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Abstract

We are interested here in small perturbations of electromagnetic waves in a saturated ferromagnetic media. By means of an asymptotic expansion we prove that the solution remains close on long times of the one of the Khokhlov-Zabolotskaya equation.

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1 Introduction

In this article, we show that small perturbations of equilibrium states in ferromagnetic media give rise to standing and travelling waves that are stable for long times. The evolution of their profiles is governed by the Khokhlov-Zabolotskaya equation which can be considered as a semilinear heat equation which has been perturbed in the transversal direction.
These results are obtained in the framework of micromagnetics. The basic variable is the magnetization vector M whose dynamic equation was first given by Landau and Lifschitz [8] and later in an equivalent form by Gilbert [7]:

$$\partial_t M = -M \wedge H - \frac{\gamma}{|M|} M \wedge (M \wedge H) \quad \text{in } \mathbb{R}^3. \tag{1}$$

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The equation is completed by the Maxwell equations for the electromagnetic field:

$$\begin{cases} \partial_t (H+M) - \nabla \wedge E = 0 \quad \text{in } \mathbb{R}^3, \\ \partial_t E + \nabla \wedge H = 0 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{2}$$

The Landau-Lifschitz-Maxwell equations admit stable equilibrium solutions where the magnetization is uniform and everywhere parallel to the effective magnetic field:

$$(M, H, E)_\alpha = (M_0, \alpha^{-1} M_0, 0), \quad \alpha > 0. \tag{3}$$

We are interested in small perturbations of the equilibrium states whose size is measured by a small parameter ε. A formal derivation has been given by H. Leblond and M. Manna [10]. T. Colin, C. Galusinski and H. Kaper have studied the case of $1D$-perturbations ([5]) and proved that the evolution of $1D$-wave profile is governed on a slow time scale by semilinear heat equations. We study here the case of $2D$-perturbations: we add a dependence on a slow transversal space direction y and we seek their behaviour. The perturbation is taken in the form:

$$\begin{aligned} M(t,x,y) &= M_0 + \varepsilon^2 \widetilde{M}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}), \\ H(t,x,y) &= \alpha^{-1} M_0 + \varepsilon^2 \widetilde{H}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}), \\ E(t,x,y) &= \varepsilon^2 \widetilde{E}(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}), \end{aligned}$$

where $\widetilde{t},\ \tau,\ \widetilde{x},\ \widetilde{y}$ are new rescaled variables:

$$\widetilde{t} = \varepsilon^2 t, \quad \tau = \varepsilon^4 t, \quad \widetilde{x} = \varepsilon^2 x, \quad \widetilde{y} = \varepsilon^3 y.$$

We introduce then the vector

$$U\left(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}\right) = \left(\alpha^{-1/2}\widetilde{M},\ \widetilde{H},\ \widetilde{E}\right)^t \left(\widetilde{t}, \tau, \widetilde{x}, \widetilde{y}\right),$$

and the perturbation U satisfy an equation in the form (we "forget" the $\widetilde{\ }$ on $t,\ x,\ y$):

$$\begin{cases} \partial_t U + \varepsilon^2 \partial_\tau U + A_1 \partial_x U + \varepsilon A_2 \partial_y U + \varepsilon^{-2} L U = B(U,U) + \varepsilon^2 T(U,U,U), \\ U(0,0) = U_0^0, \end{cases} \tag{4}$$

where the terms A_1, A_2, L, B and T are precised in Section 2.
Using a WKB method on U, we obtain that the leading term U_0 breaks down in five travelling and standing terms, $U_0 = \sum_{j=1}^5 u_j$, which satisfy:

$$\begin{cases} \left(\partial_t + v_j \partial_x\right) u_j = 0, \\ \partial_x \left(\partial_\tau u_j - D_j \partial_x^2 u_j + B_j(u_j, \partial_x u_j) + F_j(u_j, u_j, u_j)\right) = C_j \partial_y^2 u_j, \end{cases} \tag{5}$$

where the last equation is the Khokhlov-Zabolotskaya equation and

$$v_j \in \left\{ 0,\ \pm\left(\frac{1}{1+\alpha}\right)^{1/2},\ \pm\left(\frac{1+(1-|k\cdot M_0|^2)\alpha}{1+\alpha}\right)^{1/2} \right\}$$

is the speed of the wave in the direction k (the x-direction).
Through the study of the leading term of the expansion, we prove the following result, not yet stated to our knowledge:

Theorem 1.1 *Let $D>0$, $C\neq 0$. Let $B : (\mathbb{R}^k)^2 \to \mathbb{R}^k$ a bilinear application and*
$F : (\mathbb{R}^k)^3 \to \mathbb{R}^k$ a trilinear application. Let $u^0 \in (\mathbb{H}^s(\mathbb{R}^2))^k$, $s>1$. Then there exists a time $T>0$ and an unique function $u \in \mathcal{C}^0\left([0,T];(\mathbb{H}^s(\mathbb{R}^2))^k\right)$ solution of the Khokhlov-Zabolotskaya equation:

$$\begin{cases} \partial_x\left(\partial_\tau u - D\partial_x^2 u + B(u,\partial_x u) + F(u,u,u)\right) = C\partial_y^2 u \quad \text{in } \mathbb{R}^2, \\ u(\tau=0,x,y) = u^0(x,y) \quad \text{in } \mathbb{R}^2. \end{cases}$$

We then prove that the general solution of Eq. (4) remains close to the leading term U_0 of the asymptotic expansion on a slow time scale (*i.e.* with the change $\tau=\varepsilon^2 t$):

Theorem 1.2 *Let us assume we are in the ferromagnetic case (assumption on the structure of the equations) with an initial data $U_0^0 \in \mathbb{H}^s(\mathbb{R}^2)$ $(s>6)$. Let $T_0>0$ such that $U_0=\sum_{j=1}^5 u_j$, solution of Eqs. (5), lies in $\mathcal{C}\left([0,T_0/\varepsilon^2];\mathbb{H}^{s-1}(\mathbb{R}^2)\right)$. There exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$, there is an unique solution $\mathcal{U}^\varepsilon \in \mathcal{C}^0\left([0,T_0/\varepsilon^2];\mathbb{H}^{s-5}(\mathbb{R}^2)\right)$ of*

$$\partial_t \mathcal{U}^\varepsilon + A_1\partial_x\mathcal{U}^\varepsilon + A_2\partial_y\mathcal{U}^\varepsilon + \frac{1}{\varepsilon^2}L\mathcal{U}^\varepsilon = B(\mathcal{U}^\varepsilon,\mathcal{U}^\varepsilon) + \varepsilon^2 T(\mathcal{U}^\varepsilon,\mathcal{U}^\varepsilon,\mathcal{U}^\varepsilon),$$

with $\mathcal{U}^\varepsilon(0,x,y) = U_0^0(x,\varepsilon y)$. Moreover

$$\|\mathcal{U}^\varepsilon - \mathcal{U}_0^\varepsilon\|_{\mathbb{L}^\infty([0,T_0/\varepsilon^2]\times\mathbb{R}^2)} = o(1) \quad \text{as } \varepsilon\to 0,$$

where $\mathcal{U}_0^\varepsilon(t,x,y) = U_0(t,\varepsilon^2 t,x,\varepsilon y)$.

Remark 1.1 *To obtain the equation fulfilled by the leading term U_0, we assume that the profiles U_1 and U_2 satisfy a sublinear growth condition (cf Lemma 3.2). But the leading profile U_0 does not allow to build U_1 and U_2 with such properties. We use then low-frequency cut-offs methods ([2], [3], [11]) to get an approximation of the leading profile with which we build the remaining terms of the expansion and prove Theorem 1.2.*

In Section 2 we give the complete mathematical model. The system of partial differential equations which governs the spatio-temporal evolution of the per-

turbations belongs to a general class of hyperbolic equation for vector-valued functions that we examine in Section 3. Performing an asymptotic expansion, we show that the equation admits an asymptotic solution that exhibits standing and travelling waves. The wave profiles move around on a slow time scale (measured in units of ε^2) according to the Khokhlov-Zabolotskaya equation. By means of nonlinear optics techniques, we show that the asymptotic solution remains close to the exact solution of the hyperbolic equation on the long time scale as ε goes to zero. The main result for the general case is stated in Theorem 3.9 (Section 3.2.3), the local existence of regular solution for the Khokhlov-Zabolotskaya equation is stated in Theorem 3.4 (Section 3.2.1).

The derivation of the asymptotic solution and the proof of the convergence theorem require several hypotheses, which are satisfied in the case of the Landau-Lifschitz-Maxwell equations (Section 4). We find that the magnetization as well as the electromagnetic field variables breaks into standing waves and up to four travelling waves, whose speed of propagation varies with the equilibrium state.

2 Mathematical model

The state of a ferromagnet is described by the magnetization vector M whose evolution is governed by the Landau-Lifschitz equation:

$$\frac{\partial M}{\partial t} = -M \wedge H - \frac{\gamma}{|M|} M \wedge (M \wedge H) \quad \text{in } \mathbb{R}^3, \tag{6}$$

where γ is a dimensionless damping coefficient and H the magnetic field. We can note that the magnitude $|M|$ of M is an invariant of this equation. The electromagnetic field fulfills the Maxwell's equations:

$$\begin{cases} \dfrac{\partial (H+M)}{\partial t} - \nabla \wedge E = 0 \quad \text{in } \mathbb{R}^3, \\ \dfrac{\partial E}{\partial t} + \nabla \wedge H = 0 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{7}$$

where the equations are in dimensionless form and the coefficients have been set equal to one.

2.1 Basic solution

The system of equations (6)-(7) admits a family of constant solutions,

$$(M, H, E)_\alpha = (M_0, \alpha^{-1} M_0, 0), \quad \alpha > 0,$$

where M_0 is an arbitrary vector in $\mathbb{R}^3$ and $\alpha > 0$ assures that the solution is stable. We may assume without restriction that $|M_0| = 1$. We are interested in the spatio-temporal evolution of long-waves and transverse perturbations of such solutions. The perturbations are measured in terms of an arbitrarily small positive parameter ε and have the form:

$$\begin{aligned} M(t,x,y) &= M_0 + \varepsilon^2 \widetilde{M}(\widetilde{t},\tau,\widetilde{x},\widetilde{y}),\\ H(t,x,y) &= \alpha^{-1} M_0 + \varepsilon^2 \widetilde{H}(\widetilde{t},\tau,\widetilde{x},\widetilde{y}),\\ E(t,x,y) &= \varepsilon^2 \widetilde{E}(\widetilde{t},\tau,\widetilde{x},\widetilde{y}), \end{aligned}$$

where $\widetilde{M}$, $\widetilde{H}$ and $\widetilde{E}$ are $O(1)$ as ε goes to 0, and:

$$\widetilde{x} = \varepsilon^2 x,\ \widetilde{y} = \varepsilon^3 y,\ \widetilde{t} = \varepsilon^2 t,\ \tau = \varepsilon^4 t.$$

Remark 2.1 *We look for solutions propagating in the x-direction (still unknown) for which the dispersion and the non-linear effects occur at the same long time scale. This assumption sets the size of the non-linearity and the scales of $\widetilde{t}$, $\widetilde{x}$ and τ. We want to study here transversal deformations of a quasi-plane wave. This case corresponds to the scaling $\widetilde{y} = \varepsilon^p y$ with $p > 2$. There is a dominant diffraction effect that occurs for the smallest value of p for which we can "solve" the equation, which is here $p = 3$. If we take $2 < p < 3$, the diffraction effect prevails over the dispersion and prevents us to observe the non-linear effects. On the contrary, if $p > 3$ the diffraction is negligible.*

If (M, H, E) is a solution of equations (6)-(7) then $\widetilde{M}$, $\widetilde{H}$ and $\widetilde{E}$ have to fulfill:

$$\begin{aligned} \varepsilon^2 \partial_{\widetilde{t}} \widetilde{M} + \varepsilon^4 \partial_\tau \widetilde{M} = &-M_0 \wedge \widetilde{H} + \alpha^{-1} M_0 \wedge \widetilde{M} - \varepsilon^2 \widetilde{M} \wedge \widetilde{H}\\ &-\frac{\gamma}{|M|}\Big[M_0 \wedge (M_0 \wedge \widetilde{H}) - \alpha^{-1} M_0 \wedge (M_0 \wedge \widetilde{M})\\ &+\varepsilon^2 M_0 \wedge (\widetilde{M} \wedge \widetilde{H}) + \varepsilon^2 \widetilde{M} \wedge (M_0 \wedge \widetilde{H})\\ &-\varepsilon^2 \alpha^{-1} \widetilde{M} \wedge (M_0 \wedge \widetilde{M}) + \varepsilon^4 \widetilde{M} \wedge (\widetilde{M} \wedge \widetilde{H})\Big], \end{aligned} \tag{8}$$

$$\begin{cases} \varepsilon^2 \partial_{\widetilde{t}} \widetilde{H} + \varepsilon^4 \partial_\tau \widetilde{H} - \varepsilon^2 \widetilde{\nabla}_{\widetilde{x}} \wedge \widetilde{E} - \varepsilon^3 \widetilde{\nabla}_{\widetilde{y}} \wedge \widetilde{E} = -\varepsilon^2 \partial_{\widetilde{t}} \widetilde{M} - \varepsilon^4 \partial_\tau \widetilde{M},\\ \varepsilon^2 \partial_{\widetilde{t}} \widetilde{E} + \varepsilon^4 \partial_\tau \widetilde{E} + \varepsilon^2 \widetilde{\nabla}_{\widetilde{x}} \wedge \widetilde{H} + \varepsilon^3 \widetilde{\nabla}_{\widetilde{y}} \wedge \widetilde{H} = 0, \end{cases} \tag{9}$$

In Eq. (8) we have left the denominator's term $|M|$ unchanged. It induces technical complications that are non-essential for the arguments to be presented. Then we will change slightly the equation to completely avoid these complications: we replace the term $|M|$ by $|M_0| = 1$ since this quantity is conserved.

We are interested in solutions of Eqs. (6)-(7) that describe a travelling wave propagating in the direction $\widetilde{k}$ with transversal perturbations in the direction $\widetilde{l}$ where $\widetilde{k}$ and $\widetilde{l}$ are fixed unit vectors, $\widetilde{k}$ is not parallel or antiparallel to M_0

and $\widetilde{l}$ is not parallel or antiparallel to $\widetilde{k}$. By assuming that $\widetilde{x}$ is the coordinate in the $\widetilde{k}$-direction and $\widetilde{y}$ in the $\widetilde{l}$-direction we may perform the substitutions:

$$\widetilde{\nabla}_{\widetilde{x}} = \widetilde{k}\partial_{\widetilde{x}} \quad \text{and} \quad \widetilde{\nabla}_{\widetilde{y}} = \widetilde{l}\partial_{\widetilde{y}}.$$

Henceforth we omit the tilde, so the equations to be considered are:

$$\begin{aligned}\varepsilon^2\partial_t M + \varepsilon^4\partial_\tau M = & -M_0 \wedge H + \alpha^{-1} M_0 \wedge M - \varepsilon^2 M \wedge H \\ & -\gamma\Big[M_0 \wedge (M_0 \wedge H) - \alpha^{-1} M_0 \wedge (M_0 \wedge M) \\ & +\varepsilon^2 M_0 \wedge (M \wedge H) + \varepsilon^2 M \wedge (M_0 \wedge H) \\ & -\varepsilon^2\alpha^{-1} M \wedge (M_0 \wedge M) + \varepsilon^4 M \wedge (M \wedge H)\Big],\end{aligned} \tag{10}$$

$$\varepsilon^2\partial_t H + \varepsilon^4\partial_\tau H - \varepsilon^2 k \wedge \partial_x E - \varepsilon^3 l \wedge \partial_y E = -\varepsilon^2\partial_t M - \varepsilon^4\partial_\tau M, \tag{11}$$

$$\varepsilon^2\partial_t E + \varepsilon^4\partial_\tau E + \varepsilon^2 k \wedge \partial_x H + \varepsilon^3 l \wedge \partial_y H = 0. \tag{12}$$

2.2 Vector Formulation

The following formulation and properties are obtained following the work of T. Colin, C. Galusinski and H. Kaper ([5]).
The system of equations (10)-(12) can be written as a single equation for a function:
$U \ : \ \mathbb{R}_t^{+*} \times [0,T] \times \mathbb{R}_x \times \mathbb{R}_y \to \mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$,

$$U(t,\tau,x,y) = \begin{pmatrix} \alpha^{-1/2} M(t,\tau,x,y) \\ H(t,\tau,x,y) \\ E(t,\tau,x,y) \end{pmatrix}, \ t \geq 0, \ \tau \in [0,T], \ x \in \mathbb{R}, \ y \in \mathbb{R}.$$

The factor $\alpha^{-1/2}$ is introduced so that the problem have certain symmetry properties (*cf* Section 3). After having divided once more by ε^2, U has to fulfill for all $t \in \mathbb{R}^{+*}$, $\tau \in [0,T]$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$:

$$\partial_t U + \varepsilon^2\partial_\tau U + A_1\partial_x U + \varepsilon A_2\partial_y U + \varepsilon^{-2}(L_0 + L_1)U = B(U,U) + \varepsilon^2 T(U,U,U), \tag{13}$$

where A_1, A_2, L_0 and L_1 are linear operators in $\mathbb{R}^9$,

$$A_1 u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -k \wedge \cdot \\ 0 & k \wedge \cdot & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad A_2 u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -l \wedge \cdot \\ 0 & l \wedge \cdot & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

$$L_0 u = \begin{pmatrix} -\alpha^{-1} M_0 \wedge \cdot & \alpha^{-1/2} M_0 \wedge \cdot & 0 \\ \alpha^{-1/2} M_0 \wedge \cdot & -M_0 \wedge \cdot & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

$$L_1 u = \gamma \begin{pmatrix} -\alpha^{-1} M_0 \wedge (M_0 \wedge \cdot) & \alpha^{-1/2} M_0 \wedge (M_0 \wedge \cdot) & 0 \\ \alpha^{-1/2} M_0 \wedge (M_0 \wedge \cdot) & -M_0 \wedge (M_0 \wedge \cdot) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

B is a bilinear map on $\mathbb{R}^9 \times \mathbb{R}^9$,

$$B(u, v) = \begin{pmatrix} B_1(u, v) \\ -\alpha^{1/2} B_1(u, v) \\ 0 \end{pmatrix},$$

with

$$\begin{aligned} B_1(u, v) = &-\frac{1}{2} (u_1 \wedge v_2 + v_1 \wedge u_2) - \frac{1}{2} \gamma M_0 \wedge (u_1 \wedge v_2 + v_1 \wedge u_2) \\ &-\frac{1}{2} \gamma \left[u_1 \wedge \left(M_0 \wedge (v_2 - \alpha^{-1/2} v_1) \right) + v_1 \wedge \left(M_0 \wedge (u_2 - \alpha^{-1/2} u_1) \right) \right], \end{aligned}$$

and T is a trilinear map on $\mathbb{R}^9 \times \mathbb{R}^9 \times \mathbb{R}^9$,

$$T(u, v, w) = \alpha^{1/2} \gamma \begin{pmatrix} T_1(u, v, w) \\ -\alpha^{1/2} T_1(u, v, w) \\ 0 \end{pmatrix},$$

with

$$\begin{aligned} T_1(u, v, w) = \frac{1}{6} &[u_1 \wedge (v_2 \wedge w_1) + u_1 \wedge (w_2 \wedge v_1) + w_1 \wedge (v_2 \wedge u_1) \\ &+ w_1 \wedge (u_2 \wedge v_1) + v_1 \wedge (u_2 \wedge w_1) + v_1 \wedge (w_2 \wedge u_1)]. \end{aligned}$$

Here u, v and w are arbitrary vectors in $\mathbb{R}^9$, $u = (u_1, u_2, u_3)^t$, $v = (v_1, v_2, v_3)^t$, $w = (w_1, w_2, w_3)^t$ with $u_i,\ v_i,\ w_i \in \mathbb{R}^3,\ i = 1, 2, 3$.

2.3 Auxiliary properties

Since the vector product is antisymmetric, the operators A_1 and A_2 are symmetric with respect to the usual scalar product in $\mathbb{R}^9$: $A_i u \cdot v = u \cdot A_i v\ ,\ \ u, v \in \mathbb{R}^9,\ i = 1, 2$.

The operators L_0 and L_1 are respectively antisymmetric and symmetric with respect to the scalar product in $\mathbb{R}^9$: $L_0 u \cdot v = -u \cdot L_0 v,\ L_1 u \cdot v = u \cdot L_1 v,\ \ u, v \in \mathbb{R}^9$.

The bilinear map B and the trilinear map T are symmetric, *i.e.* $B(u, v) = B(v, u)$ and $T(u, v, w) = T(\pi(u, v, w))$ for all $u, v, w \in \mathbb{R}^9$ and any permutation π.

Lemma 2.1 *The operator $L = L_0 + L_1$ induces an orthogonal decomposition,*

$$\mathbb{R}^9 = \text{Ker } L \oplus \text{Im } L.$$

We have:

$$\begin{aligned}
\text{Ker } L &= \left\{ v = (v_1, v_2, v_3)^t \in \mathbb{R}^9,\ (\alpha^{-1/2} v_1 - v_2) \wedge M_0 = 0 \right\}, \\
\text{Im } L &= \left\{ v = (v_1, v_2, v_3)^t \in \mathbb{R}^9,\ v_1 \cdot M_0 = 0,\ v_2 = -\alpha^{1/2} v_1,\ v_3 = 0 \right\}.
\end{aligned}$$

Let P and Q be respectively the orthogonal projectors on Ker L and Im L, and let R be the inverse of L on Im L, trivially extended to $\mathbb{R}^9$. Then $RL = LR = I - P = Q$. Furthermore, if $Lu = v$ for $(u, v) \in \mathbb{R}^9$ then $Pv = 0$ and $Qu = Rv$. We also have the following lemma:

Lemma 2.2 *The operator L_1 is coercive on* Im L*,*

$$(L_1 Q v) \cdot (Q v) = \gamma(1 + \alpha^{-1})(Q v) \cdot (Q v), \quad v \in \mathbb{R}^9.$$

The maps B and T are transparent on Ker L*,*

$$PB(Pu, Pv) = 0, \quad PT(Pu, Pv, Pw) = 0, \quad u, v, w \in \mathbb{R}^9.$$

3 A general equation

Equation (13) is a special case of the general partial diferential equation:

$$\partial_t U + \varepsilon^2 \partial_\tau U + A_1 \partial_x U + \varepsilon A_2 \partial_y U + \varepsilon^{-2} LU = B(U,U) + \varepsilon^2 T(U,U,U) \quad (14)$$

in $\mathbb{R}^n$ where A_1 and A_2 are symmetric linear operators, L is a linear operator , B a symmetric bilinear map and T a symmetric trilinear map. In this section we consider Equation (14), the application to the special case of Equation (13) follows in Section 4. We first build an asymptotic expansion of Equation (14) using formal power series expansion in the small parameter ε (Section 3.1). Then we give precise asymptotic estimates of the various terms of the asymptotic solution (Section 3.2.1). Finally we show that the asymptotic solution actually converges to the solution of Equation (14) on the slow time scale as ε goes to 0 (Section 3.2.3).

3.1 Formal Asymptotic Expansion

We will now search an asymptotic formal expansion of $U \equiv U(t,\tau,x,y)$ in the form

$$U \equiv \left(U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \dots\right).$$

This construction requires three hypotheses:

Hypothesis 1. $\mathbb{R}^n = \text{Ker } L \oplus \text{Im } L$.

Hypothesis 2. There exists $C > 0$ such that for all $u \in \mathbb{R}^n$, $(Lu) \cdot u \geq C\|Qu\|^2$, *i.e.* L is coercive on its image.

Hypothesis 3. $PB(Pu, Pv) = 0$ and $PT(Pu, Pv, Pw) = 0$ for all $u, v, w \in \mathbb{R}^n$.

Here P and Q are respectively the orthogonal projections on Ker L and Im L. Let R the partial inverse of L on ImL, trivially extended to all of $\mathbb{R}^n$. Then $RL = LR = Q$ and we have:

Lemma 3.1 *If $Lu = v$ for some $u, v \in \mathbb{R}^n$ then $Pv = 0$ (solvability condition) and $Qu = Rv$.*

We substitute the asymptotic expansion of U in (14) and we assume that $U_0 = O(1)$, $\varepsilon U_i = o(1)$, $i = 1, \dots, 4$, as ε goes to 0. We then get:

At the order ε^{-2}: $LU_0 = 0$, *i.e.* $QU_0 = 0$ and therefore $U_0 = PU_0$.

At the order ε^{-1}: $LU_1 = 0$, *i.e.* $QU_1 = 0$ and therefore $U_1 = PU_1$.

At the order ε^0: $LU_2 = V_2(U_0) = B(U_0, U_0) - (\partial_t + A_1\partial_x)\, U_0$.
Since $U_0 = PU_0$ and B is transparent on Ker L, the solvability condition $PV_2 = 0$ reduces to

$$(\partial_t + PA_1P\partial_x)\, U_0 = 0$$

The operator PA_1P is symmetric. Then there exists k projections P_j and k numbers v_j $(j = 1, \ldots, k,\ k \leq n)$ such that:

$$P = \sum_{j=1}^{k} P_j, \quad PA_1PP_j = v_jP_j, \quad j = 1, \ldots, k.$$

Hence the solvability condition is met if

$$(\partial_t + v_j\partial_x)\, P_jU_0 = 0, \qquad j = 1, \ldots, k. \tag{15}$$

As $U_0 = PU_0$ and $R = 0$ on Ker L the equation $QU_2 = RV_2$ reduces to:

$$QU_2 = RB(U_0, U_0) - RA_1\partial_xU_0. \tag{16}$$

At the order ε: $LU_3 = V_3(U_0, U_1) = 2B(U_0, U_1) - (\partial_t + A_1\partial_x)\, U_1 - A_2\partial_yU_0$.
By the same way, the solvability condition $PV_3 = 0$ becomes:

$$(\partial_t + PA_1P\partial_x)\, U_1 + PA_2P\partial_yU_0 = 0,$$

and then,

$$(\partial_t + v_j\partial_x)\, P_jU_1 + P_jA_2\sum_{i=1}^{k}\partial_yP_iU_0 = 0, \qquad j = 1, \ldots, k. \tag{17}$$

Remark 3.1 *These equations allow us to determine U_1.*

The equation $QU_3 = RV_3$ also reads:

$$QU_3 = 2RB(U_0, U_1) - RA_1\partial_xU_1 - RA_2\partial_yU_0.$$

At the order ε^2:

$$\begin{aligned} LU_4 = V_4(U_0, U_1, U_2) = {} & 2B(U_0, U_2) + B(U_1, U_1) + T(U_0, U_0, U_0) \\ & -\partial_\tau U_0 - A_2\partial_yU_1 - (\partial_t + A_1\partial_x)\, U_2. \end{aligned}$$

Since B and T are transparent on Ker L the solvability condition $PV_4 = 0$ reduces to:

$$\partial_\tau U_0 + PA_2\partial_yU_1 + (\partial_tPU_2 + PA_1\partial_xU_2) = 2PB(U_0, U_2)$$

We rewrite this equation using Eq. (16) and the transparency condition on Ker L,

$$\begin{aligned}&\partial_\tau U_0 + (\partial_t + PA_1P\partial_x)\,PU_2 + PA_2P\partial_y U_1 - PA_1RA_1P\partial_x^2 U_0\\&= 2PB(U_0, RB(U_0,U_0)) - 2PB(U_0, RA_1\partial_x U_0) - PA_1R\partial_x B(U_0,U_0).\end{aligned}$$

This equation represents a system of k equations,

$$\begin{aligned}&\partial_\tau P_jU_0 + (\partial_t + v_j\partial_x)\,P_jU_2 + P_jA_2\sum_{i=1}^k \partial_y P_iU_1 - P_jA_1RA_1\sum_{i=1}^k \partial_x^2 P_iU_0\\&= 2P_jB(U_0, RB(U_0,U_0)) - 2P_jB(U_0, RA_1\partial_x U_0)\\&-P_jA_1R\partial_x B(U_0,U_0), \qquad j=1,\dots k.\end{aligned} \tag{18}$$

The j-th equation involves the rate of change of P_jU_0 on the slow (τ) time scale as well as the rate of change of P_iU_1, $i=1,\dots,k$ in the transversal direction (y) and the rate of change of P_jU_2 along the characteristic determined by v_j on the regular (t) time scale. We can separate the first two effects from the last one if U_2 fulfills a sublinear growth condition,

$$\lim_{t\to\infty}\frac{1}{t}\|U_2(t,\tau,\cdot,\cdot)\|_{\mathbb{H}^s} = 0, \tag{19}$$

uniformly on $[0,T]$ for some sufficiently large s. ($\mathbb{H}^s$ is the usual Sobolev space of order s.) The condition (19) also implies that $\varepsilon\|U_2\|_{H^s} = o(1)$ as ε goes to 0. The separation is accomplished by averaging over t along characteristics. Formally,

$$G_v u(t,x) = \lim_{T\to+\infty}\int_0^T u(x+vs, t+s)\,\mathrm{d}s, \quad v\in\mathbb{R}, \tag{20}$$

whenever the limit exists. The following lemma is taken from [9].

Lemma 3.2

(i) If $(\partial_t + v\partial_x)u = 0$ then $G_{v'}u$ exists for all v'. Moreover $G_{v'}u = u$ if $v' = v$ and $G_{v'}u = 0$ otherwise.

(ii) If $(\partial_t + v\partial_x)u = 0$ and $(\partial_t + v'\partial_x)u' = 0$ then $G_{v''}(uu') = uu'$ if $v'' = v' = v$ and $G_{v''}(uu') = 0$ otherwise.

(iii) If u satisfies a sublinear growth condition, $\lim_{t\to+\infty}\frac{1}{t}\|u(t,.)\|_{\mathbb{L}^\infty(\mathbb{R}^n)} = 0$ then $G_v(\partial_t + v\partial_x)u$ exists and $G_v(\partial_t + v\partial_x)u = 0$.

The application of G_{v_j} to both sides of Equation (18) eliminates the transport term and reduces the equation to:

$$\begin{cases} \partial_\tau P_j U_0 + P_j A_2 \sum_{i=1}^{k} \partial_y G_{v_j} P_i U_1 - P_j A_1 R A_1 P_j \partial_x^2 P_j U_0 \\ \qquad = 2P_j B(P_j U_0, RB(P_j U_0, P_j U_0)) - 2P_j B(P_j U_0, RA_1 \partial_x P_j U_0) \\ \qquad -P_j A_1 R \partial_x B(P_j U_0, P_j U_0), \quad j = 1, \dots k. \end{cases} \tag{21}$$

We have now to determine the term $G_{v_j} P_i U_1$. We then use Equation (17) and we assume that U_1 satisfies the same sublinear growth condition as U_2. We get:

$$\begin{aligned} & G_{v_j} \left[(\partial_t + v_i \partial_x) P_i U_1 + P_i A_2 \sum_{l=1}^{k} \partial_y P_l U_0 \right] \\ & = (v_i - v_j) \partial_x G_{v_j} P_i U_1 + P_i A_2 P_j \partial_y P_j U_0 = 0, \end{aligned} \tag{22}$$

for all $i, j = 1, \dots, k$. We then get a compatibility condition which is $P_i A_2 P_i = 0$ for all $i = 1, \dots, k$.

Remark 3.2 *This compatibility condition gives information on the direction of the transversal perturbation as we can see it in Section 4. We obtain:* $l \cdot k = l \cdot M_0 = 0$.

We also remark that the operator $P_j A_1 R A_1 P_j$ is nonnegative because of Hypothesis 2 and proportional to P_j, $P_j A_1 R A_1 P_j = D_j P_j$ where D_j is a scalar, $D_j \geq 0$.
After taking the derivative of Eq. (21) with respect to x we get that the solvability condition $PV_4 = 0$ thus yields a system of k nonlinear Khokhlov-Zabolotskaya type equations on the slow (τ) time scale, for $j = 1, \dots, k$,

$$\begin{aligned} \partial_x \Big(\partial_\tau P_j U_0 - D_j \partial_x^2 P_j U_0 + B_j(P_j U_0, \partial_x P_j U_0) + F_j(P_j U_0, P_j U_0, P_j U_0) \Big) \\ = C_j \partial_y^2 P_j U_0, \end{aligned} \tag{23}$$

where B_j is a bilinear map, $B_j(u, v) = 2P_j B(u, RA_1 v) + 2P_j A_1 RB(u, v)$ (thanks to the symmetry of B), F_j a trilinear map, $F_j(u, v, w) = 2P_j B(u, RB(v, w))$, and C_j the operator $C_j = \sum_{i \neq j} \frac{1}{v_i - v_j} P_j A_2 P_i A_2 P_j$.

Remark 3.3 *The Khokhlov-Zabolotskaya equation arises for the first time in gas theory [13] and describes nonlinear diffractive waves [4,6,10].*

Furthermore if we use Equation (21) to eliminate the τ derivative in Eq. (18) we find that the solvability condition $PV_4 = 0$ also yields a system of k

transport equations for P_jU_2 on the regular (t) time scale,

$$(\partial_t + v_j\partial_x)\, P_jU_2 = S_j(U_0, U_1), \quad j = 1, \ldots, k, \tag{24}$$

where:

$$\begin{aligned} S_j(U_0, U_1) = {} & P_jA_1RA_1 \sum_{i\neq j} \partial_x^2 P_iU_0 - P_jA_2 \sum_{i\neq j} \Big(\partial_y P_iU_1 - \partial_y G_{v_j} P_iU_1\Big) \\ & +2P_j\Big(B(U_0, RB(U_0, U_0)) - B(P_jU_0, RB(P_jU_0, P_jU_0))\Big) \\ & -2P_j\Big(B(U_0, RA_1\partial_x U_0) - B(P_jU_0, RA_1\partial_x P_jU_0)\Big) \\ & -P_jA_1R\partial_x\Big(B(U_0, U_0) - B(P_jU_0, P_jU_0)\Big). \end{aligned}$$

If Equations (23)-(24) are fulfilled, then $QU_4 = RV_4$. The equation reduces to:

$$QU_4 = 2RB(U_0, U_2)+RB(U_1, U_1)+RT(U_0, U_0, U_0)-RA_2\partial_y U_1-R\partial_t U_2-RA_1\partial_x U_2.$$

Lemma 3.3 *If U_1 and U_2 fulfill the sublinear growth condition (19), then $U_0 = \sum_{j=1}^{k} P_jU_0$ where each P_jU_0 satisfies a homogeneous transport equation on the regular (t) time scale (Eq. (15)) and a nonlinear Khoklov-Zabolotskaya equation on the slow (τ) time scale (Eq. (23)).*

3.2 Asymptotic Estimates

For the convergence proof in the next section we need asymptotic estimates of the coefficients U_0, U_1, U_2, U_3 and U_4. The estimates require an additional hypothesis:

Hypothesis 4. For all $j \in \{1, \ldots, k\}$, either $D_j > 0$ or, if $D_j = 0$, the term involving x derivative in Eq. (23), *i.e.* B_j, is zero.

Hypothesis 5. $U(t = 0, \tau = 0, x, y) = U_0^0(x, y)$.

Our first concern is the existence and uniqueness of U_0.

3.2.1 Existence of the profiles

Theorem 3.4 *Under the hypothesis 4 and if $U_0^0 = PU_0^0 \in \mathbb{H}^s(\mathbb{R}^2)$, $s > 1$, there exists a time $T > 0$ and an unique function*

$$U_0 \in \mathcal{C}^0\left([0, T]; \bigcap_{1\leq l\leq s} \mathbb{W}^{l,\infty}\left(\mathbb{R}_t; \mathbb{H}^{s-l}(\mathbb{R}^2)\right)\right),$$

such that $U_0 = \sum_{j=1}^k P_j U_0 = \sum_{j=1}^k u_j$, *where the functions* u_j, $j = 1, \dots, k$ *satisfy Eqs. (15) and (23):*

$$\begin{cases} (\partial_t + v_j \partial_x)\, u_j = 0 \quad \textit{(Transport equation)}, \\ \partial_x \Big(\partial_\tau u_j - D_j \partial_x^2 u_j + B_j(u_j, \partial_x u_j) + F_j(u_j, u_j, u_j) \Big) = C_j \partial_y^2 u_j, \end{cases}$$

where the second equation is the Khokhov-Zabolotskaya equation when $D_j > 0$ *and* $C_j \neq 0$.
Furthermore $U_0(0, 0, \cdot) = U_0^0$.

Proof. Let $u_j = P_j U_0$ and $u_j^0 = P_j U_0^0$.

We have to solve Eqs. (15) and (23). We will at first only consider Eq. (23) since Eq. (15) is an equation of transport. We denote $\widetilde{u_j}$ the solution of Eq. (23).
The (formal) operator $\partial_x^{-1} \partial_y^2$ raises some problems. We define a unitary group on $\mathbb{H}^s(\mathbb{R}^2)$ to get rid of this term. Let $S_j(\tau)$ the operator defined by:

$$\mathcal{F}_{x,y}\Big(S_j(\tau)u\Big)(\xi, \eta) = e^{i\tau C_j \frac{\eta^2}{\xi}} \Big(\mathcal{F}_{x,y}(u)\Big)(\xi, \eta), \qquad \tau \in \mathbb{R}, \quad (\xi, \eta) \in \mathbb{R}^* \times \mathbb{R},$$

where $\mathcal{F}_{x,y}$ is the $2D$-Fourier transform. We consider now the function $w_j(\tau, x, y) = S_j(-\tau)\widetilde{u_j}(\tau, x, y)$ which satisfies:

$$\partial_\tau w_j - D_j \partial_x^2 w_j + \widetilde{A}_j(\tau, w_j)\partial_x w_j + \widetilde{F}_j(\tau, w_j) = 0, \tag{25}$$

where

$$\widetilde{A}_j(\tau, u)v = S_j(-\tau) B_j(S_j(\tau)u, S_j(\tau)v),$$
$$\widetilde{F}_j(\tau, v) = S_j(-\tau) F_j(S(\tau)v, S(\tau)v, S(\tau)v).$$

We denote by Λ the pseudodifferential operator with the symbol:

$$\sigma(\zeta) = (1 + |\zeta|^2)^{1/2}, \quad \zeta = (\zeta_1, \zeta_2). \tag{26}$$

If we denote by $(\ ,\ )$ the scalar product in $\mathbb{L}^2(\mathbb{R}^2)$, the scalar product in $\mathbb{H}^s(\mathbb{R}^2)$ is defined by $(u, v)_s = (\Lambda^s u, \Lambda^s v)$.
Because of Hypothesis 4, if $D_j = 0$, Eq. (25) reduces to: $\partial_\tau w_j + \widetilde{F}_j(\tau, w_j) = 0$. Thanks to the Cauchy-Lipschitz theorem we get the existence of a time $T_j > 0$ and of
$w_j \in \mathcal{C}^0([0, T_j[; \mathbb{H}^s(\mathbb{R}^2))$.
Now if $D_j > 0$, we can approximate the solution of Eq. (25) thanks to the Galerkin method on $[-L, L]^2$ with the Dirichlet boundary condition. We then have to perform some energy estimates on v_j in $\mathbb{H}^s(\mathbb{R}^2)$ before taking the limit (we perform the estimates on v_j rather than on its approximate solution to lighten the proof). We already have:

$$\|\widetilde{A}_j(\tau, u)v\|_{\mathbb{H}^s} \leq C\|u\|_{\mathbb{H}^s}\|v\|_{\mathbb{H}^s}, \qquad \|\widetilde{F}_j(\tau, v)\|_{\mathbb{H}^s} \leq C\|v\|_{\mathbb{H}^s}^3,$$

where C does not depend on τ nor on L. By applying Λ^s to Eq. (25) and performing the estimates we obtain:

$$\frac{d}{d\tau}\|w_j\|^2_{\mathbb{H}^s(\mathbb{R}^2)} + D_j\|\partial_x w_j\|^2_{\mathbb{H}^s(\mathbb{R}^2)} \leq C\left(\|w_j\|^2_{\mathbb{H}^s(\mathbb{R}^2)} + \|w_j\|^4_{\mathbb{H}^s(\mathbb{R}^2)}\right).$$

So there exists $T_j > 0$ and $v_j \in \mathcal{C}^0([0, T_j[; \mathbb{H}^s(\mathbb{R}^2))$ solution of Eq. (25). Back to $\widetilde{u_j}$, we have $\widetilde{u_j} \in \mathcal{C}^0(0, T_j; \mathbb{H}^s(\mathbb{R}^2))$ and the transport equation gives:

$$u_j(t, \tau, x, y) = \widetilde{u}_j(\tau, x - v_j t, y).$$

So we get u_j thanks to a translation in time of $\widetilde{u}_j$. Finally,

$$u_j \in \mathcal{C}^0\left([0, T_j[; \bigcap_{0\leq l\leq s} \mathbb{W}^{l,\infty}(\mathbb{R}_t; \mathbb{H}^{s-l}(\mathbb{R}^2))\right).$$

□

Remark 3.4 *This also proves the theorem 1.1 on the Khokhlov-Zabolotskaya equation given in the introduction.*
Following the work of Ukai [12] on the KP equation we can also prove the existence of solutions to the Khokhlov-Zabolotskaya (KZ) equation but we have then to impose too many conditions on the initial data particularly non physical ones.

The equations on the other profiles pose some problems: we want to obtain their existence in spaces derived from $\mathbb{H}^s(\mathbb{R}^2)$ but it comes the operator ∂_x^{-1} which imposes the existence in $\mathbb{H}^s_{loc}(\mathbb{R}^2)$ only and prevents to get profiles (U_1 and U_2) which satisfy the sublinear growth condition. To get rid of this problem we use a technique of low-frequency cut-offs (see [2], [3], [11]) to work with an approximated "well-prepared" leading profile. We then modify the equations fulfilled by U^1, U^2,... so that they depend on the truncation of the main profile rather than the main profile itself. We next use these modified profiles to obtain an approximate solution of Eq. (14) and then prove the stability of the approximation.

Definition 3.1 (Low-frequency truncation) *Let χ be a smooth function in $\mathbb{R}_x$, such that $|\chi| \leq 1$, $\chi(x) = 0$ for $|x| < 1$ and $\chi(x) = 1$ for $|x| \geq 2$. Define the Fourier multiplier $\chi^\delta(D_x)$ as the operator acting on $\mathbb{H}^s(\mathbb{R}^2)$ by:*

$$\chi^\delta(D_x) \;:\; f \mapsto \mathcal{F}^{-1}_{\xi,\eta\to x,y}\left(\chi\left(\frac{\xi}{\delta}\right)\hat{f}(\xi,\eta)\right).$$

The dominate convergence theorem shows that:

$$\chi^\delta f - f \to 0 \quad \text{in } \mathbb{H}^s.$$

Moreover we have the following lemma:

Lemma 3.5 *If* $f \in \mathbb{L}^\infty\left(0,T;\mathbb{H}^s(\mathbb{R}^2)\right)$ *then*

$$\lim_{\delta\to 0}\sup_{t\in[0,T]}\|\chi^\delta f(t,\cdot)-f(t,\cdot)\|_{\mathbb{H}^s(\mathbb{R}^2)}=0.$$

Proof. let $I^\delta(t)=\|\chi^\delta f(t,\cdot)-f(t,\cdot)\|_{\mathbb{H}^s(\mathbb{R}^2)}$. The sequence $(I^\delta)_\delta$ is a non increasing sequence of nonnegative continuous functions which simply goes to zero as δ goes to zero. The Dini's theorem gives us the result. □

Let us introduce $X_{s,t}$ the space of functions defined on $[0,T]\times\mathbb{R}_t\times\mathbb{R}^2_{x,y}$:

$$X_{s,T}=\left\{u,\ \sup\left\{\|\partial_t^\alpha\partial_x^\beta\partial_y^\gamma u(t,\tau,\cdot,\cdot)\|_{\mathbb{L}^2(\mathbb{R}^2)}\ :t\in[0,\infty),\ \tau\in[0,T]\right\}<\infty\right\},$$

for all $\alpha,\beta,\gamma\geq 0$ sucht that $0\leq\alpha+\beta+\gamma\leq s$.

Lemma 3.6 *If* $U_0^0\in\mathbb{H}^s(\mathbb{R}^2)$, $s>1$, *there exists an unique* $U_1^\delta\in X_{s-1,T}$ *such that:*

$$\begin{cases} U_1^\delta=PU_1^\delta=\displaystyle\sum_{j=1}^k P_jU_1^\delta,\\ \left(\partial_t+v_j\partial_x\right)P_jU_1^\delta+P_jA_2\displaystyle\sum_{i\neq j}\partial_yP_iU_0^\delta=0,\quad\forall j=1,\dots,k.\end{cases}$$

Moreover $\|U_1^\delta\|_{X_{s-1,T}}\leq\dfrac{C}{\delta}\|U_0\|_{X_{s,T}}$, *where* C *does not depend on* δ *nor on* $\|U_0\|_{X_{s,T}}$.

Proof. As we have truncated U_0, we have that

$$P_jU_0^\delta(t,\tau,x,y)=(\partial_x\varphi_j^\delta)(t,\tau,x,y)=(\partial_x\widetilde{\varphi}_j^\delta)(\tau,x-v_jt,y),$$

with $\|\varphi^\delta(t,\tau,\cdot,\cdot)\|_{\mathbb{H}^s(\mathbb{R}^2)}\leq\dfrac{1}{\delta}\|U_0^\delta(t,\tau,\cdot,\cdot)\|_{\mathbb{H}^s(\mathbb{R}^2)}$.
As we assume that $U_1(t=0)=0$ we have for all $j=1,\dots,k$,

$$\begin{aligned}P_jU_1^\delta&=-\sum_{i\neq j}P_jA_2P_i\int_0^t\partial_yP_iU_0^\delta(s,\tau,x-v_j(t-s),y)\,ds\\&=-\sum_{i\neq j}P_jA_2P_i\int_0^t\partial_y\partial_x\widetilde{\varphi}_i^\delta(\tau,x-v_jt+(v_j-v_i)s,y)\,ds\\&=-\sum_{i\neq j}\frac{1}{v_j-v_i}P_jA_2P_i\left(\partial_y\widetilde{\varphi}_i^\delta(\tau,x-v_it,y)-\partial_y\widetilde{\varphi}_i^\delta(\tau,x-v_jt,y)\right).\end{aligned}$$

We then deduce that $P_jU_1^\delta\in X_{s-1,T}$ and $\|U_1^\delta\|_{X_{s-1,T}}\leq\dfrac{C}{\delta}\|U_0\|_{X_{s,T}}$. □

By the same way we prove:

Lemma 3.7 *If we furthermore assume that $U_0^0 \in \mathbb{H}^s(\mathbb{R}^2)$, $s > 3$ and $\delta < 1$, we have*

$$\|QU_2^\delta\|_{X_{s-1,T}} \leq C, \ \|PU_2^\delta\|_{X_{s-2,T}} \leq \frac{C}{\delta^2},$$
$$\|QU_3^\delta\|_{X_{s-2,T}} \leq \frac{C}{\delta}, \ \|QU_4^\delta\|_{X_{s-3,T}} \leq \frac{C}{\delta^2},$$

where the constant C only depends on $\|U_0\|_{X_{s,T}}$.

3.2.2 Estimate for the residual

We will now consider the truncated asymptotic expansion:

$$u^{\varepsilon,\delta}(t,\tau,x,y) = U_0^\delta(t,\tau,x,y) + \varepsilon U_1^\delta(t,\tau,x,y) + \varepsilon^2 U_2^\delta(t,\tau,x,y) + \varepsilon^3 QU_3^\delta(t,\tau,x,y) + \varepsilon^4 QU_4^\delta(t,\tau,x,y),$$

and we will inject it in Equation (14) and prove the smallness of the residuals $R^{\varepsilon,\delta}(t,\tau,x,y) = \sum_{i=-2}^{14} \varepsilon^i R_i^\delta(t,\tau,x,y)$.

$$\begin{aligned}
R_{-2}^\delta &= LU_0^\delta = 0,\\
R_{-1}^\delta &= LU_1^\delta = 0,\\
R_0^\delta &= \partial_t U_0^\delta + A_1\partial_x U_0^\delta - B(U_0^\delta, U_0^\delta) + LQU_2^\delta = 0,\\
R_1^\delta &= \partial_t U_1^\delta + A_1\partial_x U_1^\delta + A_2\partial_y U_0^\delta + LQU_3^\delta - 2B(U_0^\delta, U_1^\delta) = 0,\\
R_2^\delta &= \partial_t U_2^\delta + A_1\partial_x U_2^\delta + A_2\partial_y U_1^\delta + LQU_4^\delta - B(U_1^\delta, U_1^\delta) - 2B(U_0^\delta, U_2^\delta)\\
&\quad + \partial_\tau U_0^\delta - T(U_0^\delta, U_0^\delta, U_0^\delta)\\
&= \sum_j \left(B_j(P_jU_0^\delta, \partial_x P_jU_0^\delta) + F_j(P_jU_0^\delta, P_jU_0^\delta, P_jU_0^\delta) \right)\\
&\quad - \chi^\delta \left(\textstyle\sum_j \left(B_j(P_jU_0, \partial_x P_jU_0) + F_j(P_jU_0, P_jU_0, P_jU_0) \right) \right),
\end{aligned}$$

We do not have $R_2^\delta = 0$ since U_0^δ does not solve Equation (23) but:

$$\partial_x \left(\partial_\tau P_jU_0^\delta - D_j\partial_x^2 P_jU_0^\delta + \chi^\delta \left(B_j(P_jU_0, P_jU_0) + F_j(P_jU_0, P_jU_0, P_jU_0) \right) \right) = C_j\partial_y^2 P_jU_0^\delta$$

The next residuals $(R_i^\delta)_{i=3,\dots,14}$ depend polynomially on U_0^δ, U_1^δ, U_2^δ, QU_3^δ, QU_4^δ and on their derivatives with respect to the slow time variable τ.

Proposition 3.8 *Assume $s > 5$. Then the residual fulfills the following estimates:*

$$\sup_{t\in[0,T/\varepsilon^2]} \|R^{\varepsilon,\delta}(t,\varepsilon^2\tau,\cdot,\cdot)\|_{\mathbb{H}^{s-5}(\mathbb{R}^2)} = \left(f(\delta)+\frac{\varepsilon}{\delta^6}\right)O(\varepsilon^2)+\varepsilon^2 g(\varepsilon),$$

and if $s > 6$,

$$\|\mathcal{R}^{\varepsilon,\delta}\|_{\mathbb{L}^\infty([0,T/\varepsilon^2]\times\mathbb{R}^2)} = \left(f(\delta)+\frac{\varepsilon}{\delta^6}\right)O(\varepsilon^2)+\varepsilon^2 g(\varepsilon)$$

where $\mathcal{R}^{\varepsilon,\delta}(t,x,y) = R^{\varepsilon,\delta}(t,\varepsilon^2 t,x,\varepsilon y)$, f is a positive nonincreasing function such that $f(\delta)\to 0$ as $\delta\to 0$, and g is a positive function such that $g(\varepsilon)\to 0$ as $\varepsilon\to 0$.

Proof. As said above, $R^\delta_{-2} = R^\delta_{-1} = R^\delta_0 = R^\delta_1 = 0$. The lemma 3.5 gives $\sup_{t\in[0,T/\varepsilon^2]}\|R^\delta_2(t,\varepsilon^2\tau,\cdot,\cdot)\|_{\mathbb{H}^s(\mathbb{R}^2)} = f(\delta)$ where f is as stated in the proposition 3.8.
In order to estimate the following terms we need some bounds on $\partial_\tau U_0$, $\partial_\tau U_1$,... If we consider Eq. (23) we get $\|\partial_\tau U_0\|_{X_{s-2,T}} \le \frac{C}{\delta}$. Following Lemmas 3.6 and 3.7 we get:

$$\|\partial_\tau U_1^\delta\|_{X_{s-3,T}} \le \frac{C}{\delta^2},\quad \|\partial_\tau Q U_2^\delta\|_{X_{s-3,T}} \le \frac{C}{\delta},\ \|\partial_\tau P U_2^\delta\|_{X_{s-4,T}} \le \frac{C}{\delta^3},$$
$$\|\partial_\tau Q U_3^\delta\|_{X_{s-4,T}} \le \frac{C}{\delta^2},\ \|\partial_\tau Q U_4^\delta\|_{X_{s-5,T}} \le \frac{C}{\delta^3}.$$

We can now estimate each term in the residual. We finally obtain the announced result. As $s > 6$ the classical Sobolev embedding yields the $\mathbb{L}^\infty$-estimate. □

3.2.3 Stability for the KZ approximation

We have shown so far that there exits a truncated approximate solution $u^{\varepsilon,\delta}$ to Eq. (14) for times of order $O\left(\frac{1}{\varepsilon^2}\right)$ and whose residual is small. We now prove that the untruncated leading term U_0 remains close to the exact solution for times of order $O\left(\frac{1}{\varepsilon^2}\right)$. More precisely, following [3], we show that for any $T > 0$ such that the leading term exists, the exact solution of Eq. (14) exists on $[0, T/\varepsilon^2]$ and remains close to the leading term.

Theorem 3.9 *Let Hypotheses 1-5 be satisfied. Let $U_0^0 = PU_0^0 \in \mathbb{H}^s(\mathbb{R}^2)$, $s > 6$. Let $T_0 > 0$ such that $U_0 = \sum_{i=1}^k P_j U_0$ lies in $\mathcal{C}([0,T_0/\varepsilon^2];\mathbb{H}^{s-1}(\mathbb{R}^2))$, where $(P_jU_0)_{j=1,\dots,k}$ fulfill Eqs. (15) and (23). There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there is an unique solution $\mathcal{U}^\varepsilon \in \mathcal{C}^0([0,T_0/\varepsilon^2];\mathbb{H}^{s-5}(\mathbb{R}^2))$ of*

$$\partial_t\mathcal{U}^\varepsilon + A_1\partial_x\mathcal{U}^\varepsilon + A_2\partial_y\mathcal{U}^\varepsilon + \frac{1}{\varepsilon^2}L\mathcal{U}^\varepsilon = B(\mathcal{U}^\varepsilon,\mathcal{U}^\varepsilon) + \varepsilon^2 T(\mathcal{U}^\varepsilon,\mathcal{U}^\varepsilon,\mathcal{U}^\varepsilon),$$

with $\mathcal{U}^\varepsilon(0,x,y) = U_0^0(x,\varepsilon y)$. *Moreover*

$$\|\mathcal{U}^\varepsilon - \mathcal{U}_0^\varepsilon\|_{\mathbb{L}^\infty([0,T_0/\varepsilon^2]\times\mathbb{R}^2)} = o(1) \quad \textit{as } \varepsilon \to 0,$$

where $\mathcal{U}_0^\varepsilon(t,x,y) = U_0(t,\varepsilon^2 t, x, \varepsilon y)$.

Proof. We will look for the exact solution $\mathcal{U}^\varepsilon$ as a perturbation of the truncated asymptotic expansion $\mathcal{U}^{\varepsilon,\delta}$:

$$\mathcal{U}^\varepsilon(t,x,y) = \mathcal{U}^{\varepsilon,\delta}(t,x,\varepsilon y) + \widetilde{U}^\delta(t,x,\varepsilon y),$$

where $\mathcal{U}^{\varepsilon,\delta}(t,x,y) = u^{\varepsilon,\delta}(t,\varepsilon^2 t,x,y)$. The error term fulfills:

$$\begin{aligned}\partial_t\widetilde{U}^\delta + A_1\partial_x\widetilde{U}^\delta + \varepsilon A_2\partial_y\widetilde{U}^\delta + \frac{1}{\varepsilon^2}L\widetilde{U}^\delta = {} & 2B(\mathcal{U}^{\varepsilon,\delta},\widetilde{U}^\delta) + B(\widetilde{U}^\delta,\widetilde{U}^\delta) \\ & +3\varepsilon^2 T(\mathcal{U}^{\varepsilon,\delta},\mathcal{U}^{\varepsilon,\delta},\widetilde{U}^\delta) \\ & +3\varepsilon^2 T(\mathcal{U}^{\varepsilon,\delta},\widetilde{U}^\delta,\widetilde{U}^\delta) \\ & +\varepsilon^2 T(\widetilde{U}^\delta,\widetilde{U}^\delta,\widetilde{U}^\delta) + \mathcal{R}^{\varepsilon,\delta},\end{aligned} \tag{27}$$

with the initial data $\widetilde{U}^\delta(t=0,x,y) = (U_0^0 - \chi^\delta U_0^0)(x,y)$.
Local existence of $\widetilde{U}^\delta$ in $\mathcal{C}([0,T];\mathbb{H}^{s-5}(\mathbb{R}^2)) \cap \mathcal{C}^1([0,T];\mathbb{H}^{s-6}(\mathbb{R}^2))$ can be proved using classical techniques (see [1] for example).
To obtain the $\mathbb{H}^{s-5}$ estimates we apply the previously defined operator Λ^{s-5} (see (26)) to (27) and we take the scalar product in $\mathbb{L}^2$ with $\Lambda^{s-5}\widetilde{U}^\delta$. As $\mathbb{H}^l(\mathbb{R}^2)$ is an algebra as soon as $l > 1$, there exists some constants $C > 0$ such that:

$$\begin{aligned}(B(V,V'),V'')_{s-5} \le {} & C\|V\|_{\mathbb{H}^{s-5}}\|V'\|_{\mathbb{H}^{s-5}}\|QV''\|_{\mathbb{H}^{s-5}} \\ & +C\|V\|_{\mathbb{H}^{s-5}}\|V''\|_{\mathbb{H}^{s-5}}\|QV'\|_{\mathbb{H}^{s-5}} \\ & +C\|V'\|_{\mathbb{H}^{s-5}}\|V''\|_{\mathbb{H}^{s-5}}\|QV\|_{\mathbb{H}^{s-5}}, \\ \|T(V,V',V'')\|_{\mathbb{H}^{s-5}} \le {} & C\|V\|_{\mathbb{H}^{s-5}}\|V'\|_{\mathbb{H}^{s-5}}\|V''\|_{\mathbb{H}^{s-5}}, \\ (LV,V)_{s-5} \ge {} & C\|QV\|^2_{\mathbb{H}^{s-5}} \quad \text{(Hypothesis 2)}.\end{aligned}$$

We obtain:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} + \frac{C}{\varepsilon^2}\|Q\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} \le {} & C\|\mathcal{U}^{\varepsilon,\delta}\|_{\mathbb{H}^{s-5}}\|\widetilde{U}^\delta\|_{\mathbb{H}^{s-5}}\|Q\widetilde{U}^\delta\|_{\mathbb{H}^{s-5}} \\ & +C\|Q\mathcal{U}^{\varepsilon,\delta}\|_{\mathbb{H}^{s-5}}\|\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} + C\|\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}}\|Q\widetilde{U}^\delta\|_{\mathbb{H}^{s-5}} \\ & +C\varepsilon^2\|\mathcal{U}^{\varepsilon,\delta}\|^2_{\mathbb{H}^{s-5}}\|\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} + C\varepsilon^2\|\mathcal{U}^{\varepsilon,\delta}\|_{\mathbb{H}^{s-5}}\|\widetilde{U}^\delta\|^3_{\mathbb{H}^{s-5}} \\ & +C\varepsilon^2\|\widetilde{U}^\delta\|^4_{\mathbb{H}^{s-5}} + \|R^{\varepsilon,\delta}(t,\varepsilon^2 t,\cdot,\cdot)\|_{\mathbb{H}^{s-5}}\|\widetilde{U}^\delta\|_{\mathbb{H}^{s-5}}\end{aligned}$$

Following [3] we introduce $t_0(\varepsilon,\delta)$ defined as:

$$t_0(\varepsilon,\delta) = \sup\left\{ \mathrm{t} \in \left[0, \mathrm{T}_0/\varepsilon^2\right], \|\widetilde{\mathrm{U}}^\delta(\mathrm{t})\|_{\mathbb{H}^{s-5}} \leq 1 \right\}.$$

We also remark that:

$$\|\mathcal{U}^{\varepsilon,\delta}\|_{\mathbb{H}^{s-5}} \leq C\left(1+\frac{\varepsilon^4}{\delta^4}\right), \quad \|Q\mathcal{U}^{\varepsilon,\delta}\|_{\mathbb{H}^{s-5}} \leq C\varepsilon^2\left(1+\frac{\varepsilon^2}{\delta^2}\right).$$

By means of Young's inequality, absorbing every term involving $Q\widetilde{U}^\delta$, we obtain:

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} + \frac{C}{\varepsilon^2}\|Q\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} \leq C\varepsilon^2\left(1+\frac{\varepsilon^8}{\delta^8}\right)\|\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} + C\varepsilon^2\left(f(\delta)+\frac{\varepsilon}{\delta^6}+g(\varepsilon)\right),$$

where f and g are as stated in Prop. 3.8. By Gronwall's lemma,

$$\begin{aligned}\|\widetilde{U}^\delta\|^2_{\mathbb{H}^{s-5}} \leq \left(\|\widetilde{U}^\delta(t=0)\|^2_{\mathbb{H}^{s-5}} + C\left(f(\delta)+\frac{\varepsilon}{\delta^6}+g(\varepsilon)\right)\right) \\ \exp\left(C\varepsilon^2\left(1+\frac{\varepsilon^8}{\delta^8}\right)t\right).\end{aligned} \tag{28}$$

Since $\|\widetilde{U}^\delta(t=0)\|_{\mathbb{H}^{s-5}} \to 0$ when $\delta \to 0$, we can take for example $\delta = \varepsilon^{1/7}$ and for ε small enough, we deduce from Eq. (28) that $t_0(\varepsilon,\delta) = \dfrac{T_0}{\varepsilon^2}$.

Now we have to prove that the main profile $\mathcal{U}_0^\varepsilon$ remains close from the approximate solution $\mathcal{U}^{\varepsilon,\delta}$ on the interval $[0, T_0/\varepsilon^2]$:

$$\begin{aligned}\sup_{t\in[0,T_0/\varepsilon^2]}\|\mathcal{U}^{\varepsilon,\delta} - U_0(t,\varepsilon^2 t,\cdot,\cdot)\|_{\mathbb{H}^{s-5}} &\leq \sup_{t\in[0,T_0/\varepsilon^2]}\|\mathcal{U}^{\varepsilon,\delta} - U_0^\delta(t,\varepsilon^2 t,\cdot,\cdot)\|_{\mathbb{H}^{s-5}} \\ &\quad + \sup_{t\in[0,T_0/\varepsilon^2]}\|U_0^\delta(t,\varepsilon^2 t,\cdot,\cdot) - U_0(t,\varepsilon^2 t,\cdot,\cdot)\|_{\mathbb{H}^{s-5}} \\ &\leq C\varepsilon^{6/7} + \sup_{t\in\mathbb{R},\tau\in[0,T_0]}\|U_0^\delta(t,\tau,\cdot,\cdot) - U_0(t,\tau,\cdot,\cdot)\|_{\mathbb{H}^{s-5}},\end{aligned}$$

To estimate the last term, we go back to the definition of U_0:

$$U_0(t,\tau,x,y) = \sum_{j=1,\ldots,k}\widetilde{u_j}(\tau, x-v_j t, y)$$

and we let $I_j^\delta(\tau) = \|\widetilde{u_j} - \chi^\delta\widetilde{u_j}\|_{\mathbb{H}^{s-5}}$ for $j = 1,\ldots,k$. Following the proof of lemma 3.5 we obtain that for all $j = 1,\ldots,k$, $\sup_{\tau\in[0,T_0]} I_j^\delta(\tau) \to 0$ as $\delta \to 0$. By the same way we obtained the regularity in the variable t before, we get that, when $\delta \to 0$,

$$\sup_{t\in\mathbb{R},\tau\in[0,T_0]}\|U_0^\delta(t,\tau,\cdot,\cdot) - U_0(t,\tau,\cdot,\cdot)\|_{\mathbb{H}^{s-5}} \to 0.$$

Finally, as $s > 6$ we use the classical Sobolev's imbedding to obtain the announced $\mathbb{L}^\infty$ estimates. $\square$

We now return to Eq. (13) and the system of partial differential equations of micromagnetics (10)-(12).
As we observed in Section 2.2, Eq. (13) is a special case of the general equation (14). Hypotheses 1-3 are satisfied (see Section 2.3). We will verify the remaining hypothesis 4 once we have found the coefficients D_j and the applications B_j. The asymptotic expansion is therefore unique and valid on the slow time scale. The way the asymptotic approximation has been built is irrelevant. This observation is important because it allows us to use the Landau-Lifschitz equation in the form given by Gilbert:

$$\partial_t M = -M \wedge H + \frac{\gamma}{|M|} \left(M \wedge \partial_t M \right). \tag{29}$$

This equation, known as Landau-Lifschitz-Gilbert equation is equivalent with the Landau-Lifschitz (6) equation, except for a rescaling of time by a factor $(1 + \gamma^2)$. As it turns out, Eq. (29) is more convenient for constructing the asymptotic expansions.

We perform one more change in the equation: as we did in Section 2 we simplify the equation (29) by replacing the term $|M|$ in the denominator of the damping term by $|M_0| = 1$ so that the factor of the damping term is γ. Thus we consider the following system:

$$\begin{aligned} \varepsilon^2 \partial_t M + \varepsilon^4 \partial_\tau M = &-M_0 \wedge H + \alpha^{-1} M_0 \wedge M - \varepsilon^2 M \wedge H + \gamma \left[\varepsilon^2 M_0 \wedge \partial_t M \right. \\ &\left. + \varepsilon^4 M_0 \wedge \partial_\tau M + \varepsilon^4 M \wedge \partial_t M + \varepsilon^6 M \wedge \partial_\tau M \right], \end{aligned} \tag{30}$$

$$\varepsilon^2 \partial_t H + \varepsilon^4 \partial_\tau H - \varepsilon^2 k \wedge \partial_x E - \varepsilon^3 l \wedge \partial_y E = -\varepsilon^2 \partial_t M - \varepsilon^4 \partial_\tau M, \tag{31}$$

$$\varepsilon^2 \partial_t E + \varepsilon^4 \partial_\tau E + \varepsilon^2 k \wedge \partial_x H + \varepsilon^3 l \wedge \partial_y H = 0. \tag{32}$$

We perform now an asymptotic expansion of Eqs. (30)-(31) along the lines of Section 3.1:

$$\begin{aligned} M &= M_1 + \varepsilon M_2 + \varepsilon^3 M_3 + \dots \\ H &= H_1 + \varepsilon H_2 + \varepsilon^3 H_3 + \dots \\ M &= E_1 + \varepsilon E_2 + \varepsilon^3 E_3 + \dots \end{aligned} \tag{33}$$

4.1 The equations of order $O(\varepsilon^0)$

To leading order, Eqs. (30)-(32) reduce to a single equation,

$$-M_0 \wedge (H_1 - \alpha^{-1} M_1) = 0, \tag{34}$$

which gives an expression for M_1 in terms of $M_1 \cdot M_0$ and H_1,

$$M_1 = (M_1 \cdot M_0)\, M_0 - \alpha M_0 \wedge (M_0 \wedge H_1). \tag{35}$$

4.2 The equations of Order $O(\varepsilon)$

To first order we get,

$$-M_0 \wedge (H_2 - \alpha^{-1} M_2) = 0, \tag{36}$$

and then an expression for M_2,

$$M_2 = (M_2 \cdot M_0)\, M_0 - \alpha M_0 \wedge (M_0 \wedge H_2). \tag{37}$$

4.3 The equations of Order $O(\varepsilon^2)$

$$\begin{cases} \partial_t M_1 = -M_0 \wedge (H_3 - \alpha^{-1} M_3) - M_1 \wedge H_1 + \gamma M_0 \wedge \partial_t M_1, \\ \partial_t H_1 - k \wedge \partial_x E_1 = -\partial_t M_1 \\ \partial_t E_1 + k \wedge \partial_x H_1 = 0. \end{cases} \tag{38}$$

Taking the scalar product of (38) with M_0 and adding to it the scalar product of (34) with M_1 we find that $\partial_t (M_1 \cdot M_0) = 0$, so:

$$M_1 \cdot M_0 = f_0, \qquad f_0 \equiv f_0(x, y, \tau). \tag{39}$$

(Note that $M_1 \cdot M_0$ is the $O(\varepsilon^2)$ term in the expansion of $|M|^2$, which is constant) If we take the vector product instead of the scalar product we obtain an expression for M_3 in terms of $M_3 \cdot M_0$ and H_2.

$$M_3 = (M_3 \cdot M_0) M_0 - \alpha M_0 \wedge (M_0 \wedge H_3) + \alpha M_0 \wedge q_1, \tag{40}$$

where the vector q_1 is given by:

$$q_1 = -\partial_t M_1 + \gamma M_0 \wedge \partial_t M_1 - M_1 \wedge H_1.$$

We substitute the expression (35) in the second equation of (38), use the fact that $\partial_t (M_1 \cdot M_0) = 0$ and solve the resulting equation for $\partial_t H_1$ to obtain a system of equations for H_1 and E_1,

$$\begin{cases} \partial_t H_1 + \dfrac{\alpha}{\alpha + 1} \left(k \cdot (M_0 \wedge \partial_x E_1)\right) M_0 - \dfrac{1}{1+\alpha} k \wedge \partial_x E_1 = 0, \\ \partial_t E_1 + k \wedge \partial_x H_1 = 0. \end{cases} \tag{41}$$

4.3.1 Choice of coordinates

The system of equations (41) is most easily solved if we adopt a coordinate system in $\mathbb{R}^3$ that is spanned by k, $k \wedge M_0$ and M_0 (we rely on the assumption that k and M_0 are not parallel or antiparallel.) Given any vector $v \in \mathbb{R}^3$ we define:

$$v_a = v \cdot M_0, \quad v_b = v \cdot (k \wedge M_0), \quad v_c = v \cdot k. \tag{42}$$

Then

$$v = \frac{1}{1-k_a^2}\left[(v_a - k_a v_c)M_0 + v_b(k \wedge M_0) + (v_c - k_a v_a)k\right], \quad v \in \mathbb{R}^3,$$

where $k_a = M_0 \cdot k$. An easy computation shows that:

$$u \cdot v = \frac{1}{1-k_a^2}\left[u_a v_a + u_b v_b + u_c v_c - k_a(u_a v_c + u_c v_a)\right], \quad u, v \in \mathbb{R}^3,$$

$$u \wedge v = \frac{1}{1-k_a^2}\begin{vmatrix} M_0 & k \wedge M_0 & k \\ u_a & u_b & u_c \\ v_a & v_b & v_c \end{vmatrix}, \quad u, v \in \mathbb{R}^3.$$

The system of equations (41) becomes:

$$\partial_t u_1 + K \partial_x u_1 = 0, \tag{43}$$

where $u_1 = (H_{1a}, H_{1b}, H_{1c}, E_{1a}, E_{1b}, E_{1c})^t$ and

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(1+\alpha)^{-1} & 0 & k_a(1+\alpha)^{-1} \\ 0 & 0 & 0 & 0 & k_a\alpha(1+\alpha)^{-1} & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -k_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

4.3.2 Solution of Equation (43)

The characteristic polynomial of K is $\det(\lambda I - K) = (\lambda^2 - v_0^2)(\lambda^2 - v_1^2)(\lambda^2 - v_2^2)$, where

$$v_0 = 0, \quad v_1 = \left(\frac{1}{1+\alpha}\right)^{1/2}, \quad v_2 = \left(\frac{1+(1-k_a^2)\alpha}{1+\alpha}\right)^{1/2}.$$

Then the eigenvalues of K are $v_0 = 0$ (algebraic multiplicity 2), $\pm v_1$ and $\pm v_2$. We also have $v_0 < v_1 < v_2$ and $1 - v_2^2 = k_a^2(1 - v_1^2)$. The matrix K is diagonalized by the linear transformation F, $K = F^{-1}VF$, $V = \mathrm{diag}(v_0, v_0, v_1, -v_1, v_2, -v_2)$ and

$$F = \begin{pmatrix} k_a(1-v_1^{-2}) & 0 & v_1^{-2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -v_1 & 0 & k_a v_1 \\ 0 & 1 & 0 & v_1 & 0 & -k_a v_1 \\ v_2^{-1} & 0 & -k_a v_2^{-1} & 0 & 1 & 0 \\ -v_2^{-1} & 0 & k_a v_2^{-1} & 0 & 1 & 0 \end{pmatrix}.$$

Applying F to Eq. (43) we obtain a diagonal system,

$$(\partial_t + V\partial_x)Fu_1 = 0.$$

This system corresponds to Eq. (15). The equations are decoupled and each equation can be integrated along its characteristics. We find $u_1 = F^{-1}f$ where $f = (f_1, f_2, f_3, f_4, f_5, f_6)^t$ and

$$\begin{aligned} &f_1 \equiv f_1(\tau, x, y), && f_2 \equiv f_2(\tau, x, y), \\ &f_3 \equiv f_3(\tau, x - v_1 t, y), && f_4 \equiv f_4(\tau, x + v_1 t, y), \\ &f_5 \equiv f_5(\tau, x - v_2 t, y), && f_6 \equiv f_6(\tau, x + v_2 t, y), \end{aligned}$$

The functions f_1 and f_2 represent standing waves, f_3 and f_4 are travelling waves propagating respectively with the velocities v_1 and $-v_1$, and f_5 and f_6 are travelling waves propagating respectively with the velocities v_2 and $-v_2$. The component of M_1 are deduced from Equations (35)-(39) and the expression of u_1,

$$M_{1a} = f_0,\ M_{1b} = \frac{1-v_1^2}{2v_1^2}(f_3+f_4),\ M_{1c} = k_a f_0 + \frac{v_2^2 - v_1^2}{v_2^2} f_1 - k_a \frac{1-v_1^2}{2v_2}(f_5 - f_6). \tag{44}$$

This completes the analysis of the second order approximation. We now know that the coefficients of order 0 in the expansions (33) are linear combinations of standing ($v_0 = 0$) and travelling ($\pm v_1,\ \pm v_2$) waves. In the next sections we will see how the profile functions $f_0, \ldots, f_6$ evolve on the slow time scale (τ) and on the transverse direction (y).

To the third order Equations (30)-(32) yield the differential equations:

$$\begin{cases} \partial_t M_2 = -M_0 \wedge (H_4 - \alpha^{-1} M_4) - M_1 \wedge H_2 - M_2 \wedge H_1 + \gamma M_0 \wedge \partial_t M_2, \\ \partial_t H_2 - k \wedge \partial_x E_2 - l \wedge \partial_y E_1 = -\partial_t M_2, \\ \partial_t E_2 + k \wedge \partial_x H_2 + l \wedge \partial_y H_1 = 0. \end{cases} \tag{45}$$

We follow the same procedure as in the previous section. We take the scalar product of the first equation of (45) with M_0 and we use Eqs. (35) and (37) to get:

$$\partial_t (M_2 \cdot M_0) = 2\alpha M_0 \cdot H_1 \wedge H_2. \tag{46}$$

We now take the vector product with M_0,

$$M_4 = (M_0 \cdot M_4) M_0 - \alpha M_0 \wedge (M_0 \wedge H_4) + \alpha M_0 \wedge q_2, \tag{47}$$

where the vector q_2 is given by:

$$q_2 = -\partial_t M_2 - M_1 \wedge H_2 - M_2 \wedge H_1 + \gamma M_0 \wedge \partial_t M_2.$$

We substitute the expression (37) in the second equation of (45) and we solve the resulting equation for $\partial_t H_2$ to obtain a system of equations for H_2 and E_2,

$$\begin{cases} \partial_t H_2 + \dfrac{\alpha}{1+\alpha} \left(k \cdot M_0 \wedge \partial_x E_2 + l \cdot M_0 \wedge \partial_y E_1\right) M_0 \\ \qquad\quad - \dfrac{1}{1+\alpha} \left(k \wedge \partial_x E_2 + l \wedge \partial_y E_1\right) + 2\alpha (M_0 \cdot H_1 \wedge H_2) M_0 = 0, \\ \partial_t E_2 + k \wedge \partial_x H_2 + l \wedge \partial_y H_1 = 0. \end{cases} \tag{48}$$

4.4.1 Coordinate representation

We use the coordinate system introduced in Section 4.3.1 with the abbreviations defined in Eq. (42). The equations (48) correspond to the following system of equations:

$$\partial_t u_2 + K \partial_x u_2 + K' \partial_y u_1 + A(u_1) u_2 = 0, \tag{49}$$

where K is the matrix defined in Section 4.3.1, $u_2 = (H_{2a}, H_{2b}, H_{2c}, E_{2a}, E_{2b}, E_{2c})^t$,

$$K' = \frac{1}{1-k_a^2}\begin{pmatrix} 0 & 0 & 0 & k_a l_b & l_c - k_a l_a & -l_b \\ 0 & 0 & 0 & -\dfrac{l_c}{1+\alpha} & 0 & \dfrac{l_a}{1+\alpha} \\ 0 & 0 & 0 & \dfrac{(1+\alpha k_a^2) l_b}{1+\alpha} & k_a l_c - \dfrac{(1+\alpha k_a^2) l_a}{1+\alpha} & -k_a l_b \\ -k_a l_b & k_a l_a - l_c & l_b & 0 & 0 & 0 \\ l_c & 0 & -l_a & 0 & 0 & 0 \\ -l_b & l_a - k_a l_c & k_a l_b & 0 & 0 & 0 \end{pmatrix},$$

and

$$A(u_1) = \frac{2\alpha}{1-k_a^2}\begin{pmatrix} -k_a H_{1b} & k_a H_{1a} - H_{1c} & H_{1b} & 0\ 0\ 0 \\ 0 & 0 & 0 & 0\ 0\ 0 \\ -k_a^2 H_{1b} & k_a^2 H_{1a} - k_a H_{1c} & k_a H_{1b} & 0\ 0\ 0 \\ 0 & 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0\ 0\ 0 \\ 0 & 0 & 0 & 0\ 0\ 0 \end{pmatrix}$$

By applying the linear transformation F we obtain:

$$\partial_t F u_2 + V \partial_x F u_2 + F K' F^{-1} \partial_y f + \left(F A(F^{-1} f) F^{-1} \right) F u_2 = 0. \qquad (50)$$

Following the scheme given in Section 3.1 we assume that u_2 fulfills the sub-linear growth condition (19) and we take the average of Equation (50) along the characteristics we used in Section 4.3.2. We let $g = F u_2 = (g_1, \ldots, g_6)^t$ and $V' = F K' F^{-1}$ and we get:

$$(V - v\mathrm{Id})\, \partial_x G_v g + V' \partial_y G_v f + F A(F^{-1} G_v f) F^{-1} G_v g = 0, \qquad (51)$$

with $v \in \{0, \pm v_1, \pm v_2\}$, where

$$V' = \frac{1}{1-k_a^2}\begin{pmatrix} 0 & 0 & -a_1 & a_1 & a_2 & a_2 \\ 0 & 0 & -a_2 & -a_2 & -a_3 & a_3 \\ -a_4 & a_5 & a_6 & -a_7 & -a_8 & a_8 \\ a_4 & a_5 & a_7 & -a_6 & a_8 & -a_8 \\ -a_9 & -a_{10} & -a_{11} & a_{11} & a_{12} & -a_{13} \\ -a_9 & a_{10} & a_{11} & -a_{11} & a_{13} & -a_{12} \end{pmatrix}$$

with

$$
\begin{aligned}
&a_1 = \frac{l_b}{2v_1}, && a_2 = \frac{k_a l_c - l_a}{2}, && a_3 = \frac{l_b v_2}{2},\\
&a_4 = l_b \frac{v_1^3}{v_2^2}(1-k_a^2), && a_5 = v_1^2(l_a - k_a l_c), && a_6 = \frac{l_c v_1}{2}(2-k_a^2),\\
&a_7 = \frac{l_c v_1}{2}k_a^2, && a_8 = \frac{k_a l_b}{2}\frac{v_1}{v_2}(v_2^2 - v_1^2), && a_9 = (l_a - k_a l_c)\frac{v_1^2}{v_2^2},\\
&a_{10} = l_b v_2(1-k_a^2), && a_{11} = \frac{l_b k_a}{2}\frac{v_2^2 - v_1^2}{v_1 v_2}, &&\\
&a_{12} = \frac{1}{2}\frac{2-k_a^2}{v_2}\left(l_c - l_a k_a(1-v_1^2)\right), && a_{13} = \frac{k_a^2}{2v_2}\left(l_c - l_a k_a(1-v_1^2)\right). &&
\end{aligned}
$$

Two of the 29 equations (the third equation when we apply G_{v_1} and the fourth when we apply G_{-v_1}) we can deduce from Eqs. (51) give the same compatibility condition:

$$
\begin{cases}
\dfrac{l_c v_1}{2}(2-k_a^2)\partial_y f_3 &= 0,\\
-\dfrac{l_c v_1}{2}(2-k_a^2)\partial_y f_4 &= 0,
\end{cases}
$$

which is:

$$l_c = l \cdot k = 0,$$

which means that the direction of the transversal perturbations is orthogonal to the direction of propagation. This compatibility condition is one of the two that Leblond and Manna found in [10].

4.5 The equation of Order $O(\varepsilon^4)$

To the fourth order Eqs. (30)-(32) yield the differential equations

$$
\begin{cases}
\partial_t M_3 + \partial_\tau M_1 = -M_0 \wedge (H_5 - \alpha^{-1} M_5) - M_1 \wedge H_3 - M_2 \wedge H_2 - M_3 \wedge H_1 \\
\qquad\qquad +\gamma\left[M_0 \wedge \partial_t M_3 + M_0 \wedge \partial_\tau M_1 + M_1 \wedge \partial_t M_1\right],\\
\partial_t H_3 + \partial_\tau H_1 - k \wedge \partial_x E_3 - l \wedge \partial_y E_2 = -\partial_t M_3 - \partial_\tau M_1,\\
\partial_t E_3 + \partial_\tau E_1 + k \wedge \partial_x H_3 + l \wedge \partial_y H_2 = 0.
\end{cases}
\tag{52}
$$

We follow the same procedure as in the previous section. We take the scalar product of the first equation of (52) with M_0 and we use Eqs. (34), (36) and (40) to get

$$\partial_t\left(M_3 \cdot M_0 + \frac{1}{2}|M_1|^2\right) + \partial_\tau(M_1 \cdot M_0) = 0. \tag{53}$$

Now we have $M_0 \cdot M_1 = f_0$ where f_0 does not depend on t. Hence Eq. (53) implies that $M_3 \cdot M_0 + \dfrac{1}{2}|M_1|^2$ grows linearly with t as $t \to +\infty$, unless f_0 is

independent not only of t but also of τ. We avoid this type of secular behaviour by imposing the condition $f_0 \equiv f_0(x,y)$. If $M_1 \cdot M_0 = 0$ at $t = 0$ then this condition gives $f_0 \equiv 0$, and the expressions (44) simplify,

$$M_{1a} = 0,\ M_{1b} = \frac{1-v_1^2}{2v_1^2}(f_3+f_4),\ M_{1c} = \frac{v_2^2-v_1^2}{v_2^2}f_1 - k_a\frac{1-v_1^2}{2v_2}(f_5-f_6).$$

If M_1 and M_0 are not orthogonal at $t = 0$, a constant nonzero component must be added to M_{1a}, M_{1c} and the quantities derived from them.
The scalar product of Eq. (52) with M_0 thus yields the relation:

$$2M_3 \cdot M_0 + |M_1|^2 = g_0, \quad g_0 \equiv g_0(\tau,x,y). \tag{54}$$

We now take the vector product with M_0,

$$M_5 = (M_0 \cdot M_5)M_0 - \alpha M_0 \wedge (M_0 \wedge H_5) + \alpha M_0 \wedge q_3, \tag{55}$$

where the vector q_3 is given by:

$$\begin{aligned} q_3 = -\partial_t M_3 - \partial_\tau M_1 - M_1 \wedge H_3 - M_2 \wedge H_2 - M_3 \wedge H_1 + \gamma M_0 \wedge \partial_t M_3 \\ +\gamma M_1 \wedge \partial_t M_1 + \gamma M_0 \wedge \partial_\tau M_1. \end{aligned}$$

We substitute the expression (35) in the second equation of (52) and we solve the resulting equation for $\partial_t H_3$ to obtain a system of equations for H_3 and E_3,

$$\begin{cases} \partial_t H_3 + \dfrac{\alpha}{1+\alpha}\left(k \cdot (M_0 \wedge \partial_x E_3) + l \cdot (M_0 \wedge \partial_y E_2)\right) M_0 \\ \qquad - \dfrac{1}{1+\alpha}\left(k \wedge \partial_x E_3 + l \wedge \partial_y E_2\right) \\ \qquad = \dfrac{\alpha}{1+\alpha}(M_0 \cdot v_H)M_0 + \dfrac{1}{1+\alpha}v_H, \\ \partial_t E_2 + k \wedge \partial_x H_2 + l \wedge \partial_y H_1 = v_E, \end{cases} \tag{56}$$

where the vectors v_H and v_E are:

$$\begin{aligned} v_H &= -\partial_\tau(H_1 + M_1) + \partial_t\left(\frac{1}{2}|M_1|^2 M_0 - \alpha M_o \wedge q_1\right), \\ v_E &= -\partial_\tau E_1. \end{aligned}$$

4.5.1 Coordinate representation

We use the coordinate system introduced in Section 4.3.1 with the abbreviations defined in Eq. (42). Eqs. (56) correspond to the following system of equations:

$$\partial_t u_3 + K\partial_x u_3 + K'\partial_y u_2 = -\partial_\tau u_1 + \partial_t r, \tag{57}$$

where K and K' are the matrix defined in sections 4.3.1 and 4.4.1 , $u_3 = (H_{3a}, H_{3b}, H_{3c}, E_{3a}, E_{3b}, E_{3c})^t$ and

$$r = \left(\frac{1}{2}|M_1|^2 \ , \ \frac{\alpha}{1+\alpha} q_{1c} \ , \ \frac{1}{2} k_a |M_1|^2 - \frac{\alpha}{1+\alpha} q_{1b} \ , \ 0 \ , \ 0 \ , \ 0 \right)^t .$$

The elements of r are known (in terms of f_1 and f_3 through f_6; f_2 does not enter). Notice, however that f_1 does not depend on t and that the derivatives of f_3 through f_6 with respect to t can be expressed in terms of their derivatives with respect to x.

4.5.2 Solution of Equation (57)

We apply the transformation F defined in Section 4.3.2 to both sides of Eq. (57) and absorb the t-derivative term in the left member, compensating with an x-derivative term in the right member,

$$(\partial_t + V\partial_x)\, F(u_3 - r) = -\partial_\tau f - \partial_x (VFr) - V'\partial_y g. \tag{58}$$

Since V is diagonal, Eq. (58) decouples into six first-order hyperbolic equations with constant coefficients, which can be integrated along their characteristics. If the solution is to remain bounded, the right member must be such that it does not lead to secular behaviour. This condition imposes constraints, which we can find by following the averaging strategy of Section 3.1, Lemma 3.2. We decompose VFr, separating the terms that are constant along the characteristics from those that are not,

$$VFr = -D_1 \partial_x f + D_2 f^2 + w.$$

The first two terms are constant along the characteristics; D_1 and D_2 are diagonal matrices with nonnegative entries that are readily found from equations (44) and (4.5.1),

$$D_1 = \frac{1}{2} g \left(1 - v_1^2\right) \mathrm{diag}(0, 0, 1 - v_1^2, 1 - v_1^2, 1 - v_2^2, 1 - v_2^2),$$
$$D_2 = \frac{3(1 - v_1)^2 (1 - v_2^2)}{8 v_2^2} \mathrm{diag}(0, 0, 0, 0, 1, 1),$$

where $f^2 = (f_1^2, f_2^2, f_3^2, f_4^2, f_5^2, f_6^2)^t$ and the remainder term w consists exclusively of terms that vary along the characteristics: its first and second component involve at least one of f_3 through f_6, its third component at least one of f_1 and f_4 through f_6, and so on. Thus Eq. (58) becomes

$$(\partial_t + V\partial_x)\, F(u_2 - r) = -\left[\partial_\tau f - D_1 \partial_x^2 f + D_2 \partial_x f^2 - V'\partial_y g\right] + w. \tag{59}$$

Application of the averaging operator to each component yields the equation

$$\partial_\tau f - D_1 \partial_x^2 f + D_2 \partial_x f^2 = \partial_y \widetilde{V'} g, \tag{60}$$

where $\widetilde{h} = (G_0 h_1, G_0 h_2, G_{v_1} h_3, G_{-v_1} h_4, G_{v_2} h_5, G_{-v_2} h_6)^t$.

We use the equations we get in Section 4.4.1. After some small computations we get:

$$\partial_x \widetilde{V' g} = - \begin{pmatrix} L_1 \left(W_0 \; C_1 \; \partial_y f_1 + W_0 \; C_2 \; \partial_y f_2 \right) \\ L_2 \left(W_0 \; C_1 \; \partial_y f_1 + W_0 \; C_2 \; \partial_y f_2 \right) \\ L_3 \; W_1 \; C_3 \; \partial_y f_3 \\ L_4 \; W_{-1} \; C_4 \; \partial_y f_4 \\ L_5 \; W_2 \; \left(C_5 - a_{12} u_1 \right) \partial_y f_5 + a_{12} \; \partial_x G_{v_2} g_5 \\ L_6 \; W_{-2} \; \left(C_6 + a_{12} u_2 \right) \partial_y f_6 - a_{12} \; \partial_x G_{-v_2} g_6 \end{pmatrix}, \tag{61}$$

where C_i and L_i are the columns and the lines of the matrix V', W_i is the inverse of the diagonal matrix $V - v_i \mathrm{Id}$ restrained to Im $(V - v_i \mathrm{Id})$ and trivially extended by 0 on Ker $(V - v_i \mathrm{Id})$ for $i \in \{0, \pm v_1, \pm v_2\}$ and $u_1 = (\beta, 0, 0, 0, 0, -1)^t$, $u_2 = (-\beta, 0, 0, 0, -1, 0)^t$ where β is a coefficient we can determine thanks to the equation of Section 4.4.1 and wich allow us to get rid of the second member in the equation on $G_{\pm v_2} g$.
In the equation (61) we see that $\partial_x \widetilde{V' g}$ still depends on $\partial_x G_{v_2} g_5$ and $G_{-v_2} g_6$, but we have no control over these two terms. We impose then a new condition, $a_{12} = 0$ which gives

$$l_a = l \cdot M_0 = 0.$$

Remark 4.1 *We have then $l_a = l_c = 0$, i.e. the direction of the transverse perturbation is orthogonal to the direction of propagation k and to the magnetic moment M_0. These two conditions are the compatibility conditions Leblond and Manna found in [10].*

By taking the derivative of Equation (60) with respect to x we obtain the Khokhlov-Zabolotskaya equation,

$$\partial_x \left(\partial_\tau f - D_1 \partial_x^2 f + D_2 \partial_x f^2 \right) = -B \partial_y^2 f, \tag{62}$$

where

$$B = \mathrm{diag} \left(0, 0, -\frac{l_b^2 v_1}{2(1-k_a^2)}, \frac{l_b^2 v_1}{2(1-k_a^2)}, -\frac{l_b^2}{2v_2(1-k_a^2)}, \frac{l_b^2}{2v_2(1-k_a^2)} \right).$$

Thus a necessary condition for the solution of Eq. (58) to remain bounded for long times as ε goes to zero is that the first order profile functions f_1 through f_6 fulfill the Khokhlov-Zabolotskaya equation on the (slow) time scale of τ. The equations for f_1 and f_2 are simple: $\partial_\tau f_1 = 0$, $\partial_\tau f_2 = 0$, so f_1 and f_2 must

be constant on the slow time scale and $f_i \equiv f_i(x)$, $i = 1, 2$. The equations for f_3 and f_4 are linear, those for f_5 and f_6 nonlinear with a quadratic nonlinearity.

Remark 4.2 *Equation (62) corresponds to Eq. (23). The nonzero entries of D_1 are positive, and the equations for f_1 and f_2 which involves the zero entries of D_1 are trivial. This observation validates Hypothesis 4.*

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