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# Long waves in ferromagnetic media, Khokhlov-Zabolotskaya equation 

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#### Abstract

We are interested here in small perturbations of electromagnetic waves in a saturated ferromagnetic media. By means of an asymptotic expansion we prove that the solution remains close on long times of the one of the Khokhlov-Zabolotskaya equation.


Key words: Micromagnetics, ferromagnets, Maxwell equations, Landau-Lifschitz equation, Landau-Lifschitz-Gilbert equation, travelling waves, asymptotic analysis, perturbation theory, longwaves, Khokhlov-Zabolotskaya equation.
1991 MSC: 35Q60, 78M35, 34E20.

## 1 Introduction

In this article, we show that small perturbations of equilibrium states in ferromagnetic media give rise to standing and travelling waves that are stable for long times. The evolution of their profiles is governed by the KhokhlovZabolotskaya equation which can be considered as a semilinear heat equation which has been perturbed in the transversal direction.
These results are obtained in the framework of micromagnetics. The basic variable is the magnetization vector $M$ whose dynamic equation was first given by Landau and Lifschitz [8] and later in an equivalent form by Gilbert [7]:

$$
\begin{equation*}
\partial_{t} M=-M \wedge H-\frac{\gamma}{|M|} M \wedge(M \wedge H) \quad \text { in } \mathbb{R}^{3} \tag{1}
\end{equation*}
$$

[^0]The equation is completed by the Maxwell equations for the electromagnetic field:

$$
\left\{\begin{array}{l}
\partial_{t}(H+M)-\nabla \wedge E=0 \quad \text { in } \mathbb{R}^{3}  \tag{2}\\
\partial_{t} E+\nabla \wedge H=0 \quad \text { in } \mathbb{R}^{3}
\end{array}\right.
$$

The Landau-Lifschitz-Maxwell equations admit stable equilibrium solutions where the magnetization is uniform and everywhere parallel to the effective magnetic field:

$$
\begin{equation*}
(M, H, E)_{\alpha}=\left(M_{0}, \alpha^{-1} M_{0}, 0\right), \quad \alpha>0 . \tag{3}
\end{equation*}
$$

We are interested in small perturbations of the equilibrium states whose size is measured by a small parameter $\varepsilon$. A formal derivation has been given by H. Leblond and M. Manna [10]. T. Colin, C. Galusinski and H. Kaper have studied the case of $1 D$-perturbations ([5]) and proved that the evolution of $1 D$-wave profile is governed on a slow time scale by semilinear heat equations. We study here the case of $2 D$-perturbations: we add a dependence on a slow transversal space direction $y$ and we seek their behaviour. The perturbation is taken in the form:

$$
\begin{aligned}
& M(t, x, y)=M_{0}+\varepsilon^{2} \widetilde{M}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}) \\
& H(t, x, y)=\alpha^{-1} M_{0}+\varepsilon^{2} \widetilde{H}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}) \\
& E(t, x, y)=\varepsilon^{2} \widetilde{E}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y})
\end{aligned}
$$

where $\tilde{t}, \tau, \widetilde{x}, \widetilde{y}$ are new rescaled variables:

$$
\tilde{t}=\varepsilon^{2} t, \quad \tau=\varepsilon^{4} t, \quad \widetilde{x}=\varepsilon^{2} x, \quad \tilde{y}=\varepsilon^{3} y .
$$

We introduce then the vector

$$
U(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y})=\left(\alpha^{-1 / 2} \widetilde{M}, \widetilde{H}, \widetilde{E}\right)^{t}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y})
$$

and the perturbation $U$ satisfy an equation in the form (we "forget" the ${ }^{\sim}$ on $t, x, y)$ :

$$
\left\{\begin{array}{l}
\partial_{t} U+\varepsilon^{2} \partial_{\tau} U+A_{1} \partial_{x} U+\varepsilon A_{2} \partial_{y} U+\varepsilon^{-2} L U=B(U, U)+\varepsilon^{2} T(U, U, U)  \tag{4}\\
U(0,0)=U_{0}^{0}
\end{array}\right.
$$

where the terms $A_{1}, A_{2}, L, B$ and $T$ are precised in Section 2.
Using a WKB method on $U$, we obtain that the leading term $U_{0}$ breaks down in five travelling and standing terms, $U_{0}=\sum_{j=1}^{5} u_{j}$, which satisfy:

$$
\left\{\begin{array}{l}
\left(\partial_{t}+v_{j} \partial_{x}\right) u_{j}=0  \tag{5}\\
\partial_{x}\left(\partial_{\tau} u_{j}-D_{j} \partial_{x}^{2} u_{j}+B_{j}\left(u_{j}, \partial_{x} u_{j}\right)+F_{j}\left(u_{j}, u_{j}, u_{j}\right)\right)=C_{j} \partial_{y}^{2} u_{j}
\end{array}\right.
$$

where the last equation is the Khokhlov-Zabolotskaya equation and

$$
v_{j} \in\left\{0, \pm\left(\frac{1}{1+\alpha}\right)^{1 / 2}, \pm\left(\frac{1+\left(1-\left|k \cdot M_{0}\right|^{2}\right) \alpha}{1+\alpha}\right)^{1 / 2}\right\}
$$

is the speed of the wave in the direction $k$ (the $x$-direction).
Through the study of the leading term of the expansion, we prove the following result, not yet stated to our knowledge:

Theorem 1.1 Let $D>0, C \neq 0$. Let $B:\left(\mathbb{R}^{k}\right)^{2} \rightarrow \mathbb{R}^{k}$ a bilinear application and
$F:\left(\mathbb{R}^{k}\right)^{3} \rightarrow \mathbb{R}^{k}$ a trilinear application. Let $u^{0} \in\left(\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)\right)^{k}, s>1$. Then there exists a time $T>0$ and an unique function $u \in \mathcal{C}^{0}\left([0, T] ;\left(\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)\right)^{k}\right)$ solution of the Khokhlov-Zabolotskaya equation:

$$
\left\{\begin{array}{l}
\partial_{x}\left(\partial_{\tau} u-D \partial_{x}^{2} u+B\left(u, \partial_{x} u\right)+F(u, u, u)\right)=C \partial_{y}^{2} u \quad \text { in } \mathbb{R}^{2} \\
u(\tau=0, x, y)=u^{0}(x, y) \quad \text { in } \mathbb{R}^{2}
\end{array}\right.
$$

We then prove that the general solution of Eq. (4) remains close to the leading term $U_{0}$ of the asymptotic expansion on a slow time scale (i.e. with the change $\left.\tau=\varepsilon^{2} t\right):$

Theorem 1.2 Let us assume we are in the ferromagnetic case (assumption on the structure of the equations) with an initial data $U_{0}^{0} \in \mathbb{H}^{s}\left(\mathbb{R}^{2}\right)(s>6)$. Let $T_{0}>0$ such that $U_{0}=\sum_{j=1}^{5} u_{j}$, solution of Eqs. (5), lies in $\mathcal{C}\left(\left[0, T_{0} / \varepsilon^{2}\right] ; \mathbb{H}^{s-1}\left(\mathbb{R}^{2}\right)\right)$. There exists $\varepsilon_{0}>0$ such that for all $0<\varepsilon<\varepsilon_{0}$, there is an unique solution $\mathcal{U}^{\varepsilon} \in \mathcal{C}^{0}\left(\left[0, T_{0} / \varepsilon^{2}\right] ; \mathbb{H}^{s-5}\left(\mathbb{R}^{2}\right)\right)$ of

$$
\partial_{t} \mathcal{U}^{\varepsilon}+A_{1} \partial_{x} \mathcal{U}^{\varepsilon}+A_{2} \partial_{y} \mathcal{U}^{\varepsilon}+\frac{1}{\varepsilon^{2}} L \mathcal{U}^{\varepsilon}=B\left(\mathcal{U}^{\varepsilon}, \mathcal{U}^{\varepsilon}\right)+\varepsilon^{2} T\left(\mathcal{U}^{\varepsilon}, \mathcal{U}^{\varepsilon}, \mathcal{U}^{\varepsilon}\right)
$$

with $\mathcal{U}^{\varepsilon}(0, x, y)=U_{0}^{0}(x, \varepsilon y)$. Moreover

$$
\left\|\mathcal{U}^{\varepsilon}-\mathcal{U}_{0}^{\varepsilon}\right\|_{\mathbb{L}^{\infty}\left(\left[0, T_{0} / \varepsilon^{2}\right] \times \mathbb{R}^{2}\right)}=o(1) \quad \text { as } \varepsilon \rightarrow 0
$$

where $\mathcal{U}_{0}^{\varepsilon}(t, x, y)=U_{0}\left(t, \varepsilon^{2} t, x, \varepsilon y\right)$.
Remark 1.1 To obtain the equation fulfilled by the leading term $U_{0}$, we assume that the profiles $U_{1}$ and $U_{2}$ satisfy a sublinear growth condition (cf Lemma 3.2). But the leading profile $U_{0}$ does not allow to build $U_{1}$ and $U_{2}$ with such properties. We use then low-frequency cut-offs methods ([2], [3], [11]) to get an approximation of the leading profile with which we build the remaining terms of the expansion and prove Theorem 1.2.

In Section 2 we give the complete mathematical model. The system of partial differential equations which governs the spatio-temporal evolution of the per-
turbations belongs to a general class of hyperbolic equation for vector-valued functions that we examine in Section 3. Performing an asymptotic expansion, we show that the equation admits an asymptotic solution that exhibits standing and travelling waves. The wave profiles move around on a slow time scale (measured in units of $\varepsilon^{2}$ ) according to the Khokhlov-Zabolotskaya equation. By means of nonlinear optics techniques, we show that the asymptotic solution remains close to the exact solution of the hyperbolic equation on the long time scale as $\varepsilon$ goes to zero. The main result for the general case is stated in Theorem 3.9 (Section 3.2.3), the local existence of regular solution for the Khokhlov-Zabolotskaya equation is stated in Theorem 3.4 (Section 3.2.1).

The derivation of the asymptotic solution and the proof of the convergence theorem require several hypotheses, which are satisfied in the case of the Landau-Lifschitz-Maxwell equations (Section 4). We find that the magnetization as well as the electromagnetic field variables breaks into standing waves and up to four travelling waves, whose speed of propagation varies with the equilibrium state.

## 2 Mathematical model

The state of a ferromagnet is described by the magnetization vector $M$ whose evolution is governed by the Landau-Lifschitz equation:

$$
\begin{equation*}
\frac{\partial M}{\partial t}=-M \wedge H-\frac{\gamma}{|M|} M \wedge(M \wedge H) \quad \text { in } \mathbb{R}^{3} \tag{6}
\end{equation*}
$$

where $\gamma$ is a dimensionless damping coefficient and $H$ the magnetic field. We can note that the magnitude $|M|$ of $M$ is an invariant of this equation. The electromagnetic field fulfills the Maxwell's equations:

$$
\left\{\begin{array}{l}
\frac{\partial(H+M)}{\partial t}-\nabla \wedge E=0 \quad \text { in } \mathbb{R}^{3}  \tag{7}\\
\frac{\partial E}{\partial t}+\nabla \wedge H=0 \quad \text { in } \mathbb{R}^{3}
\end{array}\right.
$$

where the equations are in dimensionless form and the coefficients have been set equal to one.

### 2.1 Basic solution

The system of equations (6)-(7) admits a family of constant solutions,

$$
(M, H, E)_{\alpha}=\left(M_{0}, \alpha^{-1} M_{0}, 0\right), \quad \alpha>0
$$

where $M_{0}$ is an arbitrary vector in $\mathbb{R}^{3}$ and $\alpha>0$ assures that the solution is stable. We may assume without restriction that $\left|M_{0}\right|=1$. We are interested in the spatio-temporal evolution of long-waves and transverse perturbations of such solutions. The perturbations are measured in terms of an arbitrarily small positive parameter $\varepsilon$ and have the form:

$$
\begin{aligned}
& M(t, x, y)=M_{0}+\varepsilon^{2} \widetilde{M}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}) \\
& H(t, x, y)=\alpha^{-1} M_{0}+\varepsilon^{2} \widetilde{H}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}) \\
& E(t, x, y)=\varepsilon^{2} \widetilde{E}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y})
\end{aligned}
$$

where $\widetilde{M}, \widetilde{H}$ and $\widetilde{E}$ are $O(1)$ as $\varepsilon$ goes to 0 , and:

$$
\widetilde{x}=\varepsilon^{2} x, \widetilde{y}=\varepsilon^{3} y, \tilde{t}=\varepsilon^{2} t, \tau=\varepsilon^{4} t
$$

Remark 2.1 We look for solutions propagating in the $x$-direction (still unknown) for which the dispersion and the non-linear effects occur at the same long time scale. This assumption sets the size of the non-linearity and the scales of $\tilde{t}, \widetilde{x}$ and $\tau$. We want to study here transversal deformations of a quasi-plane wave. This case corresponds to the scaling $\widetilde{y}=\varepsilon^{p} y$ with $p>2$. There is a dominant diffraction effect that occurs for the smallest value of $p$ for which we can "solve" the equation, which is here $p=3$. If we take $2<p<3$, the diffraction effect prevails over the dispersion and prevents us to observe the non-linear effects. On the contrary, if $p>3$ the diffraction is negligible.

If $(M, H, E)$ is a solution of equations (6)-(7) then $\widetilde{M}, \widetilde{H}$ and $\widetilde{E}$ have to fulfill:

$$
\begin{align*}
& \varepsilon^{2} \partial_{\overparen{t}} \widetilde{M}+\varepsilon^{4} \partial_{\tau} \widetilde{M}=-M_{0} \wedge \widetilde{H}+\alpha^{-1} M_{0} \wedge \widetilde{M}-\varepsilon^{2} \widetilde{M} \wedge \widetilde{H} \\
&-\frac{\gamma}{|M|}\left[M_{0} \wedge\left(M_{0} \wedge \widetilde{H}\right)-\alpha^{-1} M_{0} \wedge\left(M_{0} \wedge \widetilde{M}\right)\right.  \tag{8}\\
&+\varepsilon^{2} M_{0} \wedge(\widetilde{M} \wedge \widetilde{H})+\varepsilon^{2} \widetilde{M} \wedge\left(M_{0} \wedge \widetilde{H}\right) \\
&\left.-\varepsilon^{2} \alpha^{-1} \widetilde{M} \wedge\left(M_{0} \wedge \widetilde{M}\right)+\varepsilon^{4} \widetilde{M} \wedge(\widetilde{M} \wedge \widetilde{H})\right] \\
&\left\{\begin{aligned}
& \varepsilon^{2} \partial_{\overparen{t}} \widetilde{H}+\varepsilon^{4} \partial_{\tau} \widetilde{H}- \varepsilon^{2} \widetilde{\nabla}_{\widetilde{x}} \wedge \widetilde{E}-\varepsilon^{3} \widetilde{\nabla}_{\widetilde{y}} \wedge \widetilde{E}=-\varepsilon^{2} \partial_{t} \widetilde{M}-\varepsilon^{4} \partial_{\tau} \widetilde{M} \\
& \varepsilon^{2} \partial_{\overparen{t}} \widetilde{E}+\varepsilon^{4} \partial_{\tau} \widetilde{E}+\varepsilon^{2} \widetilde{\nabla}_{\widetilde{x}} \wedge \widetilde{H}+\varepsilon^{3} \widetilde{\nabla}_{\widetilde{y}} \wedge \widetilde{H}=0
\end{aligned}\right. \tag{9}
\end{align*}
$$

In Eq. (8) we have left the denominator's term $|M|$ unchanged. It induces technical complications that are non-essential for the arguments to be presented. Then we will change slightly the equation to completely avoid these complications: we replace the term $|M|$ by $\left|M_{0}\right|=1$ since this quantity is conserved.
We are interested in solutions of Eqs. (6)-(7) that describe a travelling wave propagating in the direction $\widetilde{k}$ with transversal perturbations in the direction $\widetilde{l}$ where $\widetilde{k}$ and $\widetilde{l}$ are fixed unit vectors, $\widetilde{k}$ is not parallel or antiparallel to $M_{0}$
and $\tilde{l}$ is not parallel or antiparallel to $\widetilde{k}$. By assuming that $\widetilde{x}$ is the coordinate in the $\widetilde{k}$-direction and $\widetilde{y}$ in the $\widetilde{l}$-direction we may perform the substitutions:

$$
\widetilde{\nabla}_{\widetilde{x}}=\widetilde{k} \partial_{\widetilde{x}} \quad \text { and } \quad \widetilde{\nabla}_{\widetilde{y}}=\widetilde{l} \partial_{\widetilde{y}}
$$

Henceforth we omit the tilde, so the equations to be considered are:

$$
\begin{align*}
& \varepsilon^{2} \partial_{t} M+\varepsilon^{4} \partial_{\tau} M=-M_{0} \wedge H+\alpha^{-1} M_{0} \wedge M-\varepsilon^{2} M \wedge H \\
&-\gamma\left[M_{0} \wedge\left(M_{0} \wedge H\right)-\alpha^{-1} M_{0} \wedge\left(M_{0} \wedge M\right)\right.  \tag{10}\\
&+\varepsilon^{2} M_{0} \wedge(M \wedge H)+\varepsilon^{2} M \wedge\left(M_{0} \wedge H\right) \\
&\left.-\varepsilon^{2} \alpha^{-1} M \wedge\left(M_{0} \wedge M\right)+\varepsilon^{4} M \wedge(M \wedge H)\right] \\
& \varepsilon^{2} \partial_{t} H+\varepsilon^{4} \partial_{\tau} H-\varepsilon^{2} k \wedge \partial_{x} E-\varepsilon^{3} l \wedge \partial_{y} E=-\varepsilon^{2} \partial_{t} M-\varepsilon^{4} \partial_{\tau} M  \tag{11}\\
& \varepsilon^{2} \partial_{t} E+ \varepsilon^{4} \partial_{\tau} E+\varepsilon^{2} k \wedge \partial_{x} H+\varepsilon^{3} l \wedge \partial_{y} H=0 \tag{12}
\end{align*}
$$

### 2.2 Vector Formulation

The following formulation and properties are obtained following the work of T. Colin, C. Galusinski and H. Kaper ([5]).

The system of equations (10)-(12) can be written as a single equation for a function:

$$
\begin{aligned}
& U: \mathbb{R}_{t}^{+*} \times[0, T] \times \mathbb{R}_{x} \times \mathbb{R}_{y} \rightarrow \mathbb{R}^{9}=\mathbb{R}^{3} \times \mathbb{R}^{3} \times \mathbb{R}^{3}, \\
& \\
& U(t, \tau, x, y)=\left(\begin{array}{c}
\alpha^{-1 / 2} M(t, \tau, x, y) \\
H(t, \tau, x, y) \\
E(t, \tau, x, y)
\end{array}\right), t \geq 0, \tau \in[0, T], x \in \mathbb{R}, y \in \mathbb{R} .
\end{aligned}
$$

The factor $\alpha^{-1 / 2}$ is introduced so that the problem have certain symmetry properties (cf Section 3). After having divided once more by $\varepsilon^{2}, U$ has to fulfill for all $t \in \mathbb{R}^{+*}, \tau \in[0, T], x \in \mathbb{R}$ and $y \in \mathbb{R}$ :

$$
\begin{equation*}
\partial_{t} U+\varepsilon^{2} \partial_{\tau} U+A_{1} \partial_{x} U+\varepsilon A_{2} \partial_{y} U+\varepsilon^{-2}\left(L_{0}+L_{1}\right) U=B(U, U)+\varepsilon^{2} T(U, U, U) \tag{13}
\end{equation*}
$$

where $A_{1}, A_{2}, L_{0}$ and $L_{1}$ are linear operators in $\mathbb{R}^{9}$,

$$
\begin{aligned}
& A_{1} u=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -k \wedge \cdot \\
0 & k \wedge \cdot & 0
\end{array}\right)\left(\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array}\right), \quad A_{2} u=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -l \wedge \cdot \\
0 & l \wedge \cdot & 0
\end{array}\right)\left(\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array}\right), \\
& L_{0} u=\left(\begin{array}{ccc}
-\alpha^{-1} M_{0} \wedge \cdot \alpha^{-1 / 2} M_{0} \wedge \cdot & 0 \\
\alpha^{-1 / 2} M_{0} \wedge \cdot & -M_{0} \wedge \cdot & 0 \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array}\right), \\
& L_{1} u=\gamma\left(\begin{array}{ccc}
-\alpha^{-1} M_{0} \wedge\left(M_{0} \wedge \cdot\right) & \alpha^{-1 / 2} M_{0} \wedge\left(M_{0} \wedge \cdot\right) & 0 \\
\alpha^{-1 / 2} M_{0} \wedge\left(M_{0} \wedge \cdot\right) & -M_{0} \wedge\left(M_{0} \wedge \cdot\right) & 0 \\
0 & 0 & 0
\end{array}\right)\left(\begin{array}{l}
u_{1} \\
u_{2} \\
u_{3}
\end{array}\right),
\end{aligned}
$$

$B$ is a bilinear map on $\mathbb{R}^{9} \times \mathbb{R}^{9}$,

$$
B(u, v)=\left(\begin{array}{c}
B_{1}(u, v) \\
-\alpha^{1 / 2} B_{1}(u, v) \\
0
\end{array}\right)
$$

with

$$
\begin{aligned}
B_{1}(u, v)=- & \frac{1}{2}\left(u_{1} \wedge v_{2}+v_{1} \wedge u_{2}\right)-\frac{1}{2} \gamma M_{0} \wedge\left(u_{1} \wedge v_{2}+v_{1} \wedge u_{2}\right) \\
& -\frac{1}{2} \gamma\left[u_{1} \wedge\left(M_{0} \wedge\left(v_{2}-\alpha^{-1 / 2} v_{1}\right)\right)+v_{1} \wedge\left(M_{0} \wedge\left(u_{2}-\alpha^{-1 / 2} u_{1}\right)\right)\right]
\end{aligned}
$$

and $T$ is a trilinear map on $\mathbb{R}^{9} \times \mathbb{R}^{9} \times \mathbb{R}^{9}$,

$$
T(u, v, w)=\alpha^{1 / 2} \gamma\left(\begin{array}{c}
T_{1}(u, v, w) \\
-\alpha^{1 / 2} T_{1}(u, v, w) \\
0
\end{array}\right)
$$

with

$$
\begin{aligned}
T_{1}(u, v, w)= & \frac{1}{6}\left[u_{1} \wedge\left(v_{2} \wedge w_{1}\right)+u_{1} \wedge\left(w_{2} \wedge v_{1}\right)+w_{1} \wedge\left(v_{2} \wedge u_{1}\right)\right. \\
& \left.+w_{1} \wedge\left(u_{2} \wedge v_{1}\right)+v_{1} \wedge\left(u_{2} \wedge w_{1}\right)+v_{1} \wedge\left(w_{2} \wedge u_{1}\right)\right]
\end{aligned}
$$

Here $u, v$ and $w$ are arbitrary vectors in $\mathbb{R}^{9}, u=\left(u_{1}, u_{2}, u_{3}\right)^{t}, v=\left(v_{1}, v_{2}, v_{3}\right)^{t}$, $w=\left(w_{1}, w_{2}, w_{3}\right)^{t}$ with $u_{i}, v_{i}, w_{i} \in \mathbb{R}^{3}, i=1,2,3$.

### 2.3 Auxiliary properties

Since the vector product is antisymmetric, the operators $A_{1}$ and $A_{2}$ are symmetric with respect to the usual scalar product in $\mathbb{R}^{9}: A_{i} u \cdot v=u \cdot A_{i} v, u, v \in$ $\mathbb{R}^{9}, i=1,2$.

The operators $L_{0}$ and $L_{1}$ are respectively antisymmetric and symmetric with respect to the scalar product in $\mathbb{R}^{9}: \quad L_{0} u \cdot v=-u \cdot L_{0} v, L_{1} u \cdot v=u \cdot L_{1} v, \quad u, v \in$ $\mathbb{R}^{9}$.

The bilinear map $B$ and the trilinear map $T$ are symmetric, i.e. $B(u, v)=$ $B(v, u)$ and $T(u, v, w)=T(\pi(u, v, w))$ for all $u, v, w \in \mathbb{R}^{9}$ and any permutation $\pi$.

Lemma 2.1 The operator $L=L_{0}+L_{1}$ induces an orthogonal decomposition,

$$
\mathbb{R}^{9}=\operatorname{Ker} L \oplus \operatorname{Im} L
$$

We have:

$$
\begin{aligned}
& \operatorname{Ker} L=\left\{v=\left(v_{1}, v_{2}, v_{3}\right)^{t} \in \mathbb{R}^{9},\left(\alpha^{-1 / 2} v_{1}-v_{2}\right) \wedge M_{0}=0\right\} \\
& \operatorname{Im} L=\left\{v=\left(v_{1}, v_{2}, v_{3}\right)^{t} \in \mathbb{R}^{9}, v_{1} \cdot M_{0}=0, v_{2}=-\alpha^{1 / 2} v_{1}, v_{3}=0\right\}
\end{aligned}
$$

Let $P$ and $Q$ be respectively the orthogonal projectors on Ker $L$ and $\operatorname{Im} L$, and let $R$ be the inverse of $L$ on $\operatorname{Im} L$, trivially extended to $\mathbb{R}^{9}$. Then $R L=$ $L R=I-P=Q$. Furthermore, if $L u=v$ for $(u, v) \in \mathbb{R}^{9}$ then $P v=0$ and $Q u=R v$. We also have the following lemma:

Lemma 2.2 The operator $L_{1}$ is coercive on $\operatorname{Im} L$,

$$
\left(L_{1} Q v\right) \cdot(Q v)=\gamma\left(1+\alpha^{-1}\right)(Q v) \cdot(Q v), \quad v \in \mathbb{R}^{9}
$$

The maps $B$ and $T$ are transparent on Ker $L$,

$$
P B(P u, P v)=0, \quad P T(P u, P v, P w)=0, \quad u, v, w \in \mathbb{R}^{9} .
$$

## 3 A general equation

Equation (13) is a special case of the general partial diferential equation:

$$
\begin{equation*}
\partial_{t} U+\varepsilon^{2} \partial_{\tau} U+A_{1} \partial_{x} U+\varepsilon A_{2} \partial_{y} U+\varepsilon^{-2} L U=B(U, U)+\varepsilon^{2} T(U, U, U) \tag{14}
\end{equation*}
$$

in $\mathbb{R}^{n}$ where $A_{1}$ and $A_{2}$ are symmetric linear operators, $L$ is a linear operator, $B$ a symmetric bilinear map and $T$ a symmetric trilinear map. In this section we consider Equation (14), the application to the special case of Equation (13) follows in Section 4. We first build an asymptotic expansion of Equation (14) using formal power series expansion in the small parameter $\varepsilon$ (Section 3.1). Then we give precise asymptotic estimates of the various terms of the asymptotic solution (Section 3.2.1). Finally we show that the asymptotic solution actually converges to the solution of Equation (14) on the slow time scale as $\varepsilon$ goes to 0 (Section 3.2.3).

### 3.1 Formal Asymptotic Expansion

We will now search an asymptotic formal expansion of $U \equiv U(t, \tau, x, y)$ in the form

$$
U \equiv\left(U_{0}+\varepsilon U_{1}+\varepsilon^{2} U_{2}+\ldots\right)
$$

This construction requires three hypotheses:
Hypothesis 1. $\mathbb{R}^{n}=\operatorname{Ker} L \oplus \operatorname{Im} L$.
Hypothesis 2. There exists $C>0$ such that for all $u \in \mathbb{R}^{n},(L u) \cdot u \geq$ $C\|Q u\|^{2}$, i.e. $L$ is coercive on its image.

Hypothesis 3. $P B(P u, P v)=0$ and $P T(P u, P v, P w)=0$ for all $u, v, w \in$ $\mathbb{R}^{n}$.

Here $P$ and $Q$ are respectively the orthogonal projections on Ker $L$ and $\operatorname{Im} L$. Let $R$ the partial inverse of $L$ on $\operatorname{Im} L$, trivially extended to all of $\mathbb{R}^{n}$. Then $R L=L R=Q$ and we have:

Lemma 3.1 If $L u=v$ for some $u, v \in \mathbb{R}^{n}$ then $P v=0$ (solvability condition) and $Q u=R v$.

We substitute the asymptotic expansion of $U$ in (14) and we assume that $U_{0}=O(1), \varepsilon U_{i}=o(1), i=1, \ldots, 4$, as $\varepsilon$ goes to 0 . We then get:

At the order $\varepsilon^{-2}: L U_{0}=0$, i.e. $Q U_{0}=0$ and therefore $U_{0}=P U_{0}$.

At the order $\varepsilon^{-1}: L U_{1}=0$, i.e. $Q U_{1}=0$ and therefore $U_{1}=P U_{1}$.
At the order $\varepsilon^{0}: L U_{2}=V_{2}\left(U_{0}\right)=B\left(U_{0}, U_{0}\right)-\left(\partial_{t}+A_{1} \partial_{x}\right) U_{0}$.
Since $U_{0}=P U_{0}$ and $B$ is transparent on Ker $L$, the solvability condition $P V_{2}=0$ reduces to

$$
\left(\partial_{t}+P A_{1} P \partial_{x}\right) U_{0}=0
$$

The operator $P A_{1} P$ is symmetric. Then there exists $k$ projections $P_{j}$ and $k$ numbers $v_{j}(j=1, \ldots, k, k \leq n)$ such that:

$$
P=\sum_{j=1}^{k} P_{j}, \quad P A_{1} P P_{j}=v_{j} P_{j}, \quad j=1, \ldots, k
$$

Hence the solvability condition is met if

$$
\begin{equation*}
\left(\partial_{t}+v_{j} \partial_{x}\right) P_{j} U_{0}=0, \quad j=1, \ldots, k \tag{15}
\end{equation*}
$$

As $U_{0}=P U_{0}$ and $R=0$ on Ker $L$ the equation $Q U_{2}=R V_{2}$ reduces to:

$$
\begin{equation*}
Q U_{2}=R B\left(U_{0}, U_{0}\right)-R A_{1} \partial_{x} U_{0} \tag{16}
\end{equation*}
$$

At the order $\varepsilon: L U_{3}=V_{3}\left(U_{0}, U_{1}\right)=2 B\left(U_{0}, U_{1}\right)-\left(\partial_{t}+A_{1} \partial_{x}\right) U_{1}-A_{2} \partial_{y} U_{0}$. By the same way, the solvability condition $P V_{3}=0$ becomes:

$$
\left(\partial_{t}+P A_{1} P \partial_{x}\right) U_{1}+P A_{2} P \partial_{y} U_{0}=0
$$

and then,

$$
\begin{equation*}
\left(\partial_{t}+v_{j} \partial_{x}\right) P_{j} U_{1}+P_{j} A_{2} \sum_{i=1}^{k} \partial_{y} P_{i} U_{0}=0, \quad j=1, \ldots, k \tag{17}
\end{equation*}
$$

Remark 3.1 These equations allow us to determine $U_{1}$.
The equation $Q U_{3}=R V_{3}$ also reads:

$$
Q U_{3}=2 R B\left(U_{0}, U_{1}\right)-R A_{1} \partial_{x} U_{1}-R A_{2} \partial_{y} U_{0}
$$

At the order $\varepsilon^{2}$ :

$$
\begin{aligned}
L U_{4}=V_{4}\left(U_{0}, U_{1}, U_{2}\right)= & 2 B\left(U_{0}, U_{2}\right)+B\left(U_{1}, U_{1}\right)+T\left(U_{0}, U_{0}, U_{0}\right) \\
& -\partial_{\tau} U_{0}-A_{2} \partial_{y} U_{1}-\left(\partial_{t}+A_{1} \partial_{x}\right) U_{2}
\end{aligned}
$$

Since $B$ and $T$ are transparent on Ker $L$ the solvability condition $P V_{4}=0$ reduces to:

$$
\partial_{\tau} U_{0}+P A_{2} \partial_{y} U_{1}+\left(\partial_{t} P U_{2}+P A_{1} \partial_{x} U_{2}\right)=2 P B\left(U_{0}, U_{2}\right)
$$

We rewrite this equation using Eq. (16) and the transparency condition on Ker $L$,

$$
\begin{aligned}
& \partial_{\tau} U_{0}+\left(\partial_{t}+P A_{1} P \partial_{x}\right) P U_{2}+P A_{2} P \partial_{y} U_{1}-P A_{1} R A_{1} P \partial_{x}^{2} U_{0} \\
& =2 P B\left(U_{0}, R B\left(U_{0}, U_{0}\right)\right)-2 P B\left(U_{0}, R A_{1} \partial_{x} U_{0}\right)-P A_{1} R \partial_{x} B\left(U_{0}, U_{0}\right)
\end{aligned}
$$

This equation represents a system of $k$ equations,

$$
\begin{align*}
& \partial_{\tau} P_{j} U_{0}+\left(\partial_{t}+v_{j} \partial_{x}\right) P_{j} U_{2}+P_{j} A_{2} \sum_{i=1}^{k} \partial_{y} P_{i} U_{1}-P_{j} A_{1} R A_{1} \sum_{i=1}^{k} \partial_{x}^{2} P_{i} U_{0} \\
& =2 P_{j} B\left(U_{0}, R B\left(U_{0}, U_{0}\right)\right)-2 P_{j} B\left(U_{0}, R A_{1} \partial_{x} U_{0}\right)  \tag{18}\\
& -P_{j} A_{1} R \partial_{x} B\left(U_{0}, U_{0}\right), \quad j=1, \ldots k
\end{align*}
$$

The $j$-th equation involves the rate of change of $P_{j} U_{0}$ on the slow $(\tau)$ time scale as well as the rate of change of $P_{i} U_{1}, i=1, \ldots, k$ in the transversal direction $(y)$ and the rate of change of $P_{j} U_{2}$ along the characteristic determined by $v_{j}$ on the regular $(t)$ time scale. We can separate the first two effects from the last one if $U_{2}$ fulfills a sublinear growth condition,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t}\left\|U_{2}(t, \tau, \cdot, \cdot)\right\|_{\mathbb{H}^{s}}=0 \tag{19}
\end{equation*}
$$

uniformly on $[0, T]$ for some sufficiently large $s$. $\left(\mathbb{H}^{s}\right.$ is the usual Sobolev space of order s.) The condition (19) also implies that $\varepsilon\left\|U_{2}\right\|_{H^{s}}=o(1)$ as $\varepsilon$ goes to 0 . The separation is accomplished by averaging over $t$ along characteristics. Formally,

$$
\begin{equation*}
G_{v} u(t, x)=\lim _{T \rightarrow+\infty} \int_{0}^{T} u(x+v s, t+s) \mathrm{d} s, \quad v \in \mathbb{R} \tag{20}
\end{equation*}
$$

whenever the limit exists. The following lemma is taken from [9].

## Lemma 3.2

(i) If $\left(\partial_{t}+v \partial_{x}\right) u=0$ then $G_{v^{\prime}} u$ exists for all $v^{\prime}$. Moreover $G_{v^{\prime}} u=u$ if $v^{\prime}=v$ and $G_{v^{\prime}} u=0$ otherwise.
(ii) If $\left(\partial_{t}+v \partial_{x}\right) u=0$ and $\left(\partial_{t}+v^{\prime} \partial_{x}\right) u^{\prime}=0$ then $G_{v^{\prime \prime}}\left(u u^{\prime}\right)=u u^{\prime}$ if $v^{\prime \prime}=v^{\prime}=v$ and $G_{v^{\prime \prime}}\left(u u^{\prime}\right)=0$ otherwise.
(iii) If $u$ satisfies a sublinear growth condition, $\lim _{t \rightarrow+\infty} \frac{1}{t}\|u(t, .)\|_{\mathbb{L}^{\infty}\left(\mathbb{R}^{n}\right)}=0$ then $G_{v}\left(\partial_{t}+v \partial_{x}\right) u$ exists and $G_{v}\left(\partial_{t}+v \partial_{x}\right) u=0$.

The application of $G_{v_{j}}$ to both sides of Equation (18) eliminates the transport term and reduces the equation to:

$$
\left\{\begin{align*}
\partial_{\tau} P_{j} U_{0}+ & P_{j} A_{2} \sum_{i=1}^{k} \partial_{y} G_{v_{j}} P_{i} U_{1}-P_{j} A_{1} R A_{1} P_{j} \partial_{x}^{2} P_{j} U_{0}  \tag{21}\\
= & 2 P_{j} B\left(P_{j} U_{0}, R B\left(P_{j} U_{0}, P_{j} U_{0}\right)\right)-2 P_{j} B\left(P_{j} U_{0}, R A_{1} \partial_{x} P_{j} U_{0}\right) \\
& -P_{j} A_{1} R \partial_{x} B\left(P_{j} U_{0}, P_{j} U_{0}\right), \quad j=1, \ldots k
\end{align*}\right.
$$

We have now to determine the term $G_{v_{j}} P_{i} U_{1}$. We then use Equation (17) and we assume that $U_{1}$ satisfies the same sublinear growth condition as $U_{2}$. We get:

$$
\begin{align*}
& G_{v_{j}}\left[\left(\partial_{t}+v_{i} \partial_{x}\right) P_{i} U_{1}+P_{i} A_{2} \sum_{l=1}^{k} \partial_{y} P_{l} U_{0}\right]  \tag{22}\\
& =\left(v_{i}-v_{j}\right) \partial_{x} G_{v_{j}} P_{i} U_{1}+P_{i} A_{2} P_{j} \partial_{y} P_{j} U_{0}=0
\end{align*}
$$

for all $i, j=1, \ldots, k$. We then get a compatibility condition which is $P_{i} A_{2} P_{i}=$ 0 for all $i=1, \ldots, k$.

Remark 3.2 This compatibility condition gives information on the direction of the transversal perturbation as we can see it in Section 4. We obtain: $l \cdot k=$ $l \cdot M_{0}=0$.

We also remark that the operator $P_{j} A_{1} R A_{1} P_{j}$ is nonnegative because of Hypothesis 2 and proportional to $P_{j}, P_{j} A_{1} R A_{1} P_{j}=D_{j} P_{j}$ where $D_{j}$ is a scalar, $D_{j} \geq 0$.
After taking the derivative of Eq. (21) with respect to $x$ we get that the solvability condition $P V_{4}=0$ thus yields a system of $k$ nonlinear KhokhlovZabolotskaya type equations on the slow $(\tau)$ time scale, for $j=1, \ldots, k$,

$$
\begin{array}{r}
\partial_{x}\left(\partial_{\tau} P_{j} U_{0}-D_{j} \partial_{x}^{2} P_{j} U_{0}+B_{j}\left(P_{j} U_{0}, \partial_{x} P_{j} U_{0}\right)+F_{j}\left(P_{j} U_{0}, P_{j} U_{0}, P_{j} U_{0}\right)\right)  \tag{23}\\
=C_{j} \partial_{y}^{2} P_{j} U_{0}
\end{array}
$$

where $B_{j}$ is a bilinear map, $B_{j}(u, v)=2 P_{j} B\left(u, R A_{1} v\right)+2 P_{j} A_{1} R B(u, v)$ (thanks to the symmetry of $B), F_{j}$ a trilinear map, $F_{j}(u, v, w)=2 P_{j} B(u, R B(v, w)$ ), and $C_{j}$ the operator $C_{j}=\sum_{i \neq j} \frac{1}{v_{i}-v_{j}} P_{j} A_{2} P_{i} A_{2} P_{j}$.

Remark 3.3 The Khokhlov-Zabolotskaya equation arises for the first time in gas theory [13] and describes nonlinear diffractive waves [4,6,10].

Furthermore if we use Equation (21) to eliminate the $\tau$ derivative in Eq. (18) we find that the solvability condition $P V_{4}=0$ also yields a system of $k$
transport equations for $P_{j} U_{2}$ on the regular $(t)$ time scale,

$$
\begin{equation*}
\left(\partial_{t}+v_{j} \partial_{x}\right) P_{j} U_{2}=S_{j}\left(U_{0}, U_{1}\right), \quad j=1, \ldots, k, \tag{24}
\end{equation*}
$$

where:

$$
\begin{aligned}
S_{j}\left(U_{0}, U_{1}\right)= & P_{j} A_{1} R A_{1} \sum_{i \neq j} \partial_{x}^{2} P_{i} U_{0}-P_{j} A_{2} \sum_{i \neq j}\left(\partial_{y} P_{i} U_{1}-\partial_{y} G_{v_{j}} P_{i} U_{1}\right) \\
& +2 P_{j}\left(B\left(U_{0}, R B\left(U_{0}, U_{0}\right)\right)-B\left(P_{j} U_{0}, R B\left(P_{j} U_{0}, P_{j} U_{0}\right)\right)\right) \\
& -2 P_{j}\left(B\left(U_{0}, R A_{1} \partial_{x} U_{0}\right)-B\left(P_{j} U_{0}, R A_{1} \partial_{x} P_{j} U_{0}\right)\right) \\
& -P_{j} A_{1} R \partial_{x}\left(B\left(U_{0}, U_{0}\right)-B\left(P_{j} U_{0}, P_{j} U_{0}\right)\right) .
\end{aligned}
$$

If Equations (23)-(24) are fulfilled, then $Q U_{4}=R V_{4}$. The equation reduces to:
$Q U_{4}=2 R B\left(U_{0}, U_{2}\right)+R B\left(U_{1}, U_{1}\right)+R T\left(U_{0}, U_{0}, U_{0}\right)-R A_{2} \partial_{y} U_{1}-R \partial_{t} U_{2}-R A_{1} \partial_{x} U_{2}$.
Lemma 3.3 If $U_{1}$ and $U_{2}$ fulfill the sublinear growth condition (19), then $U_{0}=\sum_{j=1}^{k} P_{j} U_{0}$ where each $P_{j} U_{0}$ satisfies a homogeneous transport equation on the regular (t) time scale (Eq. (15)) and a nonlinear Khoklov-Zabolotskaya equation on the slow ( $\tau$ ) time scale (Eq. (23)).

### 3.2 Asymptotic Estimates

For the convergence proof in the next section we need asymptotic estimates of the coefficients $U_{0}, U_{1}, U_{2}, U_{3}$ and $U_{4}$. The estimates require an additional hypothesis:

Hypothesis 4. For all $j \in\{1, \ldots, k\}$, either $D_{j}>0$ or, if $D_{j}=0$, the term involving $x$ derivative in Eq. (23), i.e. $B_{j}$, is zero.

Hypothesis 5. $U(t=0, \tau=0, x, y)=U_{0}^{0}(x, y)$.
Our first concern is the existence and uniqueness of $U_{0}$.

### 3.2.1 Existence of the profiles

Theorem 3.4 Under the hypothesis \& and if $U_{0}^{0}=P U_{0}^{0} \in \mathbb{H}^{s}\left(\mathbb{R}^{2}\right), s>1$, there exists a time $T>0$ and an unique function

$$
U_{0} \in \mathcal{C}^{0}\left([0, T] ; \bigcap_{1 \leq l \leq s} \mathbb{W}^{l, \infty}\left(\mathbb{R}_{t} ; \mathbb{H}^{s-l}\left(\mathbb{R}^{2}\right)\right)\right)
$$

such that $U_{0}=\sum_{j=1}^{k} P_{j} U_{0}=\sum_{j=1}^{k} u_{j}$, where the functions $u_{j}, j=1, \ldots, k$ satisfy Eqs. (15) and (23):

$$
\left\{\begin{array}{l}
\left(\partial_{t}+v_{j} \partial_{x}\right) u_{j}=0 \quad \text { (Transport equation) } \\
\partial_{x}\left(\partial_{\tau} u_{j}-D_{j} \partial_{x}^{2} u_{j}+B_{j}\left(u_{j}, \partial_{x} u_{j}\right)+F_{j}\left(u_{j}, u_{j}, u_{j}\right)\right)=C_{j} \partial_{y}^{2} u_{j}
\end{array}\right.
$$

where the second equation is the Khokhov-Zabolotskaya equation when $D_{j}>0$ and $C_{j} \neq 0$.
Furthermore $U_{0}(0,0, \cdot)=U_{0}^{0}$.

Proof. Let $u_{j}=P_{j} U_{0}$ and $u_{j}^{0}=P_{j} U_{0}^{0}$.
We have to solve Eqs. (15) and (23). We will at first only consider Eq. (23) since Eq. (15) is an equation of transport. We denote $\widetilde{u_{j}}$ the solution of Eq. (23).

The (formal) operator $\partial_{x}^{-1} \partial_{y}^{2}$ raises some problems. We define a unitary group on $\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)$ to get rid of this term. Let $S_{j}(\tau)$ the operator defined by:

$$
\mathcal{F}_{x, y}\left(S_{j}(\tau) u\right)(\xi, \eta)=e^{i \tau C_{j} \frac{\eta^{2}}{\xi}}\left(\mathcal{F}_{x, y}(u)\right)(\xi, \eta), \quad \tau \in \mathbb{R}, \quad(\xi, \eta) \in \mathbb{R}^{*} \times \mathbb{R}
$$

where $\mathcal{F}_{x, y}$ is the $2 D$-Fourier transform. We consider now the function $w_{j}(\tau, x, y)=S_{j}(-\tau) \widetilde{u_{j}}(\tau, x, y)$ which satisfies:

$$
\begin{equation*}
\partial_{\tau} w_{j}-D_{j} \partial_{x}^{2} w_{j}+\widetilde{A}_{j}\left(\tau, w_{j}\right) \partial_{x} w_{j}+\widetilde{F}_{j}\left(\tau, w_{j}\right)=0 \tag{25}
\end{equation*}
$$

where

$$
\begin{aligned}
& \widetilde{A}_{j}(\tau, u) v=S_{j}(-\tau) B_{j}\left(S_{j}(\tau) u, S_{j}(\tau) v\right) \\
& \widetilde{F}_{j}(\tau, v)=S_{j}(-\tau) F_{j}(S(\tau) v, S(\tau) v, S(\tau) v)
\end{aligned}
$$

We denote by $\Lambda$ the pseudodifferential operator with the symbol:

$$
\begin{equation*}
\sigma(\zeta)=\left(1+|\zeta|^{2}\right)^{1 / 2}, \quad \zeta=\left(\zeta_{1}, \zeta_{2}\right) \tag{26}
\end{equation*}
$$

If we denote by (, ) the scalar product in $\mathbb{L}^{2}\left(\mathbb{R}^{2}\right)$, the scalar product in $\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)$ is defined by $(u, v)_{s}=\left(\Lambda^{s} u, \Lambda^{s} v\right)$.
Because of Hypothesis 4, if $D_{j}=0$, Eq. (25) reduces to: $\partial_{\tau} w_{j}+\widetilde{F}_{j}\left(\tau, w_{j}\right)=0$. Thanks to the Cauchy-Lipschitz theorem we get the existence of a time $T_{j}>0$ and of $w_{j} \in \mathcal{C}^{0}\left(\left[0, T_{j}\left[; \mathbb{H}^{s}\left(\mathbb{R}^{2}\right)\right)\right.\right.$.
Now if $D_{j}>0$, we can approximate the solution of Eq. (25) thanks to the Galerkin method on $[-L, L]^{2}$ with the Dirichlet boundary condition. We then have to perform some energy estimates on $v_{j}$ in $\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)$ before taking the limit (we perform the estimates on $v_{j}$ rather than on its approximate solution to lighten the proof). We already have:

$$
\left\|\widetilde{A}_{j}(\tau, u) v\right\|_{\mathbb{H}^{s}} \leq C\|u\|_{\mathbb{H}^{s}}\|v\|_{\mathbb{H}^{s}}, \quad\left\|\widetilde{F}_{j}(\tau, v)\right\|_{\mathbb{H}^{s}} \leq C\|v\|_{\mathbb{H}^{s}}^{3}
$$

where $C$ does not depend on $\tau$ nor on $L$. By applying $\Lambda^{s}$ to Eq. (25) and performing the estimates we obtain:

$$
\frac{d}{d \tau}\left\|w_{j}\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}^{2}+D_{j}\left\|\partial_{x} w_{j}\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}^{2} \leq C\left(\left\|w_{j}\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}^{2}+\left\|w_{j}\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}^{4}\right)
$$

So there exists $T_{j}>0$ and $v_{j} \in \mathcal{C}^{0}\left(\left[0, T_{j}\left[; \mathbb{H}^{s}\left(\mathbb{R}^{2}\right)\right)\right.\right.$ solution of Eq. (25). Back to $\widetilde{u_{j}}$, we have $\widetilde{u_{j}} \in \mathcal{C}^{0}\left(0, T_{j} ; \mathbb{H}^{s}\left(\mathbb{R}^{2}\right)\right)$ and the transport equation gives:

$$
u_{j}(t, \tau, x, y)=\widetilde{u}_{j}\left(\tau, x-v_{j} t, y\right)
$$

So we get $u_{j}$ thanks to a translation in time of $\widetilde{u}_{j}$. Finally,

$$
u_{j} \in \mathcal{C}^{0}\left(\left[0, T_{j}\left[; \bigcap_{0 \leq l \leq s} \mathbb{W}^{l, \infty}\left(\mathbb{R}_{t} ; \mathbb{H}^{s-l}\left(\mathbb{R}^{2}\right)\right)\right)\right.\right.
$$

Remark 3.4 This also proves the theorem 1.1 on the Khokhlov-Zabolotskaya equation given in the introduction.
Following the work of Ukai [12] on the KP equation we can also prove the existence of solutions to the Khokhlov-Zabolotskaya (KZ) equation but we have then to impose too many conditions on the initial data particularly non physical ones.

The equations on the other profiles pose some problems: we want to obtain their existence in spaces derived from $\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)$ but it comes the operator $\partial_{x}^{-1}$ which imposes the existence in $\mathbb{H}_{l o c}^{s}\left(\mathbb{R}^{2}\right)$ only and prevents to get profiles $\left(U_{1}\right.$ and $U_{2}$ ) which satisfy the sublinear growth condition. To get rid of this problem we use a technique of low-frequency cut-offs (see [2], [3], [11]) to work with an approximated "well-prepared" leading profile. We then modify the equations fulfilled by $U^{1}, U^{2}, \ldots$ so that they depend on the truncation of the main profile rather than the main profile itself. We next use these modified profiles to obtain an approximate solution of Eq. (14) and then prove the stability of the approximation.

Definition 3.1 (Low-frequency truncation) Let $\chi$ be a smooth function in $\mathbb{R}_{x}$, such that $|\chi| \leq 1, \chi(x)=0$ for $|x|<1$ and $\chi(x)=1$ for $|x| \geq 2$. Define the Fourier multiplier $\chi^{\delta}\left(D_{x}\right)$ as the operator acting on $\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)$ by:

$$
\chi^{\delta}\left(D_{x}\right): \quad f \mapsto \mathcal{F}_{\xi, \eta \rightarrow x, y}^{-1}\left(\chi\left(\frac{\xi}{\delta}\right) \hat{f}(\xi, \eta)\right)
$$

The dominate convergence theorem shows that:

$$
\chi^{\delta} f-f \rightarrow 0 \quad \text { in } \mathbb{H}^{s}
$$

Moreover we have the following lemma:
Lemma 3.5 If $f \in \mathbb{L}^{\infty}\left(0, T ; \mathbb{H}^{s}\left(\mathbb{R}^{2}\right)\right)$ then

$$
\lim _{\delta \rightarrow 0} \sup _{t \in[0, T]}\left\|\chi^{\delta} f(t, \cdot)-f(t, \cdot)\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}=0
$$

Proof. let $I^{\delta}(t)=\left\|\chi^{\delta} f(t, \cdot)-f(t, \cdot)\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}$. The sequence $\left(I^{\delta}\right)_{\delta}$ is a non increasing sequence of nonnegative continuous functions which simply goes to zero as $\delta$ goes to zero. The Dini's theorem gives us the result.

Let us introduce $X_{s, t}$ the space of functions defined on $[0, T] \times \mathbb{R}_{t} \times \mathbb{R}_{x, y}^{2}$ :

$$
X_{s, T}=\left\{u, \sup \left\{\left\|\partial_{t}^{\alpha} \partial_{x}^{\beta} \partial_{y}^{\gamma} u(t, \tau, \cdot, \cdot)\right\|_{\mathbb{L}^{2}\left(\mathbb{R}^{2}\right)}: t \in[0, \infty), \tau \in[0, T]\right\}<\infty\right\}
$$

for all $\alpha, \beta, \gamma \geq 0$ sucht that $0 \leq \alpha+\beta+\gamma \leq s$.
Lemma 3.6 If $U_{0}^{0} \in \mathbb{H}^{s}\left(\mathbb{R}^{2}\right), s>1$, there exists an unique $U_{1}^{\delta} \in X_{s-1, T}$ such that:

$$
\left\{\begin{array}{l}
U_{1}^{\delta}=P U_{1}^{\delta}=\sum_{j=1}^{k} P_{j} U_{1}^{\delta} \\
\left(\partial_{t}+v_{j} \partial_{x}\right) P_{j} U_{1}^{\delta}+P_{j} A_{2} \sum_{i \neq j} \partial_{y} P_{i} U_{0}^{\delta}=0, \quad \forall j=1, \ldots, k .
\end{array}\right.
$$

Moreover $\left\|U_{1}^{\delta}\right\|_{X_{s-1, T}} \leq \frac{C}{\delta}\left\|U_{0}\right\|_{X_{s, T}}$, where $C$ does not depend on $\delta$ nor on $\left\|U_{0}\right\|_{X_{s, T}}$.

Proof. As we have truncated $U_{0}$, we have that

$$
P_{j} U_{0}^{\delta}(t, \tau, x, y)=\left(\partial_{x} \varphi_{j}^{\delta}\right)(t, \tau, x, y)=\left(\partial_{x} \widetilde{\varphi}_{j}^{\delta}\right)\left(\tau, x-v_{j} t, y\right)
$$

with $\left\|\varphi^{\delta}(t, \tau, \cdot, \cdot)\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)} \leq \frac{1}{\delta}\left\|U_{0}^{\delta}(t, \tau, \cdot, \cdot)\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}$.
As we assume that $U_{1}(t=0)=0$ we have for all $j=1, \ldots, k$,

$$
\begin{aligned}
P_{j} U_{1}^{\delta} & =-\sum_{i \neq j} P_{j} A_{2} P_{i} \int_{0}^{t} \partial_{y} P_{i} U_{0}^{\delta}\left(s, \tau, x-v_{j}(t-s), y\right) d s \\
& =-\sum_{i \neq j} P_{j} A_{2} P_{i} \int_{0}^{t} \partial_{y} \partial_{x} \widetilde{\varphi}_{i}^{\delta}\left(\tau, x-v_{j} t+\left(v_{j}-v_{i}\right) s, y\right) d s \\
& =-\sum_{i \neq j} \frac{1}{v_{j}-v_{i}} P_{j} A_{2} P_{i}\left(\partial_{y} \widetilde{\varphi}_{i}^{\delta}\left(\tau, x-v_{i} t, y\right)-\partial_{y} \widetilde{\varphi}_{i}^{\delta}\left(\tau, x-v_{j} t, y\right)\right) .
\end{aligned}
$$

We then deduce that $P_{j} U_{1}^{\delta} \in X_{s-1, T}$ and $\left\|U_{1}^{\delta}\right\|_{X_{s-1, T}} \leq \frac{C}{\delta}\left\|U_{0}\right\|_{X_{s, T}}$.

By the same way we prove:
Lemma 3.7 If we furthermore assume that $U_{0}^{0} \in \mathbb{H}^{s}\left(\mathbb{R}^{2}\right), s>3$ and $\delta<1$, we have

$$
\begin{aligned}
& \left\|Q U_{2}^{\delta}\right\|_{X_{s-1, T}} \leq C, \quad\left\|P U_{2}^{\delta}\right\|_{X_{s-2, T}} \leq \frac{C}{\delta^{2}} \\
& \left\|Q U_{3}^{\delta}\right\|_{X_{s-2, T}} \leq \frac{C}{\delta},\left\|Q U_{4}^{\delta}\right\|_{X_{s-3, T}} \leq \frac{C}{\delta^{2}}
\end{aligned}
$$

where the constant $C$ only depends on $\left\|U_{0}\right\|_{X_{s, T}}$.

### 3.2.2 Estimate for the residual

We will now consider the truncated asymptotic expansion:

$$
\begin{aligned}
u^{\varepsilon, \delta}(t, \tau, x, y)= & U_{0}^{\delta}(t, \tau, x, y)+\varepsilon U_{1}^{\delta}(t, \tau, x, y)+\varepsilon^{2} U_{2}^{\delta}(t, \tau, x, y) \\
& +\varepsilon^{3} Q U_{3}^{\delta}(t, \tau, x, y)+\varepsilon^{4} Q U_{4}^{\delta}(t, \tau, x, y)
\end{aligned}
$$

and we will inject it in Equation (14) and prove the smallness of the residuals

$$
\begin{aligned}
& R^{\varepsilon, \delta}(t, \tau, x, y)=\sum_{i=-2}^{14} \varepsilon^{i} R_{i}^{\delta}(t, \tau, x, y) \\
& R_{-2}^{\delta}= L U_{0}^{\delta}=0 \\
& R_{-1}^{\delta}= L U_{1}^{\delta}=0 \\
& R_{0}^{\delta}= \partial_{t} U_{0}^{\delta}+A_{1} \partial_{x} U_{0}^{\delta}-B\left(U_{0}^{\delta}, U_{0}^{\delta}\right)+L Q U_{2}^{\delta}=0 \\
& R_{1}^{\delta}= \partial_{t} U_{1}^{\delta}+A_{1} \partial_{x} U_{1}^{\delta}+A_{2} \partial_{y} U_{0}^{\delta}+L Q U_{3}^{\delta}-2 B\left(U_{0}^{\delta}, U_{1}^{\delta}\right)=0 \\
& R_{2}^{\delta}= \partial_{t} U_{2}^{\delta}+A_{1} \partial_{x} U_{2}^{\delta}+A_{2} \partial_{y} U_{1}^{\delta}+L Q U_{4}^{\delta}-B\left(U_{1}^{\delta}, U_{1}^{\delta}\right)-2 B\left(U_{0}^{\delta}, U_{2}^{\delta}\right) \\
&+\partial_{\tau} U_{0}^{\delta}-T\left(U_{0}^{\delta}, U_{0}^{\delta}, U_{0}^{\delta}\right) \\
&= \sum_{j}\left(B_{j}\left(P_{j} U_{0}^{\delta}, \partial_{x} P_{j} U_{0}^{\delta}\right)+F_{j}\left(P_{j} U_{0}^{\delta}, P_{j} U_{0}^{\delta}, P_{j} U_{0}^{\delta}\right)\right) \\
&-\chi^{\delta}\left(\sum_{j}\left(B_{j}\left(P_{j} U_{0}, \partial_{x} P_{j} U_{0}\right)+F_{j}\left(P_{j} U_{0}, P_{j} U_{0}, P_{j} U_{0}\right)\right)\right)
\end{aligned}
$$

We do not have $R_{2}^{\delta}=0$ since $U_{0}^{\delta}$ does not solve Equation (23) but:

$$
\begin{array}{r}
\partial_{x}\left(\partial_{\tau} P_{j} U_{0}^{\delta}-D_{j} \partial_{x}^{2} P_{j} U_{0}^{\delta}+\chi^{\delta}\left(B_{j}\left(P_{j} U_{0}, P_{j} U_{0}\right)+F_{j}\left(P_{j} U_{0}, P_{j} U_{0}, P_{j} U_{0}\right)\right)\right) \\
=C_{j} \partial_{y}^{2} P_{j} U_{0}^{\delta}
\end{array}
$$

The next residuals $\left(R_{i}^{\delta}\right)_{i=3, \ldots, 14}$ depend polynomially on $U_{0}^{\delta}, U_{1}^{\delta}, U_{2}^{\delta}, Q U_{3}^{\delta}, Q U_{4}^{\delta}$ and on their derivatives with respect to the slow time variable $\tau$.

Proposition 3.8 Assume $s>5$. Then the residual fulfills the following estimates:

$$
\sup _{t \in\left[0, T / \varepsilon^{2}\right]}\left\|R^{\varepsilon, \delta}\left(t, \varepsilon^{2} \tau, \cdot, \cdot\right)\right\|_{\mathbb{H}^{s-5}\left(\mathbb{R}^{2}\right)}=\left(f(\delta)+\frac{\varepsilon}{\delta^{6}}\right) O\left(\varepsilon^{2}\right)+\varepsilon^{2} g(\varepsilon)
$$

and if $s>6$,

$$
\left\|\mathcal{R}^{\varepsilon, \delta}\right\|_{\mathbb{L}^{\infty}\left(\left[0, T / \varepsilon^{2}\right] \times \mathbb{R}^{2}\right)}=\left(f(\delta)+\frac{\varepsilon}{\delta^{6}}\right) O\left(\varepsilon^{2}\right)+\varepsilon^{2} g(\varepsilon)
$$

where $\mathcal{R}^{\varepsilon, \delta}(t, x, y)=R^{\varepsilon, \delta}\left(t, \varepsilon^{2} t, x, \varepsilon y\right)$, $f$ is a positive nonincreasing function such that $f(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, and $g$ is a positive function such that $g(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. As said above, $R_{-2}^{\delta}=R_{-1}^{\delta}=R_{0}^{\delta}=R_{1}^{\delta}=0$. The lemma 3.5 gives $\sup _{t \in\left[0, T / \varepsilon^{2}\right]}\left\|R_{2}^{\delta}\left(t, \varepsilon^{2} \tau, \cdot, \cdot\right)\right\|_{\mathbb{H}^{s}\left(\mathbb{R}^{2}\right)}=f(\delta)$ where $f$ is as stated in the proposition 3.8 .

In order to estimate the following terms we need some bounds on $\partial_{\tau} U_{0}, \partial_{\tau} U_{1}, \ldots$ If we consider Eq. (23) we get $\left\|\partial_{\tau} U_{0}\right\|_{X_{s-2, T}} \leq \frac{C}{\delta}$. Following Lemmas 3.6 and 3.7 we get:

$$
\begin{aligned}
& \left\|\partial_{\tau} U_{1}^{\delta}\right\|_{X_{s-3, T}} \leq \frac{C}{\delta^{2}}, \quad\left\|\partial_{\tau} Q U_{2}^{\delta}\right\|_{X_{s-3, T}} \leq \frac{C}{\delta},\left\|\partial_{\tau} P U_{2}^{\delta}\right\|_{X_{s-4, T}} \leq \frac{C}{\delta^{3}} \\
& \left\|\partial_{\tau} Q U_{3}^{\delta}\right\|_{X_{s-4, T}} \leq \frac{C}{\delta^{2}},\left\|\partial_{\tau} Q U_{4}^{\delta}\right\|_{X_{s-5, T}} \leq \frac{C}{\delta^{3}}
\end{aligned}
$$

We can now estimate each term in the residual. We finally obtain the announced result. As $s>6$ the classical Sobolev embedding yields the $\mathbb{L}^{\infty}$ estimate.

### 3.2.3 Stability for the KZ approximation

We have shown so far that there exits a truncated approximate solution $u^{\varepsilon, \delta}$ to Eq. (14) for times of order $O\left(\frac{1}{\varepsilon^{2}}\right)$ and whose residual is small. We now prove that the untruncated leading term $U_{0}$ remains close to the exact solution for times of order $O\left(\frac{1}{\varepsilon^{2}}\right)$. More precisely, following [3], we show that for any $T>0$ such that the leading term exists, the exact solution of Eq. (14) exists on $\left[0, T / \varepsilon^{2}\right]$ and remains close to the leading term.

Theorem 3.9 Let Hypotheses 1-5 be satisfied. Let $U_{0}^{0}=P U_{0}^{0} \in \mathbb{H}^{s}\left(\mathbb{R}^{2}\right)$, $s>6$. Let $T_{0}>0$ such that $U_{0}=\sum_{i=1}^{k} P_{j} U_{0}$ lies in $\mathcal{C}\left(\left[0, T_{0} / \varepsilon^{2}\right] ; \mathbb{H}^{s-1}\left(\mathbb{R}^{2}\right)\right)$, where $\left(P_{j} U_{0}\right)_{j=1, \ldots, k}$ fulfill Eqs. (15) and (23). There exists $\varepsilon_{0}>0$ such that for all $0<\varepsilon<\varepsilon_{0}$, there is an unique solution $\mathcal{U}^{\varepsilon} \in \mathcal{C}^{0}\left(\left[0, T_{0} / \varepsilon^{2}\right] ; \mathbb{H}^{s-5}\left(\mathbb{R}^{2}\right)\right)$ of

$$
\partial_{t} \mathcal{U}^{\varepsilon}+A_{1} \partial_{x} \mathcal{U}^{\varepsilon}+A_{2} \partial_{y} \mathcal{U}^{\varepsilon}+\frac{1}{\varepsilon^{2}} L \mathcal{U}^{\varepsilon}=B\left(\mathcal{U}^{\varepsilon}, \mathcal{U}^{\varepsilon}\right)+\varepsilon^{2} T\left(\mathcal{U}^{\varepsilon}, \mathcal{U}^{\varepsilon}, \mathcal{U}^{\varepsilon}\right)
$$

with $\mathcal{U}^{\varepsilon}(0, x, y)=U_{0}^{0}(x, \varepsilon y)$. Moreover

$$
\left\|\mathcal{U}^{\varepsilon}-\mathcal{U}_{0}^{\varepsilon}\right\|_{\mathbb{L}^{\infty}\left(\left[0, T_{0} / \varepsilon^{2}\right] \times \mathbb{R}^{2}\right)}=o(1) \quad \text { as } \varepsilon \rightarrow 0
$$

where $\mathcal{U}_{0}^{\varepsilon}(t, x, y)=U_{0}\left(t, \varepsilon^{2} t, x, \varepsilon y\right)$.

Proof. We will look for the exact solution $\mathcal{U}^{\varepsilon}$ as a perturbation of the truncated asymptotic expansion $\mathcal{U}^{\varepsilon, \delta}$ :

$$
\mathcal{U}^{\varepsilon}(t, x, y)=\mathcal{U}^{\varepsilon, \delta}(t, x, \varepsilon y)+\widetilde{U}^{\delta}(t, x, \varepsilon y)
$$

where $\mathcal{U}^{\varepsilon, \delta}(t, x, y)=u^{\varepsilon, \delta}\left(t, \varepsilon^{2} t, x, y\right)$. The error term fulfills:

$$
\begin{align*}
\partial_{t} \widetilde{U}^{\delta}+A_{1} \partial_{x} \widetilde{U}^{\delta}+\varepsilon A_{2} \partial_{y} \widetilde{U}^{\delta}+\frac{1}{\varepsilon^{2}} L \widetilde{U}^{\delta}= & 2 B\left(\mathcal{U}^{\varepsilon, \delta}, \widetilde{U}^{\delta}\right)+B\left(\widetilde{U}^{\delta}, \widetilde{U}^{\delta}\right) \\
& +3 \varepsilon^{2} T\left(\mathcal{U}^{\varepsilon, \delta}, \mathcal{U}^{\varepsilon, \delta}, \widetilde{U}^{\delta}\right)  \tag{27}\\
& +3 \varepsilon^{2} T\left(\mathcal{U}^{\varepsilon, \delta}, \widetilde{U}^{\delta}, \widetilde{U}^{\delta}\right) \\
& +\varepsilon^{2} T\left(\widetilde{U}^{\delta}, \widetilde{U}^{\delta}, \widetilde{U}^{\delta}\right)+\mathcal{R}^{\varepsilon, \delta},
\end{align*}
$$

with the initial data $\widetilde{U}^{\delta}(t=0, x, y)=\left(U_{0}^{0}-\chi^{\delta} U_{0}^{0}\right)(x, y)$.
Local existence of $\widetilde{U}^{\delta}$ in $\mathcal{C}\left([0, T] ; \mathbb{H}^{s-5}\left(\mathbb{R}^{2}\right)\right) \cap \mathcal{C}^{1}\left([0, T] ; \mathbb{H}^{s-6}\left(\mathbb{R}^{2}\right)\right)$ can be proved using classical techniques (see [1] for example).
To obtain the $\mathbb{H}^{s-5}$ estimates we apply the previously defined operator $\Lambda^{s-5}$ (see (26)) to (27) and we take the scalar product in $\mathbb{L}^{2}$ with $\Lambda^{s-5} \tilde{U}^{\delta}$. As $\mathbb{H}^{l}\left(\mathbb{R}^{2}\right)$ is an algebra as soon as $l>1$, there exists some constants $C>0$ such that:

$$
\begin{aligned}
&\left(B\left(V, V^{\prime}\right), V^{\prime \prime}\right)_{s-5} \leq C\|V\|_{\mathbb{H}^{s-5}}\left\|V^{\prime}\right\|_{\mathbb{H}^{s-5}}\left\|Q V^{\prime \prime}\right\|_{\mathbb{H}^{s-5}} \\
&+C\|V\|_{\mathbb{H}^{s-5}}\left\|V^{\prime \prime}\right\|_{\mathbb{H}^{s-5}}\left\|Q V^{\prime}\right\|_{\mathbb{H}^{s-5}} \\
&+C\left\|V^{\prime}\right\|_{\mathbb{H}^{s-5}}\left\|V^{\prime \prime}\right\|_{\mathbb{H}^{s-5}}\|Q V\|_{\mathbb{H}^{s-5}} \\
&\left\|T\left(V, V^{\prime}, V^{\prime \prime}\right)\right\|_{\mathbb{H}^{s-5}} \leq C\|V\|_{\mathbb{H}^{s-5}}\left\|V^{\prime}\right\|_{\mathbb{H}^{s-5}}\left\|V^{\prime \prime}\right\|_{\mathbb{H}^{s-5}} \\
&(L V, V)_{s-5} \geq C\|Q V\|_{\mathbb{H}^{s-5}}^{2} \quad \text { (Hypothesis 2) } .
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2}+\frac{C}{\varepsilon^{2}}\left\|Q \widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2} \leq & C\left\|\mathcal{U}^{\varepsilon, \delta}\right\|_{\mathbb{H}^{s-5}}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}\left\|Q \widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}} \\
& +C\left\|Q \mathcal{U}^{\varepsilon, \delta}\right\|_{\mathbb{H}^{s-5}}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2}+C\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2}\left\|Q \widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}} \\
& +C \varepsilon^{2}\left\|\mathcal{U}^{\varepsilon, \delta}\right\|_{\mathbb{H}^{s-5}}^{2}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2}+C \varepsilon^{2}\left\|\mathcal{U}^{\varepsilon, \delta}\right\|_{\mathbb{H}^{s-5}}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{3} \\
& +C \varepsilon^{2}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{4}+\left\|R^{\varepsilon, \delta}\left(t, \varepsilon^{2} t, \cdot, \cdot\right)\right\|_{\mathbb{H}^{s-5}}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}
\end{aligned}
$$

Following [3] we introduce $t_{0}(\varepsilon, \delta)$ defined as:

$$
t_{0}(\varepsilon, \delta)=\sup \left\{\mathrm{t} \in\left[0, \mathrm{~T}_{0} / \varepsilon^{2}\right],\left\|\widetilde{\mathrm{U}}^{\delta}(\mathrm{t})\right\|_{\mathbb{H}^{s-5}} \leq 1\right\}
$$

We also remark that:

$$
\left\|\mathcal{U}^{\varepsilon, \delta}\right\|_{\mathbb{H}^{s-5}} \leq C\left(1+\frac{\varepsilon^{4}}{\delta^{4}}\right),\left\|Q \mathcal{U}^{\varepsilon, \delta}\right\|_{\mathbb{H}^{s-5}} \leq C \varepsilon^{2}\left(1+\frac{\varepsilon^{2}}{\delta^{2}}\right)
$$

By means of Young's inequality, absorbing every term involving $Q \widetilde{U}^{\delta}$, we obtain:
$\frac{1}{2} \frac{d}{d t}\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2}+\frac{C}{\varepsilon^{2}}\left\|Q \widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2} \leq C \varepsilon^{2}\left(1+\frac{\varepsilon^{8}}{\delta^{8}}\right)\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2}+C \varepsilon^{2}\left(f(\delta)+\frac{\varepsilon}{\delta^{6}}+g(\varepsilon)\right)$,
where $f$ and $g$ are as stated in Prop. 3.8. By Gronwall's lemma,

$$
\begin{align*}
\left\|\widetilde{U}^{\delta}\right\|_{\mathbb{H}^{s-5}}^{2} \leq\left(\left\|\tilde{U}^{\delta}(t=0)\right\|_{\mathbb{H}^{s-5}}^{2}+\right. & \left.C\left(f(\delta)+\frac{\varepsilon}{\delta^{6}}+g(\varepsilon)\right)\right) \\
& \exp \left(C \varepsilon^{2}\left(1+\frac{\varepsilon^{8}}{\delta^{8}}\right) t\right) . \tag{28}
\end{align*}
$$

Since $\left\|\widetilde{U}^{\delta}(t=0)\right\|_{\mathbb{H}^{s-5}} \rightarrow 0$ when $\delta \rightarrow 0$, we can take for example $\delta=\varepsilon^{1 / 7}$ and for $\varepsilon$ small enough, we deduce from Eq. (28) that $t_{0}(\varepsilon, \delta)=\frac{T_{0}}{\varepsilon^{2}}$.

Now we have to prove that the main profile $\mathcal{U}_{0}^{\varepsilon}$ remains close from the approximate solution $\mathcal{U}^{\varepsilon, \delta}$ on the interval $\left[0, T_{0} / \varepsilon^{2}\right]$ :

$$
\begin{aligned}
\sup _{t \in\left[0, T_{0} / \varepsilon^{2}\right]}\left\|\mathcal{U}^{\varepsilon, \delta}-U_{0}\left(t, \varepsilon^{2} t, \cdot, \cdot\right)\right\|_{\mathbb{H}^{s-5}} \leq & \sup _{t \in\left[0, T_{0} / \varepsilon^{2}\right]}\left\|\mathcal{U}^{\varepsilon, \delta}-U_{0}^{\delta}\left(t, \varepsilon^{2} t, \cdot, \cdot\right)\right\|_{\mathbb{H}^{s}-5} \\
& +\sup _{t \in\left[0, T_{0} / \varepsilon^{2}\right]}\left\|U_{0}^{\delta}\left(t, \varepsilon^{2} t, \cdot, \cdot\right)-U_{0}\left(t, \varepsilon^{2} t, \cdot, \cdot\right)\right\|_{\mathbb{H}^{s-5}} \\
\leq & C \varepsilon^{6 / 7}+\sup _{t \in \mathbb{R}, \tau \in\left[0, T_{0}\right]}\left\|U_{0}^{\delta}(t, \tau, \cdot, \cdot)-U_{0}(t, \tau, \cdot, \cdot)\right\|_{\mathbb{H}^{s-5}}
\end{aligned}
$$

To estimate the last term, we go back to the definition of $U_{0}$ :

$$
U_{0}(t, \tau, x, y)=\sum_{j=1, \ldots, k} \widetilde{u_{j}}\left(\tau, x-v_{j} t, y\right)
$$

and we let $I_{j}^{\delta}(\tau)=\left\|\widetilde{u_{j}}-\chi^{\delta} \widetilde{u_{j}}\right\|_{\mathbb{H}^{s-5}}$ for $j=1, \ldots, k$. Following the proof of lemma 3.5 we obtain that for all $j=1, \ldots, k, \sup _{\tau \in\left[0, T_{0}\right]} I_{j}^{\delta}(\tau) \rightarrow 0$ as $\delta \rightarrow 0$. By the same way we obtained the regularity in the variable $t$ before, we get that, when $\delta \rightarrow 0$,

$$
\sup _{t \in \mathbb{R}, \tau \in\left[0, T_{0}\right]}\left\|U_{0}^{\delta}(t, \tau, \cdot, \cdot)-U_{0}(t, \tau, \cdot, \cdot)\right\|_{\mathbb{H}^{s-5}} \rightarrow 0
$$

Finally, as $s>6$ we use the classical Sobolev's imbedding to obtain the announced $\mathbb{L}^{\infty}$ estimates.

## 4 The Landau-Lifschitz-Maxwell Equations

We now return to Eq. (13) and the system of partial differential equations of micromagnetics (10)-(12).
As we observed in Section 2.2, Eq. (13) is a special case of the general equation (14). Hypotheses 1-3 are satisfied (see Section 2.3). We will verify the remaining hypothesis 4 once we have found the coefficients $D_{j}$ and the applications $B_{j}$. The asymptotic expansion is therefore unique and valid on the slow time scale. The way the asymptotic approximation has been built is irrelevant. This observation is important because it allows us to use the Landau-Lifschitz equation in the form given by Gilbert:

$$
\begin{equation*}
\partial_{t} M=-M \wedge H+\frac{\gamma}{|M|}\left(M \wedge \partial_{t} M\right) \tag{29}
\end{equation*}
$$

This equation, known as Landau-Lifschitz-Gilbert equation is equivalent with the Landau-Lifschitz (6) equation, except for a rescaling of time by a factor $\left(1+\gamma^{2}\right)$. As it turns out, Eq. (29) is more convenient for constructing the asymptotic expansions.

We perform one more change in the equation: as we did in Section 2 we simplify the equation (29) by replacing the term $|M|$ in the denominator of the damping term by $\left|M_{0}\right|=1$ so that the factor of the damping term is $\gamma$. Thus we consider the following system:

$$
\begin{align*}
& \varepsilon^{2} \partial_{t} M+\varepsilon^{4} \partial_{\tau} M=-M_{0} \wedge H+\alpha^{-1} M_{0} \wedge M-\varepsilon^{2} M \wedge H+\gamma\left[\varepsilon^{2} M_{0} \wedge \partial_{t} M\right. \\
&\left.\quad+\varepsilon^{4} M_{0} \wedge \partial_{\tau} M+\varepsilon^{4} M \wedge \partial_{t} M+\varepsilon^{6} M \wedge \partial_{\tau} M\right]  \tag{31}\\
& \varepsilon^{2} \partial_{t} H+\varepsilon^{4} \partial_{\tau} H-\varepsilon^{2} k \wedge \partial_{x} E-\varepsilon^{3} l \wedge \partial_{y} E=-\varepsilon^{2} \partial_{t} M-\varepsilon^{4} \partial_{\tau} M  \tag{30}\\
& \varepsilon^{2} \partial_{t} E+\varepsilon^{4} \partial_{\tau} E+\varepsilon^{2} k \wedge \partial_{x} H+\varepsilon^{3} l \wedge \partial_{y} H=0 \tag{32}
\end{align*}
$$

We perform now an asymptotic expansion of Eqs. (30)-(31) along the lines of Section 3.1:

$$
\begin{align*}
& M=M_{1}+\varepsilon M_{2}+\varepsilon^{3} M_{3}+\ldots \\
& H=H_{1}+\varepsilon H_{2}+\varepsilon^{3} H_{3}+\ldots  \tag{33}\\
& M=E_{1}+\varepsilon E_{2}+\varepsilon^{3} E_{3}+\ldots
\end{align*}
$$

### 4.1 The equations of order $O\left(\varepsilon^{0}\right)$

To leading order, Eqs. (30)-(32) reduce to a single equation,

$$
\begin{equation*}
-M_{0} \wedge\left(H_{1}-\alpha^{-1} M_{1}\right)=0 \tag{34}
\end{equation*}
$$

which gives an expression for $M_{1}$ in terms of $M_{1} \cdot M_{0}$ and $H_{1}$,

$$
\begin{equation*}
M_{1}=\left(M_{1} \cdot M_{0}\right) M_{0}-\alpha M_{0} \wedge\left(M_{0} \wedge H_{1}\right) . \tag{35}
\end{equation*}
$$

### 4.2 The equations of Order $O(\varepsilon)$

To first order we get,

$$
\begin{equation*}
-M_{0} \wedge\left(H_{2}-\alpha^{-1} M_{2}\right)=0 \tag{36}
\end{equation*}
$$

and then an expression for $M_{2}$,

$$
\begin{equation*}
M_{2}=\left(M_{2} \cdot M_{0}\right) M_{0}-\alpha M_{0} \wedge\left(M_{0} \wedge H_{2}\right) \tag{37}
\end{equation*}
$$

4.3 The equations of Order $O\left(\varepsilon^{2}\right)$

$$
\left\{\begin{array}{l}
\partial_{t} M_{1}=-M_{0} \wedge\left(H_{3}-\alpha^{-1} M_{3}\right)-M_{1} \wedge H_{1}+\gamma M_{0} \wedge \partial_{t} M_{1}  \tag{38}\\
\partial_{t} H_{1}-k \wedge \partial_{x} E_{1}=-\partial_{t} M_{1} \\
\partial_{t} E_{1}+k \wedge \partial_{x} H_{1}=0
\end{array}\right.
$$

Taking the scalar product of (38) with $M_{0}$ and adding to it the scalar product of (34) with $M_{1}$ we find that $\partial_{t}\left(M_{1} \cdot M_{0}\right)=0$, so:

$$
\begin{equation*}
M_{1} \cdot M_{0}=f_{0}, \quad f_{0} \equiv f_{0}(x, y, \tau) \tag{39}
\end{equation*}
$$

(Note that $M_{1} \cdot M_{0}$ is the $O\left(\varepsilon^{2}\right)$ term in the expansion of $|M|^{2}$, which is constant) If we take the vector product instead of the scalar product we obtain an expression for $M_{3}$ in terms of $M_{3} \cdot M_{0}$ and $H_{2}$.

$$
\begin{equation*}
M_{3}=\left(M_{3} \cdot M_{0}\right) M_{0}-\alpha M_{0} \wedge\left(M_{0} \wedge H_{3}\right)+\alpha M_{0} \wedge q_{1} \tag{40}
\end{equation*}
$$

where the vector $q_{1}$ is given by:

$$
q_{1}=-\partial_{t} M_{1}+\gamma M_{0} \wedge \partial_{t} M_{1}-M_{1} \wedge H_{1}
$$

We substitute the expression (35) in the second equation of (38), use the fact that $\partial_{t}\left(M_{1} \cdot M_{0}\right)=0$ and solve the resulting equation for $\partial_{t} H_{1}$ to obtain a system of equations for $H_{1}$ and $E_{1}$,

$$
\left\{\begin{array}{l}
\partial_{t} H_{1}+\frac{\alpha}{\alpha+1}\left(k \cdot\left(M_{0} \wedge \partial_{x} E_{1}\right)\right) M_{0}-\frac{1}{1+\alpha} k \wedge \partial_{x} E_{1}=0  \tag{41}\\
\partial_{t} E_{1}+k \wedge \partial_{x} H_{1}=0
\end{array}\right.
$$

### 4.3.1 Choice of coordinates

The system of equations (41) is most easily solved if we adopt a coordinate system in $\mathbb{R}^{3}$ that is spanned by $k, k \wedge M_{0}$ and $M_{0}$ (we rely on the assumption that $k$ and $M_{0}$ are not parallel or antiparallel.) Given any vector $v \in \mathbb{R}^{3}$ we define:

$$
\begin{equation*}
v_{a}=v \cdot M_{0}, \quad v_{b}=v \cdot\left(k \wedge M_{0}\right), \quad v_{c}=v \cdot k \tag{42}
\end{equation*}
$$

Then

$$
v=\frac{1}{1-k_{a}^{2}}\left[\left(v_{a}-k_{a} v_{c}\right) M_{0}+v_{b}\left(k \wedge M_{0}\right)+\left(v_{c}-k_{a} v_{a}\right) k\right], \quad v \in \mathbb{R}^{3},
$$

where $k_{a}=M_{0} \cdot k$. An easy computation shows that:

$$
\begin{gathered}
u \cdot v=\frac{1}{1-k_{a}^{2}}\left[u_{a} v_{a}+u_{b} v_{b}+u_{c} v_{c}-k_{a}\left(u_{a} v_{c}+u_{c} v_{a}\right)\right], \quad u, v \in \mathbb{R}^{3}, \\
u \wedge v=\frac{1}{1-k_{a}^{2}}\left|\begin{array}{ccc}
M_{0} k \wedge M_{0} & k \\
u_{a} & u_{b} & u_{c} \\
v_{a} & v_{b} & v_{c}
\end{array}\right|, u, v \in \mathbb{R}^{3} .
\end{gathered}
$$

The system of equations (41) becomes:

$$
\begin{equation*}
\partial_{t} u_{1}+K \partial_{x} u_{1}=0 \tag{43}
\end{equation*}
$$

where $u_{1}=\left(H_{1 a}, H_{1 b}, H_{1 c}, E_{1 a}, E_{1 b}, E_{1 c}\right)^{t}$ and

$$
K=\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -(1+\alpha)^{-1} & 0 & k_{a}(1+\alpha)^{-1} \\
0 & 0 & 0 & 0 & k_{a} \alpha(1+\alpha)^{-1} & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -k_{a} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

### 4.3.2 Solution of Equation (43)

The characteristic polynomial of $K$ is $\quad \operatorname{det}(\lambda I-K)=\left(\lambda^{2}-v_{0}^{2}\right)\left(\lambda^{2}-v_{1}^{2}\right)\left(\lambda^{2}-\right.$ $v_{2}^{2}$ ), where

$$
v_{0}=0, \quad v_{1}=\left(\frac{1}{1+\alpha}\right)^{1 / 2}, \quad v_{2}=\left(\frac{1+\left(1-k_{a}^{2}\right) \alpha}{1+\alpha}\right)^{1 / 2} .
$$

Then the eigenvalues of $K$ are $v_{0}=0$ (algebraic multiplicity 2 ), $\pm v_{1}$ and $\pm v_{2}$. We also have $v_{0}<v_{1}<v_{2}$ and $1-v_{2}^{2}=k_{a}^{2}\left(1-v_{1}^{2}\right)$. The matrix $K$ is diagonalized by the linear transformation $F, \quad K=F^{-1} V F, \quad V=$ $\operatorname{diag}\left(v_{0}, v_{0}, v_{1},-v_{1}, v_{2},-v_{2}\right)$ and

$$
F=\left(\begin{array}{cccccc}
k_{a}\left(1-v_{1}^{-2}\right) & 0 & v_{1}^{-2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -v_{1} & 0 & k_{a} v_{1} \\
0 & 1 & 0 & v_{1} & 0 & -k_{a} v_{1} \\
v_{2}^{-1} & 0 & -k_{a} v_{2}^{-1} & 0 & 1 & 0 \\
-v_{2}^{-1} & 0 & k_{a} v_{2}^{-1} & 0 & 1 & 0
\end{array}\right)
$$

Applying $F$ to Eq. (43) we obtain a diagonal system,

$$
\left(\partial_{t}+V \partial_{x}\right) F u_{1}=0
$$

This system corresponds to Eq. (15). The equations are decoupled and each equation can be integrated along its characteristics. We find $u_{1}=F^{-1} f$ where $f=\left(f_{1}, f_{2}, f_{3}, f_{4}, f_{5}, f_{6}\right)^{t}$ and

$$
\begin{aligned}
& f_{1} \equiv f_{1}(\tau, x, y), \\
& f_{3} \equiv f_{3}\left(\tau, x-v_{1} t, y\right), f_{4} \equiv f_{2}(\tau, x, y) \\
& f_{5} \equiv f_{5}\left(\tau, x+v_{1} t, y\right) \\
&\left(\tau-v_{2} t, y\right), f_{6} \equiv f_{6}\left(\tau, x+v_{2} t, y\right)
\end{aligned}
$$

The functions $f_{1}$ and $f_{2}$ represent standing waves, $f_{3}$ and $f_{4}$ are travelling waves propagating respectively with the velocities $v_{1}$ and $-v_{1}$, and $f_{5}$ and $f_{6}$ are travelling waves propagating respectively with the velocities $v_{2}$ and $-v_{2}$. The component of $M_{1}$ are deduced from Equations (35)-(39) and the expression of $u_{1}$,
$M_{1 a}=f_{0}, M_{1 b}=\frac{1-v_{1}^{2}}{2 v_{1}^{2}}\left(f_{3}+f_{4}\right), M_{1 c}=k_{a} f_{0}+\frac{v_{2}^{2}-v_{1}^{2}}{v_{2}^{2}} f_{1}-k_{a} \frac{1-v_{1}^{2}}{2 v_{2}}\left(f_{5}-f_{6}\right)$.
This completes the analysis of the second order approximation. We now know that the coefficients of order 0 in the expansions (33) are linear combinations of standing ( $v_{0}=0$ ) and travelling ( $\pm v_{1}, \pm v_{2}$ ) waves. In the next sections we will see how the profile functions $f_{0}, \ldots, f_{6}$ evolve on the slow time scale $(\tau)$ and on the transverse direction $(y)$.

To the third order Equations (30)-(32) yield the differential equations:

$$
\left\{\begin{array}{l}
\partial_{t} M_{2}=-M_{0} \wedge\left(H_{4}-\alpha^{-1} M_{4}\right)-M_{1} \wedge H_{2}-M_{2} \wedge H_{1}+\gamma M_{0} \wedge \partial_{t} M_{2}  \tag{45}\\
\partial_{t} H_{2}-k \wedge \partial_{x} E_{2}-l \wedge \partial_{y} E_{1}=-\partial_{t} M_{2} \\
\partial_{t} E_{2}+k \wedge \partial_{x} H_{2}+l \wedge \partial_{y} H_{1}=0
\end{array}\right.
$$

We follow the same procedure as in the previous section. We take the scalar product of the first equation of (45) with $M_{0}$ and we use Eqs. (35) and (37) to get:

$$
\begin{equation*}
\partial_{t}\left(M_{2} \cdot M_{0}\right)=2 \alpha M_{0} \cdot H_{1} \wedge H_{2} \tag{46}
\end{equation*}
$$

We now take the vector product with $M_{0}$,

$$
\begin{equation*}
M_{4}=\left(M_{0} \cdot M_{4}\right) M_{0}-\alpha M_{0} \wedge\left(M_{0} \wedge H_{4}\right)+\alpha M_{0} \wedge q_{2}, \tag{47}
\end{equation*}
$$

where the vector $q_{2}$ is given by:

$$
q_{2}=-\partial_{t} M_{2}-M_{1} \wedge H_{2}-M_{2} \wedge H_{1}+\gamma M_{0} \wedge \partial_{t} M_{2}
$$

We substitute the expression (37) in the second equation of (45) and we solve the resulting equation for $\partial_{t} H_{2}$ to obtain a system of equations for $H_{2}$ and $E_{2}$,

$$
\left\{\begin{align*}
\partial_{t} H_{2} & +\frac{\alpha}{1+\alpha}\left(k \cdot M_{0} \wedge \partial_{x} E_{2}+l \cdot M_{0} \wedge \partial_{y} E_{1}\right) M_{0}  \tag{48}\\
& \quad-\frac{1}{1+\alpha}\left(k \wedge \partial_{x} E_{2}+l \wedge \partial_{y} E_{1}\right)+2 \alpha\left(M_{0} \cdot H_{1} \wedge H_{2}\right) M_{0}=0 \\
\partial_{t} E_{2} & +k \wedge \partial_{x} H_{2}+l \wedge \partial_{y} H_{1}=0
\end{align*}\right.
$$

### 4.4.1 Coordinate representation

We use the coordinate system introduced in Section 4.3 .1 with the abbreviations defined in Eq. (42). The equations (48) correspond to the following system of equations:

$$
\begin{equation*}
\partial_{t} u_{2}+K \partial_{x} u_{2}+K^{\prime} \partial_{y} u_{1}+A\left(u_{1}\right) u_{2}=0 \tag{49}
\end{equation*}
$$

where $K$ is the matrix defined in Section 4.3.1, $u_{2}=\left(H_{2 a}, H_{2 b}, H_{2 c}, E_{2 a}, E_{2 b}, E_{2 c}\right)^{t}$,

$$
K^{\prime}=\frac{1}{1-k_{a}^{2}}\left(\begin{array}{cccccc}
0 & 0 & 0 & k_{a} l_{b} & l_{c}-k_{a} l_{a} & -l_{b} \\
0 & 0 & 0 & -\frac{l_{c}}{1+\alpha} & 0 & \frac{l_{a}}{1+\alpha} \\
0 & 0 & 0 & \frac{\left(1+\alpha k_{a}^{2}\right) l_{b}}{1+\alpha} & k_{a} l_{c}-\frac{\left(1+\alpha k_{a}^{2}\right) l_{a}}{1+\alpha} & -k_{a} l_{b} \\
-k_{a} l_{b} & k_{a} l_{a}-l_{c} & l_{b} & 0 & 0 & 0 \\
l_{c} & 0 & -l_{a} & 0 & 0 & 0 \\
-l_{b} & l_{a}-k_{a} l_{c} k_{a} l_{b} & 0 & 0 & 0
\end{array}\right),
$$

and

$$
A\left(u_{1}\right)=\frac{2 \alpha}{1-k_{a}^{2}}\left(\begin{array}{cccccc}
-k_{a} H_{1 b} & k_{a} H_{1 a}-H_{1 c} & H_{1 b} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-k_{a}^{2} H_{1 b} & k_{a}^{2} H_{1 a}-k_{a} H_{1 c} & k_{a} H_{1 b} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

By applying the linear transformation $F$ we obtain:

$$
\begin{equation*}
\partial_{t} F u_{2}+V \partial_{x} F u_{2}+F K^{\prime} F^{-1} \partial_{y} f+\left(F A\left(F^{-1} f\right) F^{-1}\right) F u_{2}=0 \tag{50}
\end{equation*}
$$

Following the scheme given in Section 3.1 we assume that $u_{2}$ fulfills the sublinear growth condition (19) and we take the average of Equation (50) along the characteristics we used in Section 4.3.2. We let $g=F u_{2}=\left(g_{1}, \ldots, g_{6}\right)^{t}$ and $V^{\prime}=F K^{\prime} F^{-1}$ and we get:

$$
\begin{equation*}
(V-v \mathrm{Id}) \partial_{x} G_{v} g+V^{\prime} \partial_{y} G_{v} f+F A\left(F^{-1} G_{v} f\right) F^{-1} G_{v} g=0 \tag{51}
\end{equation*}
$$

with $v \in\left\{0, \pm v_{1}, \pm v_{2}\right\}$, where

$$
V^{\prime}=\frac{1}{1-k_{a}^{2}}\left(\begin{array}{cccccc}
0 & 0 & -a_{1} & a_{1} & a_{2} & a_{2} \\
0 & 0 & -a_{2} & -a_{2} & -a_{3} & a_{3} \\
-a_{4} & a_{5} & a_{6} & -a_{7} & -a_{8} & a_{8} \\
a_{4} & a_{5} & a_{7} & -a_{6} & a_{8} & -a_{8} \\
-a_{9} & -a_{10} & -a_{11} & a_{11} & a_{12} & -a_{13} \\
-a_{9} & a_{10} & a_{11} & -a_{11} & a_{13} & -a_{12}
\end{array}\right)
$$

with

$$
\begin{array}{lll}
a_{1}=\frac{l_{b}}{2 v_{1}}, & a_{2}=\frac{k_{a} l_{c}-l_{a}}{2}, & a_{3}=\frac{l_{b} v_{2}}{2}, \\
a_{4}=l_{b} \frac{v_{1}^{3}}{v_{2}^{2}}\left(1-k_{a}^{2}\right), & a_{5}=v_{1}^{2}\left(l_{a}-k_{a} l_{c}\right), & a_{6}=\frac{l_{c} v_{1}}{2}\left(2-k_{a}^{2}\right), \\
a_{7}=\frac{l_{c} v_{1}}{2} k_{a}^{2}, & a_{8}=\frac{k_{a} l_{b}}{2} \frac{v_{1}}{v_{2}}\left(v_{2}^{2}-v_{1}^{2}\right), & a_{9}=\left(l_{a}-k_{a} l_{c}\right) \frac{v_{1}^{2}}{v_{2}^{2}}, \\
a_{10}=l_{b} v_{2}\left(1-k_{a}^{2}\right), & a_{11}=\frac{l_{b} k_{a}}{2} \frac{v_{2}^{2}-v_{1}^{2}}{v_{1} v_{2}}, & \\
a_{12}=\frac{1}{2} \frac{2-k_{a}^{2}}{v_{2}}\left(l_{c}-l_{a} k_{a}\left(1-v_{1}^{2}\right)\right), & a_{13}=\frac{k_{a}^{2}}{2 v_{2}}\left(l_{c}-l_{a} k_{a}\left(1-v_{1}^{2}\right)\right) .
\end{array}
$$

Two of the 29 equations (the third equation when we apply $G_{v_{1}}$ and the fourth when we apply $G_{-v_{1}}$ ) we can deduce from Eqs. (51) give the same compatibility condition:

$$
\left\{\begin{array}{l}
\frac{l_{c} v_{1}}{2}\left(2-k_{a}^{2}\right) \partial_{y} f_{3}=0 \\
-\frac{l_{c} v_{1}}{2}\left(2-k_{a}^{2}\right) \partial_{y} f_{4}=0
\end{array}\right.
$$

which is:

$$
l_{c}=l \cdot k=0
$$

which means that the direction of the transversal perturbations is orthogonal to the direction of propagation. This compatibility condition is one of the two that Leblond and Manna found in [10].

### 4.5 The equation of $\operatorname{Order} O\left(\varepsilon^{4}\right)$

To the fourth order Eqs. (30)-(32) yield the differential equations

$$
\left\{\begin{align*}
& \partial_{t} M_{3}+\partial_{\tau} M_{1}=-M_{0} \wedge\left(H_{5}-\alpha^{-1} M_{5}\right)-M_{1} \wedge H_{3}-M_{2} \wedge H_{2}-M_{3} \wedge H_{1}  \tag{52}\\
&+\gamma\left[M_{0} \wedge \partial_{t} M_{3}+M_{0} \wedge \partial_{\tau} M_{1}+M_{1} \wedge \partial_{t} M_{1}\right] \\
& \partial_{t} H_{3}+\partial_{\tau} H_{1}-k \wedge \partial_{x} E_{3}-l \wedge \partial_{y} E_{2}=-\partial_{t} M_{3}-\partial_{\tau} M_{1} \\
& \partial_{t} E_{3}+\partial_{\tau} E_{1}+k \wedge \partial_{x} H_{3}+l \wedge \partial_{y} H_{2}=0
\end{align*}\right.
$$

We follow the same procedure as in the previous section. We take the scalar product of the first equation of (52) with $M_{0}$ and we use Eqs. (34), (36) and (40) to get

$$
\begin{equation*}
\partial_{t}\left(M_{3} \cdot M_{0}+\frac{1}{2}\left|M_{1}\right|^{2}\right)+\partial_{\tau}\left(M_{1} \cdot M_{0}\right)=0 . \tag{53}
\end{equation*}
$$

Now we have $M_{0} \cdot M_{1}=f_{0}$ where $f_{0}$ does not depend on $t$. Hence Eq. (53) implies that $M_{3} \cdot M_{0}+\frac{1}{2}\left|M_{1}\right|^{2}$ grows linearly with $t$ as $t \rightarrow+\infty$, unless $f_{0}$ is
independent not only of $t$ but also of $\tau$. We avoid this type of secular behaviour by imposing the condition $f_{0} \equiv f_{0}(x, y)$. If $M_{1} \cdot M_{0}=0$ at $t=0$ then this condition gives $f_{0} \equiv 0$, and the expressions (44) simplify,

$$
M_{1 a}=0, M_{1 b}=\frac{1-v_{1}^{2}}{2 v_{1}^{2}}\left(f_{3}+f_{4}\right), M_{1 c}=\frac{v_{2}^{2}-v_{1}^{2}}{v_{2}^{2}} f_{1}-k_{a} \frac{1-v_{1}^{2}}{2 v_{2}}\left(f_{5}-f_{6}\right)
$$

If $M_{1}$ and $M_{0}$ are not orthogonal at $t=0$, a constant nonzero component must be added to $M_{1 a}, M_{1 c}$ and the quantities derived from them. The scalar product of Eq. (52) with $M_{0}$ thus yields the relation:

$$
\begin{equation*}
2 M_{3} \cdot M_{0}+\left|M_{1}\right|^{2}=g_{0}, \quad g_{0} \equiv g_{0}(\tau, x, y) \tag{54}
\end{equation*}
$$

We now take the vector product with $M_{0}$,

$$
\begin{equation*}
M_{5}=\left(M_{0} \cdot M_{5}\right) M_{0}-\alpha M_{0} \wedge\left(M_{0} \wedge H_{5}\right)+\alpha M_{0} \wedge q_{3}, \tag{55}
\end{equation*}
$$

where the vector $q_{3}$ is given by:

$$
\begin{aligned}
q_{3}=-\partial_{t} M_{3}-\partial_{\tau} M_{1}-M_{1} \wedge H_{3}-M_{2} \wedge & H_{2}-M_{3} \wedge H_{1}+\gamma M_{0} \wedge \partial_{t} M_{3} \\
& +\gamma M_{1} \wedge \partial_{t} M_{1}+\gamma M_{0} \wedge \partial_{\tau} M_{1} .
\end{aligned}
$$

We substitute the expression (35) in the second equation of (52) and we solve the resulting equation for $\partial_{t} H_{3}$ to obtain a system of equations for $H_{3}$ and $E_{3}$,

$$
\left\{\begin{align*}
\partial_{t} H_{3} & +\frac{\alpha}{1+\alpha}\left(k \cdot\left(M_{0} \wedge \partial_{x} E_{3}\right)+l \cdot\left(M_{0} \wedge \partial_{y} E_{2}\right)\right) M_{0}  \tag{56}\\
& -\frac{1}{1+\alpha}\left(k \wedge \partial_{x} E_{3}+l \wedge \partial_{y} E_{2}\right) \\
= & \frac{\alpha}{1+\alpha}\left(M_{0} \cdot v_{H}\right) M_{0}+\frac{1}{1+\alpha} v_{H}, \\
\partial_{t} E_{2} & +k \wedge \partial_{x} H_{2}+l \wedge \partial_{y} H_{1}=v_{E},
\end{align*}\right.
$$

where the vectors $v_{H}$ and $v_{E}$ are:

$$
\begin{aligned}
& v_{H}=-\partial_{\tau}\left(H_{1}+M_{1}\right)+\partial_{t}\left(\frac{1}{2}\left|M_{1}\right|^{2} M_{0}-\alpha M_{o} \wedge q_{1}\right), \\
& v_{E}=-\partial_{\tau} E_{1} .
\end{aligned}
$$

### 4.5.1 Coordinate representation

We use the coordinate system introduced in Section 4.3.1 with the abbreviations defined in Eq. (42). Eqs. (56) correspond to the following system of equations:

$$
\begin{equation*}
\partial_{t} u_{3}+K \partial_{x} u_{3}+K^{\prime} \partial_{y} u_{2}=-\partial_{\tau} u_{1}+\partial_{t} r \tag{57}
\end{equation*}
$$

where $K$ and $K^{\prime}$ are the matrix defined in sections 4.3.1 and 4.4.1, $u_{3}=\left(H_{3 a}, H_{3 b}, H_{3 c}, E_{3 a}, E_{3 b}, E_{3 c}\right)^{t}$ and

$$
r=\left(\frac{1}{2}\left|M_{1}\right|^{2}, \frac{\alpha}{1+\alpha} q_{1 c}, \frac{1}{2} k_{a}\left|M_{1}\right|^{2}-\frac{\alpha}{1+\alpha} q_{1 b}, 0,0,0\right)^{t}
$$

The elements of $r$ are known (in terms of $f_{1}$ and $f_{3}$ through $f_{6} ; f_{2}$ does not enter). Notice, however that $f_{1}$ does not depend on $t$ and that the derivatives of $f_{3}$ through $f_{6}$ with respect to $t$ can be expressed in terms of their derivatives with respect to $x$.

### 4.5.2 Solution of Equation (57)

We apply the transformation $F$ defined in Section 4.3.2 to both sides of Eq. (57) and absorb the $t$-derivative term in the left member, compensating with an $x$-derivative term in the right member,

$$
\begin{equation*}
\left(\partial_{t}+V \partial_{x}\right) F\left(u_{3}-r\right)=-\partial_{\tau} f-\partial_{x}(V F r)-V^{\prime} \partial_{y} g \tag{58}
\end{equation*}
$$

Since $V$ is diagonal, Eq. (58) decouples into six first-order hyperbolic equations with constant coefficients, which can be integrated along their characteristics. If the solution is to remain bounded, the right member must be such that it does not lead to secular behaviour. This condition imposes constraints, which we can find by following the averaging strategy of Section 3.1, Lemma 3.2. We decompose $V F r$, separating the terms that are constant along the characteristics from those that are not,

$$
\operatorname{VFr}=-D_{1} \partial_{x} f+D_{2} f^{2}+w
$$

The first two terms are constant along the characteristics; $D_{1}$ and $D_{2}$ are diagonal matrices with nonnegative entries that are readily found from equations (44) and (4.5.1),

$$
\begin{aligned}
D_{1} & =\frac{1}{2} g\left(1-v_{1}^{2}\right) \operatorname{diag}\left(0,0,1-v_{1}^{2}, 1-v_{1}^{2}, 1-v_{2}^{2}, 1-v_{2}^{2}\right), \\
D_{2} & =\frac{3\left(1-v_{1}\right)^{2}\left(1-v_{2}^{2}\right)}{8 v_{2}^{2}} \operatorname{diag}(0,0,0,0,1,1),
\end{aligned}
$$

where $f^{2}=\left(f_{1}^{2}, f_{2}^{2}, f_{3}^{2}, f_{4}^{2}, f_{5}^{2}, f_{6}^{2}\right)^{t}$ and the remainder term $w$ consists exclusively of terms that vary along the characteristics: its first and second component involve at least one of $f_{3}$ through $f_{6}$, its third component at least one of $f_{1}$ and $f_{4}$ through $f_{6}$, and so on. Thus Eq. (58) becomes

$$
\begin{equation*}
\left(\partial_{t}+V \partial_{x}\right) F\left(u_{2}-r\right)=-\left[\partial_{\tau} f-D_{1} \partial_{x}^{2} f+D_{2} \partial_{x} f^{2}-V^{\prime} \partial_{y} g\right]+w \tag{59}
\end{equation*}
$$

Application of the averaging operator to each component yields the equation

$$
\begin{equation*}
\partial_{\tau} f-D_{1} \partial_{x}^{2} f+D_{2} \partial_{x} f^{2}=\partial_{y} \widetilde{V^{\prime} g} \tag{60}
\end{equation*}
$$

where $\widetilde{h}=\left(G_{0} h_{1}, G_{0} h_{2}, G_{v_{1}} h_{3}, G_{-v_{1}} h_{4}, G_{v_{2}} h_{5}, G_{-v_{2}} h_{6}\right)^{t}$.
We use the equations we get in Section 4.4.1. After some small computations we get:

$$
\partial_{x} \widetilde{V^{\prime} g}=-\left(\begin{array}{c}
L_{1}\left(W_{0} C_{1} \partial_{y} f_{1}+W_{0} C_{2} \partial_{y} f_{2}\right)  \tag{61}\\
L_{2}\left(W_{0} C_{1} \partial_{y} f_{1}+W_{0} C_{2} \partial_{y} f_{2}\right) \\
L_{3} W_{1} C_{3} \partial_{y} f_{3} \\
L_{4} W_{-1} C_{4} \partial_{y} f_{4} \\
L_{5} W_{2}\left(C_{5}-a_{12} u_{1}\right) \partial_{y} f_{5}+a_{12} \partial_{x} G_{v_{2}} g_{5} \\
L_{6} W_{-2}\left(C_{6}+a_{12} u_{2}\right) \partial_{y} f_{6}-a_{12} \partial_{x} G_{-v_{2}} g_{6}
\end{array}\right)
$$

where $C_{i}$ and $L_{i}$ are the columns and the lines of the matrix $V^{\prime}, W_{i}$ is the inverse of the diagonal matrix $V-v_{i} \operatorname{Id}$ restrained to $\operatorname{Im}\left(V-v_{i} \mathrm{Id}\right)$ and trivially extended by 0 on $\operatorname{Ker}\left(V-v_{i} \mathrm{Id}\right)$ for $i \in\left\{0, \pm v_{1}, \pm v_{2}\right\}$ and $u_{1}=(\beta, 0,0,0,0,-1)^{t}, u_{2}=(-\beta, 0,0,0,-1,0)^{t}$ where $\beta$ is a coefficient we can determine thanks to the equation of Section 4.4.1 and wich allow us to get rid of the second member in the equation on $G_{ \pm v_{2}} g$.
In the equation (61) we see that $\partial_{x} \widetilde{V^{\prime} g}$ still depends on $\partial_{x} G_{v_{2}} g_{5}$ and $G_{-v_{2}} g_{6}$, but we have no control over these two terms. We impose then a new condition, $a_{12}=0$ which gives

$$
l_{a}=l \cdot M_{0}=0
$$

Remark 4.1 We have then $l_{a}=l_{c}=0$, i.e. the direction of the transverse perturbation is orthogonal to the direction of propagation $k$ and to the magnetic moment $M_{0}$. These two conditions are the compatibility conditions Leblond and Manna found in [10].

By taking the derivative of Equation (60) with respect to $x$ we obtain the Khokhlov-Zabolotskaya equation,

$$
\begin{equation*}
\partial_{x}\left(\partial_{\tau} f-D_{1} \partial_{x}^{2} f+D_{2} \partial_{x} f^{2}\right)=-B \partial_{y}^{2} f \tag{62}
\end{equation*}
$$

where

$$
B=\operatorname{diag}\left(0,0,-\frac{l_{b}^{2} v_{1}}{2\left(1-k_{a}^{2}\right)}, \frac{l_{b}^{2} v_{1}}{2\left(1-k_{a}^{2}\right)},-\frac{l_{b}^{2}}{2 v_{2}\left(1-k_{a}^{2}\right)}, \frac{l_{b}^{2}}{2 v_{2}\left(1-k_{a}^{2}\right)}\right)
$$

Thus a necessary condition for the solution of Eq. (58) to remain bounded for long times as $\varepsilon$ goes to zero is that the first order profile functions $f_{1}$ through $f_{6}$ fulfill the Khokhlov-Zabolotskaya equation on the (slow) time scale of $\tau$. The equations for $f_{1}$ and $f_{2}$ are simple: $\partial_{\tau} f_{1}=0, \partial_{\tau} f_{2}=0$, so $f_{1}$ and $f_{2}$ must
be constant on the slow time scale and $f_{i} \equiv f_{i}(x), i=1,2$. The equations for $f_{3}$ and $f_{4}$ are linear, those for $f_{5}$ and $f_{6}$ nonlinear with a quadratic nonlinearity.

Remark 4.2 Equation (62) corresponds to Eq. (23). The nonzero entries of $D_{1}$ are positive, and the equations for $f_{1}$ and $f_{2}$ which involves the zero entries of $D_{1}$ are trivial. This observation validates Hypothesis 4.

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