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Dedicated to the memory of Stefan Papadima

ABSTRACT. This is largely a survey paper, dealing with Cartan geometries in the complex analytic category. We first remind some standard facts going back to the seminal works of F. Klein, E. Cartan and C. Ehresmann. Then we present the concept of a *branched holomorphic Cartan geometry* which was introduced by the authors in [BD]. It generalizes to higher dimension the notion of a branched (flat) complex projective structure on a Riemann surface introduced by Mandelbaum. This new framework is much more flexible than that of the usual holomorphic Cartan geometries (e.g. all compact complex projective manifolds admit branched holomorphic projective structures). At the same time, this new definition is rigid enough to enable us to classify branched holomorphic Cartan geometries on compact simply connected Calabi-Yau manifolds.

CONTENTS

1. INTRODUCTION

An important consequence of the uniformization theorem for Riemann surfaces asserts, in particular, that any Riemann surface M admits a holomorphic atlas with coordinates in $\mathbb{C}P^1$ and transition maps in $\mathrm{PSL}(2, \mathbb{C})$. In other words, any Riemann surface is locally modelled on the complex projective line. This defines a *(flat) complex projective structure* on M. Complex projective structure on Riemann surfaces were introduced in connection with the study of the second order ordinary differential equations on complex domains and

had a very major role to play in understanding the framework of the uniformization theorem [Gu, StG].

The complex projective line $\mathbb{C}P^1$ acted on by the Möbius group $\mathrm{PSL}(2,\mathbb{C})$ is a geometry in the sense of Klein's Erlangen program in which he proposed to study all geometries in the unifying frame of the homogeneous model spaces G/H, where G is a finite dimensional Lie group and H a closed subgroup of G.

Thus homogeneous spaces could be used as good models to geometrize higher dimensional topological manifolds. Indeed, following Ehresmann [Eh], a manifold M is locally modelled on a homogeneous space G/H, if M admits an atlas with charts in G/H satisfying the condition that the transition maps are given by elements of G using the left-translation action of G on G/H. In this way M is locally endowed with the G/H-geometry and all G-invariant geometrical features of G/H have intrinsic meaning on M.

Elie Cartan generalized Klein's homogeneous model spaces to *Cartan geometries* (or *Cartan connections*) (see Definition 3.2 in Section 3.2). We recall that those are geometrical structures infinitesimally modelled on homogeneous spaces G/H. A Cartan geometry on a manifold M is equipped with a curvature tensor (see Definition 3.2) which vanishes exactly when M is locally modelled on G/H in the sense of Ehresmann [Eh]. In such a situation the Cartan geometry is called *flat*.

In [Ma1, Ma2] Mandelbaum introduced and studied *branched affine and projective structures* on Riemann surfaces. Those branched projective structures on Riemann surfaces are given by some holomorphic atlas where local charts are finite branched coverings on open sets in $\mathbb{C}P^1$ while the transition maps lie in $\mathrm{PSL}(2,\mathbb{C})$.

Inspired by the above mentioned works of Mandelbaum, we defined in [BD] a more general notion of *branched holomorphic Cartan geometry* on a complex manifold M (see Section 3.3), which is valid also in higher dimension for non necessarily flat Cartan geometries. We define and study here a more general notion of *generalized holomorphic Cartan geometry* (see Definition 3.5 and compare with [AM]).

This new notions of generalized (and branched) Cartan geometry are much more flexible than the usual one. For example, all compact complex projective manifolds admit a branched flat holomorphic projective structure (see Proposition 3.8). This will be obtain as a consequence of the fact that branched Cartan geometries are stable by pull-back through holomorphic ramified maps. Moreover, generalized Cartan geometries are stable by pull-back through any holomorphic map.

At the same time this notion is rigid enough to enable one to obtain classification results. We prove here the following results (see Corollary 4.6 and compare with [BD] for branched Cartan geometry):

Any generalized holomorphic Cartan geometry of type (G, H), with G a complex semisimple Lie group, on a compact simply connected Kähler Calabi–Yau manifold is flat.

This yields to the following corollary:

Non-projective compact simply connected Kähler Calabi–Yau manifolds do not admit any branched holomorphic projective structure.

The previous results are consequences of the following Theorem 4.1 ([Theorem 6.2, [BD]]) which is of independent interest:

If E is a holomorphic vector bundle over a compact simply connected Kähler Calabi–Yau manifold, and E admits a holomorphic connection, then E is a trivial holomorphic vector bundle equipped with the trivial connection.

This statement gives a partial answer of a more general (still open) question asked by Atiyah [At]: *Let E be a holomorphic vector bundle over a compact complex Kähler manifold. Assume that E admits a holomorphic connection. Is it true that E also admits a flat connection ?*

The question is natural since, under the previous hypothesis, the vector bundle E has trivial rational Chern classes [At] and hence there are no topological obstructions to admit a flat connection.

An original result of this paper is Proposition 4.3 which constructs nontrivial principal elliptic bundles over K3 surfaces bearing holomorphic connections. In particular, those elliptic bundles do not admit flat connections and give a counter-example to Atiyah's (generalized) question to the context of principal (elliptic) bundles over (simply connected) Kähler manifolds.

2. Klein geometries and (G, X)-structures

Following F. Klein's point of view in his Erlangen's address (1872), a *geometry* is a space (manifold) X endowed with a transitive action of a (finite dimensional) connected Lie group G. In this context, two subsets P_1 and P_2 in X, will be considered *congruent* (also called equivalent) if there exists an element $g \in G$ such that $g \cdot P_1 = P_2$.

Let us (arbitrarily) pick-up a point $x_0 \in X$. Then X is diffeomorphic to a quotient G/H, where H is the closed subgroup of G stabilizing x_0.

Of course, the first example to think of is that of the n-dimensional Euclidean geometry for which $G = O(n, \mathbb{R}) \ltimes \mathbb{R}^n$ is the group of Euclidean motions (it is the semi-direct product of the corresponding orthogonal group $O(n, \mathbb{R})$ with the translation group for the standard action of the orthogonal group) and $H = O(n, \mathbb{R})$ is the stabilizer of the origin.

In the complex analytic category, it will be suitable to consider instead the *complex Euclidean space* $(X = \mathbb{C}^n; dz_1^2 + \ldots + dz_n^2)$ for which G is the semi-direct product $O(n, \mathbb{C}) \ltimes \mathbb{C}^n$ for the standard action of $O(n, \mathbb{C})$ on $\mathbb{C}^n$ (it is the group of complex Euclidean motions) and $H = O(n, \mathbb{C})$ (the stabilizer of the origin). The non-degenerate complex quadratic form $dz_1^2 + \ldots + dz_n^2$ is a flat holomorphic Riemannian metric (see [Gh, Du, DZ]).

One also studies classically the complex affine space $X = \mathbb{C}^n$ under the action of the complex affine group $G = GL(n, \mathbb{C}) \ltimes \mathbb{C}^n$ (the semi-direct product for the standard action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$) which sends lines parametrized at constant speed on (possibly other) lines parametrized at constant speed and the complex projective space. Recall that the complex projective space $\mathbb{C}P^n$ acted on by the complex projective group $PGL(n+1, \mathbb{C})$ is a geometry where the symmetry group preserves (complex) collinearness.

In a program which is a vast generalization of Klein's geometries, E. Cartan created the framework of *Cartan geometries* which are infinitesimal versions of those G/H-homogeneous spaces (exactly in the same way as Riemannian geometry is the infinitesimal version of the Euclidean space). Those Cartan geometries which will be presented in Section 3 admit a curvature tensor which will be zero exactly when the Cartan geometry is locally isomorphic to a Klein geometry. The Cartan geometries for which the curvature tensor vanishes are called *flat*, and they were nicely described from the global geometric point of view by C. Ehresmann in [Eh].

Ehresmann's definition of those so-called *locally homogeneous spaces* (presently the terminology (G, X)-*structure* is more in use) is the following. Let X be a G–homogeneous space.

Definition 2.1. A (G, X)-structure on a manifold M is given by an open cover $(U_i)_{i \in I}$ of M with charts $\phi_i : U_i \longrightarrow X$ such that the transition maps

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$$

are given (on each connected component) by the restriction of an element in G.

Two (G, X)-structures on M will be identified if there exists a diffeomorphism of M isotopic to the identity map that sends the first (G, X)-structure to the second one.

The key point of the previous definition is the fact that any G-invariant geometric feature of X will have an intrinsic meaning on M.

Ehresmann's results in [Eh] led to the following geometric description of this situation:

Theorem 2.2 (Ehresmann)**.** *Let* $X = G/H$ *be a Klein geometry and* M *a manifold of the same dimension as* X. *Then there is a canonical one-to-one correspondence between the following structures:*

 (1) *A* (G, X)-*structure on* M.
 (2) *A fiber bundle* E_X *over* M *with fiber* X *and structure group* G, *equipped with a flat connection together with global section which is transversal to the horizontal distribution for the connection.*
 (3) *A principal* H-*bundle* E_H *over* M *and a flat connection on the associated principal* G-*bundle* $E_G = E_H \times^H G$, *such that the subbundle* E_H *is transverse to the horizontal distribution for the connection.*

Let us briefly sketch a proof of Ehresmann's theorem.

Proof. We will first prove that (1) implies (2). Consider a cover of M by open subsets U_i and local charts $\phi_i : U_i \longrightarrow X$ with the property that on intersections $U_i \cap U_j$ there exist elements $g_{ij} \in G$ such that $\phi_i \circ \phi_j^{-1} = g_{ij}$. Then g_{ij} is a locally constant cocycle with values in G and hence it defines a flat principal G-bundle E_G over M. Consider the associated flat bundle $E_X = E_G/H$, with fiber $X = G/H$. Then the charts ϕ_i are local sections of E_X which glue together compatibly to produce a global section of E_X. The flat connection of E_X is locally determined by the foliation $U_i \times \{x\}$, for $x \in X$. Since the local charts ϕ_i are local diffeomorphisms, the associated section is transverse to this foliation and hence to the

corresponding flat connection. It may be mentioned that, since $X = G/H$, a global section of E_X produces a reduction of structure group of the principal G-bundle E_G to the subgroup $H \subset G$. Indeed, for any section $\phi' : M \longrightarrow E_X$ of the projection $E_X = E_G/H \longrightarrow M$, the inverse image of $\phi'(M) \subset E_X$ under the quotient map $E_G \longrightarrow E_G/H$ is principal H–bundle on M. Therefore, from the section of $E_X \longrightarrow M$ given by the charts ϕ_i produce a reduction

$$E_H \subset E_G$$

of structure group of the principal G-bundle E_G to the subgroup H. Also since the previous section of E_X is transverse to the flat structure of E_X, we have that E_H is transverse to the flat structure of E_G. Hence we proved that (1) also implies (3). The same proof also shows that (2) implies (3).

Let us finish the proof by showing that (3) implies (1). Consider a cover of M by simply connected open sets U_i. Since the connection on E_G is flat, over each U_i, the principal G-bundle E_G equipped with the flat connection must be isomorphic to the trivial principal G-bundle $U_i \times G \longrightarrow U_i$ equipped with the trivial connection. Therefore, over each U_i, the reduction of structure group $E_H|_{U_i} \subset E_G|_{U_i}$ is defined by a smooth map

$$\phi_i : U_i \longrightarrow X = G/H \,.$$

Now the transversality condition implies that ϕ_i is a local diffeomorphism. On $U_i \cap U_j$ the two maps ϕ_i and ϕ_j agree up to multiplication by an element $g_{ij} \in G$, because any two local trivializations of a principal G-bundle E_G equipped with a flat connection differ by the left-multiplication (on the trivial principal G-bundle) by an element of G. This completes the proof. $\qquad \square$

Let $\widetilde{M} \longrightarrow M$ be a universal cover. Let us pull-back to $\widetilde{M}$ the flat bundle E_X in description (2) in Theorem 2.2. This pull-back is isomorphic to the trivial fiber bundle $\widetilde{M} \times G/H \longrightarrow \widetilde{M}$ equipped with the trivial connection. Fix an isomorphism between these two flat fiber bundles. The section of the fiber bundle $\widetilde{M} \times G/H \longrightarrow \widetilde{M}$, given by the pull-back of the section of $E_X \longrightarrow M$ using the chosen isomorphism of flat bundles, is called the developing map dev of the (G, X)-structure. It is a map from $\widetilde{M}$ to X, which is unique up to post-composition with an element in G, because we had to choose a trivialization of the flat fiber bundle over $\widetilde{M}$. This developing map is a local diffeomorphism because of the transversality condition.

Let $\rho : \pi_1(M) \longrightarrow G$ be the monodromy representation for the flat connection on E_X over M; since the structure group of E_X is G, the monodromy representation is a homomorphism to G. The construction of ρ also involves choosing a trivialization of the pullback of the flat bundle E_X to the universal cover $\widetilde{M}$. So the homomorphism ρ is unique up to inner automorphism by an element of G. The fiber bundle E_X is a quotient of $\widetilde{M} \times (G/H)$ through the identification

$$(m, x) = (\gamma \cdot m, \rho(\gamma) \cdot x) \,,$$

with $\gamma \in \pi_1(M)$ acting on $\widetilde{M}$ by deck-transformation and on X through ρ. Since developing map is the pull-back of a section of E_X, it is equivariant with respect to the monodromy morphism $dev(\gamma \cdot x) = \rho(\gamma) \cdot dev(m))$.

In the description (3) in Theorem 2.2, one can obtain the developing map and the monodromy homomorphism in the following way. Pull-back the flat principal G-bundle E_G to the universal cover $\widetilde{M}$ of M. Since $\widetilde{M}$ is simply connected this pull-back is isomorphic to the trivial principal G-bundle $\widetilde{M} \times G \longrightarrow \widetilde{M}$ equipped with the trivial connection. Once again E_G is isomorphic to a quotient of $\widetilde{M} \times G$ through the identification $(m, g) = (\gamma \cdot m, \rho(\gamma) \cdot g)$, with $\gamma \in \pi_1(M)$ acting on $\widetilde{M}$ by deck-transformation and on G through the monodromy morphism $\rho : \pi_1(M) \longrightarrow G$.

The pull-back of the subbundle E_H is given by a smooth map from $\widetilde{M}$ to $X = G/H$ which is a local diffeomorphism (by the transversality condition): it is the developing map.

3. Holomorphic Cartan geometries

It is a very stringent condition for a compact complex manifold to admit a holomorphic Cartan geometry. In order to illustrate this phenomenon, let us remind a theorem of Wang [Wa] which gives the classification of those complex compact manifolds admitting a holomorphic Cartan geometry with model a complex Lie group G (see the general Definition in Section 3.2).

3.1. **Parallelizable manifolds.**

Theorem 3.1 (Wang). *Let M be a compact complex manifold of dimension n and*

$$\omega : TM \longrightarrow \mathbb{C}^n$$

a holomorphic trivialization of the holomorphic tangent bundle. Then the universal cover of M is biholomorphic to a complex Lie group L and the pull-back of ω on L coincides with the Maurer-Cartan form of L. Consequently, $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

Moreover, M is Kähler if and only if L is abelian (and M is a complex torus).

Proof. Consider $X_1, X_2, \ldots, X_n$ global holomorphic ω-constant vector fields on M which span TM. Then, for all $1 \leq i, j \leq n$, we have

$$[X_i, X_j] = f_1^{ij} X_1 + f_2^{ij} X_2 + \ldots + f_n^{ij} X_n$$

with f_k^{ij} being holomorphic functions on M. Since M is compact, these functions have to be constant and, consequently, $X_1, X_2, \cdots, X_n$ generate a n-dimensional Lie algebra $\mathfrak{L}$. If we consider on the target vector space of ω the Lie algebra structure of $\mathfrak{L}$, then ω becomes a Lie algebra isomorphism.

By Lie's theorem there exists a unique connected simply connected complex Lie group L corresponding to $\mathfrak{L}$. The holomorphic parallelization by ω-constant holomorphic vector fields is locally isomorphic to the parallelization given by left translation-invariant vector fields on the Lie group L.

Since M is compact, the X_i are complete and they define a holomorphic transitive action of L on M (with discrete kernel). Hence M is biholomorphic to a quotient of L by a cocompact discrete subgroup Γ in L.

The Lie-Cartan formula

$$d\omega(X_i, X_j) \;=\; X_i \cdot \omega(X_j) - X_j \cdot \omega(X_i) - \omega([X_i, X_j]) \;=\; -\omega([X_i, X_j]) \;=\; -[\omega(X_i), \omega(X_j)]$$

shows that ω satisfies the Maurer-Cartan equation of the Lie group L, which can also be put under the more formal expression $d\omega + \frac{1}{2}[\omega,\omega]_L = 0$.

Assume now M is Kähler. Then, any holomorphic form on M has to be closed. The Maurer-Cartan formula shows that the one-forms composing ω are all closed if and only if L is abelian and thus M is a complex torus. $\qquad\square$

3.2. Holomorphic Cartan geometry.

We recall now the definition of a Cartan geometry in the category of complex analytic manifolds. Let G be a connected complex Lie group and $H \subset G$ a connected complex Lie subgroup. The Lie algebras of H and G will be denoted by $\mathfrak{h}$ and $\mathfrak{g}$ respectively.

The following definition generalizes the standard fibration $G \longrightarrow G/H$ seen as a H-(right) principal bundle and equipped with the left-invariant Maurer-Cartan form of G (see also [Sh]).

Definition 3.2. A holomorphic Cartan geometry of type (G, H) on a complex manifold M is a principal H–bundle $\pi_M : E_H \longrightarrow M$ endowed with a $\mathfrak{g}$–valued holomorphic 1–form ω on E_H satisfying the following conditions:

(1) $\omega : TE_H \longrightarrow E_H \times \mathfrak{g}$ is an isomorphism.
(2) ω is H–equivariant with H acting on $\mathfrak{g}$ via conjugation.
(3) the restriction of ω to each fiber of f coincides with the Maurer–Cartan form associated to the action of H on E_H.

The Cartan geometry is called *flat* if its curvature $K(\omega) = d\omega + \frac{1}{2}[\omega,\omega]$ vanishes.

Notice that ω-constant holomorphic vector fields on E_H form a holomorphic parallelization of the holomorphic tangent bundle of E_H. Consider the family of ω-constant vector fields on E_H with values in $\mathfrak{h}$. The form ω takes the Lie bracket of two such vector fields to the Lie product of the corresponding elements of $\mathfrak{h}$. Moreover, ω is a Lie algebra isomorphism from the family of ω-constant vector fields to $\mathfrak{g}$ if and only if the Cartan geometry defined by ω is flat.

In this case we have the following classical:

Theorem 3.3 (Cartan). *The curvature $d\omega + \frac{1}{2}[\omega,\omega]$ vanishes if and only if (E_H, ω) is locally isomorphic to the left-invariant Maurer-Cartan form on $G \longrightarrow G/H$.*

A flat Cartan geometry on M, gives a $(G, X = G/H)$-structure on M in Ehresmann's sense.

Let

$$E_G := E_H \times^H G \xrightarrow{f_G} M \tag{3.1}$$

be the holomorphic principal G–bundle on M obtained by extending the structure group of E_H using the inclusion of H in G. So, E_G is the quotient of $E_H \times G$ where two points $(c_1, g_1), (c_2, g_2) \in E_H \times G$ are identified if there is an element $h \in H$ such that $c_2 = c_1 h$

and $g_2 = h^{-1} g_1$. The projection f_G in (3.1) is induced by the map $E_H \times G \longrightarrow M$, $(c, g) \longmapsto \pi_M(c)$. The action of G on E_G is induced by the action of G on $E_H \times G$ given by the right–translation action of G on itself. Let $\mathrm{ad}(E_H) = E_H \times^H \mathfrak{h}$ and $\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g}$ be the adjoint vector bundles for E_H and E_G respectively. We recall that $\mathrm{ad}(E_H)$ (respectively, $\mathrm{ad}(E_G)$) is the quotient of $E_H \times \mathfrak{h}$ (respectively, $E_G \times \mathfrak{g}$) where two points (z_1, v_1) and (z_2, v_2) are identified if there is an element $g \in H$ (respectively, $g \in G$) such that $z_2 = z_1 g$ and v_1 is taken to v_2 by the automorphism of the Lie algebra $\mathfrak{h}$ (respectively, $\mathfrak{g}$) given by automorphism of the Lie group H (respectively, G) defined by $y \longmapsto g^{-1} y g$. We have a short exact sequence of holomorphic vector bundles on X

$$0 \longrightarrow \mathrm{ad}(E_H) \xrightarrow{\iota_1} \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_G)/\mathrm{ad}(E_H) \longrightarrow 0 . \tag{3.2}$$

The holomorphic tangent bundle of a complex manifold Y will be denoted by TY. Let

$$\mathrm{At}(E_H) = (TE_H)/H \longrightarrow M \quad \text{and} \quad \mathrm{At}(E_G) = (TE_G)/G \longrightarrow M$$

be the Atiyah bundles for E_H and E_G respectively; see [At]. Let

$$0 \longrightarrow \mathrm{ad}(E_H) \xrightarrow{\iota_2} \mathrm{At}(E_H) \xrightarrow{q_H} TM \longrightarrow 0 \tag{3.3}$$

and

$$0 \longrightarrow \mathrm{ad}(E_G) \xrightarrow{\iota_0} \mathrm{At}(E_G) \xrightarrow{q_G} TM \longrightarrow 0 \tag{3.4}$$

be the Atiyah exact sequences for E_H and E_G respectively; see [At]. The projection q_H (respectively, q_G) is induced by the differential of the map π_M (respectively, f_G). A holomorphic connection on a holomorphic principal bundle is defined to be a holomorphic splitting of the Atiyah exact sequence associated to the principal bundle [At]. Therefore, a holomorphic connection on E_G is a holomorphic homomorphism

$$\psi : \mathrm{At}(E_G) \longrightarrow \mathrm{ad}(E_G)$$

such that $\psi \circ \iota_0 = \mathrm{Id}_{\mathrm{ad}(E_G)}$.

It is straight-forward to check that a holomorphic Cartan geometry on X of type (G, H) is a pair (E_H, θ), where E_H is a holomorphic principal H–bundle on M and

$$\theta : \mathrm{At}(E_H) \longrightarrow \mathrm{ad}(E_G)$$

is a holomorphic isomorphism of vector bundles such that $\theta \circ \iota_2 = \iota_1$ (see (3.3) and (3.2) for ι_2 and ι_1 respectively). Therefore, we have the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota_2} & \mathrm{At}(E_H) & \xrightarrow{q_H} & TM & \longrightarrow & 0 \\
 & & \| & & \downarrow{\theta} & & \downarrow{\phi} & & \\
0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota_1} & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{ad}(E_G)/\mathrm{ad}(E_H) & \longrightarrow & 0
\end{array}
$$

the above homomorphism ϕ induced by θ is evidently an isomorphism.

The following proposition is classical (see also [Sh]):

Proposition 3.4. *The G-principal bundle E_G admits a holomorphic connection.*

Proof. Let us consider on $E_H \times G$ the holomorphic one-form with values in $\mathfrak{g}$:

$$\widetilde{\omega}(c, g) = Ad(g^{-1}) \pi_1^*(\omega) + \pi_2^*(\omega_G)$$

where π_1 and π_2 are the projections on the first and the second factor respectively and ω_G is the left-invariant Maurer-Cartan form on G.

One verify that $\widetilde{\omega}$ is invariant by the previous H-action and vanishes on the fibers of the fibration $E_H \times G \longrightarrow E_H \times^H G$.

This implies that $\widetilde{\omega}$ is basic (it is the pull-back of a form on $E_H \times^H G$, also denoted by $\widetilde{\omega}$).

The one-form $\widetilde{\omega}$ is a holomorphic connection on the principal G-bundle E_G. $\qquad\square$

Consider $f_G = E_G \longrightarrow M$ and a second projection map $h_G : E_G \longrightarrow G/H$ which associates to each class $(c, g) \in E_G$ the element $g^{-1}H \in G/H$.

The differentials of the projections f_G and h_G map the horizontal space of the connection $\widetilde{\omega}$ at $(c, g) \in E_G$ isomorphically onto $T_{f_G(c)}M$ and onto $T_{g^{-1}H}(G/H)$ respectively. This defines an isomorphism between $T_{f_G(c)}M$ and $T_{g^{-1}H}(G/H)$. Hence a Cartan geometry provides a family of 1-jets identifications of the manifold X with the model space G/H.

Also, the connection $\widetilde{\omega}$ defines a way to lift differentiable curves in M to $\widetilde{\omega}$-horizontal curves in E_G. Moreover this lift is unique once one specifies the starting point of the lifted curve.

When $\widetilde{\omega}$ is flat, its kernel is a foliation. In this case one can lift to E_G not only curves, but also simply connected open sets in M (their lift is tangent to the foliation given by the kernel of $\widetilde{\omega}$). Then their projection h_G to the model G/H is a local biholomorphism. The projections in G/H of two different lifts (in two different leaves of the foliation) are related by the action of an element in G acting on the model G/H. This procedure furnishes a (G, X)-structure on M (compatible with the complex structure on M).

Moreover, if M is simply connected one can lift to E_G all of the manifold M. This defines a global section of the principal bundle E_G which must be trivial: $E_G = M \times G$. In this trivialization our section is $M \times \{e\}$ (with e the identity element in G). The restriction of the projection $h_G : E_G \longrightarrow G/H$ to the section $M \times \{e\}$ defines a local biholomorphism from M to the model G/H, which is a developing map.

If M is no simply connected, one can still pull-back the flat Cartan geometry to the universal cover $\widetilde{M}$ of M and define in the same way the developing map which is a local biholomorphism from $\widetilde{M}$ to the model G/H.

3.3. **Generalized holomorphic Cartan geometry.** This more general framework (similar to that of [AM]) is given by the following:

Definition 3.5. A generalized holomorphic Cartan geometry of type (G, H) on a complex manifold M is a principal H–bundle $\pi_M : E_H \longrightarrow M$ endowed with a $\mathfrak{g}$–valued holomorphic 1–form ω on E_H satisfying:

(1) ω is H–equivariant with H acting on $\mathfrak{g}$ via conjugation.
(2) the restriction of ω to each fiber of f coincides with the Maurer–Cartan form associated to the action of H on E_H.

Therefore, generalized holomorphic Cartan geometry of type (G, H) is defined by a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota_2} & \mathrm{At}(E_H) & \xrightarrow{q_H} & TM & \longrightarrow & 0 \\
& & \| & & \downarrow{\theta} & & \downarrow{\phi} & & \\
0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota_1} & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{ad}(E_G)/\mathrm{ad}(E_H) & \longrightarrow & 0
\end{array}
\tag{3.5}
$$

of holomorphic vector bundles on M. We note that the homomorphism ϕ in (3.5) is induced by θ.

Suppose we are given θ as above. We can embed $\mathrm{ad}(E_H)$ in $\mathrm{At}(E_H) \oplus \mathrm{ad}(E_G)$ by sending any v to $(\iota_2(v), -\iota_1(v))$ (see (3.3), (3.2)). The Atiyah bundle $\mathrm{At}(E_G)$ is the quotient $(\mathrm{At}(E_H) \oplus \mathrm{ad}(E_G))/\mathrm{ad}(E_H)$ for this embedding. The inclusion of $\mathrm{ad}(E_G)$ in $\mathrm{At}(E_G)$ in (3.4) is given by the inclusion ι_1 or ι_2 of $\mathrm{ad}(E_G)$ in $\mathrm{At}(E_H) \oplus \mathrm{ad}(E_G)$ (note that they give the same homomorphism to the quotient bundle $(\mathrm{At}(E_H) \oplus \mathrm{ad}(E_G))/\mathrm{ad}(E_H)$). The homomorphism

$$
\mathrm{At}(E_H) \oplus \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_G), \quad (v, w) \longmapsto \theta(v) + w
$$

produces a homomorphism

$$
\theta' : \mathrm{At}(E_G) = (\mathrm{At}(E_H) \oplus \mathrm{ad}(E_G))/\mathrm{ad}(E_H) \longrightarrow \mathrm{ad}(E_G)
$$

which satisfies the condition that $\theta' \circ \iota_0 = \mathrm{Id}_{\mathrm{ad}(E_G)}$, meaning θ' is a holomorphic splitting of (3.4). Therefore, θ' is a holomorphic connection on the principal G–bundle E_G.

The Cartan geometry is called flat if its curvature $K(\omega) = d\omega + \frac{1}{2}[\omega, \omega]$ vanishes. Note that $K(\omega)$ is the curvature of the above connection θ', where θ is the homomorphism corresponding to the Cartan geometry defined by ω.

Notice that the proof of Proposition 3.4 still works for generalized holomorphic Cartan geometries and endows the principal G-bundle E_G with a holomorphic connection. This connection is flat if and only if $K(\omega) = 0$.

In this case we get a generalized (G, X)-structure on M in the sense of the following definition:

Definition 3.6. A generalized (G, X)-structure on a complex manifold M is given by an open cover $(U_i)_{i \in I}$ of M with maps $\phi_i : U_i \longrightarrow X$ such that for each pair (i, j) and for each nontrivial connected component of $U_i \cap U_j$ there exists an element $g_{ij} \in G$ such that $\phi_i = g_{ij} \circ \phi_j$ and $g_{ij} g_{jk} g_{ki} = 1$.

We note that the cocycle condition $g_{ij} g_{jk} g_{ki} = 1$ is automatically satisfied if the element g_{ij} such that $\phi_i = g_{ij} \circ \phi_j$ is unique. It may be clarified that in the previous definition G/H and M might have different dimensions.

The proof of Theorem 2.2 adapts to this situation and give the following description:

Theorem 3.7. *Let $X = G/H$ be a Klein geometry and M a manifold. Then there is a canonical one-to-one correspondence between the following structures.*

  (1). *A generalized (G,X)-structure on M.*
  (2). *A fiber bundle E_X over M with structure group G and fiber X, equipped with a flat connection and a global section.*

(3). *A principal H-bundle E_H over M and a flat connection on the associated principal G-bundle $E_G = E_H \times^H G$.*

Proof. Let us first prove that (1) implies (2). Consider a cover of M by open subsets U_i and local charts $\phi_i : U_i \longrightarrow X$ such that on intersections $U_i \cap U_j$ there exists elements $g_{ij} \in G$ such that $\phi_i \circ \phi_j^{-1} = g_{ij}$ and $g_{ij}g_{jk}g_{ki} = 1$. Then g_{ij} is a locally constant cocycle with values in G and hence it defines a flat G-bundle E_G over M. Consider the associated flat bundle E_X, with fiber $X = G/H$. Then the maps ϕ_i are local sections of E_X which glue together in a global section of E_X. The flat connection of E_X is locally determined by the foliation $U_i \times \{x\}$, for $x \in X$.

Remark that, since the fiber X is the homogeneous space G/H, the existence of a global section of E_X implies that the structural bundle of E_G reduces to H. Equivalently E_G admits a H-principal subbundle E_H. Hence we proved that (1) also implies (3). The same proof shows also that (2) implies (3).

Let us finish the proof by showing that (3) implies (1). Consider a cover of M by simply connected open sets U_i. Since E_G is flat it must be trivial, isomorphic to $U_i \times G$, over all simply connected open sets U_i. In restriction to each U_i the subbundle E_H is defined by a map $\phi_i : U_i \longrightarrow X = G/H$. On $U_i \cap U_j$ the two maps ϕ_i and ϕ_j agree up to multiplication by the element $g_{ij} \in G$. This achieves the proof. $\qquad\square$

Moreover, one gets a developing map of a generalized (G, X)-structure on M. Indeed, if M is simply connected the flat bundle E_X in (2) of previous theorem is trivial and the global section is a map from M to G/H.

In general, one can consider the pull-back of the generalized (G, X)-structure to the universal cover $\widetilde{M}$ of M and the previous global section is the developing map dev of the generalized (G, X)-geometry. Hence for holomorphic generalized Cartan geometries, this is a holomorphic map dev from the universal cover $\widetilde{M}$ to X. Notice that the rank of the differential of the developing map coincides with the rank of ω.

If ω is an isomorphism on an open dense set of E_H we call the corresponding generalized Cartan geometry: *branched* Cartan geometry. In this case the Klein geometry G/H and M must have the same dimension. For a flat branched Cartan geometry, the maps ϕ_i in Definition 3.6 are branched holomorphic maps (e.g. their differential is an isomorphism at the generic point) and elements g_{ij} are unique. In this case the global section in point (2) of Theorem 3.7 is transverse to the flat connection at the generic point (e.g. over an open dense set in M). Equivalently, in point (3) of Theorem 3.7 the H-subbundle E_H of E_G is transverse to the flat connection at the generic point.

We note that a branched holomorphic Cartan geometry is defined by a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota_2} & \mathrm{At}(E_H) & \xrightarrow{q_H} & TM & \longrightarrow & 0 \\
& & \| & & \downarrow{\theta} & & \downarrow{\phi} & & \\
0 & \longrightarrow & \mathrm{ad}(E_H) & \xrightarrow{\iota_1} & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{ad}(E_G)/\mathrm{ad}(E_H) & \longrightarrow & 0
\end{array}
\tag{3.6}
$$

of holomorphic vector bundles on M, such that θ is an isomorphism over a nonempty open subset of M; the homomorphism ϕ in (3.6) is induced by θ.

3.4. **Branched affine and projective structures.** A *holomorphic affine structure* (or equivalently *holomorphic affine connection*) on a complex manifold M of dimension d is a holomorphic Cartan geometry of type $(\mathbb{C}^d \rtimes \mathrm{GL}(d,\mathbb{C}), \mathrm{GL}(d,\mathbb{C}))$ (see also [MM, Sh]). A branched holomorphic Cartan geometry of type $(\mathbb{C}^d \rtimes \mathrm{GL}(d,\mathbb{C}), \mathrm{GL}(d,\mathbb{C}))$ will be called a *branched holomorphic affine structure* or a *branched holomorphic affine connection*.

We also recall that a *holomorphic projective structure* (or a *holomorphic projective connection*) on a complex manifold M of dimension d is a holomorphic Cartan geometry of type $(\mathrm{PGL}(d+1,\mathbb{C}), Q)$, where $Q \subset \mathrm{PGL}(d+1,\mathbb{C})$ is the maximal parabolic subgroup that fixes a given point for the standard action of $\mathrm{PGL}(d+1,\mathbb{C})$ on $\mathbb{C}P^d$ (the space of lines in $\mathbb{C}^{d+1}$). In particular, there is a standard holomorphic projective structure on $\mathrm{PGL}(d+1,\mathbb{C})/Q = \mathbb{C}P^d$ (see also [Sh, OT]).

We will call a branched holomorphic Cartan geometry of type $(\mathrm{PGL}(d+1,\mathbb{C}), Q)$ a *branched holomorphic projective structure* or a *branched holomorphic projective connection*.

Proposition 3.8. *Every compact complex projective manifold M admits a branched flat holomorphic projective structure.*

Proof. Let M be a compact complex projective manifold of complex dimension d. Then there exists a finite surjective algebraic, hence holomorphic, morphism

$$\gamma : M \longrightarrow \mathbb{C}P^d.$$

Indeed, one proves that the smallest integer N for which there exists a finite morphism f from M to $\mathbb{C}P^N$ is d. If $N > d$, then there exists $P \in \mathbb{C}P^N \setminus f(M)$; now consider the projection $\pi : \mathbb{C}P^N \setminus \{P\} \longrightarrow \mathbb{C}P^{N-1}$. The fibers of $\pi \circ f$ must be finite (otherwise $f(M)$ would contain a line through P, hence P). Since $\pi \circ f$ is a proper morphism with finite fibers, it must be finite.

Now we can pull back the standard holomorphic projective structure on $\mathbb{C}P^d$ using the above map γ to get a branched holomorphic projective structure on M. $\square$

Proposition 3.9.

  (i) *Simply connected compact complex manifolds do not admit any branched flat holomorphic affine structure.*
  (ii) *Simply connected compact complex manifolds admitting a branched flat holomorphic projective structure are Moishezon.*

Proof. (i) If, by contradiction, a simply connected compact complex manifold M admits a branched flat holomorphic affine structure, then the developing map $dev : M \longrightarrow \mathbb{C}^d$ is holomorphic and nonconstant, which is a contradiction.

(ii) If M is a simply connected manifold of complex dimension d admitting a branched flat holomorphic projective structure, then its developing map is a holomorphic map $dev : M \longrightarrow \mathbb{C}P^d$ which is a local biholomorphism away from a divisor D in M. Thus, the

algebraic dimension of M must be d; consequently, M is Moishezon (e.g. meromorphic functions on M separate points in general position) [Mo].   □

Since any given compact Kähler manifold is Moishezon if and only if it is projective [Mo], one gets the following:

Corollary 3.10. *A non-projective simply connected compact Kähler manifold does not admit any branched flat holomorphic projective structure.*

In particular, non-projective $K3$ surfaces do not admit any branched flat holomorphic projective structure.

Notice that the classification of compact complex surfaces bearing branched holomorphic affine connections and branched holomorphic projective connections is not known. In particular, we do not know if every compact complex surface bearing a branched holomorphic affine connection (respectively, a branched holomorphic projective connection) also admits a branched flat holomorphic affine connection (respectively, a branched flat holomorphic projective connection). Recall that in the classical (unbranched) setting, the corresponding questions were settled and answered in a positive way in [IKO, KO, KO2].

Also we do not know the classification of compact complex surfaces bearing *branched holomorphic Riemannian metrics*.

Recall that a branched holomorphic Riemannian metric on a complex manifold of dimension d is a branched Cartan geometry with model (G, H), where $H = \mathrm{O}(d, \mathbb{C})$ and $G = \mathbb{C}^d \rtimes \mathrm{O}(d, \mathbb{C})$.

A natural related question is to address the corresponding Cartan equivalence problem in the branched case: among the holomorphic sections of the bundle of complex quadratic forms $S^2(T^*M)$ which are nondegenerate over an open dense set in M, characterize those who define a branched holomorphic Riemannian metric on (entire) M. A more general question is to characterize holomorphic symmetric differentials (see the study in [BO, BO1, BO2]) which are given by branched holomorphic Cartan geometries.

4. Holomorphic Connections and Calabi-Yau manifolds

Recall that Kähler Calabi–Yau manifolds are compact complex Kähler manifolds with the property that the first Chern class (with real coefficients) of the holomorphic tangent bundle vanishes. By Yau's theorem proving Calabi's conjecture, those manifolds admit Kähler metrics with vanishing Ricci curvature [Ya]. Compact Kähler manifolds admitting a holomorphic affine connection have vanishing real Chern classes [At]; it was proved in [IKO] using Yau's result that they must admit finite unramified coverings which are complex tori.

The aim of this section is to give a simple proof of the following result of [BD] in the case of the *projective* simply connected Calabi-Yau manifolds and to deduce some consequences about generalized Cartan geometries on those manifolds (Corollary 4.6).

Theorem 4.1 ([BD])**.** *Let M be a compact simply connected Calabi-Yau manifold and E a holomorphic vector bundle over M. If E is equipped with a holomorphic connection ∇, then E is trivial and the connection ∇ coincides with the trivialization.*

Corollary 4.2. *Let P be a holomorphic G-principal bundle over a compact simply connected Calabi-Yau manifold M. Assume that complex Lie group G is semi-simple or simply connected. Then, if P admits a holomorphic connection, then P is trivial.*

Let us first show that Theorem 4.1 implies the Corollary.

Proof. Let $\rho : G \longrightarrow \mathrm{GL}(N, \mathbb{C})$ be a linear representation of G with discrete kernel. The corresponding Lie algebra representation ρ' is an injective Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(\mathfrak{N}, \mathbb{C})$. For G simply-connected those representations exist by Ado's theorem. For G complex semi-simple those representations do also exist (see Theorem 3.2, chapter XVII in [Ho]).

Consider the associated holomorphic vector bundle P_ρ with fiber type $\mathbb{C}^N$. Then P_ρ inherits a holomorphic connection ∇_ρ and, by Theorem 4.1, this connection ∇_ρ must be flat. Since the curvature of ∇_ρ is the image of the curvature of ∇ through ρ' and ρ' is injective, it follows that ∇ is also flat. Since M is simply-connected the flat principal bundle P has to be trivial. $\qquad\square$

In order to see that the hypothesis on G is necessary, let us prove the following proposition inspired by a construction of non integrable holomorphic one-forms in [Br].

Proposition 4.3. *There exists a nontrivial holomorphic principal elliptic bundle over a projective $K3$ surface which admits a non-flat holomorphic connection, but does not admit any flat holomorphic connection.*

Proof. Let us first remark that there exists projective $K3$-surfaces with holomorphic 2-forms Ω whose periods belong to a lattice in $\mathbb{C}$. Indeed, one can consider $\mathbb{C}^2$ and let Λ be the standard lattice generated by $(1, 0), (\sqrt{-1}, 0), (0, 1), (0, \sqrt{-1})$. Consider the involution

$$i : \mathbb{C}^2/\Lambda \longrightarrow \mathbb{C}^2/\Lambda$$

of the torus $Y = \mathbb{C}^2/\Lambda$ defined by $(x, y) \longmapsto (-x, -y)$. The fixed points of i are the 16 points of order 2 of Y. Let S be the blow-up of Y/i at these 16 points. This smooth complex projective surface S is a K3 surface (in particular, it is simply connected): a holomorphic 2-form Ω on it is given by the pull-back of the constant form $dz_1 \wedge dz_2$ on $\mathbb{C}^2$.

The cohomology group $H^2(S, \mathbb{Z})$ is generated by the 16 copies of $\mathbb{C}P^1$ (the exceptional divisors for the above mentioned blow-up) and by the generators of $H^2(Y, \mathbb{Z})$. Now the evaluation of Ω on each exceptional divisor is zero because the form Ω is pulled back from Y/i and hence vanishes on the exceptional divisors. On each generator of $H^2(Y, \mathbb{Z})$ the evaluation of Ω is one of $1/4, -1/4, \sqrt{-1}/4, -\sqrt{-1}/4$. Therefore, the periods of Ω lie in a lattice in $\mathbb{C}$.

Now consider any projective K3 surface S with a nontrivial holomorphic 2-form Ω whose periods belong to a lattice Λ in $\mathbb{C}$. The elliptic curve $\mathbb{C}/\Lambda$ will be denoted by E.

We can choose an open cover $\{U_i\}$ of S, in analytic topology, such that all U_i and all connected components of $U_i \cap U_j$ are contractible. Then on each U_i there exists a holomorphic one-form ω_i such that $\Omega_i = d\omega_i$. On intersections $U_i \cap U_j$, we get $\omega_i - \omega_j = dF_{ij}$, with F_{ij} a holomorphic function defined on $U_i \cap U_j$.

On intersections $U_i \cap U_j \cap U_k$, the sum $F_{ij} + F_{jk} + F_{ki}$ is a locally constant cocycle which represents the class of Ω in $H^2(S, \mathbb{C})$. Hence we can choose the forms ω_i and the associated functions F_{ij} in such a way that $F_{ij} + F_{jk} + F_{ki}$ belongs to the lattice of periods Λ for every triple i, j, k.

Consider then every holomorphic function F_{ij} as taking values in the translations group of E and form the holomorphic principal elliptic bundle $W_E \longrightarrow E$ with typical fiber E associated to this one-cocycle. The local forms on $U_i \times E$ which are $\pi_1^*(\omega_i) + \pi_2^* dt$, where π_1 and π_2 are the projections on the first and the second factor respectively and t is a translation invariant coordinate on the elliptic curve, glue to a global holomorphic one-form ω which is a holomorphic connection on this principal elliptic bundle W_E. The differential $d\omega$ projects on S to Ω and does not vanish. Hence, the holomorphic connection ω is not flat: its curvature is Ω.

Moreover, the above holomorphic principal bundle W_E with fiber type E does not admit any holomorphic flat connection. Indeed, since the base S is simply connected if there exists a flat connection on W_E, then the holomorphic principal bundle W_E is trivial, biholomorphic to $E \times S \longrightarrow S$. In this case the total space of the bundle W_E must be Kähler and all holomorphic one-forms on W_E would be closed: a contradiction since $d\omega \neq 0$. $\qquad \square$

Let us go back to the proof of Theorem 4.1.

Let M be a compact Kähler manifold of complex dimension m endowed with a Kähler metric ω. Recall that the degree of a torsionfree coherent analytic sheaf F on M is defined to be

$$\operatorname{degree}(F) := \int_M c_1(\det F) \wedge \omega^{m-1} \in \mathbb{R},$$

where $\det F$ is the determinant line bundle for F [Ko, Ch. V, § 6]. This degree is well-defined. Indeed, any two first Chern forms for $\det F$ differ by an exact 2–form on M, and

$$\int_M (d\alpha) \wedge \omega^{m-1} = - \int_M \alpha \wedge d\omega^{m-1} = 0.$$

In fact, this show that degree is a topological invariant. Define

$$\mu(F) := \frac{\operatorname{degree}(F)}{\operatorname{rank}(F)} \in \mathbb{R},$$

which is called the *slope* of F. A torsionfree coherent analytic sheaf F on M is called *stable* (respectively, *semi-stable*) if for every coherent analytic subsheaf $V \subset F$ such the $\operatorname{rank}(V) \in [1, \operatorname{rank}(F) - 1]$, the inequality $\mu(V) < \mu(F)$ (respectively, $\mu(V) \leq \mu(F)$) holds (see [Ko, Ch. V, § 7]).

First recall the following very useful Lemma from [Bi]:

Lemma 4.4 ([Bi]). *Let E be a holomorphic vector bundle over a compact Calabi-Yau manifold. Assume that E admits a holomorphic connection, then E is semi-stable.*

Proof. We assume, by contradiction, that E is not semi-stable. Then there exists a maximal semi-stable coherent subsheaf W. This W is destabilizing : it has the property that for any

coherent subsheaf F of E such that $W \subset F$ we have the inequality of slopes $\mu(W) > \mu(F)$. This is equivalent to the property that for any subsheaf F' of E/W we have $\mu(W) > \mu(F')$.

Now we show that W is ∇-stable. We consider the second fundamental form

$$S : W \longrightarrow \Omega^1(M) \otimes E/W$$

which associates to any local section s of W the projection of $\nabla.s$ on E/W.

Assume, by contradiction, that the image $S(W)$ is nontrivial. Since W is semi-stable and $S(W)$ is a quotient of W we get $\mu(S(W)) \geq \mu(W)$. On the other hand, $\mu(S(W)$ is at most the slope of the maximal semi-stable subsheaf of $\Omega^1(M) \otimes E/W$ which is less than $\mu_1 + \mu_2$, with μ_1 the slope of the maximal semi-stable subsheaf of $\Omega^1(M)$ and μ_2 the slope of the maximal semi-stable subsheaf of E/W. But the holomorphic tangent bundle of a Calabi-Yau manifold is semi-stable, meaning that all quotients of TM have nonnegative slope. By duality, this implies that any subsheaf of $\Omega^1(M)$ have nonpositive slope and, consequently, $\mu_1 \leq 0$. On the other hand, we have seen that $\mu_2 < \mu(W)$.

It follows that $\mu(W) \leq \mu(S(W)) \leq \mu_1 + \mu_2 < \mu(W)$: a contradiction.

This implies $S(W)$ is trivial and, thus W is ∇-stable.

Then the following Lemma 4.5 will finish the proof. Indeed, Lemma 4.5 implies that W is a subbundle endowed with a holomorphic connection. Then degree of W must be zero and we get $\mu(W) = \mu(E) = 0$: a contradiction. $\qquad\square$

Lemma 4.5. *Let E be a holomorphic vector bundle over a complex manifold M and $W \subset E$ a coherent subsheaf. Assume that E admits a holomorphic connection such that W is ∇-stable. Then W is a subbundle of E.*

Proof. Pick any $m \in M$. Since W is a coherent subsheaf one can consider local sections s_i in the neighborhood of m such that $s_i(m)$ span the fiber W_m at m. By Nakayama's lemma (see [GH], p. 680) the sections s_i span W in the neighborhood of m. We are left to prove that the sections s_i are linearly independent over the ring of local holomorphic functions.

Assume, by contradiction, that there exists a a nontrivial relation $\sum_i f_i \cdot s_i = 0$, with f_i local holomorphic functions.

It follows that $f_i(m) = 0$, for all i. Consider the integer $n > 0$ which is the minimum of vanishing orders of f_i at m (we can assume that this minimum is the vanishing order of f_1 at m).

We will construct a nontrivial relation for which the minimum of vanishing orders is strictly smaller.

In local coordinates, one can find a directional derivative (in the direction of some local vector field v), such that the holomorphic function $df_1(v)$, has vanishing order $< n$ at m. Then applying ∇ to our relation we get

$$\nabla_v(\sum_i f_i \cdot s_i) = \sum_i s_i df_i(v) + \sum_i f_i \nabla_v(s_i) = 0\,.$$

Since W is ∇-stable, for all i one gets local holomorphic functions g_{ij} such that $\nabla_v s_i = \sum_j g_{ij} s_j$.

We get then the relation

$$\sum_i s_i(df_i(v) + f_i \sum_j g_{ij}s_j) = 0$$

for which the minimum of the vanishing orders of the coefficients at m is $< n$.

This is a contradiction. Consequently, W is locally free. It is a subbundle of E. $\qquad\square$

To finish the proof of Theorem 4.1 we make use of a result of Simpson ([Si2, p. 40, Corollary 3.10]) which states that:

A holomorphic vector bundle E on a complex projective manifold M admits a flat holomorphic connection if E is semi-stable with degree$(E) = 0 = ch_2(E) \wedge \omega^{m-2}$. Notice that in this statement ω is a rational Kähler class of M and ch_2 is the second Chern class.

Consequently, if the Calabi–Yau manifolds M in Theorem 4.1 is *projective*, and the cohomology class of the Kähler form ω is rational, then that E admits a flat holomorphic connection, because E is semi-stable with vanishing Chern classes. Moreover since M is simply connected all flat bundles on M must be trivial. Therefore, E is the trivial vector bundle. Since $H^0(M, \Omega_M^1) = 0$, the trivial holomorphic vector bundle has exactly one holomorphic connection, namely the trivial connection. Hence ∇ is the trivial connection on E.

Theorem 4.1 is now proved. This theorem has the following consequence.

Corollary 4.6. *Let M be a compact simply-connected Calabi-Yau manifold endowed with a generalized holomorphic Cartan geometry (E_H, ω) modelled on (G, H), with G complex semi-simple Lie group (or G simply connected complex Lie group). Then (E_H, ω) is flat. If ω is a branched Cartan geometry, G/H must be compact and the branching set must be nontrivial.*

Proof. Let E_G be the associated G-principal constructed from E_H by extension of the structure group. Now Corollary 4.2 shows that the induced connection $\widetilde{\omega}$ on E_G is flat. Consequently, the geometry (E_H, ω) is flat.

Assume that the Cartan geometry is branched. The associated developing map is a holomorphic map $dev : M \longrightarrow G/H$, which is a submersion away from the branching set. Since M is compact, the image of the developing map must be closed (and open) and it is all of the model G/H. That implies that G/H is compact.

Assume, by contradiction, that the branching set is empty. Then the developing map is a local biholomorphism, so the developing map must be a covering map. It follows that M is also a homogeneous space. Consider global holomorphic vector fields $X_1, X_2, \cdots, X_n$ on M which are linearly independent and together span the holomorphic tangent bundle TM at the generic point. Then $X_1 \bigwedge X_2 \bigwedge \cdots \bigwedge X_n$ is a holomorphic section of $\bigwedge^n TM$. The vanishing divisor of this section is the first Chern class of TM which is zero. Consequently, the section $X_1 \bigwedge X_2 \bigwedge \cdots \bigwedge X_n$ does not vanish, and the vector fields $(X_1, \cdots, X_n)$ trivialize the holomorphic tangent bundle TM. By Wang theorem, the complex manifold M is biholomorphic to a quotient of a complex connected Lie group L by a lattice $\Gamma \subset L$. Now since M is Kähler, and it is biholomorphic to L/Γ, it follows that M must be a complex

torus, in particular the fundamental group of M is infinite. This is a contradiction because M is simply connected. $\qquad\square$

From Corollary 4.6 and from Corollary 3.10 we get the following:

Corollary 4.7. *Non-projective compact simply connected Calabi–Yau manifolds do not admit branched holomorphic projective structures.*

Observe that Corollary 4.2 and Corollary 4.6 stand for all complex Lie groups G which admit linear representations with discrete kernel.

REFERENCES

[AM]   D. V. Alekseevsky and P. W. Michor, Differential geometry of Cartan connections, *Publ. Math. Debrecen*, **47(3-4)**, (1995), 349–375.

[At]   M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957), 181–207.

[BD]   I. Biswas and S. Dumitrescu, Branched holomorphic Cartan geometries and Calabi-Yau manifolds, *Int. Math. Res. Not.*, DOI: 10.1093/imrn/rny003, arxiv:1706.04407.

[Bi]   I. Biswas, Vector bundles with holomorphic connection over a projective manifold with tangent bundle of nonnegative degree, *Proc. Amer. Math. Soc.* **126** (1998), 2827–2834.

[BM]   I. Biswas and B. McKay, Holomorphic Cartan geometries, Calabi–Yau manifolds and rational curves, *Diff. Geom. Appl.* **28** (2010), 102–106.

[BO]   F. Bogomolov, B. De Oliveira, Symmetric differentials of rank 1 and holomorphic maps, *Pure and Appl. Math. Quartely* **7** (2011), 1085–1104.

[BO1]   F. Bogomolov, B. De Oliveira, Closed symmetric 2-differentials of the 1st kind, *Pure and Appl. Math. Quartely* **9** (2013), 613–643.

[BO2]   F. Bogomolov, B. De Oliveira, Local structure of closed symmetric 2-differentials, *Foliation theory in algebraic geometry*, 53–71, Simons Symp., Springer, Cham, 2016.

[Br]   M. Brunella, On holomorphic forms on compact complex threefolds, *Comment. Math. Helv.* **74** (1999), 642–656.

[Co]   K. Corlette, Flat G-bundles with canonical metrics, *Jour. Diff. Geom.* **28** (1988), 361–382.

[Du]   S. Dumitrescu, Homogénéité locale pour les métriques riemanniennes holomorphes en dimension 3, *Ann. Instit. Fourier* **57** (2007), 739–773.

[DZ]   S. Dumitrescu and A. Zeghib, Global rigidity of holomorphic Riemannian metrics on compact complex 3-manifolds, *Math. Ann.* **345** (2009), 53–81.

[Eh]   C. Ehresmann, Sur les espaces localement homogènes, *L'Enseign. Math.* **35** (1936), 317–333.

[Gh]   E. Ghys, Déformations des structures complexes sur les espaces homogènes de $SL(2, \mathbb{C})$, *Jour. Reine Angew. Math.* **468**, (1995), 113–138.

[GH]   P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley Classics Library, (1994).

[Gu]   R. C. Gunning, *On uniformization of complex manifolds: the role of connections*, Princeton Univ. Press, 1978.

[Ho]   G. P. Hochschild, *Basic theory of algebraic groups and Lie algebras*, Springer, 1981.

[IKO]   M. Inoue, S. Kobayashi and T. Ochiai, Holomorphic affine connections on compact complex surfaces, *Jour. Fac. Sci. Univ. Tokyo*, Sect. IA Math. **27** (1980), 247–264.

[Ko]   S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, Princeton, NJ, Iwanami Shoten, Tokyo, 1987.

[KO]   S. Kobayashi and T. Ochiai, Holomorphic projective structures on compact complex surfaces, *Math. Ann.* **249** (1980), 75–94.

[KO2]   S. Kobayashi and T. Ochiai, Holomorphic projective structures on compact complex surfaces II, *Math. Ann.* **255** (1981), 519–521.

[Ma1]   R. Mandelbaum, Branched structures on Riemann surfaces, *Trans. Amer. Math. Soc.* **163** (1972), 261–275.

[Ma2]   R. Mandelbaum, Branched structures and affine and projective bundles on Riemann surfaces, *Trans. Amer. Math. Soc.* **183** (1973), 37–58.

[Mo]   B. Moishezon, On n dimensional compact varieties with n independent meromorphic functions, *Amer. Math. Soc. Transl.* **63** (1967), 51–77.

[MM]   R. Molzon and K. P. Mortensen, The Schwarzian derivative for maps between manifolds with complex projective connections, *Trans. Amer.Math. Soc.* **348** (1996), 3015–3036.

[OT]   V. Ovsienko and S. Tabachnikov, *Projective Differential Geometry Old and New*, **165**, Cambridge Univ. Press, 2005.

[Sh]   R. W. Sharpe, *Differential Geometry : Cartan's Generalization of Klein's Erlangen Program*, Graduate Text Math., 166, Springer-Verlag, New York, Berlin, Heidelberg, 1997.

[Si1]   C. T. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, *Jour. Amer. Math. Soc.* **1** (1988), 867–918.

[Si2]   C. Simpson, Higgs bundles and local systems, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95.

[StG]   H. P. de Saint Gervais, *Uniformization of Riemann Surfaces. Revisiting a hundred year old theorem*, E.M.S., 2016.

[Wa]   H.-C. Wang, Complex Parallelisable manifolds, *Proc. Amer. Math. Soc.* **5** (1954), 771–776.

[Ya]   S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978), 339–411.

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