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Abstract. In this paper we study the behaviour of Landau-Lifschitz equation in a thin layer. As the thickness of the domain and the exchange coefficient of the equation simultaneously tend to zero we perform an asymptotic expansion to precise the solution for well prepared initial condition.

1 Introduction

In this paper we consider a periodic thin layer $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon, \varepsilon)$ of ferromagnetic material and we study the behaviour of magnetic moment u when ε goes to zero. The magnetic moment u, defined on the domain $\Omega_\varepsilon \subset \mathbb{R}^3$ in which the material is confined, satisfies $|u| \equiv 1$ and links the magnetic field H and the magnetic induction B by the relation $B = H + \overline{u}$, where $\overline{u}$ is the extension of u by zero outside Ω_ε.
The magnetic field H satisfies curl $H = 0$ by static Maxwell's equations, and by the law of Faraday we have div $B =$ div $(H + \overline{u}) = 0$. Hence the magnetic moment u induces a magnetic field $H(u)$ given by:

$$\begin{cases} H(u) \in \mathbb{L}^2(\mathbb{R}^3), \\ \text{curl } H(u) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ \text{div } (H(u) + \overline{u}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3). \end{cases} \tag{1.1}$$

We will consider a quasi-stationary model in which the behaviour of u is governed by the following Landau-Lifschitz type equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u \wedge \left(\varepsilon^2 \Delta u + H(u)\right) - u \wedge \left(u \wedge (\varepsilon^2 \Delta u + H(u))\right) \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_\varepsilon, \\ u(0,.) = u_0 \text{ on } \Omega_\varepsilon, \end{cases} \tag{1.2}$$

where n denotes the outward unitary normal on $\partial\Omega_\varepsilon$. The coefficient ε^2 is an exchange coefficient. As it is smaller than all other characteristic sizes of the problem, it is intended to tend to zero.

When the exchange coefficient is fixed and the domain Ω_ε does not depend on ε, Carbou and Fabrie prove in [5] the local existence of a regular solution u of (1.2):

Theorem 1.1 (Carbou, Fabrie). *If $u_0 \in \mathbb{H}^2(\Omega)$, $\dfrac{\partial u_0}{\partial n} = 0$ on $\partial\Omega$ and $|u_0| \equiv 1$, there exists a time $T^* > 0$ and there exists an unique $u \in \mathcal{C}\left([0,T]; \mathbb{H}^2(\Omega)\right) \cap \mathbb{L}^2(0,T; \mathbb{H}^3(\Omega))$ for all $T < T^*$ such that u satisfies (1.2).*

But when the exchange coefficient ε^2 goes to zero (and the domain still does not depend on ε), the time interval given by the previous theorem also tends to zero. In [7], Carbou, Fabrie and Guès prove that the time of existence goes in fact to infinity when ε goes to zero for the initial data in $\mathbb{H}^5(\Omega)$ by showing that a boundary layer appears. Moreover they get that the solution u tends to the solution of the hyperbolic system formally obtained by taking $\varepsilon = 0$ in (1.2).

We are interested here in the case of thin layers. In [4], Carbou has studied the limit of the magnetic moment u when the thickness ε of the domain goes to 0 and the exchange coefficient is fixed. We want to precise here the behaviour of u when the exchange coefficient is ε^2 and in an ε-thick sheet $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon, \varepsilon)$ which is periodic on $\mathbb{R}^2$ by making an asymptotic expansion of u. We will have to take into account an interaction between the boundary layer, whose characteristic thickness is ε, and the thin layer. We note $I_\varepsilon = (-\varepsilon, \varepsilon)$ and $\omega = (0,1)^2$, the periodic domain. $\mathbb{H}^l_p(\omega)$ is then the subspace of $\mathbb{H}^l(\omega)$ whose functions are periodic on $\mathbb{R}^2$ and $\mathbb{L}^2_p(\omega)$ is the closure of $\mathbb{H}^1_p(\omega)$ in $\mathbb{L}^2(\omega)$. The index p denotes each time the periodicity on ω. The equations we will consider are:

$$\begin{cases} \dfrac{\partial u}{\partial t} = u \wedge \left(\varepsilon^2 \Delta u + H(u)\right) - u \wedge \left(u \wedge \left(\varepsilon^2 \Delta u + H(u)\right)\right) \text{ in } \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \omega \times \{\pm\varepsilon\}, \\ u(0,.) = u_0 \text{ on } \Omega_\varepsilon, \\ u \text{ periodic on } \omega \end{cases} \tag{1.3}$$

and

$$\begin{cases} H(u) \in \mathbb{L}^2_p(\omega \times \mathbb{R}), \\ \text{curl } H(u) = 0 \text{ in } \omega \times \mathbb{R}, \\ \text{div } (H(u) + \overline{u}) = 0 \text{ in } \omega \times \mathbb{R}. \end{cases} \tag{1.4}$$

As we will study independently H inside and outside Ω_ε we need some transmission conditions that we can deduce from (1.4):

$$\begin{cases} [H(u) \wedge e_3] = 0 \text{ on } \omega \times \{\pm\varepsilon\}, \\ [(H(u) + \overline{u}) \cdot e_3] = 0 \text{ on } \omega \times \{\pm\varepsilon\}, \end{cases} \tag{1.5}$$

where e_3 denotes the unitary normal to $\{z = 0\}$ *i.e.* $e_3 = (0,0,1)$ and $[v] = v\left(z^+\right) - v\left(z^-\right)$ denotes the jump of v at an interface $\omega \times \{\pm\varepsilon\}$. We also note $I_1 = (-1,1)$.

In order to perform the asymptotic expansion and get the profiles we work on the rescaled equation.

Assumptions 1. *The initial data u_0^ε satisfies $\dfrac{\partial u_0^\varepsilon}{\partial n} = 0$ on $\partial\Omega_\varepsilon$ and can be written in the form:*

$$u_0^\varepsilon(x,z) = U_0^0(x) + \varepsilon U_0^1\left(x, \frac{z}{\varepsilon}\right) + \varepsilon^2 U_0^2\left(x, \frac{z}{\varepsilon}\right) + \varepsilon^2 r_0^\varepsilon(x,z), \quad \forall (x,z) \in \Omega_\varepsilon,$$

where

- $U_0^0 \in \mathbb{W}_p^{6,\infty}(\omega)$ *such that* $|U_0^0| \equiv 1$,
- $U_0^1 \in \mathbb{H}_p^5(\omega) \otimes \mathbb{H}^5(-1,1)$,
- $U_0^2 \in \mathbb{H}_p^4(\omega) \otimes \mathbb{H}^5(-1,1)$,
- $r_0^\varepsilon \in \mathbb{H}_p^2(\Omega_\varepsilon)$ *is the initial data for the remainder term* r^ε *and fulfills the additional condition:*

$$\|r_0^\varepsilon\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\nabla r_0^\varepsilon\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \varepsilon^2 \|\Delta r_0^\varepsilon\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} \leq C,$$

where C *is an* ε*-independent constant.*

We get the following result:

Theorem 1.2. *Under the assumptions 1, if* T^ε *is the maximum time of existence of the regular solution* u^ε *of the equation (1.3) then* $\lim_{\varepsilon \to 0} T^\varepsilon = +\infty$ *and for all* $T < T^\varepsilon$ *we have:*

$$u^\varepsilon(t,x,z) = U^0(t,x) + \varepsilon U^1\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon^2 U^2\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon^2 r^\varepsilon(t,x,z), \quad (x,z) \in \Omega_\varepsilon,\ t < T^\varepsilon,$$

where

- $U^0 \in \mathcal{C}^\infty\left(\mathbb{R}_t^+; \mathbb{W}_p^{6,\infty}(\omega)\right)$ *is solution of the equation (1.3) with* $\varepsilon = 0$, *i.e.*

$$\begin{cases} \dfrac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge \left(U^0 \wedge H^0\right) \ \text{ in } \omega, \\ \\ U^0(0,.) = u_0 \ \text{ in } \omega, \end{cases}$$

  where $H^0 = -U_3^0 e^3$ *is a local operator,*

- $U^1 \in \bigcap_{k=0}^{2} \left[\mathbb{W}^{k,\infty}\left(0,T; \mathbb{H}_p^5(\omega) \otimes \mathbb{H}^{5-2k}(I_1)\right) \cap \mathbb{H}^k\left(0,T; \mathbb{H}_p^5(\omega) \otimes \mathbb{H}^{6-2k}(I_1)\right)\right]$ *for all* $0 < T < T^\varepsilon$,
- $U^2 \in \bigcap_{k=0}^{2} \left[\mathbb{W}^{k,\infty}\left(0,T; \mathbb{H}_p^4(\omega) \otimes \mathbb{H}^{5-2k}(I_1)\right) \cap \mathbb{H}^k\left(0,T; \mathbb{H}_p^4(\omega) \otimes \mathbb{H}^{6-2k}(I_1)\right)\right]$ *for all* $0 < T < T^\varepsilon$,
- *for all* $T < T^\varepsilon$ *the function* r^ε *is bounded in* $\mathbb{L}^\infty\left(0,T; \mathbb{H}^1(\Omega_\varepsilon)\right)$ *and the function* $\varepsilon r^\varepsilon$ *is bounded in* $\mathbb{L}^\infty\left(0,T; \mathbb{H}^2(\Omega_\varepsilon)\right) \cap \mathbb{L}^2\left(0,T, \mathbb{H}^3(\Omega_\varepsilon)\right)$.

Remark 1.1. *The space* $\mathbb{L}^2(\Omega_\varepsilon)$ *is supplied with the classical norm* $\|u\|_{\mathbb{L}^2(\Omega_\varepsilon)} = \displaystyle\int_{\Omega_\varepsilon} |u|^2\, dx$.

This paper is organized in the following way. The first part consists in the reminder and proof of some technical lemmas and in the study of the operator H outside Ω_ε. In the second part we formally build the asymptotic expansion of u^ε and we prove its existence for all time and the regularity of its terms. The equations satisfied by the remainder term of u^ε and H^ε are explained in the third part and the fourth part contains the energy estimates which conclude the proof of theorem 1.2.

2 Preliminary results

2.1 Some results of regularity

Lemma 2.1. *Let $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon,\varepsilon)$. There exists a constant c_0 independent of ε such that*

$$\|u\|_{\mathbb{H}^2(\Omega_\varepsilon)} \leq c_0 \left(\|u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\Delta u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} \right)^{\frac{1}{2}},$$

for all $u \in \mathbb{H}^2(\Omega_\varepsilon)$ such that $\dfrac{\partial u}{\partial z} = 0$ on $(0,1)^2 \times \{\pm\varepsilon\}$ and u is periodic in $x \in (0,1)^2$. Moreover if $u \in \mathbb{H}^3(\Omega_\varepsilon)$,

$$\|\nabla u\|_{\mathbb{H}^2(\Omega_\varepsilon)} \leq c_0 \left(\|\nabla u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\Delta u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\nabla \Delta u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} \right)^{\frac{1}{2}}.$$

Proof: we take the Fourier coefficient of u and the proof follows for the first estimate. For the second one, we want to apply the inequality (see [8])

$$\|u\|_{\mathbb{H}^m(\Omega)} \leq c(\Omega) \left[\|u\|_{\mathbb{L}^2(\Omega)} + \|\text{div } u\|_{\mathbb{H}^{m-1}(\Omega)} + \|\text{curl } u\|_{\mathbb{H}^{m-1}(\Omega)} + \|u \cdot n\|_{\mathbb{H}^{m-1/2}(\partial\Omega)} \right]$$

Since we need the dependence of $c(\Omega)$ in ε we make the proof for the open set we consider:

$$\|\nabla u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} = \|\text{div } u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{curl } u\|^2_{\mathbb{L}^2(\Omega_\varepsilon)} + \int_{\omega \times \{\pm\varepsilon\}} \left[\text{curl } u \cdot (u \wedge n) - (u \cdot n)\ \text{div } u + u \cdot \frac{\partial u}{\partial n} \right] d\gamma$$

First step: We assume that $u \cdot n = u \cdot e_3 = 0$. The boundary term then satisfies:

$$I_{\partial\Omega_\varepsilon} = \int_{\partial\Omega_\varepsilon} \left[\text{curl } u \cdot (u \wedge n) + u \cdot \frac{\partial u}{\partial n} \right] d\gamma = \int_{\partial\Omega_\varepsilon} (u_1 \partial_x + u_2 \partial_y)\, u_3\ d\gamma.$$

Since $u_3 = 0$ on $\omega \times \{\pm\varepsilon\}$, $I_{\partial\Omega_\varepsilon} = 0$ and

$$\|u\|_{\mathbb{H}^1(\Omega_\varepsilon)} \leq \|u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{div } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{curl } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} \tag{2.1}$$

Second step: We assume that $u \cdot n = g \in \mathbb{H}^{1/2}(\partial\Omega_\varepsilon)$ on $\partial\Omega_\varepsilon$. We then take a lifting $\varphi = Rg \in \mathbb{H}^2(\Omega_\varepsilon)$ of g such that $\dfrac{\partial \varphi}{\partial n} = g \in \mathbb{H}^{1/2}(\partial\Omega_\varepsilon)$. So $v = u - \nabla\varphi$ fulfills $v \cdot n = 0$. We apply (2.1) and we get:

$$\begin{aligned} \|u\|_{\mathbb{H}^1(\Omega_\varepsilon)} \leq\ & \|u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{div } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{curl } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} \\ & + \|\nabla\varphi\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\Delta\varphi\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\nabla\varphi\|_{\mathbb{H}^1(\Omega_\varepsilon)} \end{aligned}$$

Since $\|\varphi\|_{\mathbb{H}^2(\Omega_\varepsilon)} \leq C(\Omega_\varepsilon)\ \|g\|_{\mathbb{H}^{1/2}(\partial\Omega_\varepsilon)}$ we have:

$$\|u\|_{\mathbb{H}^1(\Omega_\varepsilon)} \leq \|u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{div } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{curl } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + C(\Omega_\varepsilon)\|u \cdot n\|_{\mathbb{H}^{1/2}(\partial\Omega_\varepsilon)}$$

Third step: We now apply this inequality to all partial derivatives of u to get the inequality in $\mathbb{H}^2$. Since $(\partial_i u) \cdot n = (\partial_i u) \cdot e_3 = \partial_i (u \cdot e_3)$ we finally get:

$$\|u\|_{\mathbb{H}^2(\Omega_\varepsilon)} \leq c_0 \left(\|u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{div } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|\text{curl } u\|_{\mathbb{L}^2(\Omega_\varepsilon)} \right) + C(\Omega_\varepsilon)\|u \cdot n\|_{\mathbb{H}^{3/2}(\partial\Omega_\varepsilon)},$$

where c_0 is independent of Ω_ε. We get the result by applying this inequality to ∇u (such that $\dfrac{\partial u}{\partial n} = 0$). $\square$

In the following $\dfrac{\partial}{\partial x_i}$ denotes the derivative in x_i in the anisotropic inequalities where the three coordinates are (x_1, x_2, x_3). In the thin domains $\dfrac{\partial}{\partial x_i}$ is the derivative in x_i where (x_1, x_2) belongs to $(0,1)^2$ and $\dfrac{\partial}{\partial z}$ is the derivative in the thickness z of the thin layer.

Theorem 2.2 (Anisotropic Agmon's inequality). *(cf [17])*
Let $\Omega_0 = (0,1)^3$ then there exists an absolute constant c_0 such that

$$\|u\|_{\mathbb{L}^\infty(\Omega_0)} \leq c_0 \|u\|^{\frac{1}{4}}_{\mathbb{L}^2(\Omega_0)} \prod_{i=1}^{3} \left(\left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{\mathbb{L}^2(\Omega_0)} + \left\| \frac{\partial u}{\partial x_i} \right\|_{\mathbb{L}^2(\Omega_0)} + \|u\|_{\mathbb{L}^2(\Omega_0)} \right)^{\frac{1}{4}},$$

for all $u \in \mathbb{H}^2(\Omega_0)$.

Corollary 2.3 (Agmon's inequality in thin domains). *(cf [17])*
Let $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon, \varepsilon)$. There exists a positive constant c_0 independent of ε such that for all $u \in \mathbb{H}^2(\Omega_\varepsilon)$,

$$\|u\|_{\mathbb{L}^\infty(\Omega_\varepsilon)} \leq c_0 \|u\|^{\frac{1}{4}}_{\mathbb{L}^2(\Omega_\varepsilon)} \left(\left\| \frac{\partial^2 u}{\partial z^2} \right\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon} \left\| \frac{\partial u}{\partial z} \right\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \frac{1}{\varepsilon^2} \|u\|_{\mathbb{L}^2(\Omega_\varepsilon)} \right)^{\frac{1}{4}}$$
$$\times \prod_{i=1}^{2} \left(\left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \left\| \frac{\partial u}{\partial x_i} \right\|_{\mathbb{L}^2(\Omega_\varepsilon)} + \|u\|_{\mathbb{L}^2(\Omega_\varepsilon)} \right)^{\frac{1}{4}}.$$

Corollary 2.4. *Let $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon, \varepsilon)$. There exists a positive constant c_0 independent of ε such that for all $u \in \mathbb{H}^2(\Omega_\varepsilon)$ and for all $2 \leq p \leq +\infty$,*

$$\|u\|_{\mathbb{L}^p(\Omega_\varepsilon)} \leq c_0 \varepsilon^{-(\frac{1}{2} - \frac{1}{p})} \|u\|^{\frac{1}{4} + \frac{3}{2p}}_{\mathbb{L}^2(\Omega_\varepsilon)} \|u\|^{\frac{3}{4} - \frac{3}{2p}}_{\mathbb{H}^2(\Omega_\varepsilon)}.$$

Theorem 2.5 (Anisotropic Ladyzhenskaya's inequality). *(cf [17])*
Let $\Omega_0 = (0,1)^3$. There exists an absolute constant c_0 such that

$$\|u\|_{\mathbb{L}^6(\Omega_0)} \leq c_0 \prod_{i=1}^{3} \left(\|u\|_{\mathbb{L}^2(\Omega_0)} + \left\| \frac{\partial u}{\partial x_i} \right\|_{\mathbb{L}^2(\Omega_0)} \right)^{\frac{1}{3}}, \qquad \forall u \in \mathbb{H}^1(\Omega_0).$$

Corollary 2.6. *Let $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon, \varepsilon)$. There exists a positive constant c_0 independent of ε such that for all $u \in \mathbb{H}^1(\Omega_\varepsilon)$ and for all $2 \leq p \leq 6$,*

$$\|u\|_{\mathbb{L}^p(\Omega_\varepsilon)} \leq c_0 \varepsilon^{-(\frac{1}{2} - \frac{1}{p})} \|u\|^{\frac{3}{p} - \frac{1}{2}}_{\mathbb{L}^2(\Omega_\varepsilon)} \|u\|^{\frac{3}{2} - \frac{3}{p}}_{H^1(\Omega_\varepsilon)}.$$

Theorem 2.7. *Let $\Omega_\varepsilon = (0,1)^2 \times (-\varepsilon, \varepsilon)$. There exists a positive constant c_0 independent of ε, such that $\forall u \in \mathbb{H}^2(\Omega_\varepsilon)$,*

$$\|u\|_{\mathbb{L}^{12}(\Omega_\varepsilon)} \leq c_0 \varepsilon^{-\frac{5}{12}} \|u\|^{\frac{3}{4}}_{\mathbb{H}^1(\Omega_\varepsilon)} \|u\|^{\frac{1}{4}}_{\mathbb{H}^2(\Omega_\varepsilon)}.$$

Proof: This theorem is an anisotropic version in thin layers of the classical embeddings from $\mathbb{H}^{\frac{5}{4}}(\Omega)$ into $\mathbb{L}^{12}(\Omega)$. We prove in the appendix A the anisotropic embedding. The proof follows the scheme of that of theorem 2.2 given in [17], the proof of the thin layers version is the same as that of corollary 2.3 also given in [17]. □

We also need a 1D estimate:

Proposition 2.8. *There exists a constant c_0 independent of ε such that for all $u \in \mathbb{H}^1(-\varepsilon,\varepsilon)$ we have:*

$$\|u\|_{\mathbb{L}^\infty(-\varepsilon,\varepsilon)} \leq c_0 \ \|u\|^{\frac{1}{2}}_{\mathbb{L}^2(-\varepsilon,\varepsilon)} \left(\|\partial_x u\|_{\mathbb{L}^2(-\varepsilon,\varepsilon)} + \frac{1}{\varepsilon}\|u\|_{\mathbb{L}^2(-\varepsilon,\varepsilon)} \right)^{\frac{1}{2}} \leq c_0 \varepsilon^{-\frac{1}{2}} \|u\|_{\mathbb{H}^1(-\varepsilon,\varepsilon)}.$$

2.2 Behaviour of H outside Ω_ε

We denote now by H^+ (resp. H^-) the restriction of the function H on $\omega \times (\varepsilon, +\infty)$ (resp. $\omega \times (-\infty, -\varepsilon)$) and by J_ε^+ (resp. J_ε^-) the domain $(\varepsilon, +\infty)$ (resp.$(-\infty, -\varepsilon)$). We are interested in the behaviour of H outside $\omega \times (-\varepsilon, \varepsilon)$. H fulfills:

$$\begin{cases} H^\pm \in \mathbb{L}^2_p\left(\omega \times J_\varepsilon^\pm\right), \\ \text{curl } H^\pm = 0 \text{ in } \omega \times J_\varepsilon^\pm, \\ \text{div } H^\pm = 0 \text{ in } \omega \times J_\varepsilon^\pm, \\ H^\pm \cdot e_3 = f^\pm \text{ or } H^\pm \wedge e_3 = g^\pm \text{ on } \omega \times \{\pm\varepsilon\}, \end{cases} \tag{2.2}$$

where $f^\pm \in \mathbb{H}^{l+\frac{1}{2}}_p(\omega \times \{\pm\varepsilon\}; \mathbb{R})$, $g^\pm \in \mathbb{H}^{l+\frac{1}{2}}_p\left(\omega \times \{\pm\varepsilon\}; \mathbb{R}^3\right)$, $g^\pm$ is orthogonal to e_3 and $l \in \mathbb{N}$.

We want to get an expression of $H^\pm \wedge e_3$ (resp. $H^\pm \cdot e_3$) thanks to $H^\pm \cdot e_3$ (resp. $H^\pm \wedge e_3$) on $\omega \times \{\pm\varepsilon\}$.

Lemma 2.9. *For $f \in \mathbb{H}^{\frac{1}{2}}_p(\omega; \mathbb{R})$ we consider $H \in \mathbb{L}^2_p\left(\omega \times \mathbb{R}^+; \mathbb{R}^3\right)$, solution of*

$$\begin{cases} \text{curl } H = 0 \ \textit{in } \omega \times \mathbb{R}^+, \\ \text{div } H = 0 \ \textit{in } \omega \times \mathbb{R}^+, \\ H \cdot e_3 = f \ \textit{ on } \omega \times \{0\}, \end{cases} \tag{2.3}$$

and we define the operator $\varphi : \mathbb{H}^{\frac{1}{2}}_p(\omega; \mathbb{R}) \mapsto \mathbb{H}^{\frac{1}{2}}_p\left(\omega; \mathbb{R}^3\right)$ which maps $(H \cdot e_3)_{|z=0}$ to $(H \wedge e_3)_{|z=0}$. φ is continuous and given by (in Fourier):

$$c_k(\varphi(f)) = c_k(f) \frac{i}{|k|} \begin{pmatrix} -k_2 \\ k_1 \\ 0 \end{pmatrix} \quad \forall k \in \mathbb{Z}^2 \setminus \{0\}, \qquad c_0(\varphi(f)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In the same way for $g \in \mathbb{H}^{\frac{1}{2}}_p(\omega; \mathbb{R}^3)$, g orthogonal to e_3 we consider H solution of

$$\begin{cases} \text{curl } H = 0 \ \textit{in } \omega \times \mathbb{R}^+, \\ \text{div } H = 0 \ \textit{in } \omega \times \mathbb{R}^+, \\ H \wedge e_3 = g \ \textit{ on } \omega \times \{0\}, \end{cases} \tag{2.4}$$

and we define the operator $\Psi : \mathbb{H}_p^{\frac{1}{2}}\left(\omega;\mathbb{R}^3\right) \mapsto \mathbb{H}_p^{\frac{1}{2}}\left(\omega;\mathbb{R}\right)$ *which maps* $(H\wedge e_3)_{|z=0}$ *to* $(H\cdot e_3)_{|z=0}$. Ψ *is continuous and given by (in Fourier):*

$$c_k\left(\Psi\begin{pmatrix}g_1\\g_2\\0\end{pmatrix}\right)=\frac{i}{|k|}\left(k_2c_k(g_1)-k_1c_k(g_2)\right)\quad\forall k\in\mathbb{Z}^2\setminus\{0\},\qquad c_0\left(\Psi\begin{pmatrix}g_1\\g_2\\0\end{pmatrix}\right)=0.$$

Proof: As the problem is linear and the boundary conditions are periodic, we consider the Fourier series of H, f and g:

$$V=\sum_{k\in\mathbb{Z}^2}c_ke^{2\pi i(k\cdot x)}\text{ in }\mathbb{L}_p^2\left(\omega\right),$$

where $\quad c_k=\dfrac{1}{2}\displaystyle\int_\omega V(x)e^{-2\pi i(k\cdot x)}\,dx,\ \forall k\in\mathbb{Z}^2.$

We then get for all $k\in\mathbb{Z}^2$ a linear differential equation:

$$\frac{d}{dz}c_k(H)+Ac_k(H)=0\text{ in }\omega\times\mathbb{R}^+,$$

where $A=2i\pi\begin{pmatrix}0&0&-k_1\\0&0&-k_2\\k_1&k_2&0\end{pmatrix}$. Let us diagonalize A: $A=\begin{pmatrix}2\pi|k|&0&0\\0&-2\pi|k|&0\\0&0&0\end{pmatrix}$ for all $k\neq(0,0)$ with the eigenvectors

$$e_1'=\left(-i\frac{k_1}{|k|},-i\frac{k_2}{|k|},1\right),\ e_2'=\left(i\frac{k_1}{|k|},i\frac{k_2}{|k|},1\right)\text{ and }e_3'=\left(\frac{-k_2}{|k|},\frac{k_1}{|k|},0\right).$$

As we search solutions in $\mathbb{L}_p^2\left(\omega\times\mathbb{R}^+\right)$ only the component of e_1' does not vanish. Finally we get:

- if $(H\cdot e_3)_{|z=0}=f$, for all $k\in\mathbb{Z}^2\setminus\{0\}$

$$c_k(H)=c_k(f)e^{-2\pi z|k|}\begin{pmatrix}-i\frac{k_1}{|k|}\\-i\frac{k_2}{|k|}\\1\end{pmatrix},\qquad c_0(H)=0,$$

- if $(H\wedge e_3)_{|z=0}=g=(g_1,g_2,0)$, for all $k\in\mathbb{Z}^2\setminus\{0\}$ (as $k_1c_k(H_2)=k_2c_k(H_1)$, *cf* (2.3))

$$c_k(H)=e^{-2\pi z|k|}\begin{pmatrix}-c_k(g_2)\\c_k(g_1)\\\frac{i}{|k|}\left(k_2c_k(g_1)-k_1c_k(g_2)\right)\end{pmatrix},\qquad c_0(H)=0.\quad\square$$

So we have the

Lemma 2.10. *There exists some applications*

- $\varphi^{\pm}:\mathbb{H}_p^{l+\frac{1}{2}}\left(\omega;\mathbb{R}\right)\mapsto\mathbb{H}_p^{l+\frac{1}{2}}\left(\omega;\mathbb{R}^3\right)$ *which maps* $(H\cdot e_3)_{|z=\pm\varepsilon/2}$ *to* $(H\wedge e_3)_{|z=\pm\varepsilon/2}$,
- $\Psi^{\pm}:\mathbb{H}_p^{l+\frac{1}{2}}\left(\omega;\mathbb{R}^3\right)\mapsto\mathbb{H}_p^{l+\frac{1}{2}}\left(\omega;\mathbb{R}\right)$ *which maps* $(H\wedge e_3)_{|z=\pm\varepsilon/2}$ *to* $(H\cdot e_3)_{|z=\pm\varepsilon/2}$,

where H is solution of (2.2). Moreover for all $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$c_k\left(\varphi^{\pm}(f)\right) = \pm c_k(f)\frac{i}{|k|}\begin{pmatrix}-k_2\\ k_1\\ 0\end{pmatrix}, \qquad c_k\left(\Psi^{\pm}\begin{pmatrix}g_1\\ g_2\\ 0\end{pmatrix}\right) = \pm\frac{i}{|k|}\left(k_2 c_k(g_1) - k_1 c_k(g_2)\right).$$

and $\quad c_0\left(\varphi^{\pm}(f)\right) = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}, \quad c_0\left(\Psi^{\pm}\begin{pmatrix}g_1\\ g_2\\ 0\end{pmatrix}\right) = 0.$

Proof: We deduce these relationships from the previous lemma. □

3 Formal asymptotic expansion

We will now search an asymptotic expansion of u^ε and H^ε in the form:

- in Ω_ε,

$$\begin{aligned} u^\varepsilon(t,x,z) &= U^0\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon U^1\left(t,x,\frac{z}{\varepsilon}\right) + \dots \\ H^\varepsilon(t,x,z) &= H^0\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon H^1\left(t,x,\frac{z}{\varepsilon}\right) + \dots \end{aligned} \tag{3.1}$$

- in $(\omega \times \mathbb{R}) \setminus \Omega_\varepsilon$,

$$H^\varepsilon(t,x,z) = H^0(t,x,z) + \varepsilon H^1(t,x,z) + \dots \tag{3.2}$$

where $x = (x_1, x_2) \in \omega$.

3.1 Equations of the profiles

As it is proved in [5], the equation (1.3) is equivalent for regular enough solutions to:

$$\frac{\partial u}{\partial t} - \varepsilon^2\Delta u = \varepsilon^2|\nabla u|^2 u + \varepsilon^2 u \wedge \Delta u + u \wedge H(u) - u \wedge (u \wedge H(u)). \tag{3.3}$$

Since this equation isolates the dissipative term we will use it in the following.
We substitute to u and H their asymptotic expansion in (1.3)-(1.4) in order to get the expression satisfied by the term of the expansion for $(x,z) \in \omega \times I_1$ or $(x,z) \in (\omega \times \mathbb{R}) \setminus \Omega_\varepsilon$.
As the equations (1.4) are linear, each H^i fulfills (1.4) outside Ω_ε. By lemma 2.10 we have then a relationship between the normal and the tangential component of H^i outside Ω_ε. The transmission conditions allow then us to fully determine H^i since the conditions become for all $i \in \mathbb{N}$:

$$\begin{cases} \left[H^i \wedge e_3\right] = 0 \text{ on } \omega \times \{\pm\varepsilon\}, \\ \left[\left(H^i + \overline{U^i}\right) \cdot e_3\right] = 0 \text{ on } \omega \times \{\pm\varepsilon\}. \end{cases} \tag{3.4}$$

The Neumann condition on u also becomes $\left(\dfrac{\partial}{\partial z}U^i\right)_{|z=\pm 1} = 0$ for all $i \in \mathbb{N}$.

Preliminary calculus:

In Ω_ε, we have the following formal expansion of curl V and div V (with $V = (V_1, V_2, V_3)$):

$$\text{curl } V(t,x,z) = \frac{1}{\varepsilon}\frac{\partial}{\partial z}\left(e_3 \wedge V^0\right) + \sum_{i\geq 0} \varepsilon^i \left(\text{curl}_x\, V^i + \frac{\partial}{\partial z}\left(e_3 \wedge V^{i+1}\right)\right),$$

$$\text{div } V(t,x,z) = \frac{1}{\varepsilon}\frac{\partial}{\partial z} V_3^0 + \sum_{i\geq 0} \varepsilon^i \left(\text{div}_x\, V^i + \frac{\partial}{\partial z} V_3^{i+1}\right),$$

where $\text{div}_x\, V = \dfrac{\partial V_1}{\partial x_1} + \dfrac{\partial V_2}{\partial x_2}$ and $\text{curl}_x\, V = \left(\dfrac{\partial}{\partial x_1},\ \dfrac{\partial}{\partial x_2},\ 0\right)^t \wedge V$. In the following $\dfrac{\partial V}{\partial z}$ is denoted by V_z. Back in (1.3)-(1.4) the terms of the equations of the different order in ε vanish:

Terms in ε^{-1}: in $\omega \times I_1$,

$$\begin{cases} \dfrac{\partial}{\partial z}\left(H_3^0 + U_3^0\right) = 0, \\ \dfrac{\partial}{\partial z}\left(e_3 \wedge H^0\right) = 0. \end{cases}$$

So $H^0 + U_3^0\ e_3$ does not depend on z. By applying lemma 2.10 we get

$$c_k(H^0 \wedge e_3) = \pm c_k\left(H_3^0 + U_3^0\right)\frac{i}{|k|}\begin{pmatrix} -k_2 \\ k_1 \\ 0 \end{pmatrix} \text{ for } k \neq 0 \text{ and } c_0(H^0 \wedge e_3) = 0,$$

So $H^0 \wedge e_3 = 0$ and $H_3^0 + U_3^0 = 0$,

$$H^0 = -U_3^0\ e_3 \tag{3.5}$$

Remark 3.1. *Although H fulfills a global equation in $\mathbb{R}^3 \setminus \Omega$, it becomes a* **local operator** *at the 0-order.*

Terms in ε^0: in $\omega \times I_1$,

$$\frac{\partial U^0}{\partial t} - U_{zz}^0 = |U_z^0|^2 U^0 + U^0 \wedge U_{zz}^0 + U^0 \wedge H^0 - U^0 \wedge \left(U^0 \wedge H^0\right), \tag{3.6}$$

As we assume that U_0^0 does not depend on z, we ca n prove formally that U^0 does not depend on z and is solution of

$$\begin{cases} \dfrac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge \left(U^0 \wedge H^0\right) & \text{on } \omega, \\ U^0(t=0) = U_0^0 & \text{on } \omega, \end{cases}$$

as a solution of the previous equation.

$$\begin{cases} \dfrac{\partial}{\partial z}\left(H_3^1 + U_3^1\right) + \text{div}_x\ \left(H^0 + U^0\right) = 0, \\ \dfrac{\partial}{\partial z}\left(e_3 \wedge H^1\right) + \text{curl}_x\, H^0 = 0. \end{cases} \tag{3.7}$$

Terms in ε: in $\omega \times I_1$,

$$\begin{aligned}\frac{\partial U^1}{\partial t} - U^1_{zz} = |U^0_z|^2 U^1 + 2(U^0_z \cdot U^1_z)U^0 + U^0 \wedge U^1_{zz} + U^1 \wedge U^0_{zz} \\ + U^1 \wedge H^0 + U^0 \wedge H^1 - U^0 \wedge \left(U^0 \wedge H^1\right) \\ - U^0 \wedge \left(U^1 \wedge H^0\right) - U^1 \wedge \left(U^0 \wedge H^0\right).\end{aligned} \tag{3.8}$$

$$\begin{cases} \dfrac{\partial}{\partial z}\left(H^2_3 + U^2_3\right) + \mathrm{div}_x\ \left(H^1 + U^1\right) = 0, \\ \dfrac{\partial}{\partial z}\left(e_3 \wedge H^2\right) + \mathrm{curl}_x\ H^1 = 0. \end{cases} \tag{3.9}$$

Terms in ε^2: in $\omega \times I_1$,

$$\begin{aligned}\frac{\partial U^2}{\partial t} - U^2_{zz} = |U^0_z|^2 U^2 + 2(U^0_z \cdot U^2_z)U^0 + U^0 \wedge U^2_{zz} + U^2 \wedge U^0_{zz} \\ + U^2 \wedge H^0 + U^0 \wedge H^2 - U^0 \wedge \left(U^0 \wedge H^2\right) \\ - U^0 \wedge \left(U^2 \wedge H^0\right) - U^2 \wedge \left(U^0 \wedge H^0\right) \\ + \Delta_x U^0 + |U^1_z|^2 U^0 + 2(U^0_z \cdot U^1_z)U^1 + \left|\nabla_x U^0\right|^2 U^0 \\ + U^0 \wedge \Delta_x U^0 + U^1 \wedge U^1_{zz} + U^1 \wedge H^1 - U^0 \wedge (U^1 \wedge H^1) \\ - U^1 \wedge (U^0 \wedge H^1) - U^1 \wedge (U^1 \wedge H^0),\end{aligned} \tag{3.10}$$

3.2 Existence of the asymptotic expansion

3.2.1 Existence of U^0

Theorem 3.1. *Given U^0_0 such that $|U^0_0| \equiv 1$ and $U^0_0 \in \mathbb{W}^{6,\infty}_p(\omega)$, there exists an unique solution to (3.11):*

$$\begin{cases} \dfrac{\partial U^0}{\partial t} = U^0 \wedge H^0 - U^0 \wedge \left(U^0 \wedge H^0\right) \quad \text{in } \omega, \\ U^0(0) = U^0_0 \quad \text{in } \omega \\ H^0 = -(U^0 \cdot e_3)e_3 \quad \text{in } \omega, \end{cases} \tag{3.11}$$

such that $|U^0| = 1$ and $U^0 \in \mathcal{C}^\infty\left(\mathbb{R}^+; \mathbb{W}^{6,\infty}_p(\omega)\right)$.

Remark 3.2. *If we choose an initial data U^0_0 which depends on z and if we want to get the existence of the solution on $\mathbb{R}^+_t$ we need the z-dependent part of U^0_0 i.e.*
$\widetilde{U^0_0} = U^0_0 - \overline{U^0_0} = U^0_0 - \displaystyle\int_{-1}^{1} U^0_0(x,z)\,dz$ to be smaller than $\overline{U^0_0}$. In order to perform the estimates on the remainder term of u we will need that $\widetilde{U^0_0} = \mathcal{O}(\varepsilon)$ which gives us a term of the same order as U^1 and which fulfills the same equation. We take then a z-independent initial data for U^0_0.

Proof: As $\mathbb{W}^{6,\infty}_p(\omega;\mathbb{R}^3)$ is an algebra we have the existence and uniqueness of U^0, solution of the previous ordinary differential equation, in $\mathcal{C}^\infty\left(0,T;\mathbb{W}^{6,\infty}_p(\omega)\right)$. As $|U(t)| = 1$ for all t we get the result. □

3.2.2 Existence of U^1

As U^0 and H^0 are known, we are now able to determine H^1. We also remark that

$$\mathbb{L}^\infty(\omega) \subset \mathbb{L}^2(\omega).$$

We have the

Lemma 3.2. *If U_0^0 satisfies the assumptions of theorem 3.1, (i.e. $U_0^0 \in \mathbb{W}^6(\omega)$, $|U_0^0| \equiv 1$) and $U^1 \in \mathbb{H}_p^l(\omega) \otimes \mathbb{H}^{l'}(I_1)$ there exists then $H^1 \in \mathbb{H}_p^k(\omega) \otimes \mathbb{H}^{l'}(I_1)$ with $k = \min(5, l)$, solution of (3.7), and:*

$$H^1 = -U_3^1\ e_3 + q,$$

where $q \in \mathcal{C}^\infty\left(\mathbb{R}^+; \mathbb{H}_p^5(\omega) \otimes \mathcal{C}^\infty(I_1)\right)$.

Proof: The equations (3.7) give:

$$\begin{cases} \left(H_3^1 + U_3^1\right) + (z+1)\left(\mathrm{div}_x\ U^0\right)(z') = a(x), \\ \\ e_3 \wedge H^1 + (z+1)\left(\mathrm{curl}_x\ H^0\right)(z') = b(x), \end{cases}$$

where a and b are two periodic functions that we determine thanks to lemma 2.10. For all $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$c_k(a) = 2i\pi\left[k \cdot c_k(U^0) - i|k| c_k(H^0) \cdot e_3\right],$$

$$c_k(b) = 2i\pi\left[k \wedge c_k(H^0) - i\frac{k}{|k|} \cdot c_k(U^0) \begin{pmatrix} -k1 \\ k2 \\ 0 \end{pmatrix}\right],$$

$$c_0(a) = 0, \quad c_0(b) = 0. \ \square$$

Theorem 3.3. *Under the assumptions of theorem 3.1 and with the additional assumptions that $U_0^1 \in \mathbb{H}^5(\omega) \otimes \mathbb{H}^5(I_1)$ and $U_{0,z}^1 = 0$ on $\omega \times \{\pm 1\}$, there exists an unique solution to*

$$\begin{cases} \dfrac{\partial U^1}{\partial t} - U_{zz}^1 = U^0 \wedge U_{zz}^1 + U^1 \wedge H^0 + U^0 \wedge H^1 - U^0 \wedge \left(U^0 \wedge H^1\right) \\ \qquad\qquad - U^0 \wedge \left(U^1 \wedge H^0\right) - U^1 \wedge \left(U^0 \wedge H^0\right) \ \text{in } \omega \times I_1, \\ \\ U_z^1 = 0 \ \text{on } \omega \times \{\pm 1\}, \\ \\ U^1(0,.) = U_0^1(.) \ \text{in } \omega \times I_1, \end{cases} \tag{3.12}$$

such that for all $T > 0$,

$$U^1 \in \bigcap_{k=0}^{2} \left[\mathbb{W}^{k,\infty}\left(0, T; \mathbb{H}_p^5(\omega) \otimes \mathbb{H}^{5-2k}(I_1)\right) \cap \mathbb{H}^k\left(0, T; \mathbb{H}_p^5(\omega) \otimes \mathbb{H}^{6-2k}(I_1)\right)\right].$$

Proof: According to lemma 3.2 let $H^1 = p(U^1) + q(z, x)$, where $p(V) = -V_3\ e_3$ and $q \in \mathcal{C}^\infty\left(\mathbb{R}^+; \mathbb{H}_p^5(\omega)\right)$. We have then:

$$\begin{aligned} \frac{\partial U^1}{\partial t} - U_{zz}^1 = {} & U^0 \wedge U_{zz}^1 + U^1 \wedge H^0 + U^0 \wedge p(U^1) - U^0 \wedge \left(U^0 \wedge p(U^1)\right) \\ & - U^0 \wedge \left(U^1 \wedge H^0\right) - U^1 \wedge \left(U^0 \wedge H^0\right) + f(t, x, z) \text{ in } \omega \times I_1, \end{aligned} \tag{3.13}$$

where $f(t,x,z) = U^0 \wedge\ q(t,z,x) - U^0 \wedge \left(U^0 \wedge q(t,x,z)\right) \in \mathcal{C}^\infty\left(\mathbb{R}^+; \mathbb{H}^5_p(\omega) \otimes \mathcal{C}^\infty(-1,1)\right)$.
The variable $x \in \omega$ is only a parameter here. We use Galerkin's method on I_1: let $(w_i)_i$ be a basis of eigenfunctions of the operator $A = I - \dfrac{\partial^2}{\partial z^2}$ with domain $D(A) = \mathbb{H}^2(I_1)$ and Neumann conditions on the boundary. Let $W_N = Vect(w_1, \dots w_n)$. We have then the existence of solutions v^N in $\mathcal{C}^\infty\left(0,T; \mathbb{H}^5_p(\omega) \otimes W_N\right)$ for all $T > 0$.
Let us make *a priori* estimates on the approximated solution:

$\mathbb{L}^2$ **estimate:** we get

$$\frac{1}{2}\frac{d}{dt}\|v^N\|^2_{\mathbb{L}^2(\omega\times I_1)} + \|v^N_z\|^2_{\mathbb{L}^2(\omega\times I_1)} \leq C\|v^N\|^2_{\mathbb{L}^2(\omega\times I_1)} + \|f\|_{\mathbb{L}^2(\omega\times I_1)}\|v^N\|_{\mathbb{L}^2(\omega\times I_1)},$$

so

$$\frac{d}{dt}\|v^N\|^2_{\mathbb{L}^2(\omega\times I_1)} + \|v^N_z\|^2_{\mathbb{L}^2(\omega\times I_1)} \leq C\left(1 + \|v^N\|^2_{\mathbb{L}^2(\omega\times I_1)}\right).$$

By applying Gronwall's lemma we get

$$v^N \in \mathbb{L}^\infty\left(0,T; \mathbb{L}^2_p(\omega\times I_1)\right), \quad v^N_z \in \mathbb{L}^2\left(0,T; \mathbb{L}^2_p(\omega\times I_1)\right).$$

$\mathbb{H}^1$ **estimate:** we take the product by v^N_{zz},

$$\frac{1}{2}\frac{d}{dt}\|v^N_z\|^2_2 + \|v^N_{zz}\|^2_2 \leq C\|v^N\|_2\|v^N_{zz}\|_2 + \|v^N_{zz}\|_2\|f\|_2.$$

By the same way we get

$$v^N_z \in \mathbb{L}^\infty\left(0,T; \mathbb{L}^2_p(\omega\times I_1)\right), \quad v^N_{zz} \in \mathbb{L}^2\left(0,T; \mathbb{L}^2_p(\omega\times I_1)\right).$$

and finally $\quad v^N \in \mathbb{L}^\infty\left(0,T; \mathbb{L}^2_p(\omega) \otimes \mathbb{H}^1(I_1)\right) \cap \mathbb{L}^2\left(0,T; \mathbb{L}^2_p(\omega) \otimes \mathbb{H}^2(I_1)\right)$.

The variable x was as far as here only a parameter. We will now get some regularity for $x \in \omega$. We can derivate in x the equation (3.13) and perform the same *a priori* estimates for z since the equation is linear in the term with the higher derivative in x. We get:

$$v^N \in \mathbb{L}^\infty\left(0,T; \mathbb{H}^5_p(\omega) \otimes \mathbb{H}^1(I_1)\right) \cap \mathbb{L}^2\left(0,T; \mathbb{H}^5_p(\omega) \otimes \mathbb{H}^2(I_1)\right).$$

We will now derivate in time the equation (3.13). We get in $\omega \times I_1$:

$$\begin{aligned}\frac{\partial^2 v^N}{\partial t^2} - \frac{\partial v^N_{zz}}{\partial t} &= U^0 \wedge \frac{\partial v^N_{zz}}{\partial t} + \frac{\partial v^N}{\partial t} \wedge H^0 + U^0 \wedge p\left(\frac{\partial v^N}{\partial t}\right) - U^0 \wedge \left(U^0 \wedge p\left(\frac{\partial v^N}{\partial t}\right)\right) \\ &\quad - U^0 \wedge \left(\frac{\partial v^N}{\partial t} \wedge H^0\right) - \frac{\partial v^N}{\partial t} \wedge \left(U^0 \wedge H^0\right) + A(v^N) + \frac{\partial f}{\partial t}(t,x,z),\end{aligned}$$

where

$$\begin{aligned}A(v^N) &= \frac{\partial U^0}{\partial t} \wedge v^N_{zz} + v^N \wedge \frac{\partial H^0}{\partial t} + \frac{\partial U^0}{\partial t} \wedge p(v^N) - \frac{\partial U^0}{\partial t} \wedge \left(U^0 \wedge p(v^N)\right) \\ &\quad - U^0 \wedge \left(\frac{\partial U^0}{\partial t} \wedge p(v^N)\right) - \frac{\partial U^0}{\partial t} \wedge \left(v^N \wedge H^0\right) - U^0 \wedge \left(v^N \wedge \frac{\partial H^0}{\partial t}\right) \\ &\quad - v^N \wedge \left(\frac{\partial U^0}{\partial t} \wedge H^0\right) - v^N \wedge \left(U^0 \wedge \frac{\partial H^0}{\partial t}\right).\end{aligned}$$

By using the previous estimates on v^N and the same method we can deduce estimates on $\dfrac{\partial v^N}{\partial t}$:

$$v^N \in \mathbb{W}^{1,\infty}\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^1(I_1)\right)\cap\mathbb{H}^1\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^2(I_1)\right).$$

By the same way we get for $0\leq k\leq 2$ (we are limited by the regularity of U_0^1):

$$v^N \in \mathbb{W}^{k,\infty}\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^1(I_1)\right)\cap\mathbb{H}^k\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^2(I_1)\right),$$

By using the weak-$\star$ compactness of $\mathbb{L}^\infty$ and the weak compactness of $\mathbb{L}^2$ we can take the limit in N. We have for $0\leq k\leq 2$:

$$U^1 \in \ \mathbb{W}^{k,\infty}\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^1(I_1)\right)\cap\mathbb{H}^k\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^2(I_1)\right).$$

If we rewrite now the equation (3.13) we get

$$\begin{aligned} U_{zz}^1+U^0\wedge U_{zz}^1 &= \frac{\partial U^1}{\partial t}-U^1\wedge H^0-U^0\wedge p(U^1)+U^0\wedge\left(U^0\wedge p(U^1)\right)\\ &\quad +U^0\wedge\left(U^1\wedge H^0\right)+U^1\wedge\left(U^0\wedge H^0\right)-f(t,x,z)\text{ in }\omega\times I_1. \end{aligned}$$

As $\left(I+U^0\wedge\right)$ is invertible and its inverse is $I-\dfrac{1}{2}(U^0\wedge.)+\dfrac{1}{2}U^0\wedge\left(U^0\wedge.\right)$, we get for $k=0,1$

$$U_{zz}^1 \in \mathbb{W}^{k,\infty}\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^1(I_1)\right)\cap\mathbb{H}^k\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^2(I_1)\right).$$

We finally get for all $T>0$ that

$$U^1 \in \bigcap_{k=0}^{2}\left[\mathbb{W}^{k,\infty}\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^{5-2k}(I_1)\right)\cap\mathbb{H}^k\left(0,T;\mathbb{H}_p^5(\omega)\otimes\mathbb{H}^{6-2k}(I_1)\right)\right].$$

□

3.2.3 Existence of U^2

Theorem 3.4. *Under the assumptions of theorems 3.1 and 3.3 and with the extra assumptions $U_0^2\in\mathbb{H}^4(\omega)\otimes\mathbb{H}^5(I_1)$ and $U_{0,z}^2=0$ on $\omega\times\{\pm1\}$, there exists an unique solution to*

$$\begin{cases} \dfrac{\partial U^2}{\partial t}-U_{zz}^2 = U^0\wedge U_{zz}^2+U^2\wedge H^0+U^0\wedge H^2-U^0\wedge\left(U^0\wedge H^2\right)\\ \qquad -U^0\wedge\left(U^2\wedge H^0\right)-U^2\wedge\left(U^0\wedge H^0\right)+F \text{ in } \omega\times I_1,\\ U_z^2=0 \text{ on } \omega\times\{\pm1\},\\ U^2(0,.)=U_0^2(.) \text{ in } \omega\times I_1, \end{cases} \tag{3.14}$$

where

$$\begin{cases} \dfrac{\partial}{\partial z}\left(H_3^2+U_3^2\right)+\operatorname{div}_x\left(H^1+U^1\right)=0,\\ \dfrac{\partial}{\partial z}\left(e_3\wedge H^2\right)+\operatorname{curl}_x H^1=0, \end{cases}$$

and

$$\begin{aligned} F=\Delta_x U^0+|U_z^1|^2U^0+\left|\nabla_x U^0\right|^2U^0+U^0\wedge\Delta_x U^0+U^1\wedge U_{zz}^1+U^1\wedge H^1\\ -U^0\wedge(U^1\wedge H^1)-U^1\wedge(U^0\wedge H^1)-U^1\wedge(U^1\wedge H^0), \end{aligned}$$

such that for all $T > 0$,

$$U^2 \in \bigcap_{k=0}^{2} \left[\mathbb{W}^{k,\infty}\left(0,T;\mathbb{H}^4_p(\omega)\otimes \mathbb{H}^{5-2k}(I_1)\right) \cap \mathbb{H}^k\left(0,T;\mathbb{H}^4_p(\omega)\otimes \mathbb{H}^{6-2k}(I_1)\right)\right].$$

Proof: As we did in lemma 3.2 we get that

$$H^2 = -U_3^2 e_3 + q',$$

where $q' \in \mathbb{H}^2\left(0,T;\mathbb{H}^4_p(\omega)\otimes \mathbb{H}^2(I_1)\right)$.
According to theorems 3.1, 3.3 and lemma 3.2 we get that

$$F \in \mathbb{H}^2\left(0,T;\mathbb{H}^4_p(\omega)\otimes \mathbb{L}^2(I_1)\right).$$

The proof of theorem 3.3 gives that for all $T > 0$,

$$U^2 \in \bigcap_{k=0}^{2} \left[\mathbb{W}^{k,\infty}\left(0,T;\mathbb{H}^4_p(\omega)\otimes \mathbb{H}^{5-2k}(I_1)\right) \cap \mathbb{H}^k\left(0,T;\mathbb{H}^4_p(\omega)\otimes \mathbb{H}^{6-2k}(I_1)\right)\right].$$

□

4 Equations satisfied by the remainder term

- In Ω_ε let

$$\begin{cases} u(t,x,z) = U^0(t,x) + \varepsilon U^1\left(t,x,\dfrac{z}{\varepsilon}\right) + \varepsilon^2 U^2\left(t,x,\dfrac{z}{\varepsilon}\right) + \varepsilon^2 r^\varepsilon(t,x,z), \\ \\ H(t,x,z) = H^0(t,x) + \varepsilon H^1\left(t,x,\dfrac{z}{\varepsilon}\right) + \varepsilon^2 H^2\left(t,x,\dfrac{z}{\varepsilon}\right) + \varepsilon^2 Q^\varepsilon(t,x,z), \end{cases}$$

- In $(\omega \times \mathbb{R}) \setminus \Omega_\varepsilon$

$$H(t,x,z) = H^0(t,x) + \varepsilon H^1(t,x,z) + \varepsilon^2 H^2(t,x,z) + \varepsilon^2 Q^\varepsilon(t,x,z),$$

where r^ε and Q^ε are the remainder terms of u and H.

4.1 Remainder term of H

Back in the equations (1.4)-(1.5) we simplify by using the equations satisfied by the profiles and we get:

Lemma 4.1. *Under the assumptions of theorems 3.1, 3.3 and 3.4 and lemma 3.2, if* $r^\varepsilon \in \mathbb{H}^p$ *then there exists an unique* Q^ε, *remainder term of* H *which fulfills*

$$\begin{cases} Q^\varepsilon \in \mathbb{L}^2(\omega \times \mathbb{R}), \\ \\ \text{curl } Q^\varepsilon + \text{curl}_x H^2 = 0 \text{ in } w \times \mathbb{R}, \\ \\ \text{div } (Q^\varepsilon + \overline{r^\varepsilon}) + \text{div}_x (H^2 + \overline{U^2}) = 0 \text{ in } \omega \times \mathbb{R}. \end{cases} \tag{4.1}$$

$$\begin{cases} [Q^\varepsilon \wedge e_3] = 0 \ \textit{on} \ \omega \times \{\pm\varepsilon\}\,, \\ \\ [(Q^\varepsilon + \overline{r^\varepsilon}) \cdot e_3] = 0 \ \textit{on} \ \omega \times \{\pm\varepsilon\}\,, \end{cases} \tag{4.2}$$

Moreover Q^ε can be written under the form $R^\varepsilon + S^\varepsilon$ where R^ε is linear in r^ε and S^ε does not depend on r^ε and there exists some ε-independent constant c_0 such that for all $i \in \mathbb{N}^*$,

$$\|R^\varepsilon\|_{\mathbb{H}^p_p(\Omega_\varepsilon)} \leq c_0 \|r^\varepsilon\|_{\mathbb{H}^p_p(\Omega_\varepsilon)},$$

$$\|S^\varepsilon\|_{\mathbb{H}^4_p(\omega)\otimes\mathbb{H}^i(-\varepsilon,\varepsilon)} \leq c_0 \left(\|H^2\|_{\mathbb{H}^4_p(\omega)\otimes\mathbb{H}^{i-1}(-\varepsilon,\varepsilon)} + \|U^2\|_{\mathbb{H}^4_p(\omega)\otimes\mathbb{H}^{i-1}(-\varepsilon,\varepsilon)} \right).$$

Proof: outside Ω_ε we can solve (4.2) by using lemma 2.10. In Ω_ε we take the Fourier coefficients of the equations (4.2) and we get:

$$\frac{d}{dz} c_k(V) + A c_k(V) = -\left(B_1 c_k(r^\varepsilon) + A c_k(H^1) + B_2 c_k(U^1) \right), \tag{4.3}$$

where

$$V = Q^\varepsilon + \begin{pmatrix} 0 \\ 0 \\ r_3^\varepsilon \end{pmatrix}, \ A = 2i\pi \begin{pmatrix} 0 & 0 & -k_1 \\ 0 & 0 & -k_2 \\ k_1 & k_2 & 0 \end{pmatrix}, \ B_1 = 2i\pi \begin{pmatrix} 0 & 0 & k_1 \\ 0 & 0 & k_2 \\ k_1 & k_2 & 0 \end{pmatrix}$$
$$\text{and } B_2 = 2i\pi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & 0 \end{pmatrix}.$$

According to theorem 3.3 the second member of (4.3) has the regularity of r^ε in z. Let us assume that $r^\varepsilon \in \mathbb{H}^p(\Omega_\varepsilon)$. We consider now the basis of eigenvectors of A which was determined in 2.2. The index i will here denote the component of e_i'. We get:

$$\frac{d}{dz} c_k \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} + 2\pi|k| \begin{pmatrix} c_k(V_1) \\ -c_k(V_2) \\ 0 \end{pmatrix} = -2\pi|k| \begin{pmatrix} c_k(r_2^\varepsilon) - c_k(H_1^2) + \frac{1}{2}(c_k(U_2^2) - c_k(U_1^2)) \\ -c_k(r_1^\varepsilon) + c_k(H_2^2) + \frac{1}{2}(c_k(U_2^2) - c_k(U_1^2)) \\ 0 \end{pmatrix}. \tag{4.4}$$

Thanks to lemma 2.10 we know that $c_k(V)$ is colinear to e_2' in $-\varepsilon$ and to e_1' in ε. So $c_0(V) = 0$. Let $2\pi|k|L = (2\pi|k|L_1, 2\pi|k|L_2, 0)$ be the second member of (4.4). For all $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$c_k(V) = \begin{pmatrix} e^{-2\pi|k|z} a_1 \\ e^{2\pi|k|z} a_2 \\ a_3 \end{pmatrix} + 2\pi|k| \int_{-\varepsilon}^{z} \begin{pmatrix} e^{-2\pi|k|(z-z')} L_1(z') \\ e^{2\pi|k|(z-z')} L_2(z') \\ 0 \end{pmatrix} dz',$$

then $a_1 = a_3 = 0$, $a_2 = -2\pi|k| \displaystyle\int_{-\varepsilon}^{\varepsilon} e^{-2\pi|k|z'} L_2(z')\ dz'$ (since $c_k(V)$ is colinear to e_1' in ε) and

$$c_k(V) = -2\pi|k| \begin{pmatrix} \displaystyle\int_{-\varepsilon}^{z} e^{-2\pi|k|(z-z')} (c_k(r_2^\varepsilon) - c_k(H_1^2) + \frac{1}{2}(c_k(U_2^2) - c_k(U_1^2)))\ dz' \\ \displaystyle -\int_{z}^{\varepsilon} e^{2\pi|k|(z-z')} (- c_k(r_1^\varepsilon) + c_k(H_2^2) + \frac{1}{2}(c_k(U_2^2) - c_k(U_1^2)))\ dz' \\ 0 \end{pmatrix}$$

We are now interested in the regularity in z of the Fourier's coefficient:

$$\begin{aligned}
\int_{-\varepsilon}^{\varepsilon} |c_k(V_1)(z)|^2 \, dz &= 4\pi^2|k|^2 \int_{-\varepsilon}^{\varepsilon} \left(\int_{-\varepsilon}^{z} e^{-2\pi|k|(z-z')} L_1(z') \, dz' \right)^2 \, dz \\
&\leq 4\pi^2|k|^2 \int_{-\varepsilon}^{\varepsilon} \left(\int_{-\varepsilon}^{z} e^{-2\pi|k|(z-z')} \, dz' \right) \left(\int_{-\varepsilon}^{z} e^{-2\pi|k|(z-z')} |L_1(z')|^2 \, dz' \right) \, dz \\
&\leq 4\pi^2|k|^2 \int_{-\varepsilon}^{\varepsilon} \frac{1}{\pi|k|} \left(\int_{-\varepsilon}^{z} e^{-2\pi|k|(z-z')} |L_1(z')|^2 \, dz' \right) \, dz \\
&\leq 4\pi|k| \int_{-\varepsilon}^{\varepsilon} |L_1(z')|^2 \left(\int_{z'}^{\varepsilon} e^{-2\pi|k|(z-z')} \, dz \right) \, dz' \\
&\leq 4 \int_{-\varepsilon}^{\varepsilon} |L_1(z')|^2 \, dz'.
\end{aligned}$$

The same result stands for $c_k(V_2)$.
We derivate $c_k(V)$ in z and we use the estimates made on $c_k(V)$ to get the regularity. □

4.2 Remainder term of u

According to lemma 4.1 we let

$$a^\varepsilon(t,x,z) = U^0(t,x) + \varepsilon U^1\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon^2 U^2\left(t,x,\frac{z}{\varepsilon}\right) \text{ in } \Omega_\varepsilon,$$

$$b^\varepsilon(t,x,z) = U^1\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon U^2\left(t,x,\frac{z}{\varepsilon}\right) \text{ in } \Omega_\varepsilon,$$

$$Q^\varepsilon = R^\varepsilon + S^\varepsilon \text{ where } R^\varepsilon \text{ is linear in } r^\varepsilon,$$

$$K^\varepsilon(t,x,z) = H^0(t,x) + \varepsilon H^1\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon^2 H^2\left(t,x,\frac{z}{\varepsilon}\right) + \varepsilon^2 S^\varepsilon \text{ in } \Omega_\varepsilon.$$

Back in the equation (3.3) we simplify by using the equations of the profiles. We get:

$$\begin{cases}
\dfrac{\partial r^\varepsilon}{\partial t} - \varepsilon^2 \Delta r^\varepsilon = T_1 + \cdots + T_8 + F^\varepsilon \text{ in } \Omega_\varepsilon, \\
\dfrac{\partial r^\varepsilon}{\partial n} = 0 \text{ on } \omega \times (-\varepsilon, \varepsilon), \\
r^\varepsilon(0,x,z) = r_0^\varepsilon(.) \text{ in } \Omega_\varepsilon,
\end{cases} \tag{4.5}$$

where

$$\begin{cases} T_1 = \varepsilon^6 |\nabla r^\varepsilon|^2 r^\varepsilon, \\ T_2 = \varepsilon^4 \left(|\nabla r^\varepsilon|^2 a^\varepsilon + 2(\nabla r^\varepsilon \cdot \nabla a^\varepsilon) r^\varepsilon \right), \\ T_3 = \varepsilon^2 \left(|\nabla a^\varepsilon|^2 r^\varepsilon + 2(\nabla r^\varepsilon \cdot \nabla a^\varepsilon) a^\varepsilon \right), \\ T_4 = \varepsilon^2 \left(r^\varepsilon \wedge \Delta a^\varepsilon + a^\varepsilon \wedge \Delta r^\varepsilon + \varepsilon^2 r^\varepsilon \wedge \Delta r^\varepsilon \right), \\ T_5 = a^\varepsilon \wedge R^\varepsilon + r^\varepsilon \wedge K^\varepsilon + \varepsilon^2 r^\varepsilon \wedge R^\varepsilon, \\ T_6 = -\left[a^\varepsilon \wedge (a^\varepsilon \wedge R^\varepsilon) + a^\varepsilon \wedge (r^\varepsilon \wedge K^\varepsilon) + r^\varepsilon \wedge (a^\varepsilon \wedge K^\varepsilon) \right], \\ T_7 = -\varepsilon^2 \left[r^\varepsilon \wedge (r^\varepsilon \wedge K^\varepsilon) + r^\varepsilon \wedge (a^\varepsilon \wedge R^\varepsilon) + a^\varepsilon \wedge (r^\varepsilon \wedge R^\varepsilon) \right], \\ T_8 = -\varepsilon^4 r^\varepsilon \wedge (r^\varepsilon \wedge R^\varepsilon), \end{cases}$$

$$\begin{aligned} F^\varepsilon = \; & \varepsilon \left[\Delta_x b^\varepsilon + \varepsilon^2 |\nabla b^\varepsilon|^2 b^\varepsilon - \varepsilon |U_z^2|^2 a^\varepsilon - 2\varepsilon (U_z^2 \cdot a_z^\varepsilon) U^2 + \varepsilon^2 |U_z^2|^2 b^\varepsilon + |\nabla_x a^\varepsilon|^2 b^\varepsilon - \varepsilon |\nabla_x b^\varepsilon|^2 a^\varepsilon \right. \\ & + (U_z^2 \cdot a_z^\varepsilon) b^\varepsilon + 2(\nabla_x a^e \cdot \nabla_x b^\varepsilon) a^\varepsilon - 2\varepsilon (\nabla_x a^\varepsilon \cdot \nabla_x b^\varepsilon) b^\varepsilon + 2\varepsilon^2 (b_z^\varepsilon \cdot U_z^2) U^2 - 4\varepsilon^2 (b_z^\varepsilon \cdot U_z^2) b^\varepsilon \\ & + 2(U_z^2 \cdot b_z^\varepsilon) a^\varepsilon + 2(a_z^\varepsilon \cdot b_z^\varepsilon) U^2 - 2\varepsilon |b_z^\varepsilon|^2 U^2 - \varepsilon b^\varepsilon \wedge \Delta_x b^\varepsilon - \varepsilon U^2 \wedge U_{zz}^2 + a^\varepsilon \wedge \Delta_x b^\varepsilon \\ & + b^\varepsilon \wedge U_{zz}^2 + U^2 \wedge b_{zz}^\varepsilon + b^\varepsilon \wedge \Delta_x a^\varepsilon + U^2 \wedge H^1 + b^\varepsilon \wedge H^2 + a^\varepsilon \wedge S^\varepsilon - \varepsilon^2 U^2 \wedge \left(U^2 \wedge H^1 \right) \\ & - \varepsilon^3 U^2 \wedge \left(U^2 \wedge H^2 \right) - a^\varepsilon \wedge \left(U^2 \wedge H^1 \right) - U^2 \wedge \left(a^\varepsilon \wedge H^1 \right) + \varepsilon U^2 \wedge \left(U^2 \wedge K^\varepsilon \right) \\ & + 2\varepsilon b^\varepsilon \wedge \left(U^2 \wedge H^1 \right) + \varepsilon^2 b^\varepsilon \wedge \left(U^2 \wedge H^2 \right) - b^\varepsilon \wedge \left(a^\varepsilon \wedge H^2 \right) - b^\varepsilon \wedge \left(U^2 \wedge K^\varepsilon \right) \\ & + 2\varepsilon U^2 \wedge \left(b^\varepsilon \wedge H^1 \right) + \varepsilon^3 U^2 \wedge \left(b^\varepsilon \wedge H^2 \right) - a^\varepsilon \wedge \left(b^\varepsilon \wedge H^2 \right) - U^2 \wedge (b^\varepsilon \wedge K^\varepsilon) \\ & \left. - b^\varepsilon \wedge \left(b^\varepsilon \wedge H^1 \right) + \varepsilon b^\varepsilon \wedge \left(b^\varepsilon \wedge H^2 \right) - a^\varepsilon \wedge (a^\varepsilon \wedge S^\varepsilon) \right]. \end{aligned}$$

According to the regularity results we get on the profiles we have the following proposition:

Proposition 4.2. *Let us assume that* $U^0 \in \mathbb{W}^{6,\infty}(\omega)$.
For all p, $1 \leq p \leq +\infty$ *and for all* $T > 0$ *there exists some* ε*-independent constants* C_p *such that for all* $\varepsilon > 0$ *and* $t \in [0, T]$,

$$\varepsilon^{-\frac{1}{p}} \|a^\varepsilon(t,.)\|_{\mathbb{W}^{1,p}(\Omega_\varepsilon)} \leq C_p,$$

$$\varepsilon^{1-\frac{1}{p}} \|D^2 a^\varepsilon(t,.)\|_{\mathbb{L}^p(\Omega_\varepsilon)} \leq C_p.$$

For all p, $2 \leq p < +\infty$ *and for all* $T > 0$ *there exists an* ε*-independent constant* C_p *such that for all* $\varepsilon > 0$ *and* $t \in [0, T]$,

$$\varepsilon^{2-\frac{1}{p}} \|\nabla \Delta a^\varepsilon(t,.)\|_{\mathbb{L}^p(\Omega_\varepsilon)} \leq C_p.$$

For all p, $1 \leq p \leq +\infty$ *and for all* $T > 0$ *there exists some* ε*-independent constants* C_p *such*

that for all $\varepsilon > 0$ *and* $t \in [0,T]$,

$$\|K^\varepsilon(t,.)\|_{\mathbb{W}^{1,p}(\Omega_\varepsilon)} \leq C_p,$$

$$\|F^\varepsilon(t,.)\|_{\mathbb{H}^1(\Omega_\varepsilon)} \leq C_2.$$

Proof: as we assume that $u_0 \in \mathbb{W}^{6,\infty}(\omega)$ the regularity results on the profiles (Theorems 3.1, 3.3 and the classical Sobolev's embeddings give that

$$\|a^\varepsilon\|_{\mathbb{W}^{1,\infty}(\omega\times I_1)} \leq C, \text{ so}$$

$$\begin{aligned}
&\varepsilon^{-\frac{1}{p}}\|a^\varepsilon(t,.)\|_{\mathbb{W}^{1,p}(\Omega_\varepsilon)} \leq C_p, \quad \forall 1 \leq p \leq +\infty,\\
&\|D^2 a^\varepsilon\|_{\mathbb{L}^p(\Omega_\varepsilon)} \leq \varepsilon^{\frac{1}{p}}\|D^2 a^\varepsilon\|_{\mathbb{L}^p(\omega\times I_1)} \leq \varepsilon^{\frac{1}{p}-1}C, \quad \forall 1 \leq p \leq +\infty,
\end{aligned}$$

$$\|D^3 a^\varepsilon\|_{\mathbb{L}^p(\Omega_\varepsilon)} \leq \varepsilon^{\frac{1}{p}}\|D^3 a^\varepsilon\|_{\mathbb{L}^p(\omega\times I_1)} \leq \varepsilon^{\frac{1}{p}-2}C, \quad \forall 2 \leq p < +\infty.$$

The estimates on K^ε are easily deduced from the previous estimates except for $\varepsilon^2 S^\varepsilon$ where we use the lemma 4.1 and 2.8:

$$\begin{aligned}
\|S^\varepsilon\|_{\mathbb{L}^\infty(\Omega_\varepsilon)} &\leq C\varepsilon^{-\frac{1}{2}}\|S^\varepsilon\|_{\mathbb{H}^2_p(\omega)\otimes\mathbb{H}^1(-\varepsilon,\varepsilon)}\\
&\leq C\varepsilon^{-\frac{1}{2}}\left[\|U^2\|_{\mathbb{H}^2_p(\omega)\otimes\mathbb{L}^2(-\varepsilon,\varepsilon)} + \|H^2\|_{\mathbb{H}^2_p(\omega)\otimes\mathbb{L}^2(-\varepsilon,\varepsilon)}\right]\\
&\leq C\left[\|U^2\|_{\mathbb{H}^2_p(\omega)\otimes\mathbb{L}^2(-1,1)} + \|H^2\|_{\mathbb{H}^2_p(\omega)\otimes\mathbb{L}^2(-1,1)}\right]\\
&\leq C\ ,
\end{aligned}$$

$$\begin{aligned}
\|\nabla S^\varepsilon\|_{\mathbb{L}^\infty(\Omega_\varepsilon)} &\leq C\varepsilon^{-\frac{1}{2}}\|\nabla S^\varepsilon\|_{\mathbb{H}^2_p(\omega)\otimes\mathbb{H}^1(-\varepsilon,\varepsilon)}\\
&\leq C\varepsilon^{-\frac{1}{2}}\left[\|U^2\|_{\mathbb{H}^3_p(\omega)\otimes\mathbb{H}^1(-\varepsilon,\varepsilon)} + \|H^2\|_{\mathbb{H}^3_p(\omega)\otimes\mathbb{H}^1(-\varepsilon,\varepsilon)}\right]\\
&\leq C\frac{1}{\varepsilon}\left[\|U^2\|_{\mathbb{H}^3_p(\omega)\otimes\mathbb{H}^1(-1,1)} + \|H^2\|_{\mathbb{H}^3_p(\omega)\otimes\mathbb{H}^1(-1,1)}\right]\\
&\leq C\varepsilon^{-1}.
\end{aligned}$$

The estimates on F^ε are easily deduced from the previous results. □

5 Proof of theorem 1.2 – Estimate on the remainder term

We will now give an estimate on the remainder term r^ε. We will so get another proof of the existence of u for an initial data with regularity $\mathbb{W}^{6,\infty}(\omega)$ instead of $\mathbb{H}^3$. This loss of regularity is typical of the asymptotic expansion methods (see [9], [3], [7]). These estimates are made on an approximation of Galerkin which will allow us to justify all integrations by parts. The approximation space we consider is built on the basis of eigenfunctions of the Laplace's operator with domain $\left\{u \in \mathbb{H}^2(\Omega_\varepsilon),\ \frac{\partial u}{\partial n} = 0 \text{ on } \omega\times\{\pm\varepsilon\}\right\}$ for Landau-Lifschitz' equation. For the magnetic field we only introduce R_n the projection of $R(r_n)$ on the approximation space, but we will not detail this step.
In the following we let

$$\mathbf{Q}(t) = \left(\|r_n^\varepsilon(t,.)\|^2_{\mathbb{L}^2} + \|\nabla r_n^\varepsilon(t,.)\|^2_{\mathbb{L}^2} + \|\varepsilon\Delta r_n^\varepsilon(t,.)\|^2_{\mathbb{L}^2}\right).$$

As we consider the Galerkin's approximation of r^ε we get the existence of $\mathbf{Q}$ on a maximum interval $\left[0, T^*_{n,\varepsilon}\right[$ where $0 < T^*_{n,\varepsilon} \leq +\infty$. We want to prove here the existence of ε_T small enough

such that we have the existence of $\mathbf{Q}$ on $[0,T]$ independently of n if $\varepsilon \leq \varepsilon_T$. We also show the estimates of the main result. We will proceed as it follows:
We let $T>0$ and thanks to the energy estimates we get by integrations by parts, we show that $\mathbf{Q}$ fulfills on $[0,T]\cap[0,T^*_{n,\varepsilon}[$ a differential inequality:

$$\mathbf{Q}'(t) \leq C_T(1+\mathbf{Q}(t)) + \varepsilon P_T(\mathbf{Q}(t)), \quad t\in[0,T]\cap[0,T^*_{n,\varepsilon}[, \tag{5.1}$$

where C_T is a constant and P_T a polynomial, both independent from n and ε (but not from T). This inequality implies that $T^*_{n,\varepsilon} \geq T$ as soon as $\varepsilon \in]0,\varepsilon_T]$ where ε_T is a constant small enough. $\mathbf{Q}(t)$ fulfills then (5.1) on the interval $[0,T]$ and we get the main theorem.

Remark 5.1. *The remainder term r^ε satisfies the same equation as the remainder that Carbou, Fabrie and Guès get in [7]. Although the thin layer induces some changes in the proof we will follow the same scheme.*

In the following estimates we will use some of the inequalities reminded in the lemma 2.1, 2.4 and 2.6.

$\mathbb{L}^2$-estimate: we multiply (4.5) by r^ε and we integrate on Ω_ε. The integrations by parts are licit since we are considering Galerkin's approximation. We get:

$$\frac{1}{2}\frac{d}{dt}\left(\|r^\varepsilon\|_2^2\right) + \varepsilon^2\|\nabla r^\varepsilon\|_2^2 \leq \int_{\Omega_\varepsilon}(T_1+\cdots+T_8+F^\varepsilon)\cdot v^\varepsilon.$$

We estimate each $\displaystyle\int_{\Omega_\varepsilon} T_i r^\varepsilon$ thanks to Hölder's inequalities then by anisotropic Sobolev's injections:

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_1 r^\varepsilon\right| &\leq \varepsilon^6\|r^\varepsilon\|^2_{\mathbb{L}^\infty}\|\nabla r^\varepsilon\|^2_{\mathbb{L}^2}\\
&\leq C\,\varepsilon^{7/2}\|r^\varepsilon\|^{5/2}_{\mathbb{H}^1}\|\varepsilon r^\varepsilon\|^{3/2}_{\mathbb{H}^2} \leq C\varepsilon^{7/2}\mathbf{Q}^2
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_2 r^\varepsilon\right| &\leq \varepsilon^4\|a^\varepsilon\|_{\mathbb{L}^\infty}\|r^\varepsilon\|_{\mathbb{L}^3}\|\nabla r^\varepsilon\|^2_{\mathbb{L}^3} + 2\varepsilon^4\|r^\varepsilon\|^2_{\mathbb{L}^3}\|\nabla a^\varepsilon\|_{\mathbb{L}^\infty}\|\nabla r^\varepsilon\|_{\mathbb{L}^3}\\
&\leq C\varepsilon^{5/2}\|r^\varepsilon\|^{1/2}_{\mathbb{L}^2}\|r^\varepsilon\|^{3/2}_{\mathbb{H}^1}\|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}\\
&\quad + C\,\varepsilon^3\|r^\varepsilon\|_{\mathbb{L}^2}\|r^\varepsilon\|^{3/2}_{\mathbb{H}^1}\|\varepsilon r^\varepsilon\|^{1/2}_{\mathbb{H}^2} \leq C(\varepsilon^{5/2}+\varepsilon^3)\mathbf{Q}^{3/2}
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_3 r^\varepsilon\right| &\leq \varepsilon^2\|r^\varepsilon\|^2_{\mathbb{L}^2}\|\nabla a^\varepsilon\|^2_{\mathbb{L}^\infty} + 2\varepsilon^2\|a^\varepsilon\|_{\mathbb{L}^\infty}\|r^\varepsilon\|_{\mathbb{L}^2}\|\nabla r^\varepsilon\|_{\mathbb{L}^2}\|\nabla a^\varepsilon\|_{\mathbb{L}^\infty}\\
&\leq C\,\varepsilon^2\,\mathbf{Q}
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_4 r^\varepsilon\right| &\leq \varepsilon^2\|r^\varepsilon\|_{\mathbb{L}^2}\|\nabla a^\varepsilon\|_{\mathbb{L}^\infty}\|\nabla r^\varepsilon\|_{\mathbb{L}^2}\\
&\leq C\varepsilon^2\|r^\varepsilon\|_{\mathbb{L}^2}\|\nabla r^\varepsilon\|_{\mathbb{L}^2} \leq C\,\varepsilon^2\,\mathbf{Q}
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_5 r^\varepsilon\right| &\leq \|a^\varepsilon\|_{\mathbb{L}^\infty}\|r^\varepsilon\|_{\mathbb{L}^2}\|R^\varepsilon\|_{\mathbb{L}^2}\\
&\leq C\|r^\varepsilon\|^2_{\mathbb{L}^2}
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_6 r^\varepsilon\right| &\leq \|a^\varepsilon\|^2_{\mathbb{L}^\infty}\|r^\varepsilon\|_{\mathbb{L}^2}\|R^\varepsilon\|_{\mathbb{L}^2} + \|a^\varepsilon\|_{\mathbb{L}^\infty}\|K^\varepsilon\|_{\mathbb{L}^\infty}\|r^\varepsilon\|^2_{\mathbb{L}^2} \\
&\leq C\|r^\varepsilon\|_{\mathbb{L}^2}\|R^\varepsilon\|_{\mathbb{L}^2} + C\|r^\varepsilon\|^2_{\mathbb{L}^2} \leq C\|r^\varepsilon\|^2_{\mathbb{L}^2}
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_7 r^\varepsilon\right| &\leq C\varepsilon^2\|a^\varepsilon\|_{\mathbb{L}^\infty}\|r^\varepsilon\|^2_{\mathbb{L}^4}\|R^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C\varepsilon^{3/2}\|r^\varepsilon\|^{1/2}_{\mathbb{L}^2}\|r^\varepsilon\|^{3/2}_{\mathbb{H}^1}\|R^\varepsilon\|_{\mathbb{L}^2} \leq C\ \varepsilon^{3/2}\ \mathbf{Q}^{3/2}
\end{aligned}$$

$$\left|\int_{\Omega_\varepsilon} T_8 r^\varepsilon\right| = 0$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} F^\varepsilon r^\varepsilon\right| &\leq \|F^\varepsilon\|_{\mathbb{L}^2}\|r^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C(1+\|r^\varepsilon\|^2_{\mathbb{L}^2})
\end{aligned}$$

By using the proposition 4.2 we get that there exists a constant C such that

$$\frac{d}{dt}\|r^\varepsilon\|^2_{\mathbb{L}^2} + \varepsilon^2\|\nabla r^\varepsilon\|^2_{\mathbb{L}^2} \leq C\left(1+\|r^\varepsilon\|^2_{\mathbb{L}^2}\right) + \varepsilon^{3/2}P(\mathbf{Q}), \tag{5.2}$$

where P is a function which increases polynomially.

$\mathbb{H}^1$-estimate: By multiplying Galerkin's approximation of the equation (4.5) by Δr^ε and by integrating by parts we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla r^\varepsilon\|^2_{\mathbb{L}^2}\right) + \varepsilon^2\|\Delta r^\varepsilon\|^2_{\mathbb{L}^2} \leq \int_{\Omega_\varepsilon}(T_1+\cdots+T_8+F^\varepsilon)\cdot\Delta r^\varepsilon$$

and we estimate each term of the second member:

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_1\Delta r^\varepsilon\right| &\leq \varepsilon^6\|r^\varepsilon\|_{\mathbb{L}^6}\|\Delta r^\varepsilon\|_{\mathbb{L}^2}\|\nabla r^\varepsilon\|^2_{\mathbb{L}^6} \\
&\leq C\varepsilon^2\|r^\varepsilon\|_{\mathbb{H}^1}\|\varepsilon r^\varepsilon\|^3_{\mathbb{H}^2} \\
&\leq C\varepsilon^2\ \mathbf{Q}^2
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_2\Delta r^\varepsilon\right| &\leq \varepsilon^4\|a^\varepsilon\|_{\mathbb{L}^\infty}\|\nabla r^\varepsilon\|^2_{\mathbb{L}^4}\|\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +2\varepsilon^4\|r^\varepsilon\|_{\mathbb{L}^6}\|\nabla r^\varepsilon\|_{\mathbb{L}^3}\|\nabla a^\varepsilon\|_{\mathbb{L}^\infty}\|\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C\varepsilon\|r^\varepsilon\|^{1/2}_{\mathbb{H}^1}\|\varepsilon r^\varepsilon\|^{5/2}_{\mathbb{H}^2} + C\varepsilon^2\|r^\varepsilon\|^{3/2}_{\mathbb{H}^1}\|\varepsilon r^\varepsilon\|^{3/2}_{\mathbb{H}^2} \\
&\leq C\ (\varepsilon+\varepsilon^2)\ \mathbf{Q}^{3/2}
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_3\Delta r^\varepsilon\right| &\leq \varepsilon^2\|r^\varepsilon\|_{\mathbb{L}^2}\|\nabla a^\varepsilon\|^2_{\mathbb{L}^\infty}\|\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +2\varepsilon^2\|a^\varepsilon\|_{\mathbb{L}^\infty}\|\nabla r^\varepsilon\|_{\mathbb{L}^2}\|\nabla a^\varepsilon\|_{\mathbb{L}^\infty}\|\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C\ \varepsilon\ \mathbf{Q}
\end{aligned}$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_4 \Delta r^\varepsilon\right| &\leq \varepsilon^2 \left\|r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla \Delta a^\varepsilon\right\|_{\mathbb{L}^3} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C \ \left\|r^\varepsilon\right\|_{\mathbb{H}^1}^2 \\
&\leq C \ \mathbf{Q}
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_5 \Delta r^\varepsilon\right| &\leq \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} + \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad + \left\|r^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} + \varepsilon^2 \left\|r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^3} \\
&\leq C \left(\|r^\varepsilon\|_{\mathbb{H}^1}^2 + \varepsilon \|r^\varepsilon\|_{\mathbb{H}^1}^{5/2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{1/2} \right) \\
&\leq C \ \mathbf{Q} \ + \ C \ \varepsilon \ \mathbf{Q}^{3/2}
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_6 \Delta r^\varepsilon\right| &\leq 2 \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad + \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty}^2 \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} + \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|r^\varepsilon\right\|_{\mathbb{L}^2} \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad + \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2}^2 \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} + 2 \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|r^\varepsilon\right\|_{\mathbb{L}^2} \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad + \left\|r^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C \|r^\varepsilon\|_{\mathbb{H}^1}^2 \\
&\leq C \ \mathbf{Q}
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_7 \Delta r^\varepsilon\right| &\leq \varepsilon^2 \left\|r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^3} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} \\
&\quad + \varepsilon^2 \left\|r^\varepsilon\right\|_{\mathbb{L}^4}^2 \left\|\nabla K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad + 2\varepsilon^2 \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|r^\varepsilon\right\|_{\mathbb{L}^4} \left\|R^\varepsilon\right\|_{\mathbb{L}^4} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad + \varepsilon^2 \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \left\|R^\varepsilon\right\|_{\mathbb{L}^4} \left\|r^\varepsilon\right\|_{\mathbb{L}^4} \\
&\quad + 2\varepsilon^2 \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^3} \\
&\leq C\varepsilon \left(\|r^\varepsilon\|_{\mathbb{H}^1}^{5/2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{1/2} + \varepsilon^{1/2} \|r^\varepsilon\|_{\mathbb{H}^1}^3 \right) \\
&\leq C \ (\varepsilon + \varepsilon^{3/2}) \ \mathbf{Q}^{3/2}
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} T_8 \Delta r^\varepsilon\right| &\leq \varepsilon^4 \left\|r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^3}^2 \left\|R^\varepsilon\right\|_{\mathbb{L}^6} \\
&\quad + \varepsilon^4 \left\|r^\varepsilon\right\|_{\mathbb{L}^6}^2 \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^3} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^3} \\
&\leq C\varepsilon^2 \|r^\varepsilon\|_{\mathbb{H}^1}^3 \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2} \\
&\leq C \ \varepsilon^2 \ \mathbf{Q}^2
\end{aligned}
$$

$$
\left|\int_{\Omega_\varepsilon} F^\varepsilon \Delta r^\varepsilon\right| \leq \left\|\nabla F^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \leq C(1 + \|\nabla r^\varepsilon\|_{\mathbb{L}^2}^2)
$$

By summing the previous inequalities we get:

$$\frac{d}{dt}\left(\|\nabla r^\varepsilon\|_{\mathbb{L}^2}^2\right) + \varepsilon^2 \|\Delta r^\varepsilon\|_{\mathbb{L}^2}^2 \leq C(1+\mathbf{Q}) + \varepsilon P(\mathbf{Q}), \tag{5.3}$$

where P is a polynomial.

$\mathbb{H}^2$-estimate: We will now estimate $\|\varepsilon\Delta r^\varepsilon\|_{\mathbb{L}^2}^2$ by multiplying 4.5 by $\varepsilon^2\Delta^2 r^\varepsilon$. On each term we use at first Hölder's inequalities then anisotropic Sobolev's injections and finally Young's inequality in order to isolate the term $\varepsilon^2 \|\varepsilon\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2}^2$.
Finally we get:

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_1 \Delta^2 r^\varepsilon\right| &\leq \varepsilon^8 \left(\|\nabla r^\varepsilon\|_{\mathbb{L}^6}^3 \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} + \|r^\varepsilon\|_{\mathbb{L}^{12}} \|\Delta r^\varepsilon\|_{\mathbb{L}^3} \|\nabla r^\varepsilon\|_{\mathbb{L}^{12}} \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2}\right) \\
&\leq C\varepsilon^2 \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^3 + C\varepsilon^2 \|r^\varepsilon\|_{\mathbb{H}^1}^{3/4} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{3/2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2}^{7/4} \\
&\leq \frac{1}{18}\left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 \;+\; C\;\varepsilon^4\;\mathbf{Q}^3 \;+\; C\;\varepsilon^{16}\;\mathbf{Q}^9
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_2 \Delta^2 r^\varepsilon\right| &\leq 2\varepsilon^6 \|\nabla a^\varepsilon\|_{\mathbb{L}^\infty} \|\nabla r^\varepsilon\|_{\mathbb{L}^4}^2 \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^6 \|a^\varepsilon\|_{\mathbb{L}^\infty} \|D^2 r^\varepsilon\|_{\mathbb{L}^3} \|\nabla r^\varepsilon\|_{\mathbb{L}^6} \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^6 \|r^\varepsilon\|_{\mathbb{L}^6} \left\|D^2 a^\varepsilon\right\|_{\mathbb{L}^6} \|\nabla r^\varepsilon\|_{\mathbb{L}^6} \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^6 \|r^\varepsilon\|_{\mathbb{L}^6} \|\nabla a^\varepsilon\|_{\mathbb{L}^\infty} \left\|D^2 r^\varepsilon\right\|_{\mathbb{L}^3} \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C\varepsilon^2 \|r^\varepsilon\|_{\mathbb{H}^1}^{1/2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{3/2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} + C\varepsilon^{3/2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^2 \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +C\varepsilon \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{3/2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2}^{3/2} + C\varepsilon^{3/2} \|r^\varepsilon\|_{\mathbb{H}^1} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +C\varepsilon^{5/2} \|r^\varepsilon\|_{\mathbb{H}^1} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} + C\varepsilon^2 \|r^\varepsilon\|_{\mathbb{H}^1} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{1/2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2}^{3/2} \\
&\leq C\;(\varepsilon^{3/2}+\varepsilon^2+\varepsilon^{5/2})\;\mathbf{Q}\left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \;+\; C\;(\varepsilon+\varepsilon^2)\;\mathbf{Q}^{3/4}\;\left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2}^{3/2} \\
&\leq \frac{1}{18}\|\varepsilon\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2}^2 \;+\; C\;(\varepsilon^3+\varepsilon^4+\varepsilon^5)\;\mathbf{Q}^2 \;+\; C\;(\varepsilon^4+\varepsilon^8)\;\mathbf{Q}^3
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_3 \Delta^2 r^\varepsilon\right| &\leq 2\varepsilon^4 \|\nabla r^\varepsilon\|_{\mathbb{L}^2} \|\nabla a^\varepsilon\|_{\mathbb{L}^\infty}^2 \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \|r^\varepsilon\|_{\mathbb{L}^6} \|\nabla a^\varepsilon\|_{\mathbb{L}^\infty} \left\|D^2 a^\varepsilon\right\|_{\mathbb{L}^3} \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \|a^\varepsilon\|_{\mathbb{L}^\infty} \|\nabla a^\varepsilon\|_{\mathbb{L}^\infty} \left\|D^2 r^\varepsilon\right\|_{\mathbb{L}^2} \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \|a^\varepsilon\|_{\mathbb{L}^\infty} \|\nabla r^\varepsilon\|_{\mathbb{L}^3} \left\|D^2 a^\varepsilon\right\|_{\mathbb{L}^6} \|\nabla\Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C(1+\varepsilon) \|r^\varepsilon\|_{\mathbb{H}^1} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +C\varepsilon^{1/2} \|r^\varepsilon\|_{\mathbb{H}^1}^{1/2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{1/2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +C\varepsilon \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2} \left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C\;(1+\varepsilon^{1/2}+\varepsilon)\;\mathbf{Q}^{1/2}\;\left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq \frac{1}{18}\left\|\varepsilon^2\nabla\Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 + C\;(1+\varepsilon+\varepsilon^2)\;\mathbf{Q}
\end{aligned}$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_4 \Delta^2 r^\varepsilon\right| &\leq \varepsilon^4 \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\Delta a^\varepsilon\right\|_{\mathbb{L}^3} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \left\|r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla \Delta a^\varepsilon\right\|_{\mathbb{L}^3} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \left\|\Delta r^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^6 \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\Delta r^\varepsilon\right\|_{\mathbb{L}^3} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C \left\|\varepsilon r^\varepsilon\right\|_{\mathbb{H}^2} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} + C \left\|r^\varepsilon\right\|_{\mathbb{H}^1} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +C\varepsilon \left\|\varepsilon r^\varepsilon\right\|_{\mathbb{H}^2} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} + C\varepsilon \left\|\varepsilon r^\varepsilon\right\|_{\mathbb{H}^2}^{3/2} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^{3/2} \\
&\leq C\ (1+\varepsilon) \mathbf{Q}^{1/2}\ \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} + C\varepsilon \mathbf{Q}^{3/4} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^{3/2} \\
&\leq \frac{1}{18} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 \ +\ C\ (1+\varepsilon^2)\ \mathbf{Q}\ +C\varepsilon^4\ \mathbf{Q}^3
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_5 \Delta^2 r^\varepsilon\right| &\leq \varepsilon^2 \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^2 \left\|r^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^2 \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^2 \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^3} \left\|R^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \left\|r^\varepsilon\right\|_{\mathbb{L}^6} \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^3} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C\|r^\varepsilon\|_{\mathbb{H}^1} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} + C\varepsilon \|r^\varepsilon\|_{\mathbb{H}^1}^{3/2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{1/2} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C\ \mathbf{Q}^{1/2}\ \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \ +\ C\ \varepsilon\ \mathbf{Q} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq \frac{1}{18} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 + C\ \mathbf{Q}\ +\ C\ \varepsilon^2\ \mathbf{Q}^2
\end{aligned}
$$

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_6 \Delta^2 r^\varepsilon\right| &\leq 2\varepsilon^2 \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^2 \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty}^2 \left\|\nabla R^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^2 \left\|\nabla a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|r^\varepsilon\right\|_{\mathbb{L}^2} \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^2 \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla r^\varepsilon\right\|_{\mathbb{L}^2} \left\|K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^2 \left\|a^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|r^\varepsilon\right\|_{\mathbb{L}^2} \left\|\nabla K^\varepsilon\right\|_{\mathbb{L}^\infty} \left\|\nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C\|r^\varepsilon\|_{\mathbb{H}^1} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq \frac{1}{18} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 \ +\ C\ \mathbf{Q}
\end{aligned}
$$

$$\begin{aligned}
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_7 \Delta^2 r^\varepsilon\right| &\leq \varepsilon^4 \|r^\varepsilon\|_{\mathbb{L}^6} \|\nabla r^\varepsilon\|_{\mathbb{L}^3} \|K^\varepsilon\|_{\mathbb{L}^\infty} \|\nabla \Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \|r^\varepsilon\|_{\mathbb{L}^4}^2 \|\nabla K^\varepsilon\|_{\mathbb{L}^\infty} \|\nabla \Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \|r^\varepsilon\|_{\mathbb{L}^4} \|\nabla a^\varepsilon\|_{\mathbb{L}^\infty} \|R^\varepsilon\|_{\mathbb{L}^4} \|\nabla \Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \|a^\varepsilon\|_{\mathbb{L}^\infty} \|\nabla r^\varepsilon\|_{\mathbb{L}^3} \|R^\varepsilon\|_{\mathbb{L}^6} \|\nabla \Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^4 \|a^\varepsilon\|_{\mathbb{L}^\infty} \|r^\varepsilon\|_{\mathbb{L}^6} \|\nabla R^\varepsilon\|_{\mathbb{L}^3} \|\nabla \Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C\varepsilon \|r^\varepsilon\|_{\mathbb{H}^1}^{3/2} \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2}^{1/2} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} + C\varepsilon^{3/2} \|r^\varepsilon\|_{\mathbb{H}^1}^2 \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C(\varepsilon + \varepsilon^{3/2}) \mathbf{Q} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq \frac{1}{18} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 \; + \; C \; (\varepsilon^2 + \varepsilon^3) \; \mathbf{Q}^2 \\
\left|\int_{\Omega_\varepsilon} \varepsilon^2 T_8 \Delta^2 r^\varepsilon\right| &\leq 2\varepsilon^6 \|r^\varepsilon\|_{\mathbb{L}^6} \|\nabla r^\varepsilon\|_{\mathbb{L}^6} \|R^\varepsilon\|_{\mathbb{L}^6} \|\nabla \Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\quad +\varepsilon^6 \|r^\varepsilon\|_{\mathbb{L}^6}^2 \|\nabla R^\varepsilon\|_{\mathbb{L}^6} \|\nabla \Delta r^\varepsilon\|_{\mathbb{L}^2} \\
&\leq C\varepsilon^2 \|r^\varepsilon\|_{\mathbb{H}^1}^2 \|\varepsilon r^\varepsilon\|_{\mathbb{H}^2} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq \frac{1}{18} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 \; + \; C \; \varepsilon^4 \; \mathbf{Q}^3 \\
\left|\int_{\Omega_\varepsilon} \varepsilon^2 F^\varepsilon \Delta^2 r^\varepsilon\right| &\leq \|\nabla F^\varepsilon\|_{\mathbb{L}^2} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq C \left\|\varepsilon \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2} \\
&\leq \frac{1}{18} \left\|\varepsilon^2 \nabla \Delta r^\varepsilon\right\|_{\mathbb{L}^2}^2 + C
\end{aligned}$$

By adding the previous inequalities we get:

$$\frac{d}{dt}\left(\|\varepsilon \Delta r^\varepsilon\|_{\mathbb{L}^2}^2\right) + \varepsilon^2 \|\varepsilon \nabla \Delta r^\varepsilon\|_{\mathbb{L}^2}^2 \leq C(1 + \mathbf{Q}) + \varepsilon^2 P(\mathbf{Q}), \tag{5.4}$$

where P is an ε-independent polynomial.
By summing (5.2), (5.3) and (5.4) we finally have:

$$\frac{d}{dt}\mathbf{Q} + \varepsilon^2 \left(\|\nabla r^\varepsilon\|_{\mathbb{L}^2}^2 + \|\Delta r^\varepsilon\|_{\mathbb{L}^2}^2 + \|\varepsilon \nabla \Delta r^\varepsilon\|_{\mathbb{L}^2}^2\right) \leq C(1 + \mathbf{Q}) + \varepsilon P(\mathbf{Q}),$$

which implies the inequality (5.1). Classically we have the existence of a time T_ε such that $\lim_{\varepsilon \to 0} T_\varepsilon = +\infty$ and r^ε is bounded independently from ε in $\mathbb{L}^\infty\left(0, T; \mathbb{H}^1(\Omega_\varepsilon)\right)$ and $\varepsilon r^\varepsilon$ is bounded independently from ε in $\mathbb{L}^\infty\left(0, T; \mathbb{H}^2(\Omega_\varepsilon)\right)$ for all $0 < T < T_\varepsilon$. □

Remark 5.2. *By applying Poincaré's lemma we also get that for all $T > 0$,*

$$\|r^\varepsilon\|_{\mathbb{L}^\infty(0,T;\mathbb{L}^2(\Omega_\varepsilon))} = \mathcal{O}(\varepsilon),$$

i.e. we get a real asymptotic expansion at the second order in $\mathbb{L}^2(\Omega_\varepsilon)$ for finite times.

Remark 5.3. *We have studied here the behaviour of Landau-Lifschitz equations with an exchange coefficient ε^2 in a thin layer of thickness ε. That is the case where the thickness of the boundary layer (if there were any, cf [7]) and the domain's thickness are of the same size. If we instead consider a thin layer with a thickness ε^α ($\alpha > 0$, $\alpha = \frac{p}{q} \in \mathbf{Q}$), we perform now an asymptotic expansion at the scale $\varepsilon^{\frac{1}{q}}$ and we see two different behaviour. If $\alpha > 1$ we get the same type of behaviour than that of the case $\alpha = 1$, i.e. the behaviour of a thin layer and the localization of H at the low orders. If $\alpha < 1$, the layer's thickness is bigger than the characteristic thickness of the boundary layer. We then obtain a boundary layer in the thin layer but with a smaller thickness ($\varepsilon^{1-\alpha}$) and we also get at the low orders the localization of H.*

A Proof of theorem 2.7

Theorem 2.7. *Let $\Omega_0 = (0,1)^3$. Then there exists an absolute constant c such that for all $u \in \mathbb{H}^2(\Omega_0)$,*

$$\|u\|_{\mathbb{L}^{12}(\Omega_0)} \leq c \prod_{i=1}^{3} \left(\sum_{j=0}^{1} \left\| \frac{\partial^j u}{\partial x_i^j} \right\|_{\mathbb{L}^2(\Omega_0)} \right)^{\frac{1}{4}} \left(\sum_{j=0}^{2} \left\| \frac{\partial^j u}{\partial x_i^j} \right\|_{\mathbb{L}^2(\Omega_0)} \right)^{\frac{1}{12}}.$$

Proof: We have the classical Sobolev's injection: $\mathbb{H}^{5/4}(\Omega_0) \subset \mathbb{L}^{12}(\Omega_0)$. We will prove this theorem in three steps.

Step 1. We replace Ω_0 by $\mathbb{R}^3$ and assume that $u \in \mathcal{C}_0^2(\mathbb{R}^3)$. We write, using the Sobolev's inclusion $\mathbb{H}^{\frac{5}{4}}(\Omega_0) \subset \mathbb{L}^{12}(\Omega_0)$,

$$\|u\|_{\mathbb{L}^{12}(\mathbb{R}^3)} \leq c_0 \|u\|_{\mathbb{H}^{\frac{5}{4}}(\mathbb{R}^3)} = c_0 \left(\int_{\mathbb{R}^d} (1 + |\xi|^{\frac{5}{2}}) |\hat{u}|^2 \, d\xi \right)^{\frac{1}{2}}, \tag{A.1}$$

where $\hat{u}$ is the Fourier transform of u. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$, $\lambda_i > 0$ and set $y_i = \lambda_i x_i$ for all $i = 1, 2, 3$. We define the function $v_\lambda \in \mathcal{C}_0^2(\mathbb{R}^3)$ as follows:

$$v_\lambda(y_1, y_2, y_3) = u\left(\frac{y_1}{\lambda_1}, \frac{y_2}{\lambda_2}, \frac{y_3}{\lambda_3} \right).$$

We have immediately

$$\begin{cases} \|v_\lambda\|_{\mathbb{L}^{12}(\mathbb{R}^3)} = \left(\prod_{i=1}^3 \lambda_i \right)^{\frac{1}{12}} \|u\|_{\mathbb{L}^{12}(\mathbb{R}^3)}, \\ \hat{v}_\lambda(\xi_1, \xi_2, \xi_3) = \left(\prod_{i=1}^3 \lambda_i \right) \hat{u}(\lambda_1 \xi_1, \lambda_2 \xi_2, \lambda_3 \xi_3). \end{cases}$$

Let $\xi' = (\lambda_1 \xi_1, \lambda_2 \xi_2, \lambda_3 \xi_3)$. We have

$$\begin{aligned} \|v_\lambda\|_{\mathbb{H}^{\frac{5}{4}}(\mathbb{R}^3)} &= \left(\prod_{i=1}^3 \lambda_i \right) \left[\int_{\mathbb{R}^3} |\hat{u}(\xi')|^2 \, d\xi + \int_{\mathbb{R}^3} |\xi|^{\frac{5}{2}} |\hat{u}(\xi')|^2 \, d\xi \right]^{\frac{1}{2}} \\ &\leq \left(\prod_{i=1}^3 \lambda_i \right)^{\frac{1}{2}} \left[\|u\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + C \sum_{i=1}^3 \int_{\mathbb{R}^3} \left(\frac{\xi_i'}{\lambda_i} \right)^{\frac{5}{2}} |\hat{u}(\xi')|^2 \, d\xi' \right]^{\frac{1}{2}}. \end{aligned}$$

Cauchy-Schwarz' lemma gives

$$\begin{aligned}\int_{\mathbb{R}^3} (\xi_i')^{\frac{5}{2}} |\hat{u}(\xi')|^2 \, d\xi' &\leq \left(\int_{\mathbb{R}^3} |\xi_i|^2 |\hat{u}|^2 \, d\xi_i \right)^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\xi_i|^4 |\hat{u}|^2 \, d\xi_i \right)^{\frac{1}{2}} \\ &\leq \left\| \frac{\partial u}{\partial x_i} \right\|^{\frac{3}{2}}_{\mathbb{L}^2(\mathbb{R}^3)} \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|^{\frac{1}{2}}_{\mathbb{L}^2(\mathbb{R}^3)}.\end{aligned}$$

Inequality (A.1) applied to v_λ yields

$$\|u\|_{\mathbb{L}^{12}(\mathbb{R}^3)} \leq \left(\prod_{i=1}^3 \lambda_i \right)^{\frac{5}{12}} \left[\|u\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + C \sum_{i=1}^3 \frac{1}{\lambda_i^{\frac{5}{2}}} \left\| \frac{\partial u}{\partial x_i} \right\|^{\frac{3}{2}}_{\mathbb{L}^2(\mathbb{R}^3)} \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|^{\frac{1}{2}}_{\mathbb{L}^2(\mathbb{R}^3)} \right]^{\frac{1}{2}}, \tag{A.2}$$

and since (A.2) is valid for all choices of $\lambda_1, \lambda_2, \lambda_3$, we can take

$$\lambda_i = C^{\frac{2}{5}} \left[\frac{\left\| \frac{\partial u}{\partial x_i} \right\|^{\frac{3}{4}}_{\mathbb{L}^2(\mathbb{R}^3)} \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|^{\frac{1}{4}}_{\mathbb{L}^2(\mathbb{R}^3)}}{\|u\|_{\mathbb{L}^2(\mathbb{R}^3)}} \right]^{\frac{4}{5}} \quad \text{for } i = 1, 2, 3,$$

Hence,

$$\|u\|_{\mathbb{L}^{12}(\mathbb{R}^3)} \leq c_1 \prod_{i=1}^3 \left\| \frac{\partial u}{\partial x_i} \right\|^{\frac{1}{4}}_{\mathbb{L}^2(\mathbb{R}^3)} \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|^{\frac{1}{12}}_{\mathbb{L}^2(\mathbb{R}^3)}.$$

Step 2. We will show that there exists an absolute constant $c_2 > 0$, such that if $u \in \mathcal{C}_0^2(\overline{\Omega_0})$, then there exists $\overline{u} \in \mathcal{C}_0^2(\overline{\Omega_1})$, where $\Omega_1 = \left(-\frac{1}{3}, \frac{4}{3} \right)^3$ such that

$$\left\| \frac{\partial^j u}{\partial x_i^j} \right\|_{\mathbb{L}^2(\Omega_1)} \leq c_2 \left\| \frac{\partial^j \overline{u}}{\partial x_i^j} \right\|_{\mathbb{L}^2(\Omega_0)}, \quad j = 0, 1, 2; \ i = 1, 2, 3.$$

We use the classical Babitch extension operators (see [1], [13]); we first extend u to the domain $\left(-\frac{1}{3}, \frac{4}{3} \right) \times (0, 1)^2$ using

$$E_k^1 u(x) = \begin{cases} u(x) & \text{for } x_1 \in [0, 1], \\ \displaystyle\sum_{j=1}^3 (-j)^l \alpha_j \ u(-j x_1, x_2, \ldots, x_d) & \text{for } x_1 \in \left[-\frac{1}{3}, 0 \right), \\ \displaystyle\sum_{j=1}^3 (-j)^l \alpha_j \ u(1 - j(x_1 - 1), x_2, \ldots, x_d) & \text{for } x_1 \in \left(1, \frac{4}{3} \right], \end{cases}$$

where $(\alpha_1, \alpha_2, \alpha_3)$ is the unique solution to the system

$$\sum_{j=1}^3 (-j)^k \alpha_j = 1, \quad k = 0, 1, 2. \tag{A.3}$$

We have immediately

$$\frac{\partial^l E_0^1 u}{\partial x_i^l} = E_l^1 \frac{\partial^l u}{\partial x_i^l}, \quad l = 0,1,2;\ i = 1,2,3,$$

and (A.3) implies that $E_0^1 u \in \mathcal{C}^2\left([-1/3,4/3]\times[0,1]^2\right)$ if $u \in \mathcal{C}^2(\overline{\Omega_0})$. Moreover

$$\int_{(-1/3,4/3)\times(0,1)^2} \left|\frac{\partial^k E_0^1 u}{\partial x_i^k}\right|^2 dx = \int_{(-1/3,4/3)\times(0,1)^2} \left|E_k^1 \frac{\partial^k u}{\partial x_i^k}\right|^2 dx$$
$$= \int_{\Omega_0} \left|\frac{\partial^k u}{\partial x_i^k}\right|^2 dx + \int_{(-1/3,0)\times(0,1)^2} \left|\sum_{j=1}^{3} (-j)^k \alpha_j \frac{\partial^k u}{\partial x_i^k}(-jx_1, x_2, \ldots, x_d)\right|^2 dx$$
$$+ \int_{(1,4/3)\times(0,1)^2} \left|\sum_{j=1}^{3} (-j)^k \alpha_j \frac{\partial^k u}{\partial x_i^k}(1 - j(x_1-1), x_2, \ldots, x_d)\right|^2 dx \le C_1 \int_{\Omega_0} \left|\frac{\partial^k u}{\partial x_i^k}\right|^2 dx.$$

The extension is complete in the direction x_1. The same argument can be applied successively in the directions x_2, x_3 so that we reach the domain Ω_1 and the extended function satisfies

$$\left\|\frac{\partial^l \overline{u}}{\partial x_i^l}\right\|^2_{\mathbb{L}^2(\Omega_1)} \le C_1^3 \left\|\frac{\partial^l u}{\partial x_i^l}\right\|^2_{\mathbb{L}^2(\Omega_0)} \qquad l = 0,1,2;\ i = 1,2,3.$$

Step 3. We will show that we can localize without mixing the directions. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that supp $\varphi \subset (-1/3, 4/3)$, $0 \le \varphi \le 1$ and $\varphi = 1$ on $[0,1]$. Set $\Phi(x_1,x_2,x_3) = \prod_{i=1}^{3} \varphi(x_i)$. We have immediately that $\Phi\overline{u} \in \mathcal{C}_0^2(\mathbb{R}^d)$ and $\|u\|_{\mathbb{L}^r(\Omega_0)} \le \|\Phi\overline{u}\|_{\mathbb{L}^r(\mathbb{R}^3)} \le c_3\|u\|_{\mathbb{L}^r(\Omega_0)}$ for all $1 \le r \le +\infty$, where $\overline{u}$ is defined in step 2 and c_3 is an absolute constant. Moreover,

$$\frac{\partial^2 \Phi\overline{u}}{\partial x_i^2} = \sum_{i=0}^{2} C_l^2 \frac{\partial^i \overline{u}}{\partial x_i^i} \frac{\partial^{2-i}\Phi}{\partial x_i^{2-i}}$$

Let $K_2 = \sup_{\mathbb{R}} \left(\sum_{i=0}^{2} C_2^i |\varphi^{(i)}|\right)$, we have

$$\left\|\frac{\partial^2 \Phi\overline{u}}{\partial x_i^2}\right\|_{\mathbb{L}^2(\mathbb{R}^3)} \le K_l \left[\sum_{i=0}^{2} \left\|\frac{\partial^i \overline{u}}{\partial x_i^i}\right\|_{\mathbb{L}^2(\Omega_1)}\right] \le K_{1,l} \left[\sum_{i=0}^{2} \left\|\frac{\partial^i u}{\partial x_i^i}\right\|_{\mathbb{L}^2(\Omega_0)}\right],$$

which concludes the proof. □

Remark A.1. *This proof can be generalized straightforward to the case of an embedding from* $\mathbb{H}^s(\Omega)$ *into* $\mathbb{L}^q(\Omega)$ *where* $\Omega = (0,1)^d \subset \mathbb{R}^d$ *and* $q = \dfrac{2d}{d-2s}$.

Acknowledgments: The author wishes to warmly thank Gilles Carbou and Pierre Fabrie for their help and their advice.

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