon$ ) we can perform the same kind of estimates that we did in [24].

These estimates are performed on a Galerkin approximation which will allow us to justify all integrations by parts. The approximation space we consider for the Landau Lifschitz equation is built on the basis of eigenfunctions of the Laplace's operator with domain

$$\left\{ u \in \mathbb{H}_\varepsilon^2, \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma \cup \Gamma_\varepsilon \right\}.$$

For the magnetic field we only introduce  $R_n$  the projection of  $R(r_n)$  on the approximation space, but we will not detail this step and only concentrate on the estimates.

In the following we let:

$$\begin{aligned} \mathbf{Q}(t) = & |r_n^\varepsilon(t, \cdot)|_{\varepsilon,2}^2 + |\nabla_{\Gamma_z} r_n^\varepsilon(t, \cdot)|_{\varepsilon,2}^2 + |\varepsilon \partial_z r_n^\varepsilon(t, \cdot)|_{\varepsilon,2}^2 \\ & + \varepsilon^2 \left( |\Delta_{\Gamma_z} r_n^\varepsilon(t, \cdot)|_{\varepsilon,2}^2 + |\varepsilon \partial_z \nabla_{\Gamma_z} r_n^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon \nabla_{\Gamma_z} \partial_z r_n^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon^2 \partial_z^2 r_n^\varepsilon(t, \cdot)|_{\varepsilon,2}^2 \right), \end{aligned}$$

and we assume that  $\mathbf{Q}(0) \leq \|r_0^\varepsilon\|_{\varepsilon,1}^2 + \varepsilon^2 \|r_0^\varepsilon\|_{\varepsilon,2}^2$  is bounded.

Since we consider a Galerkin approximation of  $r^\varepsilon$  we get the existence of  $\mathbf{Q}$  on a maximum time interval  $[0, T_{n,\varepsilon}^* [$  where  $0 < T_{n,\varepsilon}^* \leq +\infty$ . We want to prove here the existence of  $\varepsilon_T$  small enough such that we have the existence of  $\mathbf{Q}$  on  $[0, T]$  independently from  $n$  if  $\varepsilon \leq \varepsilon_T$ . We also show the estimates on the main result. We will proceed as it follows:

We let  $T > 0$  and thanks to the energy estimates we obtained by integrating by parts we show that  $\mathbf{Q}$  fulfills on  $[0, T] \cap [0, T_{n,\varepsilon}^* [$  the differential inequality:

$$\mathbf{Q}'(t) \leq C_T(1 + \mathbf{Q}(t)) + \varepsilon P_T(\mathbf{Q}(t)), \quad t \in [0, T] \cap [0, T_{n,\varepsilon}^* [, \quad (6.1)$$

where  $C_T$  is a constant and  $P_T$  a polynomial, both independent from  $n$  and  $\varepsilon$  (but not from  $T$ ). We let now

$$T^\varepsilon = \sup \left\{ T, P(\mathbf{Q}(t)) \leq \frac{1}{\varepsilon^2}, \forall t \in [0, T] \right\}.$$

For all  $t < T^\varepsilon$ ,  $\mathbf{Q}' \leq C(1 + \mathbf{Q})$ , so  $\mathbf{Q}(t) \leq (1 + \mathbf{Q}(0)) \exp(Ct) - 1$  thanks to the Gronwall lemma. We deduce then that  $P(\mathbf{Q}(t)) \leq \frac{1}{\varepsilon^2}$  for all  $t \leq a \ln \frac{1}{\varepsilon} + b$ . Finally we have that  $\lim_{\varepsilon \rightarrow 0} T_\varepsilon = +\infty$  which implies  $T_{n,\varepsilon}^* \geq T$  as soon as  $\varepsilon \in ]0, \varepsilon_T]$  where  $\varepsilon_T$  is a small enough constant. Moreover  $r^\varepsilon$  is bounded independently from  $\varepsilon$  in  $\mathbb{L}^\infty(0, T; \mathbb{H}_\varepsilon^1)$  as well as  $\varepsilon r^\varepsilon$  in  $\mathbb{L}^\infty(0, T; \mathbb{H}_\varepsilon^2)$  for all  $0 < T < T_\varepsilon$ . We then use the anisotropic Sobolev embeddings and the renormalization in the Lebesgue spaces to obtain the main result.



**Remark 6.1.** In the following inequalities the  $\nabla$  and  $\Delta$  operators are the usual operators. When we estimate them in the anisotropic Sobolev spaces we lose some powers of  $\varepsilon$ :

$$\begin{aligned}\varepsilon |\nabla u|_{\varepsilon,p} &\leq \left( |\nabla_x u|_{\varepsilon,p} + |\varepsilon \partial_z u|_{\varepsilon,p} \right), \\ \varepsilon^2 |\Delta u|_{\varepsilon,p} &\leq \left( |\Delta_x u|_{\varepsilon,p} + |\varepsilon^2 \partial_z^2 u|_{\varepsilon,p} \right).\end{aligned}$$

We also have to consider commutators between the differential operators since the tangential differential operators depend on  $z$ . These commutators are differential operators of order one less in the normal direction. For example in the case of the gradient,

$$[\partial_z, \nabla_{\Gamma_z}]u = -(I + zdn)^{-1} dn \nabla_{\Gamma_z} u.$$

These commutators appear in the second member and in the dissipative term of the following estimates. In both cases they generate tangential derivatives terms that are easy to estimate or to absorb.

To simplify the estimates we then proceed as if the differential operators commute. We also note  $D_z$  the operator  $n\partial_z$  and  $D_z^2$  the operator  $\frac{1}{\gamma_z}\partial_z(\gamma_z\partial_z)$ .

$\mathbb{L}^2$  **estimates:** we take the scalar product of (5.2) with  $\frac{1}{|\omega_\varepsilon|}r^\varepsilon$  and we integrate on  $\omega_\varepsilon$  (so that the norm  $|\cdot|_{\varepsilon,2}$  appears). The integrations by parts are licit since we work with a Galerkin approximate. We get:

$$\frac{1}{2} \frac{d}{dt} \left( |r^\varepsilon|_{\varepsilon,2}^2 \right) + \varepsilon^2 |\nabla r^\varepsilon|_{\varepsilon,2}^2 \leq \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} (T_1 + \dots + T_8 + F^\varepsilon) \cdot r^\varepsilon dx.$$

We estimate each  $\frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} T_i r^\varepsilon dx$  thanks to Hölder's inequality and the anisotropic Sobolev embeddings:

$$\begin{aligned}\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_1 r^\varepsilon dx \right| &\leq \varepsilon^8 |r^\varepsilon|_{\varepsilon,\infty}^2 |\nabla r^\varepsilon|_{\varepsilon,2}^2 \\ &\leq C \varepsilon^{9/2} |r^\varepsilon|_{\varepsilon,2}^{1/2} \|r^\varepsilon\|_{\varepsilon,1}^2 \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^{3/2} \leq C \varepsilon^{9/2} \mathbf{Q}^2 \\ &\quad \text{(thanks to Ineq. (4.5) and Rem. 6.1)}\end{aligned}$$

$$\begin{aligned}\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_2 r^\varepsilon dx \right| &\leq \varepsilon^5 |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,3} |\nabla r^\varepsilon|_{\varepsilon,3}^2 + 2\varepsilon^5 |r^\varepsilon|_{\varepsilon,3}^2 |\nabla a^\varepsilon|_{\varepsilon,\infty} |\nabla r^\varepsilon|_{\varepsilon,3} \\ &\leq C \varepsilon^2 |r^\varepsilon|_{\varepsilon,2}^{1/2} \|r^\varepsilon\|_{\varepsilon,1}^{3/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2} + C \varepsilon^{5/2} |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1}^{3/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^{1/2} \\ &\quad \text{(thanks to Prop. 5.2, Ineq. (4.4), (4.9) and Rem. 6.1)} \\ &\leq C(\varepsilon^2 + \varepsilon^{5/2}) \mathbf{Q}^{3/2}\end{aligned}$$

$$\begin{aligned}\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_3 r^\varepsilon dx \right| &\leq \varepsilon^2 |r^\varepsilon|_{\varepsilon,2}^2 |\nabla a^\varepsilon|_{\varepsilon,\infty}^2 + 2\varepsilon^2 |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} |\nabla r^\varepsilon|_{\varepsilon,2} |\nabla a^\varepsilon|_{\varepsilon,\infty} \\ &\leq C |r^\varepsilon|_{\varepsilon,2}^2 + C |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \leq C \mathbf{Q} \\ &\quad \text{(thanks to Prop. 5.2 and Rem. 6.1)}\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_4 r^\varepsilon dx \right| &\leq \varepsilon^2 |r^\varepsilon|_{\varepsilon,2} |\nabla a^\varepsilon|_{\varepsilon,\infty} |\nabla r^\varepsilon|_{\varepsilon,2} \\
&\leq C |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \leq C \mathbf{Q} \\
&\text{(thanks to Prop. 5.2 and Rem. 6.1)}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_5 r^\varepsilon dx \right| &\leq |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} |R^\varepsilon|_{\varepsilon,2} \\
&\leq C |r^\varepsilon|_{\varepsilon,2}^2 \leq C \mathbf{Q} \\
&\text{(thanks to Prop. 5.2 and Lemma 5.1)}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_6 r^\varepsilon dx \right| &\leq |a^\varepsilon|_{\varepsilon,\infty}^2 |r^\varepsilon|_{\varepsilon,2} |R^\varepsilon|_{\varepsilon,2} + |a^\varepsilon|_{\varepsilon,\infty} |K^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2}^2 \\
&\leq C |r^\varepsilon|_{\varepsilon,2}^2 \leq C \mathbf{Q} \\
&\text{(thanks to Prop. 5.2 and Lemma 5.1)}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_7 r^\varepsilon dx \right| &\leq C \varepsilon^3 |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,4}^2 |R^\varepsilon|_{\varepsilon,2} \\
&\leq C \varepsilon^3 |r^\varepsilon|_{\varepsilon,2}^{3/2} \|r^\varepsilon\|_{\varepsilon,1}^{3/2} \leq C \varepsilon^3 \mathbf{Q}^{3/2} \\
&\text{(thanks to Prop. 5.2, Ineq. (4.4) and Lemma 5.1)}
\end{aligned}$$

$$\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_8 r^\varepsilon dx \right| = 0$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} F^\varepsilon r^\varepsilon dx \right| &\leq |F^\varepsilon|_{\varepsilon,2} |r^\varepsilon|_{\varepsilon,2} \\
&\leq C(1 + |r^\varepsilon|_{\varepsilon,2}^2) \\
&\text{(thanks to Prop. 5.2 and Young's Inequality)}
\end{aligned}$$

Adding the previous inequalities we obtain the existence of a constant  $C$  and a polynomial  $P$  both independent from  $\varepsilon$  such that:

$$\frac{d}{dt} |r^\varepsilon|_{\varepsilon,2}^2 + \varepsilon^2 |\nabla r^\varepsilon|_{\varepsilon,2}^2 \leq C(1 + \mathbf{Q}) + \varepsilon^2 P(\mathbf{Q}), \tag{6.2}$$

$\mathbb{H}^1$  **estimates:** Taking the scalar product of (5.2) with

$$\frac{1}{|\omega_\varepsilon|} \left( \Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 \frac{1}{\gamma_z} \partial_z (\gamma_z \partial_z r^\varepsilon) \right) = \frac{1}{|\omega_\varepsilon|} (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon)$$

and integrating by parts, it follows:

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon D_z r^\varepsilon|_{\varepsilon,2}^2 \right) &+ \varepsilon^2 |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 + \varepsilon^2 (1 + \varepsilon^2) |\nabla_{\Gamma_z} D_z r^\varepsilon|_{\varepsilon,2}^2 \\
&\leq \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} (T_1 + \dots + T_8 + F^\varepsilon) \cdot (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) dx
\end{aligned}$$

and we estimate each term in the second member in the following way:

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_1 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) dx \right| &\leq \varepsilon^8 |r^\varepsilon|_{\varepsilon,6} |\nabla r^\varepsilon|_{\varepsilon,6}^2 \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2} \right) \\
&\leq C \varepsilon^3 \|r^\varepsilon\|_{\varepsilon,1} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^3 \leq C \varepsilon^3 \mathbf{Q}^2 \\
&\quad \text{(thanks to Ineq. (4.4), (4.9) and Rem. 6.1)} \\
\\
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_2 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) dx \right| &\leq \varepsilon^5 |a^\varepsilon|_{\varepsilon,\infty} |\nabla r^\varepsilon|_{\varepsilon,4}^2 \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2} \right) \\
&\quad + 2\varepsilon^5 |r^\varepsilon|_{\varepsilon,6} |\nabla r^\varepsilon|_{\varepsilon,3} |\nabla a^\varepsilon|_{\varepsilon,\infty} \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2} \right) \\
&\leq C \varepsilon^{1/2} \|r^\varepsilon\|_{\varepsilon,1}^{1/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^{5/2} + C \varepsilon^{3/2} \|r^\varepsilon\|_{\varepsilon,1}^{3/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^{3/2} \\
&\quad \text{(thanks to Prop. 5.2, Ineq. (4.4), (4.9) and Rem. 6.1)} \\
&\leq C (\varepsilon^{1/2} + \varepsilon^{3/2}) \mathbf{Q}^{3/2} \\
\\
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_3 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) dx \right| &\leq \varepsilon^2 |\nabla a^\varepsilon|_{\varepsilon,\infty}^2 \|r^\varepsilon\|_{\varepsilon,1}^2 \\
&\quad + 2\varepsilon^2 |a^\varepsilon|_{\varepsilon,\infty} \left( |\nabla_{\Gamma_z} \nabla a^\varepsilon|_{\varepsilon,\infty} + |\varepsilon D_z \nabla a^\varepsilon|_{\varepsilon,\infty} \right) |r^\varepsilon|_{\varepsilon,2} \times \\
&\quad \|r^\varepsilon\|_{\varepsilon,1} + 2\varepsilon^2 |\nabla a^\varepsilon|_{\varepsilon,\infty} \left( |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon,\infty} + |\varepsilon D_z a^\varepsilon|_{\varepsilon,\infty} \right) \times \\
&\quad |\nabla r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} + 2\varepsilon^2 |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \times \\
&\quad \left( |\nabla_{\Gamma_z} \nabla a^\varepsilon|_{\varepsilon,\infty} + |\varepsilon D_z \nabla a^\varepsilon|_{\varepsilon,\infty} \right) \\
&\quad + 2\varepsilon^2 |a^\varepsilon|_{\varepsilon,\infty} |\nabla a^\varepsilon|_{\varepsilon,\infty} \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} \right. \\
&\quad \left. + |D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon D_z^2 r^\varepsilon|_{\varepsilon,2} \right) \|r^\varepsilon\|_{\varepsilon,1} \\
&\leq C \left( \|r^\varepsilon\|_{\varepsilon,1}^2 + |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \right) + C \|r^\varepsilon\|_{\varepsilon,1} \times \\
&\quad \left( \varepsilon |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + (1 + \varepsilon) |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2} \right) \\
&\quad \text{(thanks to Prop. 5.2 and Rem. 6.1)} \\
&\leq C \mathbf{Q} + \frac{1}{18} \left( \varepsilon^2 |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + (1 + \varepsilon^2) |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 \right. \\
&\quad \left. + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 \right)
\end{aligned}$$

To estimate  $T_4$  we look more precisely the behaviour of the terms with  $\Delta r^\varepsilon$  thanks to integrations by parts (which are licit thanks to the splitting between the two directions):

$$\begin{aligned}
A_1 &= \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \varepsilon^2 a^\varepsilon \wedge \Delta r^\varepsilon \cdot (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) dx \\
&= \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \varepsilon^2 (a^\varepsilon \wedge \Delta_{\Gamma_z} r^\varepsilon \cdot \varepsilon^2 D_z^2 r^\varepsilon + a^\varepsilon \wedge D_z^2 r^\varepsilon \cdot \Delta_{\Gamma_z} r^\varepsilon) dx \\
&= \frac{1 - \varepsilon^2}{|\omega_\varepsilon|} \left( \int_{\omega_\varepsilon} -\varepsilon^2 \nabla_{\Gamma_z} a^\varepsilon \wedge D_z^2 r^\varepsilon \cdot \nabla_{\Gamma_z} r^\varepsilon dx + \int_{\omega_\varepsilon} \varepsilon^2 D_z a^\varepsilon \wedge D_z \nabla_{\Gamma_z} r^\varepsilon \cdot \nabla_{\Gamma_z} r^\varepsilon dx \right) \\
&\leq |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon,\infty} \|r^\varepsilon\|_{\varepsilon,1} |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2} + |\varepsilon D_z a^\varepsilon|_{\varepsilon,\infty} \|r^\varepsilon\|_{\varepsilon,1} |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} \\
\\
&\leq C \|r^\varepsilon\|_{\varepsilon,1}^2 + \frac{\varepsilon^2}{18} \left( |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 \right)
\end{aligned}$$

$$\begin{aligned}
A_2 &= \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \varepsilon^2 r^\varepsilon \wedge \Delta r^\varepsilon \cdot (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) \, dx \\
&= \frac{1}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \varepsilon^5 (r^\varepsilon \wedge \Delta_{\Gamma_z} r^\varepsilon \cdot \varepsilon^2 D_z^2 r^\varepsilon + r^\varepsilon \wedge D_z^2 r^\varepsilon \cdot \Delta_{\Gamma_z} r^\varepsilon) \, dx \\
&= \frac{1 - \varepsilon^2}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \varepsilon^5 \nabla_{\Gamma_z} r^\varepsilon \wedge D_z r^\varepsilon \cdot D_z \nabla_{\Gamma_z} r^\varepsilon \, dx \\
&\leq C \varepsilon^3 |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,4} |\varepsilon D_z r^\varepsilon|_{\varepsilon,4} |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}.
\end{aligned}$$

Back to the estimate of  $T_4$  we obtain:

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_4 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) \, dx \right| &\leq \varepsilon^2 |\Delta a^\varepsilon|_{\varepsilon,\infty} \|r^\varepsilon\|_{\varepsilon,1}^2 \\
&\quad + \varepsilon^2 \left( |\nabla_{\Gamma_z} \Delta a^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z \Delta a^\varepsilon|_{\varepsilon,3} \right) |r^\varepsilon|_{\varepsilon,6} \|r^\varepsilon\|_{\varepsilon,1} \\
&\leq C \|r^\varepsilon\|_{\varepsilon,1}^2 + C \varepsilon^{1/2} \|r^\varepsilon\|_{\varepsilon,1}^{1/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^{3/2} \left( \varepsilon |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} \right) \\
&\quad + \frac{1}{18} \left( |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 \right) \\
&\quad \text{(thanks to Prop. 5.2, Ineq. (4.4) and Rem. 6.1)} \\
&\leq C \mathbf{Q} + C \varepsilon^{1/2} \mathbf{Q}^{3/2} \\
&\quad + \frac{1}{18} \left( |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 \right)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_5 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) \, dx \right| &\leq \|a^\varepsilon\|_{\mathbb{W}_\varepsilon^{1,\infty}} |R^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} + |a^\varepsilon|_{\varepsilon,\infty} \|R^\varepsilon\|_{\varepsilon,1} \|r^\varepsilon\|_{\varepsilon,1} \\
&\quad + \|K^\varepsilon\|_{\mathbb{W}_\varepsilon^{1,\infty}} |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \\
&\quad + \varepsilon^3 |r^\varepsilon|_{\varepsilon,6} \|R^\varepsilon\|_{\varepsilon,1} \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z r^\varepsilon|_{\varepsilon,3} \right) \\
&\leq C \|r^\varepsilon\|_{\varepsilon,1}^2 + C \varepsilon^{5/2} \|r^\varepsilon\|_{\varepsilon,1}^{5/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^{1/2} \\
&\quad \text{(thanks to Prop. 5.2, Lemma 5.1 and Ineq. 4.4)} \\
&\leq C \mathbf{Q} + C \varepsilon^{5/2} \mathbf{Q}^{3/2}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_6 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) \, dx \right| &\leq 2 \|a^\varepsilon\|_{\mathbb{W}_\varepsilon^{1,\infty}} |a^\varepsilon|_{\varepsilon,\infty} |R^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \\
&\quad + |a^\varepsilon|_{\varepsilon,\infty}^2 \|R^\varepsilon\|_{\varepsilon,1} \|r^\varepsilon\|_{\varepsilon,1} \\
&\quad + 2 \|a^\varepsilon\|_{\mathbb{W}_\varepsilon^{1,\infty}} |K^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \\
&\quad + 2 \|K^\varepsilon\|_{\mathbb{W}_\varepsilon^{1,\infty}} |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} \|r^\varepsilon\|_{\varepsilon,1} \\
&\quad + |a^\varepsilon|_{\varepsilon,\infty} |K^\varepsilon|_{\varepsilon,\infty} \|r^\varepsilon\|_{\varepsilon,1}^2 \\
&\leq C (|r^\varepsilon|_{\varepsilon,2} + \|r^\varepsilon\|_{\varepsilon,1}) \|r^\varepsilon\|_{\varepsilon,1} \leq C \mathbf{Q} \\
&\quad \text{(thanks to Prop. 5.2 and Lemma 5.1)}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_7 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) \, dx \right| &\leq \varepsilon^3 \left( |r^\varepsilon|_{\varepsilon,4}^2 |K^\varepsilon|_{\varepsilon,\infty} + 2 |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,4} |R^\varepsilon|_{\varepsilon,4} \right) \\
&\quad \times \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2} \right) \\
&\leq C \varepsilon^2 |r^\varepsilon|_{\varepsilon,2}^{1/2} \|r^\varepsilon\|_{\varepsilon,1}^{3/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2} \leq C \varepsilon^2 \mathbf{Q}^{3/2} \\
&\quad \text{(thanks to Prop. 5.2, Lemma 5.1 and Ineq. (4.4))}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} T_8 (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) dx \right| &\leq \varepsilon^6 |r^\varepsilon|_{\varepsilon,6}^2 |R^\varepsilon|_{\varepsilon,6} \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2} \right) \\
&\leq C \varepsilon^5 \|r^\varepsilon\|_{\varepsilon,1}^3 \|\varepsilon r^\varepsilon\|_{\varepsilon,2} \leq C \varepsilon^5 \mathbf{Q}^2 \\
&\text{(thanks to Lemma 5.1 and Ineq. (4.4))}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} F^\varepsilon (\Delta_{\Gamma_z} r^\varepsilon + \varepsilon^2 D_z^2 r^\varepsilon) dx \right| &\leq \|F^\varepsilon\|_{\varepsilon,1} \|r^\varepsilon\|_{\varepsilon,1} \\
&\leq C(1 + \mathbf{Q}) \\
&\text{(thanks to Prop. 5.2 and Young's inequality)}
\end{aligned}$$

Adding the previous inequalities we obtain by absorbing the terms that appeared in the estimates of  $T_4$ :

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon D_z r^\varepsilon|_{\varepsilon,2}^2 \right) + \frac{1}{2} \left( \varepsilon^2 |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 \right. \\
\left. + (1 + \varepsilon^2) |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 \right) \leq C(1 + \mathbf{Q}) + \varepsilon^2 P(\mathbf{Q}),
\end{aligned} \tag{6.3}$$

where  $C$  is a constant and  $P$  a polynomial, both independent from  $\varepsilon$ .

**$\mathbb{H}^2$  estimates:** We now estimate  $\varepsilon^2 \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 \right)$  by taking the scalar product of (5.2) with  $\frac{\varepsilon^2}{|\omega_\varepsilon|} (\Delta_{\Gamma_z}^2 + \varepsilon^2 D_z^2 \Delta_{\Gamma_z} + \varepsilon^4 D_z^4) r^\varepsilon = \frac{\varepsilon^2}{|\omega_\varepsilon|} \mathbf{B}$ . For each term we integrate at first by parts, we then use Hölder's inequality, the anisotropic Sobolev embeddings and finally Young's inequality to isolate the term

$$\varepsilon^4 \mathbf{A}^2 = \varepsilon^4 |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + \varepsilon^2 |\varepsilon D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + \varepsilon^2 |\varepsilon^2 D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + \varepsilon^2 |\varepsilon^3 D_z^3 r^\varepsilon|_{\varepsilon,2}^2.$$

We have:

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_1 \mathbf{B} dx \right| &\leq \varepsilon^{10} |\nabla r^\varepsilon|_{\varepsilon,6}^2 \left( 2 |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,6} + |\varepsilon D_z r^\varepsilon|_{\varepsilon,6} \right) \mathbf{A} \\
&\quad + \varepsilon^{10} |r^\varepsilon|_{\varepsilon,12} \left( 2 |\nabla_{\Gamma_z} \nabla r^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z \nabla r^\varepsilon|_{\varepsilon,3} \right) |\nabla r^\varepsilon|_{\varepsilon,12} \mathbf{A} \\
&\leq C \varepsilon^8 \|r^\varepsilon\|_{\varepsilon,1}^{3/4} \|r^\varepsilon\|_{\varepsilon,2}^{3/2} \left( \|r^\varepsilon\|_{\varepsilon,2}^{1/4} + \mathbf{A}^{1/4} \right) \left( \|r^\varepsilon\|_{\varepsilon,2}^{1/2} + \mathbf{A}^{1/2} \right) \mathbf{A} \\
&\quad + C \varepsilon^5 \mathbf{A} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^3 \\
&\text{(thanks to Ineq. (4.4), (4.6), (4.9), (4.10), and Rem. 6.1)} \\
&\leq \frac{1}{18} \varepsilon^4 \mathbf{A}^2 + C \varepsilon^6 \mathbf{Q}^3 + C \varepsilon^{28/3} \mathbf{Q}^{11/3} + C \varepsilon^{13} \mathbf{Q}^5 + C \varepsilon^{24} \mathbf{Q}^9
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_2 \mathbf{B} dx \right| &\leq \varepsilon^7 \left( 2 |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} + |\varepsilon D_z a^\varepsilon|_{\varepsilon, \infty} \right) |\nabla r^\varepsilon|_{\varepsilon, 4}^2 \mathbf{A} \\
&\quad + 2\varepsilon^7 |a^\varepsilon|_{\varepsilon, \infty} |\nabla r^\varepsilon|_{\varepsilon, 6} \left( 2 |\nabla_{\Gamma_z} \nabla r^\varepsilon|_{\varepsilon, 3} + |\varepsilon D_z \nabla r^\varepsilon|_{\varepsilon, 3} \right) \mathbf{A} \\
&\quad + 2\varepsilon^7 |\nabla a^\varepsilon|_{\varepsilon, \infty} |\nabla r^\varepsilon|_{\varepsilon, 4} \left( 2 |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 4} + |\varepsilon D_z r^\varepsilon|_{\varepsilon, 4} \right) \mathbf{A} \\
&\quad + 2\varepsilon^7 \left( 2 |\nabla_{\Gamma_z} \nabla a^\varepsilon|_{\varepsilon, \infty} + |\varepsilon D_z a^\varepsilon|_{\varepsilon, \infty} \right) |r^\varepsilon|_{\varepsilon, 6} |\nabla r^\varepsilon|_{\varepsilon, 3} \mathbf{A} \\
&\quad + 2\varepsilon^7 |\nabla a^\varepsilon|_{\varepsilon, \infty} |r^\varepsilon|_{\varepsilon, 6} \left( 2 |\nabla_{\Gamma_z} \nabla r^\varepsilon|_{\varepsilon, 3} + |\varepsilon D_z \nabla r^\varepsilon|_{\varepsilon, 3} \right) \mathbf{A} \\
&\leq C\varepsilon^{7/2} \|r^\varepsilon\|_{\varepsilon, 1}^{1/2} \|\varepsilon r^\varepsilon\|_{\varepsilon, 2}^{3/2} \mathbf{A} + C\varepsilon^5 \|r^\varepsilon\|_{\varepsilon, 2}^{3/2} \left( \|r^\varepsilon\|_{\varepsilon, 2}^{1/2} + \mathbf{A}^{1/2} \right) \mathbf{A} \\
&\quad + C\varepsilon^{7/2} \|r^\varepsilon\|_{\varepsilon, 1}^{1/2} \|\varepsilon r^\varepsilon\|_{\varepsilon, 2}^{3/2} \mathbf{A} + C\varepsilon^{9/2} \|r^\varepsilon\|_{\varepsilon, 1}^{3/2} \|\varepsilon r^\varepsilon\|_{\varepsilon, 2}^{1/2} \mathbf{A} \\
&\quad + C\varepsilon^5 \|r^\varepsilon\|_{\varepsilon, 1} \|r^\varepsilon\|_{\varepsilon, 2}^{1/2} \left( \|r^\varepsilon\|_{\varepsilon, 2}^{1/2} + \mathbf{A}^{1/2} \right) \mathbf{A} \\
&\quad \text{(thanks to Ineq. (4.4), (4.9), (4.5), Prop. 5.2 and Rem. 6.1)} \\
&\leq \frac{1}{18} \varepsilon^4 \mathbf{A}^2 + C\varepsilon^2 \mathbf{Q}^2 + C\varepsilon^2 \mathbf{Q}^3
\end{aligned}$$

To obtain the estimates on  $T_3$  we integrate several times by parts the term:

$$\begin{aligned}
A &= \frac{2\varepsilon^4}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} (D_z a^\varepsilon \cdot D_z \nabla_{\Gamma_z} r^\varepsilon) (a^\varepsilon \cdot \nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon) dx \\
&\leq 2\varepsilon^4 |D_z^2 a^\varepsilon|_{\varepsilon, \infty} |a^\varepsilon|_{\varepsilon, \infty} |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + 2\varepsilon^4 |D_z a^\varepsilon|_{\varepsilon, \infty}^2 |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + 2\varepsilon^4 |a^\varepsilon|_{\varepsilon, \infty} |D_z \nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + 2\varepsilon^4 |D_z a^\varepsilon|_{\varepsilon, \infty} |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + 2\varepsilon^4 |D_z a^\varepsilon|_{\varepsilon, \infty} |a^\varepsilon|_{\varepsilon, \infty} |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\leq C\mathbf{Q}^{1/2} (\varepsilon^2 \mathbf{A})
\end{aligned}$$

Back in the estimate of  $T_3$ ,

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_3 \mathbf{B} dx \right| &\leq \varepsilon^4 |\nabla a^\varepsilon|_{\varepsilon, \infty}^2 \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} + |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\varepsilon^2 D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \right. \\
&\quad \left. + |\varepsilon D_z r^\varepsilon|_{\varepsilon, 2} |\varepsilon^3 D_z^3 r^\varepsilon|_{\varepsilon, 2} \right) + 2\varepsilon^4 |\nabla a^\varepsilon|_{\varepsilon, \infty} |r^\varepsilon|_{\varepsilon, 2} \times \\
&\quad \left( |\nabla_{\Gamma_z} \nabla a^\varepsilon|_{\varepsilon, \infty} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} + |\nabla_{\Gamma_z} \nabla a^\varepsilon|_{\varepsilon, \infty} |\varepsilon^2 D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \right. \\
&\quad \left. + |\varepsilon D_z \nabla a^\varepsilon|_{\varepsilon, \infty} |\varepsilon^3 D_z^3 r^\varepsilon|_{\varepsilon, 2} \right) + 2\varepsilon^4 |\nabla a^\varepsilon|_{\varepsilon, \infty} |\nabla r^\varepsilon|_{\varepsilon, 2} \times \\
&\quad \left( |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} + |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |\varepsilon^2 D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \right. \\
&\quad \left. + |\varepsilon D_z a^\varepsilon|_{\varepsilon, \infty} |\varepsilon^3 D_z^3 r^\varepsilon|_{\varepsilon, 2} \right) + 2\varepsilon^4 |a^\varepsilon|_{\varepsilon, \infty} |\nabla r^\varepsilon|_{\varepsilon, 2} \times \\
&\quad \left( |\nabla_{\Gamma_z} \nabla a^\varepsilon|_{\varepsilon, \infty} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} + |\nabla_{\Gamma_z} \nabla a^\varepsilon|_{\varepsilon, \infty} |\varepsilon^2 D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \right. \\
&\quad \left. + |\varepsilon D_z \nabla a^\varepsilon|_{\varepsilon, \infty} |\varepsilon^3 D_z^3 r^\varepsilon|_{\varepsilon, 2} \right) + 2\varepsilon^4 |a^\varepsilon|_{\varepsilon, \infty} |\nabla a^\varepsilon|_{\varepsilon, \infty} \times \\
&\quad \left( |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} + |\nabla_{\Gamma_z} \nabla r^\varepsilon|_{\varepsilon, 2} |\varepsilon^2 D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \right. \\
&\quad \left. + |\varepsilon D_z \nabla r^\varepsilon|_{\varepsilon, 2} |\varepsilon^3 D_z^3 r^\varepsilon|_{\varepsilon, 2} \right) + A \\
&\leq C(\|r^\varepsilon\|_{\varepsilon, 1} + \|\varepsilon r^\varepsilon\|_{\varepsilon, 2}) (\varepsilon^2 \mathbf{A}) \\
&\quad \text{(thanks to Ineq. (4.4), (4.9), Prop. 5.2 and Rem. 6.1)} \\
&\leq \frac{1}{18} \varepsilon^4 \mathbf{A}^4 + C\mathbf{Q}
\end{aligned}$$

To obtain the estimates on  $T_4$  we integrate several times by parts the term:

$$\begin{aligned}
A &= \frac{\varepsilon^4}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} r^\varepsilon \wedge \Delta a^\varepsilon \cdot (\Delta_{\Gamma_z}^2 r^\varepsilon + \varepsilon^2 D_z^2 \Delta_{\Gamma_z} r^\varepsilon + \varepsilon^4 D_z^4 r^\varepsilon) dx \\
&\leq |\Delta a^\varepsilon|_{\varepsilon, \infty} \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} + |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\varepsilon^2 D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \right. \\
&\quad \left. + |\varepsilon D_z r^\varepsilon|_{\varepsilon, 2} |\varepsilon^3 D_z^3 r^\varepsilon|_{\varepsilon, 2} \right) \\
B &= \frac{\varepsilon^4}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge (\Delta_{\Gamma_z} r^\varepsilon + D_z^2 r^\varepsilon) \cdot (\Delta^2 r^\varepsilon + \varepsilon^2 D_z^2 \Delta_{\Gamma_z} r^\varepsilon + \varepsilon^4 D_z^4 r^\varepsilon) dx \\
&= B_1 + B_2 + B_3 + B_4 + B_5 + B_6,
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= \frac{\varepsilon^4}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge \Delta_{\Gamma_z} r^\varepsilon \cdot \Delta_{\Gamma_z}^2 r^\varepsilon dx \leq \varepsilon^4 |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\leq \frac{1}{200} \varepsilon^4 \mathbf{A}^2 + C \varepsilon^2 \|\varepsilon r^\varepsilon\|_{\varepsilon, 2}^2 \\
B_2 &= \frac{\varepsilon^4}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge D_z^2 r^\varepsilon \cdot \Delta_{\Gamma_z}^2 r^\varepsilon dx = \frac{\varepsilon^4}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} \nabla_{\Gamma_z} a^\varepsilon \wedge D_z^2 r^\varepsilon \cdot \nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon dx \\
&\quad + \frac{\varepsilon^4}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge D_z^2 \nabla_{\Gamma_z} r^\varepsilon \cdot \nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon dx \\
&\leq \varepsilon^4 |D_z \nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |D_z r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + \varepsilon^4 |\Delta_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |D_z r^\varepsilon|_{\varepsilon, 2} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + \varepsilon^4 |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + \varepsilon^4 |D_z a^\varepsilon|_{\varepsilon, \infty} |D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\quad + \varepsilon^4 |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\leq \frac{1}{200} \varepsilon^4 \mathbf{A}^2 + C \mathbf{Q} \\
B_3 &= \frac{\varepsilon^6}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge \Delta_{\Gamma_z} r^\varepsilon \cdot D_z^2 \Delta_{\Gamma_z} r^\varepsilon dx \leq \varepsilon^6 |D_z a^\varepsilon|_{\varepsilon, \infty} |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\leq \frac{1}{200} \varepsilon^4 \mathbf{A}^2 + C \varepsilon^4 \mathbf{Q} \\
B_4 &= \frac{\varepsilon^6}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge D_z^2 r^\varepsilon \cdot D_z^2 \Delta_{\Gamma_z} r^\varepsilon dx \leq |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |D_z^2 r^\varepsilon|_{\varepsilon, 2} |D_z^2 \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} \\
&\leq \frac{1}{200} \varepsilon^4 \mathbf{A}^2 + C \mathbf{Q} \\
B_5 &= \frac{\varepsilon^8}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge \Delta_{\Gamma_z} r^\varepsilon \cdot D_z^4 r^\varepsilon dx \leq \varepsilon^8 |D_z a^\varepsilon|_{\varepsilon, \infty} |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z^3 r^\varepsilon|_{\varepsilon, 2} \\
&\quad + \varepsilon^8 |\nabla_{\Gamma_z} a^\varepsilon|_{\varepsilon, \infty} |D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |D_z^3 r^\varepsilon|_{\varepsilon, 2} \\
&\quad + \varepsilon^8 |D_z a^\varepsilon|_{\varepsilon, \infty} |D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon, 2} |\nabla_{\Gamma_z} D_z^2 r^\varepsilon|_{\varepsilon, 2} \\
&\leq \frac{1}{200} \varepsilon^4 \mathbf{A}^2 + C \varepsilon^4 \mathbf{Q} \\
B_6 &= \frac{\varepsilon^8}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} a^\varepsilon \wedge D_z^2 r^\varepsilon \cdot D_z^4 r^\varepsilon dx \leq \varepsilon^8 |D_z a^\varepsilon|_{\varepsilon, \infty} |D_z^2 r^\varepsilon|_{\varepsilon, 2} |D_z^3 r^\varepsilon|_{\varepsilon, 2} \\
&\leq \frac{1}{200} \mathbf{A} + C \varepsilon^2 \|\varepsilon r^\varepsilon\|_{\varepsilon, 2}^2.
\end{aligned}$$

In the same way

$$\begin{aligned}
C &= \frac{\varepsilon^7}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} r^\varepsilon \wedge (\Delta_{\Gamma_z} r^\varepsilon + D_z^2 r^\varepsilon) \cdot (\Delta^2 r^\varepsilon + \varepsilon^4 D_z^4 r^\varepsilon) dx \\
&\leq 2\varepsilon^7 \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,6} + |\varepsilon D_z r^\varepsilon|_{\varepsilon,6} \right) |\Delta r^\varepsilon|_{\varepsilon,3} \mathbf{A} + \varepsilon^7 |D_z r^\varepsilon|_{\varepsilon,6} |\nabla_{\Gamma_z} D_z r^\varepsilon|_{\varepsilon,3} |\nabla_{\Gamma_z} \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} \\
&\quad + \varepsilon^7 |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,6} |\nabla_{\Gamma_z} D_z r^\varepsilon|_{\varepsilon,3} |D_z \Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2} + \varepsilon^{11} |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,6} |\nabla_{\Gamma_z} D_z r^\varepsilon|_{\varepsilon,3} |D_z^3 r^\varepsilon|_{\varepsilon,2} \\
&\quad + \varepsilon^{11} |D_z r^\varepsilon|_{\varepsilon,6} |\nabla_{\Gamma_z} D_z r^\varepsilon|_{\varepsilon,3} |\nabla_{\Gamma_z} D_z^2 r^\varepsilon|_{\varepsilon,2} \\
&\leq C\varepsilon^3 \mathbf{Q} \mathbf{A} + C\varepsilon^{7/2} \mathbf{Q}^{3/4} \mathbf{A}^{3/2} \leq \frac{1}{200} \varepsilon^4 \mathbf{A}^2 + C\varepsilon^2 \mathbf{Q}^2 + C\varepsilon^2 \mathbf{Q}^3
\end{aligned}$$

Back in the estimate of  $T_4$ ,

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_4 \mathbf{B} dx \right| &\leq A + B + C \\
&\leq \frac{1}{18} \varepsilon^4 \mathbf{A}^2 + C \mathbf{Q} + C\varepsilon^2 \mathbf{Q}^2 + C\varepsilon^2 \mathbf{Q}^3 \\
\\
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_5 \mathbf{B} dx \right| &\leq \varepsilon^2 \|r^\varepsilon\|_{\varepsilon,1} |K^\varepsilon|_{\varepsilon,\infty} \mathbf{A} + \varepsilon^2 |r^\varepsilon|_{\varepsilon,2} \|K^\varepsilon\|_{\mathbb{W}_\varepsilon^1,\infty} \mathbf{A} \\
&\quad + \varepsilon^2 \|a^\varepsilon\|_{\mathbb{W}_\varepsilon^1,\infty} |R^\varepsilon|_{\varepsilon,2} \mathbf{A} + \varepsilon^2 |a^\varepsilon|_{\varepsilon,\infty} \|R^\varepsilon\|_{\varepsilon,1} \mathbf{A} \\
&\quad + \varepsilon^5 \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z r^\varepsilon|_{\varepsilon,3} \right) |R^\varepsilon|_{\varepsilon,6} \mathbf{A} \\
&\quad + \varepsilon^5 |r^\varepsilon|_{\varepsilon,6} \left( |\nabla_{\Gamma_z} R^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z R^\varepsilon|_{\varepsilon,3} \right) \mathbf{A} \\
&\leq C\varepsilon^2 \|r^\varepsilon\|_{\varepsilon,1} \mathbf{A} + C\varepsilon^5 \|r^\varepsilon\|_{\varepsilon,1} \|r^\varepsilon\|_{\varepsilon,2}^{1/2} \left( \|r^\varepsilon\|_{\varepsilon,2}^{1/2} + \mathbf{A}^{1/2} \right) \mathbf{A} \\
&\quad \text{(thanks to Ineq. (4.4), Prop. 5.2, Lem. 5.1 and Rem. 6.1)} \\
&\leq C\mathbf{Q}^{1/2} (\varepsilon^2 \mathbf{A}) + C\varepsilon^2 \mathbf{Q} (\varepsilon^2 \mathbf{A}) + C\varepsilon^{3/2} \mathbf{Q}^{3/4} (\varepsilon^2 \mathbf{A})^{3/2} \\
&\leq \frac{1}{18} \varepsilon^4 \mathbf{A}^2 + C \mathbf{Q} + C \varepsilon^4 \mathbf{Q}^2 + C\varepsilon^6 \mathbf{Q}^3 \\
\\
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_6 \mathbf{B} dx \right| &\leq 2\varepsilon^2 \|a^\varepsilon\|_{\mathbb{W}_\varepsilon^1,\infty} |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} \mathbf{A} + \varepsilon^2 |a^\varepsilon|_{\varepsilon,\infty}^2 \|r^\varepsilon\|_{\varepsilon,1} \mathbf{A} \\
&\quad + 2\varepsilon^2 \|a^\varepsilon\|_{\mathbb{W}_\varepsilon^1,\infty} |K^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} \mathbf{A} \\
&\quad + 2\varepsilon^2 \|K^\varepsilon\|_{\mathbb{W}_\varepsilon^1,\infty} |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,2} \mathbf{A} \\
&\quad + 2\varepsilon^2 |a^\varepsilon|_{\varepsilon,\infty} |K^\varepsilon|_{\varepsilon,\infty} \|r^\varepsilon\|_{\varepsilon,1} \mathbf{A} \\
&\leq C\varepsilon^2 \|r^\varepsilon\|_{\varepsilon,1} \mathbf{A} \leq \frac{1}{18} \varepsilon^4 \mathbf{A}^2 + C \mathbf{Q} \\
&\quad \text{(thanks to Prop. 5.2 and Lemma 5.1)} \\
\\
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_7 \mathbf{B} dx \right| &\leq 2\varepsilon^5 |a^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,6} \left( |\nabla_{\Gamma_z} R^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z R^\varepsilon|_{\varepsilon,3} \right) \mathbf{A} \\
&\quad + 2\varepsilon^5 |a^\varepsilon|_{\varepsilon,\infty} |R^\varepsilon|_{\varepsilon,6} \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z r^\varepsilon|_{\varepsilon,3} \right) \mathbf{A} \\
&\quad + 2\varepsilon^5 \|a^\varepsilon\|_{\mathbb{W}_\varepsilon^1,\infty} |r^\varepsilon|_{\varepsilon,4} |R^\varepsilon|_{\varepsilon,4} \mathbf{A} \\
&\quad + 2\varepsilon^5 |K^\varepsilon|_{\varepsilon,\infty} |r^\varepsilon|_{\varepsilon,6} \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,3} + |\varepsilon D_z r^\varepsilon|_{\varepsilon,3} \right) \mathbf{A} \\
&\quad + \varepsilon^5 \|K^\varepsilon\|_{\mathbb{W}_\varepsilon^1,\infty} |r^\varepsilon|_{\varepsilon,4}^2 \mathbf{A} \\
&\leq C\varepsilon^{5/2} \|r^\varepsilon\|_{\varepsilon,1}^{3/2} \|\varepsilon r^\varepsilon\|_{\varepsilon,2}^{1/2} (\varepsilon^2 \mathbf{A}) + C\varepsilon^3 |r^\varepsilon|_{\varepsilon,2}^{1/2} \|r^\varepsilon\|_{\varepsilon,1}^{3/2} (\varepsilon^2 \mathbf{A}) \\
&\quad \text{(thanks to Ineq. (4.4), Prop. 5.2 and Lemma 5.1)} \\
&\leq \frac{1}{18} \varepsilon^2 \mathbf{A}^4 + C (\varepsilon^5 + \varepsilon^6) \mathbf{Q}^2
\end{aligned}$$



$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 T_8 \mathbf{B} \, dx \right| &\leq 2\varepsilon^8 |r^\varepsilon|_{\varepsilon,6} \left( |\nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,6} + |\varepsilon D_z r^\varepsilon|_{\varepsilon,6} \right) |R^\varepsilon|_{\varepsilon,6} \mathbf{A} \\
&\quad + \varepsilon^8 |r^\varepsilon|_{\varepsilon,6}^2 \left( |\nabla_{\Gamma_z} R^\varepsilon|_{\varepsilon,6} + |\varepsilon D_z R^\varepsilon|_{\varepsilon,6} \right) \mathbf{A} \\
&\leq C\varepsilon^5 \|r^\varepsilon\|_{\varepsilon,1}^2 \|\varepsilon r^\varepsilon\|_{\varepsilon,2} (\varepsilon^2 \mathbf{A}) \\
&\quad \text{(thanks to Ineq. (4.4) and Lemma 5.1)} \\
&\leq \frac{1}{18} \varepsilon^4 \mathbf{A}^2 + C \varepsilon^{10} \mathbf{Q}^3
\end{aligned}$$

$$\begin{aligned}
\frac{1}{|\omega_\varepsilon|} \left| \int_{\omega_\varepsilon} \varepsilon^2 F^\varepsilon \mathbf{B} \, dx \right| &\leq \|F^\varepsilon\|_{\varepsilon,1} (\varepsilon^2 \mathbf{A}) \\
&\leq \frac{1}{18} \varepsilon^4 \mathbf{A}^2 + C \\
&\quad \text{(thanks to Prop. 5.2)}
\end{aligned}$$

By adding the previous inequalities we obtain:

$$\frac{d}{dt} \left( \varepsilon^2 |\Delta_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + \varepsilon^2 |\varepsilon D_z \nabla_{\Gamma_z} r^\varepsilon|_{\varepsilon,2}^2 + \varepsilon^2 |\varepsilon^2 D_z^2 r^\varepsilon|_{\varepsilon,2}^2 \right) + \varepsilon^4 \mathbf{A}^2 \leq C(1 + \mathbf{Q}) + \varepsilon^2 P(\mathbf{Q}), \quad (6.4)$$

where  $P$  is an  $\varepsilon$ -independent polynomial.

By adding the inequalities (6.2), (6.3) and (6.4) we get:

$$\frac{d}{dt} \mathbf{Q} \leq C(1 + \mathbf{Q}) + \varepsilon^2 P(\mathbf{Q}),$$

which implies the inequality (6.1) and thus the main theorem.

## A Expression of $F^\varepsilon$

Using the notations of Section 5.2 we let in  $\omega_\varepsilon$ :

$$\begin{aligned}
b^\varepsilon(t, x) &= U^1 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon U^2 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon^2 U^3 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right), \\
c^\varepsilon(t, x) &= U^2 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right) + \varepsilon U^3 \left( t, P(x), \frac{\varphi(x)}{\varepsilon} \right),
\end{aligned}$$

The expression  $F^\varepsilon$  then writes:

$$\begin{aligned}
F^\varepsilon = &\varepsilon \left[ -2\varepsilon |\partial_z b^\varepsilon|^2 U^3 + \varepsilon^2 |\nabla_{\Gamma_z} b^\varepsilon|^2 c^\varepsilon + |\partial_z c^\varepsilon|^2 a^\varepsilon + a_2 \partial_z b^\varepsilon - 4\varepsilon (\partial_z b^\varepsilon \cdot \partial_z U^3) b^\varepsilon \right. \\
&+ 2\varepsilon^{-1} (\nabla_{\Gamma_z} a^\varepsilon \cdot \nabla^r a^\varepsilon) a^\varepsilon - 2\varepsilon (\nabla_{\Gamma_z} a^\varepsilon \cdot \nabla_{\Gamma_z} b^\varepsilon) c^\varepsilon + 2\varepsilon^2 (\nabla_{\Gamma_z} b^\varepsilon \cdot \nabla_{\Gamma_z} c^\varepsilon) b^\varepsilon + |\nabla_{\Gamma_z} a^\varepsilon|^2 c^\varepsilon \\
&+ \varepsilon^2 |\partial_z c^\varepsilon|^2 c^\varepsilon + \varepsilon^2 |(\nabla_{\Gamma_{0,1}} + \nabla^r) a^\varepsilon|^2 c^\varepsilon - 2\varepsilon (\nabla_{\Gamma_z} a^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) a^\varepsilon) c^\varepsilon \\
&+ 2\varepsilon^3 ((\nabla_{\Gamma_{0,1}} + \nabla^r) b^\varepsilon \cdot \nabla^r b^\varepsilon) b^\varepsilon - 4\varepsilon^3 |(\nabla_{\Gamma_{0,1}} + \nabla^r) b^\varepsilon|^2 b^\varepsilon + 2(\partial_z b^\varepsilon \cdot \partial_z U^2) a^\varepsilon \\
&+ 2(\partial_z U^3 \cdot \partial_z a^\varepsilon) b^\varepsilon + 2(\partial_z a^\varepsilon \cdot \partial_z b^\varepsilon) U^3 + |(\nabla_{\Gamma_{0,1}} + \nabla^r) a^\varepsilon|^2 a^\varepsilon \\
&+ \varepsilon^4 |(\nabla_{\Gamma_{0,1}} + \nabla^r) b^\varepsilon|^2 c^\varepsilon - 2\varepsilon (\nabla_{\Gamma_z} a^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) c^\varepsilon) a^\varepsilon - 2\varepsilon (\nabla_{\Gamma_z} c^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) a^\varepsilon) a^\varepsilon \\
&- 2\varepsilon^3 (\nabla_{\Gamma_z} b^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) c^\varepsilon) b^\varepsilon - 2\varepsilon^3 (\nabla_{\Gamma_z} c^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) b^\varepsilon) b^\varepsilon - 2\varepsilon |\partial_z c^\varepsilon|^2 b^\varepsilon \\
&- 2\varepsilon^3 ((\nabla_{\Gamma_{0,1}} + \nabla^r) a^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) c^\varepsilon) b^\varepsilon - 2\varepsilon^2 (\nabla_{\Gamma_z} a^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) c^\varepsilon) b^\varepsilon \\
&- 2\varepsilon^2 (\nabla_{\Gamma_z} c^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) a^\varepsilon) b^\varepsilon - 2\varepsilon^2 (\nabla_{\Gamma_z} b^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) c^\varepsilon) a^\varepsilon \\
&- 2\varepsilon^2 (\nabla_{\Gamma_z} c^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) b^\varepsilon) a^\varepsilon + 2(\partial_z a^\varepsilon \cdot \partial_z c^\varepsilon) c^\varepsilon \\
&- 2\varepsilon^3 ((\nabla_{\Gamma_{0,1}} + \nabla^r) b^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) c^\varepsilon) a^\varepsilon + 2\varepsilon^4 ((\nabla_{\Gamma_{0,1}} + \nabla^r) b^\varepsilon \cdot (\nabla_{\Gamma_{0,1}} + \nabla^r) c^\varepsilon) b^\varepsilon
\end{aligned}$$

$$\begin{aligned}
& -2\varepsilon^3((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)c^\varepsilon - 2\varepsilon^3(\nabla_{\Gamma_z}b^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)c^\varepsilon \\
& -2\varepsilon^2((\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon \cdot \nabla^r b^\varepsilon)a^\varepsilon + 3\varepsilon^2|(\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon|^2a^\varepsilon \\
& +2\varepsilon^2((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)c^\varepsilon)a^\varepsilon - 2\varepsilon|(\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon|^2b^\varepsilon \\
& +2\varepsilon((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot \nabla^r a^\varepsilon)b^\varepsilon + 2\varepsilon^2(\nabla_{\Gamma_z}a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)c^\varepsilon \\
& +2\varepsilon^2(\nabla_{\Gamma_z}b^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon)c^\varepsilon - 2((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot \nabla^r a^\varepsilon)a^\varepsilon \\
& +2\varepsilon(\nabla_{\Gamma_z}a^\varepsilon \cdot \nabla^r b^\varepsilon)b^\varepsilon + 2\varepsilon(\nabla_{\Gamma_z}b^\varepsilon \cdot \nabla^r a^\varepsilon)b^\varepsilon + 6\varepsilon^2((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)b^\varepsilon \\
& -2\varepsilon^2((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot \nabla^r b^\varepsilon)b^\varepsilon - 2\varepsilon^2((\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon \cdot \nabla^r a^\varepsilon)b^\varepsilon - 2\varepsilon(\nabla_{\Gamma_z}b^\varepsilon \cdot \nabla_{\Gamma_z}c^\varepsilon)a^\varepsilon \\
& +2(\nabla_{\Gamma_z}a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon)b^\varepsilon - 2(\nabla_{\Gamma_z}a^\varepsilon \cdot \nabla^r a^\varepsilon)b^\varepsilon + 2\varepsilon((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot \nabla^r b^\varepsilon)a^\varepsilon \\
& +2\varepsilon((\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon \cdot \nabla^r a^\varepsilon)a^\varepsilon - 4\varepsilon((\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)a^\varepsilon \\
& +2(\nabla_{\Gamma_z}a^\varepsilon \cdot \nabla_{\Gamma_z}b^\varepsilon)b^\varepsilon + |\nabla_{\Gamma_z}b^\varepsilon|^2a^\varepsilon + 2(\nabla_{\Gamma_z}a^\varepsilon \cdot \nabla_{\Gamma_z}c^\varepsilon)a^\varepsilon \\
& -2\varepsilon^2(\nabla_{\Gamma_z}b^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)b^\varepsilon + 2\varepsilon(\nabla_{\Gamma_z}b^\varepsilon \cdot \nabla^r b^\varepsilon)a^\varepsilon - 2(\nabla_{\Gamma_z}a^\varepsilon \cdot \nabla^r b^\varepsilon)a^\varepsilon \\
& -2(\nabla_{\Gamma_z}b^\varepsilon \cdot \nabla^r a^\varepsilon)a^\varepsilon + 2(\nabla_{\Gamma_z}a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)a^\varepsilon + 2(\nabla_{\Gamma_z}b^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon)a^\varepsilon \\
& -2\varepsilon(\nabla_{\Gamma_z}a^\varepsilon \cdot \nabla_{\Gamma_z}c^\varepsilon)b^\varepsilon + 6\varepsilon^2(\nabla_{\Gamma_z}b^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)b^\varepsilon - 4\varepsilon(\nabla_{\Gamma_z}a^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)b^\varepsilon \\
& -4\varepsilon(\nabla_{\Gamma_z}b^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)a^\varepsilon)b^\varepsilon - 4\varepsilon(\nabla_{\Gamma_z}b^\varepsilon \cdot (\nabla_{\Gamma_0,1} + \nabla^r)b^\varepsilon)a^\varepsilon + 2\varepsilon^2(\partial_z b^\varepsilon \cdot \partial_z U^3)c^\varepsilon \\
& -2\varepsilon|\nabla_{\Gamma_z}b^\varepsilon|^2b^\varepsilon + |\partial_z b^\varepsilon|^2c^\varepsilon + 2\varepsilon^2(\partial_z b^\varepsilon \cdot \partial_z c^\varepsilon)U^3 + 2(\partial_z b^\varepsilon \cdot \partial_z c^\varepsilon)b^\varepsilon \\
& -2\varepsilon(\partial_z c^\varepsilon \cdot \partial_z a^\varepsilon)c^\varepsilon - 2\varepsilon(\partial_z a^\varepsilon \cdot \partial_z c^\varepsilon)U^3 - 4\varepsilon(\partial_z b^\varepsilon \cdot \partial_z c^\varepsilon)c^\varepsilon - 2\varepsilon(\partial_z c^\varepsilon \cdot \partial_z U^3)a^\varepsilon \\
& +2\varepsilon^2(\partial_z c^\varepsilon \cdot \partial_z U^3)b^\varepsilon + b^\varepsilon \wedge \frac{1}{\gamma_z} \partial_z(\gamma_z \partial_z U^3) + U^3 \wedge \frac{1}{\gamma_z} \partial_z(\gamma_z \partial_z b^\varepsilon) + \frac{1}{\varepsilon} \Delta^r a^\varepsilon \\
& + c^\varepsilon \wedge \frac{1}{\gamma_z} \partial_z(\gamma_z \partial_z c^\varepsilon) + a^r \partial_z a^\varepsilon + a_1 b^\varepsilon \wedge \partial_z b^\varepsilon - \varepsilon^2 a_2 b^\varepsilon \wedge \partial_z b^\varepsilon + a_2 b^\varepsilon \wedge \partial_z a^\varepsilon \\
& + \frac{1}{\varepsilon} a^\varepsilon \wedge \Delta^r a^\varepsilon + a_2 a^\varepsilon \wedge \partial_z b^\varepsilon - \varepsilon a_0 U^3 \wedge \partial_z U^1 - \varepsilon^4 a^r U^3 \wedge \partial_z U^1 - \varepsilon^2 a_1 c^\varepsilon \wedge \partial_z c^\varepsilon \\
& - \varepsilon^4 a^r c^\varepsilon \wedge \partial_z c^\varepsilon - \varepsilon^2 a_1 U^3 \wedge \partial_z U^1 - \varepsilon^3 a_2 c^\varepsilon \wedge \partial_z c^\varepsilon - \varepsilon^4 a_3 c^\varepsilon \wedge \partial_z c^\varepsilon + a_1 c^\varepsilon \wedge \partial_z a^\varepsilon \\
& + a_3 a^\varepsilon \wedge \partial_z a^\varepsilon + a_0 a^\varepsilon \wedge \partial_z U^3 + a_0 \partial_z U^3 + \Delta_{\Gamma_z} c^\varepsilon + \Delta_{\Gamma_0,1} b^\varepsilon - \varepsilon(\Delta_{\Gamma_0,1} + \Delta^r) c^\varepsilon \\
& + a^r a^\varepsilon \wedge \partial_z a^\varepsilon + a_3 \partial_z a^\varepsilon + a_0 U^3 \wedge \partial_z a^\varepsilon - \varepsilon^3 a_2 U^3 \wedge \partial_z U^1 - \varepsilon^3 a_3 U^3 \wedge \partial_z U^1 \\
& - 2\varepsilon b^\varepsilon \wedge (\Delta_{\Gamma_0,1} + \Delta^r) b^\varepsilon - a^\varepsilon \wedge \Delta^r b^\varepsilon + b^\varepsilon \wedge (\Delta_{\Gamma_0,1} + \Delta^r) a^\varepsilon - b^\varepsilon \wedge \Delta^r a^\varepsilon \\
& - \varepsilon a^\varepsilon \wedge (\Delta_{\Gamma_0,1} + \Delta^r) c^\varepsilon + \varepsilon^2 b^\varepsilon \wedge (\Delta_{\Gamma_0,1} + \Delta^r) c^\varepsilon - \varepsilon c^\varepsilon \wedge (\Delta_{\Gamma_0,1} + \Delta^r) a^\varepsilon \\
& + \varepsilon^2 c^\varepsilon \wedge (\Delta_{\Gamma_0,1} + \Delta^r) b^\varepsilon + a^\varepsilon \wedge \Delta_{\Gamma_z} c^\varepsilon - \varepsilon b^\varepsilon \wedge \Delta_{\Gamma_z} c^\varepsilon + c^\varepsilon \wedge \Delta_{\Gamma_z} a^\varepsilon \\
& - \varepsilon c^\varepsilon \wedge \Delta_{\Gamma_z} b^\varepsilon + a_1 a^\varepsilon \wedge \partial_z c^\varepsilon - a_1 b^\varepsilon \wedge \partial_z c^\varepsilon - \varepsilon a_1 c^\varepsilon \wedge \partial_z b^\varepsilon + b^\varepsilon \wedge \Delta_{\Gamma_z} b^\varepsilon \\
& + \varepsilon b^\varepsilon \wedge \Delta^r b^\varepsilon - \varepsilon c^\varepsilon \wedge \partial_z^2 U^3 - \varepsilon U^3 \wedge \frac{1}{\gamma_z} \partial_z(\gamma_z \partial_z c^\varepsilon) + a_2 \partial_z c^\varepsilon - \varepsilon a_0 b^\varepsilon \wedge \partial_z U^3 \\
& + a_0 b^\varepsilon \wedge \partial_z c^\varepsilon - \varepsilon a_0 U^3 \wedge \partial_z b^\varepsilon + a_0 c^\varepsilon \wedge \partial_z b^\varepsilon + a^\varepsilon \wedge (\Delta_{\Gamma_0,1} + \Delta^r) b^\varepsilon - 2\varepsilon a_0 c^\varepsilon \wedge \partial_z c^\varepsilon \\
& + U^3 \wedge H^1 + b^\varepsilon \wedge H^3 + c^\varepsilon \wedge H^2 - U^3 \wedge (a^\varepsilon \wedge H^1) - a^\varepsilon \wedge (U^3 \wedge H^1) \\
& - \varepsilon^4 c^\varepsilon \wedge (U^3 \wedge H^3) + \varepsilon c^\varepsilon \wedge (U^3 \wedge K^\varepsilon) + \varepsilon^3 b^\varepsilon \wedge (U^3 \wedge H^3) - b^\varepsilon \wedge (a^\varepsilon \wedge H^3) \\
& - b^\varepsilon \wedge (U^3 \wedge K^\varepsilon) - \varepsilon^4 U^3 \wedge (c^\varepsilon \wedge H^3) + \varepsilon U^3 \wedge (c^\varepsilon \wedge K^\varepsilon) + \varepsilon^3 U^3 \wedge (b^\varepsilon \wedge H^3) \\
& - U^3 \wedge (b^\varepsilon \wedge K^\varepsilon) - a^\varepsilon \wedge (b^\varepsilon \wedge H^3) + \varepsilon^3 c^\varepsilon \wedge (c^\varepsilon \wedge H^3) - c^\varepsilon \wedge (c^\varepsilon \wedge K^\varepsilon) \\
& + \varepsilon b^\varepsilon \wedge (b^\varepsilon \wedge H^3) - \varepsilon^2 c^\varepsilon \wedge (U^3 \wedge H^1) - \varepsilon^3 c^\varepsilon \wedge (U^3 \wedge H^2) - c^\varepsilon \wedge (a^\varepsilon \wedge H^2) \\
& + 2\varepsilon b^\varepsilon \wedge (U^3 \wedge H^1) + \varepsilon^2 \beta^\varepsilon \wedge (U^3 \wedge H^2) - \varepsilon^2 U^3 \wedge (c^\varepsilon \wedge H^1) - \varepsilon^3 U^3 \wedge (c^\varepsilon \wedge H^2) \\
& + a^\varepsilon (c^\varepsilon \wedge H^2) + 2\varepsilon U^3 \wedge (b^\varepsilon \wedge H^1) + \varepsilon^2 U^3 \wedge (b^\varepsilon \wedge H^2) + \varepsilon^2 c^\varepsilon \wedge (c^\varepsilon \wedge H^2) \\
& - b^\varepsilon \wedge (c^\varepsilon \wedge H^1) + 2\varepsilon c^\varepsilon \wedge (c^\varepsilon \wedge H^1) - c^\varepsilon \wedge (b^\varepsilon \wedge H^1) - b^\varepsilon \wedge (b^\varepsilon \wedge H^2) \\
& + \varepsilon b^\varepsilon \wedge (c^\varepsilon \wedge H^2) + \varepsilon c^\varepsilon \wedge (b^\varepsilon \wedge H^2)] + a^\varepsilon \wedge S^\varepsilon - a^\varepsilon \wedge (a^\varepsilon \wedge S^\varepsilon).
\end{aligned}$$

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