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LOCAL CONTROLLABILITY OF REACTION-DIFFUSION SYSTEMS AROUND NONNEGATIVE STATIONARY STATES

KÉVIN LE BALC'H

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ABSTRACT. We consider a $n \times n$ nonlinear reaction-diffusion system posed on a smooth bounded domain Ω of \mathbb{R}^N . This system models reversible chemical reactions. We act on the system through m controls ($1 \leq m < n$), localized in some arbitrary nonempty open subset ω of the domain Ω . We prove the local exact controllability to nonnegative (constant) stationary states in any time $T > 0$. A specificity of this control system is the existence of some invariant quantities in the nonlinear dynamics that prevents controllability from happening in the whole space $L^\infty(\Omega)^n$. The proof relies on several ingredients. First, an adequate affine change of variables transforms the system into a cascade system with second order coupling terms. Secondly, we establish a new null-controllability result for the linearized system thanks to a spectral inequality for finite sums of eigenfunctions of the Neumann Laplacian operator, due to David Jerison, Gilles Lebeau and Luc Robbiano and precise observability inequalities for a family of finite dimensional systems. Thirdly, the source term method, introduced by Yuning Liu, Takéo Takahashi and Marius Tucsnak, is revisited in a L^∞ -context. Finally, an appropriate inverse mapping theorem enables to go back to the nonlinear reaction-diffusion system.

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1. INTRODUCTION

1.1. **Free system.** We consider the following reversible chemical reaction:

$$(1) \quad \alpha_1 A_1 + \cdots + \alpha_n A_n \rightleftharpoons \beta_1 A_1 + \cdots + \beta_n A_n,$$

where $n \in \mathbb{N}^*$, A_1, \dots, A_n denote n chemical species and $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)$ belongs to $(\mathbb{N})^n$ and are such that for every $1 \leq i \leq n$, $\alpha_i \neq \beta_i$.

For $1 \leq i \leq n$, let $u_i(t, \cdot) : \Omega \rightarrow \mathbb{R}$ be the concentration of the chemical component A_i at time t . By using the law of mass action and Fick's law, $(u_i)_{1 \leq i \leq n}$ satisfies the following reaction-diffusion system:

$$(2) \quad \begin{cases} \forall 1 \leq i \leq n, \\ \partial_t u_i - \underbrace{d_i \Delta u_i}_{\text{diffusion}} = \underbrace{(\beta_i - \alpha_i) \left(\prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k} \right)}_{\text{reaction}} & \text{in } (0, T) \times \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ u_i(0, \cdot) = u_{i,0} & \text{in } \Omega, \end{cases}$$

where $T \in (0, +\infty)$, Ω is a bounded, connected, open subset of \mathbb{R}^N (with $N \geq 1$) of class C^2 , ν is the outer unit normal vector to $\partial\Omega$ and for every $1 \leq i \leq n$, $d_i \in (0, +\infty)$ is the diffusion coefficient of the chemical species A_i .

In general, global existence of classical solutions (in the sense of [35, Definition (1.5)]) or weak solutions (in the sense of [35, Definition (5.12)]) for (2) is an open problem.

- For particular semilinearities with a so-called triangular structure (see [35, Section 3.3]), classical solutions exist in the time interval $[0, +\infty)$ and are unique. For example, take $n = 2$, $\alpha_1 \geq 1$, $\beta_2 = 1$, $\alpha_2 = \beta_1 = 0$ and apply [35, Theorem 3.1].
- For at most quadratic nonlinearities, global existence of weak solutions holds (see [35, Theorem 5.12]). For instance, take $n = 4$, $\alpha_1 = \alpha_3 = \beta_2 = \beta_4 = 1$, $\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0$. For spatial dimension $N \geq 3$, the recent works [7] and [37] (inspired by the previous works [25] and [26]) prove that the solutions are bounded for bounded initial data, which ensure global existence of classical solutions.
- Without a priori L^1 -bound on the nonlinearities, a challenging problem is to understand whether global solutions exist. For example, take $n = 2$, $\alpha_1 = \beta_2 = 2$, $\beta_1 = \alpha_2 = 3$ (see [35, Problem 1]).

Let us also mention that global existence of renormalized solutions holds in all cases for (2) (see [19]).

1.2. **Control system and open question.** We assume that one can act on the system through controls localized on a nonempty open subset ω of Ω . From a chemical viewpoint, it means that one can add or remove chemical species at a specific location of the domain Ω . More precisely, let

$$(3) \quad J \subset \{1, \dots, n\} \text{ and } := \#J < n \text{ be the number of controls.}$$

We consider the control system:

$$(NL-U) \quad \begin{cases} \forall 1 \leq i \leq n, \\ \partial_t u_i - d_i \Delta u_i = \\ \quad (\beta_i - \alpha_i) \left(\prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k} \right) + h_i 1_\omega 1_{i \in J} & \text{in } (0, T) \times \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ u_i(0, \cdot) = u_{i,0} & \text{in } \Omega, \end{cases}$$

where $1_{i \in J} := 1$ if $i \in J$ and 0 if $i \notin J$. Here, $(u_i(t, \cdot))_{1 \leq i \leq n} : \Omega \rightarrow \mathbb{R}^n$ is the *state* to be controlled, $(h_i(t, \cdot))_{i \in J} : \Omega \rightarrow \mathbb{R}^m$ is the *control input* supported in ω .

Let $(u_i^*)_{1 \leq i \leq n}$ be a nonnegative stationary state of (2) i.e.

$$(4) \quad \forall 1 \leq i \leq n, \quad u_i^* \in [0, +\infty) \text{ and } \prod_{k=1}^n u_k^{*\alpha_k} = \prod_{k=1}^n u_k^{*\beta_k}.$$

Note that the nonnegative stationary solutions of (2) do not depend on the space variable (see Proposition A.1 in Appendix A.1). Thus, it is not restrictive to assume that $(u_i^*)_{1 \leq i \leq n} \in [0, +\infty)^n$.

The question we ask is the following one: For a given initial condition $(u_{i,0})_{1 \leq i \leq n}$, does there exist $(h_i)_{i \in J}$ such that the solution $(u_i)_{1 \leq i \leq n}$ of (NL-U) satisfies

$$(5) \quad \forall i \in \{1, \dots, n\}, \quad u_i(T, \cdot) = u_i^*?$$

Under appropriate assumptions (see Assumption 2.4 and Assumption 3.2 below), we prove the controllability of (NL-U), in an appropriate subspace of $L^\infty(\Omega)^n$, locally around $(u_i^*)_{1 \leq i \leq n}$, with controls in $L^\infty((0, T) \times \Omega)^m$ (see Theorem 4.2 below).

By an adequate affine transformation, the proof relies on the study of the null-controllability of an equivalent cascade system with second order coupling terms (see Section 3.1 below).

1.3. Bibliographical comments. In this section, we recall some known results about the null-controllability of linear and semilinear parabolic systems with Neumann boundary conditions. We investigate the case of one control, i.e., $m = 1$ in this section to simplify. We introduce the notation

$$Q_T := (0, T) \times \Omega.$$

1.3.1. Linear results. The null-controllability of the heat equation was proved independently by Gilles Lebeau, Luc Robbiano in 1995 (see [30], [24] and the survey [29]) and by Andrei Fursikov, Oleg Imanuvilov in 1996 (see [20] and [17]).

Theorem 1.1. [20, Chapter I, Theorem 2.1]

For every $z_0 \in L^2(\Omega)$, there exists $h \in L^2(Q_T)$ such that the solution z of

$$\begin{cases} \partial_t z - \Delta z = h 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ z(0, \cdot) = z_0 & \text{in } \Omega, \end{cases}$$

satisfies $z(T, \cdot) = 0$.

In the work [16], Enrique Fernández-Cara, Manuel González-Burgos, Sergio Guerrero and Jean-Pierre Puel prove the same null-controllability result for more general parabolic operators, i.e., $\partial_t z - \Delta z + B(t, x) \cdot \nabla z + a(t, x)z$ with $a \in L^\infty(Q_T)$, $B \in L^\infty(Q_T)^n$ and linear Robin conditions, i.e., $\frac{\partial z}{\partial \nu} + \beta(t, x)z = 0$ on $(0, T) \times \partial\Omega$ with $\beta \in L^\infty((0, T) \times \partial\Omega; \mathbb{R}^+)$.

Then, the null-controllability of coupled linear parabolic systems has been a challenging issue. Let us now focus on a *cascade* system with *coupling terms of zero order*. The following result comes from an easy adaptation of Manuel González-Burgos and Luz de Teresa's proof in the case of Dirichlet boundary conditions to Neumann boundary conditions.

Theorem 1.2. [22, Theorem 1.1]

Let $(d_i)_{1 \leq i \leq n} \in (0, +\infty)^n$, $(a_{i,j})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ and assume that $a_{i+1,i} \neq 0$ for every $1 \leq i \leq n$. Then, for every $(z_{i,0})_{1 \leq i \leq n} \in L^2(\Omega)^n$, there exists $h \in L^2(Q_T)$ such that the solution $(z_i)_{1 \leq i \leq n}$ of

$$\begin{cases} \partial_t z_1 - d_1 \Delta z_1 = \sum_{k=1}^n a_{1,k} z_k + h 1_\omega & \text{in } (0, T) \times \Omega, \\ \partial_t z_i - d_i \Delta z_i = a_{i,i-1} z_{i-1} & \text{in } (0, T) \times \Omega, \quad 2 \leq i \leq n, \\ \frac{\partial z_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \quad 1 \leq i \leq n, \\ z_i(0, \cdot) = z_{i,0} & \text{in } \Omega, \quad 1 \leq i \leq n, \end{cases}$$

satisfies $z_i(T, \cdot) = 0$ for every $1 \leq i \leq n$.

Roughly speaking, the component z_1 is controlled by the control input h , the component z_2 is controlled by z_1 thanks to the coupling term $a_{2,1} z_1$, \dots , the component z_n is controlled by z_{n-1} thanks to the coupling term $a_{n,n-1} z_{n-1}$.

The following result for a 2×2 linear parabolic system with a ‘‘cross-diffusion’’ term is due to Sergio Guerrero. We introduce the function space

$$L_\sigma^2(\Omega) := \left\{ z \in L^2(\Omega) ; \int_\Omega z = 0 \right\}.$$

Theorem 1.3. [21, Theorem 1]

Let $(d_1, d_2) \in (0, +\infty)^2$, $a, b \in \mathbb{R}$. Then, for every $(z_{1,0}, z_{2,0}) \in L^2(\Omega) \times L_\sigma^2(\Omega)$, there exists $h \in L^2(Q_T)$ such that the solution (z_1, z_2) of

$$\begin{cases} \partial_t z_1 - d_1 \Delta z_1 = a z_1 + b z_2 + h 1_\omega & \text{in } (0, T) \times \Omega, \\ \partial_t z_2 - d_2 \Delta z_2 = \Delta z_1 & \text{in } (0, T) \times \Omega, \\ \frac{\partial z_1}{\partial \nu} = \frac{\partial z_2}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ (z_1, z_2)(0, \cdot) = (z_{1,0}, z_{2,0}) & \text{in } \Omega, \end{cases}$$

satisfies $(z_1, z_2)(T, \cdot) = 0$.

The main difference with Theorem 1.2 is that now, the *coupling term is of second order* Δz_1 . The condition $\int_\Omega z_{2,0} = 0$ is necessary because by integrating with respect to the space variable the second equation of the system, we get

$$\frac{d}{dt} \int_\Omega z_2(t, \cdot) = 0.$$

In particular if $z_2(T, \cdot) = 0$, then we need $\int_\Omega z_{2,0} = 0$.

In view of Theorem 1.2 and Theorem 1.3, a natural question is: are cascade cross-diffusion systems (of arbitrary size $n \geq 2$) null-controllable? A byproduct of this article is a positive answer to this question.

Theorem 1.4. Let $(d_i)_{1 \leq i \leq n} \in (0, +\infty)^n$, $(a_{i,j})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ and assume that $a_{2,1} \neq 0$. Then, for every $(z_{i,0})_{1 \leq i \leq n} \in L^2(\Omega) \times L_\sigma^2(\Omega)^{n-1}$, there exists $h \in L^2(Q_T)$ such that the solution $(z_i)_{1 \leq i \leq n}$ of

$$\begin{cases} \partial_t z_1 - d_1 \Delta z_1 = \sum_{k=1}^n a_{1,k} z_k + h 1_\omega & \text{in } (0, T) \times \Omega, \\ \partial_t z_2 - d_2 \Delta z_2 = a_{2,1} z_1 & \text{in } (0, T) \times \Omega, \\ \partial_t z_i - d_i \Delta z_i = \Delta z_{i-1} & \text{in } (0, T) \times \Omega, \quad 3 \leq i \leq n, \\ \frac{\partial z_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \quad 1 \leq i \leq n, \\ z_i(0, \cdot) = z_{i,0} & \text{in } \Omega, \quad 1 \leq i \leq n, \end{cases}$$

satisfies $z_i(T, \cdot) = 0$ for every $1 \leq i \leq n$.

Theorem 1.4 is a particular case of Theorem 5.1 with $m = 1$ (see Section 5 below).

Remark 1.5. The proof strategies of Theorem 1.2 and Theorem 1.3 rely on global Carleman estimates. By combining these methods to obtain Theorem 1.4, we are facing the same difficulty as appearing in [15], i.e., we can only treat the case of $n \times n$

systems with $n \leq 5$. Inspired by the recent work [32] by Pierre Lissy and Enrique Zuazua based on the Lebeau-Robbiano method, we prove Theorem 1.4.

Remark 1.6. Note that the diffusion matrix $\begin{pmatrix} d_1 & 0 & \dots & \dots & 0 \\ 1 & d_2 & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & 1 & d_n \end{pmatrix}$ is diagonalizable

if and only if $d_i \neq d_j$ for every $i \neq j$. In this case, the Kalman condition [2, Theorem 5.3] (that can be easily extended to Neumann boundary conditions instead of Dirichlet boundary conditions) yields null-controllability for initial data in $L^2_\sigma(\Omega)^n$ which is smaller than $L^2(\Omega) \times L^2_\sigma(\Omega)^{n-1}$.

For a recent survey on the null-controllability of linear parabolic systems, see [3] and references therein.

1.3.2. *Semilinear results.* Let $(d_i) \in (0, +\infty)^n$, $(f_i)_{1 \leq i \leq n} \in C^\infty(\mathbb{R}^n; \mathbb{R})^n$, satisfying $f_i(0) = 0$ for every $1 \leq i \leq n$. For semilinear parabolic systems

$$(NL) \quad \forall 1 \leq i \leq n, \quad \begin{cases} \partial_t z_i - d_i \Delta z_i = f_i((z_j)_{1 \leq j \leq n}) + h_1 \mathbf{1}_\omega \mathbf{1}_{i \in \{1\}} & \text{in } (0, T) \times \Omega, \\ \frac{\partial z_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ z_i(0, \cdot) = z_{i,0} & \text{in } \Omega, \end{cases}$$

the usual strategy consists in deducing a *local null-controllability result* for (NL) from a (global) null-controllability result for the *linearized system* around $((\bar{z}_i)_{1 \leq i \leq n}, \bar{h}) = (0, 0)$

$$(L) \quad \forall 1 \leq i \leq n, \quad \begin{cases} \partial_t z_i - d_i \Delta z_i = \sum_{j=1}^n \partial_j f_i(0) z_j + h_1 \mathbf{1}_\omega \mathbf{1}_{i \in \{1\}} & \text{in } (0, T) \times \Omega, \\ \frac{\partial z_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ z_i(0, \cdot) = z_{i,0} & \text{in } \Omega. \end{cases}$$

In this article, we use the powerful *source term method*, introduced by Yuning Liu, Takéo Takahashi and Marius Tucsnak in [33]. This method enables to prove null-controllability in time T for

$$(L+S) \quad \begin{cases} \forall 1 \leq i \leq n \\ \partial_t z_i - d_i \Delta z_i = \sum_{j=1}^n \partial_j f_i(0) z_j + h_1 \mathbf{1}_\omega \mathbf{1}_{i \in \{1\}} + S_i(t, x) & \text{in } (0, T) \times \Omega, \\ \frac{\partial z_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ z_i(0, \cdot) = z_{i,0} & \text{in } \Omega, \end{cases}$$

where S_i ($1 \leq i \leq n$) has a prescribed decay rate at $t = T$, depending on the cost of null-controllability for (L). Then, for $(z_{i,0})_{1 \leq i \leq n}$ sufficiently small (in an appropriate norm), a fixed-point strategy in suitable spaces is applied to the map:

$$\mathcal{N} : (S_i)_{1 \leq i \leq n} \mapsto \left(f_i((z_j)_{1 \leq j \leq n}) - \sum_{j=1}^n \partial_j f_i(0) z_j \right)_{1 \leq i \leq n},$$

where $(z_i)_{1 \leq i \leq n}$ is the solution associated to the optimal control (in an appropriate norm) of (L+S). Consequently, the local null-controllability for (NL) comes from the null-controllability of **only one** linear system (L).

In this article, we adapt the source term method in a L^∞ -context in the following way.

- The source term method in L^2 enables to prove a strong observability inequality (see Corollary 6.4). This estimate looks like a global Carleman estimate (see for example [17, Lemma 1.3]), whereas the method to get it is very different.

- By using the Penalized Hilbert Uniqueness Method, introduced by Viorel Barbu in [4], we construct L^∞ -controls (see Theorem 7.1).
- We use another time the source term method in L^∞ (see Proposition 7.5).
- We conclude by an appropriate inverse mapping theorem (see Section 8).

For other results using the source term method, see for instance [6], [18] and [34].

Another strategy to get local controllability result for (NL), called the *Small L^∞ -perturbations method* is used in [1], [4], [28], [31] and [38]. This method requires the null-controllability of a family of linear parabolic systems. Thus, this type of result is proved by using global Carleman estimates that enable to treat parabolic operators as $\partial_t z - \Delta z + a(t, x)z$ with $a \in L^\infty(Q_T)$.

Nevertheless, for the linearized system around $((\bar{z}_i)_{1 \leq i \leq n}, \bar{h}) = (0, 0)$ of Section 3.2 (see below), a technical difficulty appears when we want to prove an observability inequality for the adjoint system by using global Carleman estimates when $n > 4$ (see Remark 1.5).

For degenerate cases (see Section 9.2), i.e., when (L) is not null-controllable, one can try to perform the *return method*, introduced by Jean-Michel Coron in [8] (see also [9, Chapter 6]). This method consists in finding a reference trajectory $((\bar{z}_i)_{1 \leq i \leq n}, \bar{h})$ verifying $z_i(0, \cdot) = z_i(T, \cdot) = 0$ ($1 \leq i \leq n$) of (NL) such that the linearized system of (NL) around $((\bar{z}_i)_{1 \leq i \leq n}, \bar{h})$ is null-controllable. By using the small L^∞ -perturbations method, we can obtain a local null-controllability result for (NL). See for instance [11], [10], [12] and [28].

Let us also mention the new method of [27] to prove the global null-controllability of reaction-diffusion systems of two species with only one control force by constructing controls of the heat equation behaving as odd regular functions.

2. DEFINITION AND GENERAL PROPERTIES OF THE TRAJECTORIES

In this section, we introduce the concept of *trajectory* of (NL-U) which requires a well-posedness result (see Definition-Proposition 2.2).

2.1. Usual notations. Let $k, l \in \mathbb{N}^*$. We denote by $\mathcal{M}_k(\mathbb{R})$ (respectively $\mathcal{M}_{k,l}(\mathbb{R})$) the algebra of matrices with k lines and k columns (respectively the algebra of matrices with k lines and l columns with entries in \mathbb{R}). The matrix $A^{\text{tr}} \in \mathcal{M}_{l,k}(\mathbb{R})$ denotes the transpose of the matrix $A \in \mathcal{M}_{k,l}(\mathbb{R})$. For $M \in \mathcal{M}_k(\mathbb{R})$, $Sp(M)$ is the set of complex eigenvalues of M : $Sp(M) := \{\lambda \in \mathbb{C}; \exists X \in \mathbb{C}^k \setminus \{0\}, MX = \lambda X\}$.

For $\tau > 0$, we introduce

$$Q_\tau := (0, \tau) \times \Omega.$$

For every $(a_1, \dots, a_n) \in \mathbb{R}^n$, we define

$$(6) \quad \forall 1 \leq i \leq n, f_i(a_1, \dots, a_n) := (\beta_i - \alpha_i) \left(\prod_{k=1}^n a_k^{\alpha_k} - \prod_{k=1}^n a_k^{\beta_k} \right),$$

$$(7) \quad F(a_1, \dots, a_n) := (f_i(a_1, \dots, a_n))_{1 \leq i \leq n}^{\text{tr}}.$$

We introduce

$$(8) \quad U := (u_1, \dots, u_n)^{\text{tr}}, \quad U^* := (u_1^*, \dots, u_n^*)^{\text{tr}}.$$

Up to a renumbering of $(u_i)_{1 \leq i \leq n}$, we can assume that $J = \{1, \dots, m\}$ where J is defined in (3). Hence, we define

$$(9) \quad H^J := (h_1, \dots, h_m, 0, \dots, 0)^{\text{tr}}.$$

We must be careful with the dependence on the constants appearing in the estimates with respect to T (when T is small). That is why, from now and until the end of the article, we assume that

$$(10) \quad T \in (0, 1).$$

Unless otherwise specified, we denote by C various positive constants varying from line to line.

2.2. Well-posedness results and definition of a trajectory. We define the function space

$$(11) \quad W_T := L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega))'),$$

that satisfies the continuous embedding

$$(12) \quad W_T \hookrightarrow C([0, T]; L^2(\Omega)).$$

The following result introduces the notion of solution for linear parabolic systems and provides estimates in terms of the initial data and the source term.

Definition-Proposition 2.1. *Let $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ a diagonalizable matrix such that $Sp(D) \subset (0, +\infty)$, $A \in \mathcal{M}_k(\mathbb{R})$, $U_0 \in L^2(\Omega)^k$, $S \in L^2(Q_T)^k$. The following Cauchy problem admits a unique weak solution $U \in W_T^k$*

$$(13) \quad \begin{cases} \partial_t U - D\Delta U = AU + S & \text{in } (0, T) \times \Omega, \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ U(0, \cdot) = U_0 & \text{in } \Omega. \end{cases}$$

This means that U is the unique function in W_T^k that satisfies the variational formulation

$$(14) \quad \forall V \in L^2(0, T; H^1(\Omega)^k), \\ \int_0^T (\partial_t U, V)_{(H^1(\Omega)^k)', H^1(\Omega)^k} + \int_{Q_T} D\nabla U \cdot \nabla V = \int_{Q_T} (AU + S) \cdot V,$$

and

$$(15) \quad U(0, \cdot) = U_0 \text{ in } L^2(\Omega)^k.$$

Moreover, there exists $C > 0$ independent of U_0 and S such that

$$(16) \quad \|U\|_{W_T^k} \leq C \left(\|U_0\|_{L^2(\Omega)^k} + \|S\|_{L^2(Q_T)^k} \right).$$

Finally, if $U_0 \in L^\infty(\Omega)^k$ and $S \in L^\infty(Q_T)^k$, then $U \in L^\infty(Q_T)^k$ and there exists $C > 0$ independent of U_0 and S such that

$$(17) \quad \|U\|_{L^\infty(Q_T)^k} \leq C \left(\|U_0\|_{L^\infty(\Omega)^k} + \|S\|_{L^\infty(Q_T)^k} \right).$$

The proof of Definition-Proposition 2.1 can be found in [28, Proposition 2.3].

The following result introduces the notion of trajectory associated to the nonlinear system (NL-U) (see Section 1.2).

Definition-Proposition 2.2. *Let $D = \text{diag}(d_1, \dots, d_n)$ with $d_i \in (0, +\infty)$. For every $U_0 \in L^\infty(\Omega)^n$, (U, H^J) (see (8) and (9)) is a **trajectory** of (NL-U) if*

- (1) $(U, H^J) \in \left(W_T \cap L^\infty(Q_T) \right)^n \times L^\infty(Q_T)^m$,
- (2) U is the (unique) solution of (NL-U). This means that U is the unique function in $\left(W_T \cap L^\infty(Q_T) \right)^n$ which satisfies

$$(18) \quad \forall V \in L^2(0, T; H^1(\Omega)^k), \\ \int_0^T (\partial_t U, V)_{(H^1(\Omega)^k)', H^1(\Omega)^k} + \int_{Q_T} D\nabla U \cdot \nabla V = \int_{Q_T} (F(U) + H^J 1_\omega) \cdot V,$$

with F defined in (7) and

$$(19) \quad U(0, \cdot) = U_0 \text{ in } L^\infty(\Omega)^k.$$

Moreover, (U, H^J) is a **trajectory** of (NL-U) **reaching** U^* (see (8)) in time T if

$$U(T, \cdot) = U^*.$$

The proof of the uniqueness of Definition-Proposition 2.2 comes from the fact that F is locally Lipschitz on \mathbb{R}^n (see the proof of [27, Definition-Proposition 2.4]).

2.3. Invariant quantities of the nonlinear dynamics. In this section, we show that in the system (NL-U) (see Section 1.2), some quantities are invariant. They impose some restrictions on the initial condition, for the controllability results.

Proposition 2.3. *Let $U_0 \in L^\infty(\Omega)^n$ and let (U, H^J) be a trajectory of (NL-U) reaching U^* in time T . Then, we have for every $k \neq l \in \{m+1, \dots, n\}$, $t \in [0, T]$,*

$$(20) \quad \int_{\Omega} \frac{u_k(t, x) - u_k^*}{\beta_k - \alpha_k} dx = \int_{\Omega} \frac{u_l(t, x) - u_l^*}{\beta_l - \alpha_l} dx,$$

$$(21) \quad (d_k = d_l) \Rightarrow \left(\frac{u_k(t, \cdot) - u_k^*}{\beta_k - \alpha_k} = \frac{u_l(t, \cdot) - u_l^*}{\beta_l - \alpha_l} \right).$$

In particular, for every $k \neq l \in \{m+1, \dots, n\}$,

$$(22) \quad \int_{\Omega} \frac{u_{k,0}(x) - u_k^*}{\beta_k - \alpha_k} dx = \int_{\Omega} \frac{u_{l,0}(x) - u_l^*}{\beta_l - \alpha_l} dx,$$

$$(23) \quad (d_k = d_l) \Rightarrow \left(\frac{u_{k,0} - u_k^*}{\beta_k - \alpha_k} = \frac{u_{l,0} - u_l^*}{\beta_l - \alpha_l} \right).$$

The proof of Proposition 2.3 is done in Appendix A.2. We prove (20) by integrating with respect to the space variable an appropriate linear combination of equations of (NL-U) and by using the Neumann boundary conditions. We prove (21) by the backward uniqueness of the heat equation applied to an appropriate linear combination of equations of (NL-U).

The equation (21) implies that we can reduce the number of components of $(u_i)_{1 \leq i \leq n}$ of (NL-U) when some diffusion coefficients d_i are equal for $m+1 \leq i \leq n$. This simplifies the study, thus in order to treat the more difficult case, we make the following hypothesis.

Assumption 2.4. For every $k \neq l \in \{m+1, \dots, n\}$, $d_k \neq d_l$.

Remark 2.5. It will be interesting to note that the mass condition (22) is equivalent to

$$(24) \quad \forall k \geq m+2, \int_{\Omega} \frac{u_{k,0}(x) - u_k^*}{\beta_k - \alpha_k} dx = \int_{\Omega} \frac{u_{m+1,0}(x) - u_{m+1}^*}{\beta_{m+1} - \alpha_{m+1}} dx.$$

3. AN ADEQUATE CHANGE OF VARIABLES AND LINEARIZATION

3.1. Change of variables - Cross diffusion system. The goal of this section is to transform the controlled system (NL-U) (see Section 1.2) satisfied by U into another system of *cascade* type for which we better understand the controllability properties. Roughly speaking, for $1 \leq i \leq m$, the component u_i is easy to control thanks to the localized control term $h_i 1_\omega$. Thus, the challenge is to understand how the reaction term $f_i(U)$ (see (6)) acts on the component u_i for $m+1 \leq i \leq n$.

We multiply the $(m+1)$ -th equation of (NL-U) by $1/((\beta_{m+1} - \alpha_{m+1})(d_{m+1} - d_{m+2}))$ and the $(m+2)$ -th equation of (NL-U) by $1/((\beta_{m+2} - \alpha_{m+2})(d_{m+2} - d_{m+1}))$ and we sum:

$$\partial_t v_{m+2} - d_{m+2} \Delta v_{m+2} = \frac{1}{\beta_{m+1} - \alpha_{m+1}} \Delta u_{m+1},$$

where

$$v_{m+2} = \frac{1}{(\beta_{m+1} - \alpha_{m+1})(d_{m+1} - d_{m+2})} u_{m+1} + \frac{1}{(\beta_{m+2} - \alpha_{m+2})(d_{m+2} - d_{m+1})} u_{m+2}.$$

Roughly speaking, this linear combination enables to “kill” the reaction-term and to create a coupling term of second order.

By iterating this strategy, we construct a linear transformation $V = PU$ such that u_{m+1} acts on v_{m+2} , v_{m+2} acts on v_{m+3} , ..., v_{n-1} acts on v_n through cross diffusion terms. Moreover, we transform the problem of controllability for U to U^* into a null-controllability problem for

$$(25) \quad Z := P(U - U^*),$$

where P is the invertible triangular matrix defined by:

$$(26) \quad P := \left(\begin{array}{c|c} I_m & (0) \\ \hline (0) & * \end{array} \right),$$

with

$$(27) \quad \forall k, l \geq m+1, P_{kl} := \begin{cases} \left((\beta_l - \alpha_l) \prod_{\substack{m+1 \leq r \leq k \\ r \neq l}} (d_l - d_r) \right)^{-1} & \text{if } k \geq l, \\ 0 & \text{if } k < l, \end{cases}$$

with the convention $\prod_{\emptyset} = 1$.

Proposition 3.1. *The couple (U, H^J) is a trajectory of (NL-U) if and only if (Z, H^J) satisfies*

$$(28) \quad \begin{cases} \partial_t z_i - d_i \Delta z_i = f_i(P^{-1}Z + U^*) + h_i 1_\omega & \text{in } (0, T) \times \Omega, & 1 \leq i \leq m \\ \partial_t z_{m+1} - d_{m+1} \Delta z_{m+1} = \frac{f_{m+1}(P^{-1}Z + U^*)}{\beta_{m+1} - \alpha_{m+1}} & \text{in } (0, T) \times \Omega, \\ \partial_t z_i - d_i \Delta z_i = \Delta z_{i-1} & \text{in } (0, T) \times \Omega, & m+2 \leq i \leq n, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ Z(0, \cdot) = Z_0 & \text{in } \Omega, \end{cases}$$

The proof of Proposition 3.1 is done in Appendix A.3.1.

We introduce the notations: for every $(a_1, \dots, a_n) \in \mathbb{R}^n$,

$$(29) \quad \forall 1 \leq i \leq m, g_i(a_1, \dots, a_n) := f_i(P^{-1}(a_1, \dots, a_n)^{\text{tr}} + U^*),$$

$$(30) \quad g_{m+1}(a_1, \dots, a_n) := \frac{f_{m+1}(P^{-1}(a_1, \dots, a_n)^{\text{tr}} + U^*)}{\beta_{m+1} - \alpha_{m+1}},$$

$$(31) \quad G(a_1, \dots, a_n) := (g_1(a_1, \dots, a_n), \dots, g_{m+1}(a_1, \dots, a_n), 0, \dots, 0)^{\text{tr}},$$

$$(32) \quad D_J := \left(\begin{array}{c|c} \text{diag}(d_1, \dots, d_m) & (0) \\ \hline (0) & D_{\#} \end{array} \right), \quad D_{\#} := \begin{pmatrix} d_{m+1} & 0 & \dots & \dots & 0 \\ 1 & d_{m+2} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & d_n \end{pmatrix}.$$

With these notations, the nonlinear system (28) is

$$(NL-Z) \quad \begin{cases} \partial_t Z - D_J \Delta Z = G(Z) + H^J 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ Z(0, \cdot) = Z_0 & \text{in } \Omega. \end{cases}$$

3.2. Linearization. We will work under the following hypothesis which will guarantee the null-controllability of this linearized system.

Assumption 3.2. We assume that there exists $1 \leq j \leq m$ such that

$$(33) \quad \partial_j g_{m+1}(0, \dots, 0) \neq 0.$$

By using (26) and (30), we check that (33) is equivalent to

$$(34) \quad \partial_j f_{m+1}(u_1^*, \dots, u_n^*) \neq 0.$$

When $\alpha_j, \beta_j \geq 1$, a sufficient condition to ensure (34) is

$$(35) \quad \forall 1 \leq k \leq n, u_k^* \neq 0.$$

Indeed, by using (6), (4) and $\alpha_j \neq \beta_j$, if (35) holds true then

$$\begin{aligned} \partial_j f_{m+1}(u_1^*, \dots, u_n^*) &= \alpha_j (u_j^*)^{\alpha_j - 1} \prod_{\substack{k=1 \\ k \neq j}}^n u_k^{*\alpha_k} - \beta_j (u_j^*)^{\beta_j - 1} \prod_{\substack{k=1 \\ k \neq j}}^n u_k^{*\beta_k} \\ &= \frac{\alpha_j}{u_j^*} \prod_{k=1}^n u_k^{*\alpha_k} - \frac{\beta_j}{u_j^*} \prod_{k=1}^n u_k^{*\beta_k} \\ &= \frac{\alpha_j - \beta_j}{u_j^*} \prod_{k=1}^n u_k^{*\alpha_k} \neq 0. \end{aligned}$$

Note that (35) is not equivalent to (34) as shown by the examples in Application 4.3.

The linearized system of (NL-Z) satisfied by Z around $(0, 0)$ is

$$(L-Z) \quad \begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + H^J 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ Z(0, \cdot) = Z_0 & \text{in } \Omega, \end{cases}$$

where

$$(36) \quad A_J = (a_{ik})_{1 \leq i, k \leq n}, \quad a_{ik} = \begin{cases} \partial_k g_i(0, \dots, 0) & \text{if } 1 \leq i \leq m+1, \\ 0 & \text{if } m+2 \leq i \leq n. \end{cases}$$

Up to a renumbering of the first m equations, we can assume that $j = m$. Then, by Assumption 3.2, we have

$$(37) \quad a_{m+1, m} \neq 0.$$

Roughly speaking, we summarize the controllability properties established in the following diagram:

$$\begin{array}{ccccccccccc} h_1 & \xrightarrow{\text{controls}} & z_1 & , & h_2 & \xrightarrow{\text{controls}} & z_2 & , & \dots & , & h_{m-1} & \xrightarrow{\text{controls}} & z_{m-1}, \\ h_m & \xrightarrow{\text{controls}} & z_m & \xrightarrow{a_{m+1m} z_m} & z_{m+1} & \xrightarrow{\Delta z_{m+1}} & z_{m+2} & \xrightarrow{\Delta z_{m+2}} & \dots & \xrightarrow{\Delta z_{n-1}} & z_n. \end{array}$$

4. MAIN RESULTS

The goal of this section is to state the main results of the paper. First, we prove a local null-controllability result for the system (NL-Z) (see Section 3.1). Then, we deduce a local controllability result around U^* for (NL-U) (see Section 1.2).

We have seen in Proposition 2.3 that a trajectory (U, H^J) reaching U^* has to verify the condition (20). Thus, it prevents local-controllability from happening for arbitrary initial data. This is why we introduce a notion of local controllability adapted to (22).

Let $p \in [1, +\infty]$. We introduce the following subspace of $L^p(\Omega)^n$:

$$(38) \quad L_{inv}^p := \left\{ Z_0 \in L^p(\Omega)^n ; \forall m+2 \leq i \leq n, \int_{\Omega} z_{i,0}(x) dx = 0 \right\}.$$

Theorem 4.1. *Under Assumption 2.4 and Assumption 3.2, the system (NL-Z) is locally null-controllable, i.e., there exists $r > 0$ such that for every $Z_0 \in L_{inv}^\infty$ verifying $\|Z_0\|_{L^\infty(\Omega)^n} \leq r$, there exists $H^J \in L^\infty(Q_T)^m$ such that the solution Z of (NL-Z) satisfies $Z(T, \cdot) = 0$.*

We deduce from Theorem 4.1 the following local controllability result.

Theorem 4.2. *Under Assumption 2.4 and Assumption 3.2, the system (NL-U) is locally controllable around U^* , i.e., there exists $r > 0$ such that for every $U_0 \in L^\infty$ satisfying the mass condition (22) and $\|U_0 - U^*\|_{L^\infty(\Omega)} \leq r$, there exists $H^J \in L^\infty(Q_T)^m$ such that the solution U of (NL-U) satisfies $U(T, \cdot) = U^*$.*

Theorem 4.1 is equivalent to Theorem 4.2. Indeed, it comes from Proposition 3.1 and the following equivalence

$$(39) \quad Z_0 \in L_{inv}^\infty \Leftrightarrow U_0 \text{ satisfies (22)} \Leftrightarrow U_0 \text{ satisfies (24)} \text{ (Remark 2.5).}$$

The proof of (39) is done in Appendix A.3.2.

Application 4.3. For $n = 4$, $\alpha_1 = \alpha_3 = \beta_2 = \beta_4 = 1$ and $\alpha_2 = \alpha_4 = \beta_1 = 0$, we are studying the following system:

$$(40) \quad \forall 1 \leq i \leq 4, \begin{cases} \partial_t u_i - d_i \Delta u_i = (-1)^i (u_1 u_3 - u_2 u_4) + h_i \mathbf{1}_{\omega} \mathbf{1}_{i \in J} & \text{in } (0, T) \times \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial \Omega, \\ u_i(0, \cdot) = u_{i,0} & \text{in } \Omega. \end{cases}$$

In this case, we check that Assumption 3.2 is

$$\text{for } J = \{1, 2, 3\}, \quad \left(\exists j \in \{1, 2, 3\}, \partial_j f_4(u_1^*, \dots, u_4^*) \neq 0 \right) \Leftrightarrow \left((u_1^*, u_3^*, u_4^*) \neq (0, 0, 0) \right),$$

$$\text{for } J = \{1, 2\}, \quad \left(\exists j \in \{1, 2\}, \partial_j f_3(u_1^*, \dots, u_4^*) \neq 0 \right) \Leftrightarrow \left((u_3^*, u_4^*) \neq (0, 0) \right),$$

$$\text{for } J = \{1\}, \quad \left(\partial_1 f_2(u_1^*, \dots, u_4^*) \neq 0 \right) \Leftrightarrow \left(u_3^* \neq 0 \right).$$

Thus, Theorem 4.2 recovers the result of [28, Theorem 3.2] except for the case $J = \{1, 2, 3\}$ and $(u_1^*, u_3^*, u_4^*) = (0, 0, 0)$ that the proof of the present article does not treat (see Example 9.2 for more details about the strategy of [28]).

5. LINEAR NULL-CONTROLLABILITY UNDER CONSTRAINTS IN L^2

The main result of this section, stated in the following theorem, is the null-controllability in L_{inv}^2 for the linear system (L-Z) (see Section 3.2).

Theorem 5.1. *The system (L-Z) is null-controllable in L_{inv}^2 . More precisely, there exists $C > 0$ such that for every $T > 0$ and $Z_0 \in L_{inv}^2$, there exists a control $H^J \in L^2((0, T) \times \Omega)^m$ verifying*

$$(41) \quad \|H^J\|_{L^2(Q_T)^m} \leq C_T \|Z_0\|_{L^2(\Omega)^n}, \text{ where } C_T = C e^{C/T},$$

and such that the solution $Z \in W_T^n$ of (L-Z) satisfies $Z(T, \cdot) = 0$.

The goal of the next two subsections is to prove Theorem 5.1. The proof is based on the Lebeau-Robbiano's method, introduced for the first time to prove the null-controllability of the heat equation (see [30]). First, it consists in establishing a null-controllability result in finite dimensional subspaces of L_{inv}^2 with a precise estimate of the cost of the control (see Proposition 5.2). This first step is based on two main results: the spectral inequality for eigenfunctions of the Neumann-Laplace operator (see Lemma 5.4) and precise observability estimates of linear finite dimensional systems associated to the adjoint system of (L-Z) (see Lemma 5.5). Secondly, we conclude by a time-splitting procedure: the control H^J is built as a sequence of active controls and passive controls. The passive mode allows to take advantage of the natural parabolic exponential decay of the L^2 norm of the solution. This decay enables to compensate the cost of the control which steers the low frequencies to 0

(see Section 5.2).

5.1. A null-controllability result for the low frequencies. We define $H_{N_e}^2(\Omega) := \left\{ y \in H^2(\Omega) ; \frac{\partial y}{\partial \nu} = 0 \right\}$. The unbounded operator on $L^2(\Omega)$: $(-\Delta, H_{N_e}^2(\Omega))$ is self-adjoint and has compact resolvent. Thus, we introduce the orthonormal basis $(e_k)_{k \geq 0}$ of $L^2(\Omega)$ of eigenfunctions associated to the increasing sequence of eigenvalues $(\lambda_k)_{k \geq 0}$ of the Laplacian operator, i.e., we have $-\Delta e_k = \lambda_k e_k$ and $(e_k, e_l)_{L^2(\Omega)} = \delta_{k,l}$. For $\lambda > 0$, we define the finite dimensional space $E_\lambda = \left\{ \sum_{\lambda_k \leq \lambda} c_k e_k ; c_k \in \mathbb{R}^n \right\} \subset L^2(\Omega)^n$ and the orthogonal projection Π_{E_λ} onto E_λ in $L^2(\Omega)^n$.

The goal of this section is to prove the following null-controllability result in a finite dimensional subspace of L_{inv}^2 .

Proposition 5.2. *There exist $C > 0$, $p_1 \in \mathbb{N}$ such that for every $\tau \in (0, T)$, $\lambda > 0$, $Z_0 \in E_\lambda \cap L_{inv}^2$, there exists a control function $H^J \in L^2(Q_\tau)$ verifying*

$$(42) \quad \|H^J\|_{L^2(Q_\tau)^m}^2 \leq \frac{C}{\tau^{p_1}} e^{C\sqrt{\lambda}} \|Z_0\|_{L^2(\Omega)^n}^2,$$

such that the solution Z of

$$(43) \quad \begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + H^J 1_\omega & \text{in } (0, \tau) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0, \tau) \times \partial\Omega, \\ Z(0, \cdot) = Z_0 \in E_\lambda & \text{in } \Omega, \end{cases}$$

satisfies $Z(\tau, \cdot) = 0$.

From Proposition 5.2, for every $\tau, \lambda > 0$ and $Z_0 \in E_\lambda \cap L_{inv}^2$ we introduce the notation:

$$(44) \quad H_\lambda(Z_0, 0, \tau) := H^J,$$

such that the solution Z of (43) satisfies $Z(\tau, \cdot) = 0$ and H^J is the minimal-norm element of $L^2(Q_\tau)^m$ satisfying the estimate (42). In other words, H^J is the projection of 0 in the nonempty closed convex set of controls satisfying (42) and driving the solution Z of (43) in time τ to 0.

By the Hilbert Uniqueness Method (see [9, Theorem 2.44]), in order to prove Proposition 5.2, we need to prove an observability inequality for the solution of the adjoint system of (43).

Proposition 5.3. *There exist $C > 0$, $p_1 \in \mathbb{N}$ such that for every $\tau \in (0, T)$, $\lambda > 0$ and $\varphi_\tau \in E_\lambda \cap L_{inv}^2$, the solution φ of*

$$(45) \quad \begin{cases} -\partial_t \varphi - D_J^{\text{tr}} \Delta \varphi = A_J^{\text{tr}} \varphi & \text{in } (0, \tau) \times \Omega, \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } (0, \tau) \times \partial\Omega, \\ \varphi(\tau, \cdot) = \varphi_\tau & \text{in } \Omega, \end{cases}$$

satisfies

$$(46) \quad \|\varphi(0, \cdot)\|_{L^2(\Omega)^n}^2 \leq \frac{C}{\tau^{p_1}} e^{C\sqrt{\lambda}} \sum_{i=1}^m \int_0^\tau \int_\omega |\varphi_i(t, x)|^2 dx dt.$$

Proof. The proof is inspired by [32, Section 3].

Let $\tau > 0$, $\lambda > 0$ and $\varphi_\tau \in E_\lambda \cap L_{inv}^2$. We have:

$$(47) \quad \varphi_\tau(x) = \sum_{\lambda_k \leq \lambda} \varphi_k^\tau e_k(x),$$

with $\varphi_k^\tau \in F_k$ where $F_0 := \mathbb{R}^{m+1} \times \{0\}^{n-m-1}$ because $\varphi_\tau \in L_{inv}^2$ and $F_k := \mathbb{R}^n$ for $k \geq 1$.

Then, the solution φ of (45) is

$$(48) \quad \forall (t, x) \in (0, \tau) \times \Omega, \quad \varphi(t, x) = \sum_{\lambda_k \leq \lambda} \varphi_k(t) e_k(x),$$

where φ_k is the unique solution of the ordinary differential system

$$(49) \quad \begin{cases} -\varphi_k' + \lambda_k D_J^{\text{tr}} \varphi_k = A_J^{\text{tr}} \varphi_k, & \text{in } (0, \tau), \\ \varphi_k(\tau) = \varphi_k^\tau & . \end{cases}$$

We recall the spectral inequality for eigenfunctions of the Neumann-Laplace operator.

Lemma 5.4. [24, Theorem 14.6]

There exists $C > 0$ such that for every sequence $(a_k)_{k \geq 0} \subset \mathbb{C}^{\mathbb{N}}$ and for every $\lambda > 0$, we have:

$$(50) \quad \sum_{\lambda_k \leq \lambda} |a_k|^2 = \int_{\Omega} \left| \sum_{\lambda_k \leq \lambda} a_k e_k(x) \right|^2 dx \leq C e^{C\sqrt{\lambda}} \int_{\omega} \left| \sum_{\lambda_k \leq \lambda} a_k e_k(x) \right|^2 dx.$$

By using (50) for $a_k = \varphi_{k,i}(t)$ with $1 \leq i \leq m$ and by summing on $1 \leq i \leq m$, we obtain that there exists $C > 0$ such that

$$(51) \quad \sum_{\lambda_k \leq \lambda} \sum_{i=1}^m |\varphi_{k,i}(t)|^2 \leq C e^{C\sqrt{\lambda}} \sum_{i=1}^m \int_{\omega} \left| \sum_{\lambda_k \leq \lambda} \varphi_{k,i}(t) e_k(x) \right|^2 dx.$$

By integrating with respect to the time variable between 0 and τ the inequality (51), we obtain

$$(52) \quad \int_0^\tau \sum_{\lambda_k \leq \lambda} \sum_{i=1}^m |\varphi_{k,i}(t)|^2 dt \leq C e^{C\sqrt{\lambda}} \sum_{i=1}^m \int_0^\tau \int_{\omega} \left| \sum_{\lambda_k \leq \lambda} \varphi_{k,i}(t) e_k(x) \right|^2 dx dt.$$

Now, our goal is to establish the following lemma.

Lemma 5.5. *There exist $C, p_1, p_2 \in \mathbb{N}$ such that for every $\tau \in (0, 1)$, $k \in \mathbb{N}$, $\varphi_k^\tau \in F_k$, the solution φ_k of (49) satisfies*

$$(53) \quad \|\varphi_k(0)\|^2 \leq C \left(1 + \frac{1}{\tau^{p_1}} + \lambda_k^{p_2} \right) \sum_{i=1}^m \int_0^\tau |\varphi_{k,i}(t)|^2 dt.$$

One can take for instance $p_1 = p_2 = 2(n - m + 1/2)$ in (53), we give a proof in Appendix A.4 (see also [36]).

By using (52), (53), we deduce that

$$(54) \quad \begin{aligned} \sum_{\lambda_k \leq \lambda} \|\varphi_k(0)\|^2 &\leq \sum_{\lambda_k \leq \lambda} \frac{C}{\tau^{p_1}} (1 + \lambda_k^{p_2}) \sum_{i=1}^m \int_0^\tau |\varphi_{k,i}(t)|^2 dt \\ &\leq \frac{C}{\tau^{p_1}} e^{C\sqrt{\lambda}} \sum_{i=1}^m \int_0^\tau \int_{\omega} \left| \sum_{\lambda_k \leq \lambda} \varphi_{k,i}(t) e_k(x) \right|^2 dx dt. \end{aligned}$$

By using (48), we deduce (46) from (54). \square

5.2. The Lebeau-Robbiano's method. The goal of this section is to prove Theorem 5.1.

Proof. The proof is inspired by [29, Section 6.2]. The constants C, C' will increase from line to line.

We split the interval $[0, T] = \cup_{k \in \mathbb{N}} [a_k, a_{k+1}]$ with $a_0 = 0$, $a_{k+1} = a_k + 2T_k$ and $T_k = T/2^k$ for $k \in \mathbb{N}$. We also define $\mu_k = M2^{2k}$ for $M > 0$ sufficiently large which

will be defined later and for $k \in \mathbb{N}$. Then, we define the control H^J in the following way:

- if $t \in (a_k, a_k + T_k)$, $H^J = H_{\mu_k}(\Pi_{E_{\mu_k}} Z(a_k, \cdot), a_k, T_k)$ (see the notation (44)) and $Z(t, \cdot) = S(t - a_k)Z(a_k, \cdot) + \int_{a_k}^t S(t - s)H^J(s, \cdot)ds$,
- if $t \in (a_k + T_k, a_{k+1})$, $H^J = 0$ and $Z(t, \cdot) = S(t - a_k - T_k)Z(a_k + T_k, \cdot)$,

where $S(t)$ denotes the semigroup of the parabolic system: $S(t) = e^{t(D_J \Delta + A_J)}$. In particular, by (16) and (12), $\|S(t)\|_{\mathcal{L}(L^2(\Omega)^n)} \leq C$.

By (42), the choice of H^J during the interval time $[a_k, a_k + T_k]$ implies

$$(55) \quad \|Z(a_k + T_k, \cdot)\|_{L^2(\Omega)^n}^2 \leq (C + C(2^{-k}T)^{-p_1} e^{C\sqrt{M}2^k}) \|Z(a_k, \cdot)\|_{L^2(\Omega)^n}^2 \\ \leq \frac{C}{T^{p_1}} e^{C\sqrt{M}2^k} \|Z(a_k, \cdot)\|_{L^2(\Omega)^n}^2.$$

During the passive period of the control, $t \in [a_k + T_k, a_{k+1}]$, the solution exponentially decreases:

$$(56) \quad \|Z(a_{k+1}, \cdot)\|_{L^2(\Omega)^n}^2 \leq C' e^{-C'M2^{2k}T_k} \|Z(a_k + T_k, \cdot)\|_{L^2(\Omega)^n}^2.$$

Thus, by using $2^{2k}T_k = 2^kT$, (55) and (56), we have

$$\|Z(a_{k+1}, \cdot)\|_{L^2(\Omega)^n}^2 \leq e^{C\sqrt{M}2^k - C'M2^kT + C/T} \|Z(a_k, \cdot)\|_{L^2(\Omega)^n}^2,$$

and consequently,

$$(57) \quad \|Z(a_{k+1}, \cdot)\|_{L^2(\Omega)^n}^2 \leq e^{\sum_{j=0}^k (C\sqrt{M}2^j - C'MT2^j + C/T)} \|Z_0\|_{L^2(\Omega)^n}^2 \\ \leq e^{(C\sqrt{M} - C'MT)2^{k+1} + (C/T)(k+1)} \|Z_0\|_{L^2(\Omega)^n}^2.$$

By taking M such that $C\sqrt{M} - C'MT < 0$, for instance $M \geq 2(C/C'T)^2$, we conclude by (57) that we have $\lim_{k \rightarrow +\infty} \|Z(a_k, \cdot)\| = 0$, i.e., $Z(T, \cdot) = 0$ because $t \rightarrow Z(t, \cdot) \in C([0, T]; L^2(\Omega)^n)$ because $H^J \in L^2(Q_T)^m$ (see Definition-Proposition 2.1 and (12)) as we will show now.

We have $\|H^J\|_{L^2(Q_T)^m}^2 = \sum_{k=0}^{+\infty} \|H^J\|_{L^2((a_k, a_k + T_k) \times \Omega)^m}^2$. Then, by using the estimate (42) of the control on each time interval $(a_0, a_0 + T_0)$ and the estimate (57), we get:

$$(58) \quad \|H^J\|_{L^2(Q_T)^m}^2 \leq \left(CT_0^{-p_1} e^{C\sqrt{M}} + \sum_{k \geq 1} CT_k^{-p_1} e^{C\sqrt{M}2^k} e^{-C'MT2^{(k-1)}} \right) \|Z_0\|_{L^2(\Omega)^n}^2 \\ \leq \left(CT^{-p_1} e^{C\sqrt{M}} + \sum_{k \geq 1} C(2^kT^{-1})^{p_1} e^{(C\sqrt{M} - C'MT/2)2^k} \right) \|Z_0\|_{L^2(\Omega)^n}^2.$$

By taking M such that $C\sqrt{M} - C'MT/2 < 0$, for instance $M \geq 8(C/C'T)^2 \Rightarrow C\sqrt{M} - C'MT/2 = -C''/T$ with $C'' > 0$, we deduce from (58) that $H^J \in L^2(Q_T)^m$ and

$$\|H^J\|_{L^2(Q_T)^m}^2 \leq \left(Ce^{C/T} + C \int_0^{+\infty} \left(\frac{\sigma}{T}\right)^{p_1} e^{-C''\frac{\sigma}{T}} d\sigma \right) \|Z_0\|_{L^2(\Omega)^n}^2 \\ \leq Ce^{C/T} \|Z_0\|_{L^2(\Omega)^n}^2.$$

This concludes the proof of Theorem 5.1. \square

6. THE SOURCE TERM METHOD IN L^2

We use the source term method, introduced by Yuning Liu, Takéo Takahashi and Marius Tucsnak in [33, Proposition 2.3] to deduce a local null-controllability result for a nonlinear system from the null-controllability result for only one linear system (and an estimate of the cost of the control) (see also [6]).

By Theorem 5.1, we have an estimate for the control cost in L^2 , then we fix $M > 0$ such that $C_T \leq Me^{M/T}$. Let $q \in (1, \sqrt{2})$ and $p > q^2/(2 - q^2)$. We define the weights

$$(59) \quad \rho_0(t) := M^{-p} \exp\left(-\frac{Mp}{(q-1)(T-t)}\right),$$

$$(60) \quad \rho_S(t) = M^{-1-p} \exp\left(-\frac{(1+p)q^2M}{(q-1)(T-t)}\right).$$

Remark 6.1. The assumption $p > q^2/(2 - q^2) \Leftrightarrow 2p > (1+p)q^2$ implies

$$(61) \quad \rho_0^2/\rho_S \in C([0, T]),$$

which will be useful for the estimate of the polynomial nonlinearity (see Section 8).

Let $r \in \{2, +\infty\}$. For $S \in L^r((0, T); L^r_{inv})$, $H^J \in L^r((0, T); L^r(\Omega)^m)$, $Z_0 \in L^r_{inv}$, we introduce the following system:

$$(L+S-Z) \quad \begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + S + H^J 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ Z(0, \cdot) = Z_0 & \text{in } \Omega. \end{cases}$$

Then, we define associated spaces for the source term, the state and the control

$$(62) \quad \mathcal{S}_r := \left\{ S \in L^r((0, T); L^r_{inv}) ; \frac{S}{\rho_S} \in L^r((0, T); L^r_{inv}) \right\},$$

$$(63) \quad \mathcal{Z}_r := \left\{ Z \in L^r((0, T); L^r_{inv}) ; \frac{Z}{\rho_0} \in L^r((0, T); L^r_{inv}) \right\},$$

$$(64) \quad \mathcal{H}_r := \left\{ H^J \in L^r((0, T); L^r(\Omega)^m) ; \frac{H^J}{\rho_0} \in L^r((0, T); L^r(\Omega)^m) \right\}.$$

Remark 6.2. From the behaviors near $t = T$ of ρ_S and ρ_0 , we deduce that each element of \mathcal{S}_r , \mathcal{Z}_r , \mathcal{H}_r vanishes at $t = T$.

From the abstract result: [33, Proposition 2.3], we deduce the null-controllability for (L+S-Z) in L^2_{inv} .

Proposition 6.3. *For every $S \in \mathcal{S}_2$ and $Z_0 \in L^2_{inv}$, there exists $H^J \in \mathcal{H}_2$, such that the solution Z of (L+S-Z) satisfies $Z \in \mathcal{Z}_2$. Furthermore, there exists $C > 0$, not depending on S and Z_0 , such that*

$$(65) \quad \|Z/\rho_0\|_{C([0, T]; L^2(\Omega)^n)} + \|H^J\|_{\mathcal{H}_2} \leq C_T \left(\|Z_0\|_{L^2(\Omega)^n} + \|S\|_{\mathcal{S}_2} \right),$$

where $C_T = Ce^{C/T}$. In particular, since ρ_0 is a continuous function satisfying $\rho_0(T) = 0$, the above relation (65) yields $Z(T, \cdot) = 0$.

For the sake of completeness, the proof of Proposition 6.3 is in Appendix A.5 (see Proposition A.6 applied with $r = 2$).

Now, we will deduce an observability estimate for the adjoint system:

$$(66) \quad \begin{cases} -\partial_t \varphi - D_J^{\text{tr}} \Delta \varphi = A_J^{\text{tr}} \varphi & \text{in } (0, T) \times \Omega, \\ \frac{\partial \varphi}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi(T, \cdot) = \varphi_T & \text{in } \Omega. \end{cases}$$

We have the following result which is an adaptation of [33, Corollary 2.6] or [23, Theorem 4.1] (see Appendix A.6 for a complete proof).

Corollary 6.4. *There exists $C > 0$ such that for every $\varphi_T \in L^2_{inv}$, the solution of (66) satisfies:*

$$(67) \quad \|\varphi(0, \cdot)\|_{L^2(\Omega)^n}^2 + \int_0^T \int_{\Omega} |\rho_S(t)\varphi(t, x)|^2 dt dx \leq C_T \left(\sum_{i=1}^m \int_0^T \int_{\omega} |\rho_0(t)\varphi_i(t, x)|^2 dt dx \right),$$

where $C_T = Ce^{C/T}$.

Remark 6.5. The estimate (67) looks like a global Carleman inequality for (66). But the strategy to get this type of estimate comes from the null-controllability theorem in L^2_{inv} for (L-Z) with an estimate of the cost and the source term method: Theorem 5.1 and Proposition 6.3. We insist on the fact that we do not know how to prove the null-controllability in L^2_{inv} of (L-Z) by the usual global Carleman estimates applied to each equation of (66) when $m < n - 4$ (see Section 9 and in particular Open problem 9.4).

In the next section, we take advantage of the strong observability estimate (67) to get more regularity in L^p -sense for the control H^J .

7. CONSTRUCT L^∞ -CONTROLS AND THE SOURCE TERM METHOD IN L^∞

7.1. Construct L^∞ -controls: the Penalized Hilbert Uniqueness Method.

The goal of this section is to prove a null-controllability result in L^∞ with an estimate of the cost of the control.

Theorem 7.1. *There exists $C > 0$ such that for every $T > 0$, $Z_0 \in L^2_{inv}$, there exists a control $H^J \in L^\infty((0, T) \times \Omega)^m$ verifying*

$$(68) \quad \|H^J\|_{L^\infty(Q_T)^m} \leq C_T \|Z_0\|_{L^2(\Omega)^n}, \text{ where } C_T = Ce^{C/T}.$$

and such that the solution Z of (L-Z) (see Section 3.2) satisfies $Z(T, \cdot) = 0$.

From now and until the end of the section, we will denote by C_T various positive constants which can change from line to line and such that $C_T \leq Ce^{C/T}$.

In the next four parts, we perform the usual Penalized Hilbert Uniqueness Method, introduced for the first time by Viorel Barbu in [4]. The idea is the following one: it is a well-known fact that the optimal control $H^J \in L^2((0, T) \times \Omega)^m$, i.e., the minimal-norm element in L^2 , which steers the solution Z of (L-Z) to 0 in time T can be expressed as a function of a solution of the adjoint system (66) (see [9, Section 1.4] for more details in the context of linear finite dimensional controlled systems). By using the strong observability inequality (67), we will use this link by considering a penalized problem in $\mathcal{H}_2 \subset L^2((0, T) \times \Omega)^m$: the behavior at time $t = T$ of the weight ρ_0 will be the key point to produce more regular controls in L^p -sense.

7.1.1. *The beginning of the Penalized Hilbert Uniqueness Method.* Let us fix $Z_0 \in L^2_{inv}$.

We define $P_\varepsilon : \mathcal{H}_2 \rightarrow \mathbb{R}^+$, by, for every $H^J \in \mathcal{H}_2$,

$$(69) \quad P_\varepsilon(H^J) := \frac{1}{2} \int \int_{(0, T) \times \omega} \rho_0^{-2}(t) |H^J(t, x)|^2 dx dt + \frac{1}{2\varepsilon} \|Z(T, \cdot)\|_{L^2(\Omega)^n}^2,$$

where Z is the solution to the Cauchy problem (L-Z) (see Section 3.2) associated to the control H^J .

The functional P_ε is a C^1 , coercive, strictly convex functional on the Hilbert space \mathcal{H}_2 , then P_ε has a unique minimum $H^{J, \varepsilon} \in \mathcal{H}_2$. Let Z^ε be the solution to the Cauchy problem (L-Z) with control $H^{J, \varepsilon}$ and initial data Z_0 .

The Euler-Lagrange equation gives

$$(70) \quad \forall H^J \in \mathcal{H}_2, \int \int_{(0, T) \times \omega} \rho_0^{-2} H^{J, \varepsilon} \cdot H^J dx dt + \frac{1}{\varepsilon} \int_{\Omega} Z^\varepsilon(T, x) \cdot Z(T, x) dx = 0,$$

where Z is the solution to the Cauchy problem (L-Z) associated to the control H^J and initial data $Z_0 = 0$.

We introduce φ^ε the solution to the adjoint problem (66) with final condition $\varphi^\varepsilon(T, \cdot) = -\frac{1}{\varepsilon}Z^\varepsilon(T, \cdot)$. A duality argument between Z and φ^ε gives

$$(71) \quad -\frac{1}{\varepsilon} \int_{\Omega} Z(T, x) \cdot Z^\varepsilon(T, x) dx = \int_{\Omega} Z(T, x) \cdot \varphi^\varepsilon(T, x) dx = \int \int_{(0,T) \times \omega} H^J \cdot \varphi^\varepsilon.$$

Then, we deduce from (70) and (71) that

$$\forall H^J \in \mathcal{H}_2, \int \int_{(0,T) \times \omega} \rho_0^{-2} H^{J,\varepsilon} \cdot H^J = \int \int_{(0,T) \times \omega} \varphi^\varepsilon \cdot H^J.$$

Consequently, we have

$$(72) \quad \forall i \in \{1, \dots, m\}, h_i^\varepsilon = \rho_0^2 \varphi_i^\varepsilon 1_\omega.$$

Another duality argument applied between Z^ε and φ^ε together with (72) gives

$$\begin{aligned} -\frac{1}{\varepsilon} \int_{\Omega} |Z^\varepsilon(T, x)|^2 dx &= \int_{\Omega} Z^\varepsilon(T, x) \cdot \varphi^\varepsilon(T, x) dx \\ &= \int_{\Omega} Z_0(x) \cdot \varphi^\varepsilon(0, x) dx + \int \int_{(0,T) \times \omega} H^{J,\varepsilon} \cdot \varphi^\varepsilon, \end{aligned}$$

which yields

$$(73) \quad -\frac{1}{\varepsilon} \|Z^\varepsilon(T, \cdot)\|_{L^2(\Omega)^n}^2 = \int_{\Omega} Z_0(x) \cdot \varphi^\varepsilon(0, x) dx + \sum_{i=1}^m \int \int_{(0,T) \times \omega} |\rho_0 \varphi_i^\varepsilon|^2.$$

By Young's inequality and the observability estimate (67) applied to φ^ε , for $\delta > 0$, we have:

$$(74) \quad \begin{aligned} &\left| \int_{\Omega} Z_0(x) \cdot \varphi^\varepsilon(0, x) dx \right| \\ &\leq \delta \|\varphi^\varepsilon(0, \cdot)\|_{L^2(\Omega)^n}^2 + C_\delta \|Z_0\|_{L^2(\Omega)^n} \\ &\leq \delta C_T \left(\sum_{i=1}^m \int \int_{(0,T) \times \omega} |\rho_0(t) \varphi_i^\varepsilon(t, x)|^2 dx dt \right) + C_\delta \|Z_0\|_{L^2(\Omega)^n}. \end{aligned}$$

Then, by using (72), (73), (74) and by taking δ sufficiently small, we get

$$(75) \quad \frac{1}{\varepsilon} \|Z^\varepsilon(T, \cdot)\|_{L^2(\Omega)^n}^2 + \frac{1}{2} \|\rho_0^{-1} H^{J,\varepsilon}\|_{L^2((0,T) \times \omega)^n}^2 \leq C_T \|Z_0\|_{L^2(\Omega)^n}^2.$$

Remark 7.2. The estimate (75) yields Proposition 6.3 for $S = 0$ by letting $\varepsilon \rightarrow 0$. We remark that we have only used the term $\|\varphi(0, \cdot)\|_{L^2(\Omega)^n}^2$ in the left hand side of (67). The second term in the left hand side of (67) enables to get more regularity (in L^p -sense) for the control H^J (see Section 7.1.3 below).

7.1.2. Maximal regularity theorems and Sobolev embeddings. In this part, we recall a maximal regularity theorem in L^p ($1 < p < +\infty$) for parabolic systems and an embedding result for Sobolev spaces.

We introduce the following spaces: for every $r \in [1, +\infty]$,

$$W_{Ne}^{2,r}(\Omega) := \left\{ u \in W^{2,r}(\Omega) ; \frac{\partial u}{\partial \nu} = 0 \right\}, \quad X_r := L^r(0, T; W_{Ne}^{2,r}(\Omega)) \cap W^{1,r}(0, T; L^r(\Omega)).$$

We have the following maximal regularity theorem.

Proposition 7.3. [13, Theorem 2.1]

Let $1 < r < +\infty$, $k \in \mathbb{N}^*$, $D \in \mathcal{M}_k(\mathbb{R})$ such that $Sp(D) \subset (0, +\infty)$, $A \in \mathcal{M}_k(\mathbb{R})$ and

$S \in L^r(Q_T)^k$. The following Cauchy problem admits a unique solution $U \in X_r^k$

$$\begin{cases} \partial_t U - D\Delta U = AU + S(t, x) & \text{in } (0, T) \times \Omega, \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ U(0, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

Moreover, there exists $C > 0$ independent of S such that

$$\|U\|_{X_r^k} \leq C \|S\|_{L^r(Q_T)^k}.$$

We have the following embedding result for Sobolev spaces.

Proposition 7.4. [39, Theorem 1.4.1]

Let $r \in [1, +\infty[$, we have

$$X_r \hookrightarrow \begin{cases} L^{\frac{(N+2)r}{N+2-2r}}(Q_T) & \text{if } r < \frac{N+2}{2}, \\ L^{2r}(Q_T) & \text{if } r = \frac{N+2}{2}, \\ L^\infty(Q_T) & \text{if } r > \frac{N+2}{2}. \end{cases}$$

7.1.3. Bootstrap method. In the next two parts, we will use the key identity between the control $H^{J,\varepsilon}$ and the solution of the adjoint system φ^ε , i.e., (72) in order to deduce L^p -regularity for $H^{J,\varepsilon}$ from L^p -regularity for φ^ε . This kind of regularity will come from the application of successive L^p -parabolic regularity theorems stated in Proposition 7.3 to a modification of φ^ε called $\psi^{\varepsilon,r}$ (see a precise definition in (80) below) which is bounded from below by $\rho_0^2 \varphi$. The beginning of this bootstrap argument is the strong observability inequality (67). Finally, we will pass to the limit ($\varepsilon \rightarrow 0$) in $\frac{1}{\varepsilon} \|Z^\varepsilon(T, \cdot)\|_{L^2(\Omega)^n}^2 \leq C_T \|Z_0\|_{L^2(\Omega)^n}^2$ coming from (75) and $\|H^{J,\varepsilon}\|_{L^\infty(Q_T)} \leq C_T \|Z_0\|_{L^2(\Omega)^n}$ coming from (89) (see below).

By using Remark 6.1, we introduce the positive real number

$$(76) \quad \gamma := 2p - (1+p)q^2 > 0.$$

Let us define a sequence of increasing positive real numbers $(\gamma_r)_{r \in \mathbb{N}}$ such that $\lim_{r \rightarrow +\infty} \gamma_r = \gamma$, where γ is defined in (76).

We introduce for every $r \in \mathbb{N}$,

$$(77) \quad \rho_{S,r}(t) = M^{-1-p} \exp\left(-\frac{((1+p)q^2 + \gamma_r)M}{(q-1)(T-t)}\right).$$

Then, we have from (59), for every $r \in \mathbb{N}$,

$$(78) \quad \rho_0^2 \leq C_T \rho_{S,r}.$$

We remark that we have for every $r \in \mathbb{N}$,

$$(79) \quad |\rho'_{S,r+1}(t)| \leq C_{T,r} \rho_{S,r}(t).$$

We define for every $r \in \mathbb{N}$,

$$(80) \quad \psi^{\varepsilon,r}(t, x) := \rho_{S,r}(t) \varphi^\varepsilon(t, x).$$

From (66), (77) and (80), we have for every $r \in \mathbb{N}^*$,

$$(81) \quad \begin{cases} -\partial_t \psi^{\varepsilon,r} - D_J^{\text{tr}} \Delta \psi^{\varepsilon,r} = A_J^{\text{tr}} \psi^{\varepsilon,r} - \rho'_{S,r}(t) \varphi^\varepsilon & \text{in } (0, T) \times \Omega, \\ \frac{\partial \psi^{\varepsilon,r}}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \psi^{\varepsilon,r}(T, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

By using (79), we remark that

$$(82) \quad |-\rho'_{S,r}(t) \varphi^\varepsilon| \leq C_T |\psi^{\varepsilon,r-1}|.$$

Let $(p_r)_{r \in \mathbb{N}}$ be the following sequence defined by induction

$$(83) \quad p_0 = 2,$$

$$(84) \quad p_{r+1} := \begin{cases} \frac{(N+2)p_r}{N+2-2p_r} & \text{if } p_r < \frac{N+2}{2}, \\ 2p_r & \text{if } p_r = \frac{N+2}{2}, \\ +\infty & \text{if } p_r > \frac{N+2}{2}. \end{cases}$$

There exists $l \in \mathbb{N}^*$ such that

$$(85) \quad \forall r \geq l, \quad p_r = +\infty.$$

We show, by induction, that for every $0 \leq r \leq l$, we have

$$(86) \quad \psi^{\varepsilon,r} \in L^{p_r}(Q_T)^n \text{ and } \|\psi^{\varepsilon,r}\|_{L^{p_r}(Q_T)^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

The case $r = 0$ can be deduced from the fact that $\gamma_0 > 0$ and the observability estimate (67) ($p_0 = 2$ by (83)).

Let $r \in \mathbb{N}^*$. We assume that

$$(87) \quad \psi^{\varepsilon,r-1} \in L^{p_{r-1}}(Q_T)^n \text{ and } \|\psi^{\varepsilon,r-1}\|_{L^{p_{r-1}}(Q_T)^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

Then, from (81), (82), (87) and from the maximal regularity theorem: Proposition 7.3 applied with $p_{r-1} \in (1, +\infty)$, we get

$$(88) \quad \psi^{\varepsilon,r} \in X_{p_{r-1}}^n \text{ and } \|\psi^{\varepsilon,r}\|_{X_{p_{r-1}}^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

Moreover, by the Sobolev embedding: Proposition 7.4 and (84), we have

$$\psi^{\varepsilon,r} \in L^{p_r}(Q_T)^n \text{ and } \|\psi^{\varepsilon,r}\|_{L^{p_r}(Q_T)^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

This concludes the induction.

7.1.4. *The end of the Penalized Hilbert Uniqueness Method.* Now, by applying consecutively (85) ($p_l = +\infty$), (72), (78) and (86), we have for every $i \in \{1, \dots, m\}$,

$$(89) \quad \|h_i^\varepsilon\|_{L^\infty(Q_T)} = \|\rho_0^2 \varphi_i^\varepsilon\|_{L^{p_l}(Q_T)} \leq C_T \|\rho_{S,l} \varphi_i^\varepsilon\|_{L^{p_l}(Q_T)} \leq C_T \|\psi_i^{\varepsilon,l}\|_{L^{p_l}(Q_T)} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

Therefore, from (89), $(H^{J,\varepsilon})_\varepsilon$ is bounded in $L^\infty(Q_T)^m$, then up to a subsequence, we can assume that there exists $H^J \in L^\infty(Q_T)^m$ such that

$$(90) \quad H^{J,\varepsilon} \xrightarrow[\varepsilon \rightarrow 0]{*} H^J \text{ in } L^\infty(Q_T)^m,$$

and

$$(91) \quad \|H^J\|_{L^\infty(Q_T)^m} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

From (89), Definition-Proposition 2.1 applied to (L-Z) satisfied by Z^ε , we obtain

$$(92) \quad \|Z^\varepsilon\|_{W_T^n} \leq C_T \|Z_0\|_{L^2(\Omega)^n}.$$

So, from (92), up to a subsequence, we can suppose that there exists $Z \in W_T^n$ such that

$$(93) \quad Z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} Z \text{ in } L^2(0, T; H^1(\Omega)^n), \quad \partial_t Z^\varepsilon \xrightarrow[\varepsilon \rightarrow 0]{} \partial_t Z \text{ in } L^2(0, T; (H^1(\Omega))^m),$$

and from (12),

$$(94) \quad Z^\varepsilon(0, \cdot) \xrightarrow[\varepsilon \rightarrow 0]{} Z(0, \cdot) \text{ in } L^2(\Omega)^n, \quad Z^\varepsilon(T, \cdot) \xrightarrow[\varepsilon \rightarrow 0]{} Z(T, \cdot) \text{ in } L^2(\Omega)^n.$$

Then, as we have $Z^\varepsilon(0, \cdot) = Z_0$ and $Z^\varepsilon(T, \cdot) \rightarrow 0$ from (75), we deduce that

$$(95) \quad Z(0, \cdot) = Z_0, \text{ and } Z(T, \cdot) = 0.$$

By letting $\varepsilon \rightarrow 0$, we have from (93), (90) and (95) that

$$(96) \quad \begin{cases} \partial_t Z - D_J \Delta Z = A_J Z + H^J 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ (Z(0, \cdot), Z(T, \cdot)) = (Z_0, 0) & \text{in } \Omega. \end{cases}$$

This ends the proof of Theorem 7.1 by using (91) and (96).

7.2. The come back to the source term method in L^∞ . The goal of this section is to apply the source term method in L^∞ thanks to the null-controllability result in L^∞ : Theorem 7.1.

To simplify the notations, we assume that the control cost in L^∞ of Theorem 7.1 satisfies: $C_T \leq Me^{M/T}$ where M is already defined at the beginning of Section 6.

From Proposition A.6 with $r = +\infty$ proved in Appendix A.5, we deduce the following null-controllability result for (L+S-Z) (see Section 6) in L^∞ .

Proposition 7.5. *For every $S \in \mathcal{S}_\infty$ and $Z_0 \in L_{inv}^\infty$, there exists $H^J \in \mathcal{H}_\infty$, such that the solution Z of (L+S-Z) satisfies $Z \in \mathcal{Z}_\infty$. Furthermore, there exists $C > 0$, not depending on S and Z_0 , such that*

$$(97) \quad \|Z/\rho_0\|_{L^\infty([0,T];L^\infty(\Omega)^n)} + \|H^J\|_{\mathcal{H}_\infty} \leq C \left(\|Z_0\|_{L^\infty(\Omega)^n} + \|S\|_{\mathcal{S}_\infty} \right).$$

In particular, since ρ_0 is a continuous function satisfying $\rho_0(T) = 0$, the above relation (97) yields $Z(T, \cdot) = 0$.

8. THE INVERSE MAPPING THEOREM IN APPROPRIATE SPACES

The goal of this section is to prove Theorem 4.1. The proof is based on Proposition 7.5 and an inverse mapping theorem in suitable spaces.

Proof. Let us introduce the following space (see the definitions (62), (63) and (64)):

$$(98) \quad E := \{(Z, H^J) \in \mathcal{Z}_\infty \times \mathcal{H}_\infty; \partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega \in S_\infty\}.$$

We endow E with the following norm: for every $(Z, H^J) \in E$,

$$(99) \quad \|(Z, H^J)\|_E = \|Z(0, \cdot)\|_{L^\infty} + \|Z\|_{\mathcal{Z}_\infty} + \|H^J\|_{\mathcal{H}_\infty} + \|\partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega\|_{S_\infty}.$$

Then, $(E, \|\cdot\|_E)$ is a Banach space.

For every $Z \in \mathcal{Z}_\infty$, we introduce the following polynomial nonlinearity of degree more than 2:

$$(100) \quad Q(Z) := G(Z) - A_J Z,$$

where G is defined in (31). By denoting $\gamma := \max\left(\sum_{i=1}^n \alpha_i, \sum_{i=1}^n \beta_i\right)$, we remark that for

every $Z \in \mathcal{Z}_\infty$, $Q(Z) = \sum_{i=2}^{\gamma} Q_i(Z)$ where for every $2 \leq i \leq \gamma$, $Q_i(Z)$ is a polynomial term with respect to $Z = (z_1, \dots, z_n)$ of degree i . By using (61), we deduce that $Q(Z) \in S_\infty$ and for every $2 \leq i \leq \gamma$,

$$(101) \quad \|Q_i(Z)\|_{S_\infty} = \left\| \frac{Q_i(Z)}{\rho_S} \right\|_{L^\infty(Q_T)^n} = \left\| \rho_0^{i-2} \frac{\rho_0^2}{\rho_S} \frac{Q_i(Z)}{\rho_0^i} \right\|_{L^\infty(Q_T)^n} \leq C \|Z\|_{\mathcal{Z}_\infty}^i.$$

We introduce the following mapping:

$$(102) \quad \begin{aligned} \mathcal{A} : E &\longrightarrow F := S_\infty \times L_{inv}^\infty \\ (Z, H) &\longmapsto (\partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega - Q(Z), Z(0, \cdot)). \end{aligned}$$

By using (98), the fact that for $(Z, H^J) \in E$ and $Q(Z) \in S_\infty$ by (101), we see that \mathcal{A} is well-defined. Moreover, $\mathcal{A} \in C^1(E; F)$. Indeed, all the terms in (102) are linear and continuous (thus C^∞) thanks to (99) except the term $Q(Z)$. And, for $(Z, H^J) \in E$, $Q(Z)$ is a polynomial function with respect to Z which is C^∞ . Indeed,

$$Q_i(Z) = \sum_{\substack{\gamma_1 + \dots + \gamma_n = i \\ \gamma_1, \dots, \gamma_n \geq 0}} c_{\gamma_1, \dots, \gamma_n} z_1^{\gamma_1} \dots z_n^{\gamma_n} = \sum_{\substack{\gamma_1 + \dots + \gamma_n = i \\ \gamma_1, \dots, \gamma_n \geq 0}} c_{\gamma_1, \dots, \gamma_n} \mathcal{B}_{\gamma_1, \dots, \gamma_n} \circ \mathcal{L}_{\gamma_1, \dots, \gamma_n}(Z),$$

where

$$\mathcal{L}_{\gamma_1, \dots, \gamma_n}(z_1, \dots, z_n) := \underbrace{(z_1, \dots, z_1)}_{\gamma_1 \text{ times}}, \underbrace{(z_2, \dots, z_2)}_{\gamma_2 \text{ times}}, \dots, \underbrace{(z_n, \dots, z_n)}_{\gamma_n \text{ times}},$$

$$\mathcal{B}_{\gamma_1, \dots, \gamma_n} \underbrace{(a_{1,1}, \dots, a_{1,\gamma_1}, a_{2,1}, \dots, a_{2,\gamma_2}, \dots, a_{n,1}, \dots, a_{n,\gamma_n})}_{a_{\gamma_1, \dots, \gamma_n}} := \prod_{i=1}^n \prod_{j=1}^{\gamma_i} a_{i,j}.$$

To simplify, we renote $\mathcal{L} := \mathcal{L}_{\gamma_1, \dots, \gamma_n}$ and $\mathcal{B} := \mathcal{B}_{\gamma_1, \dots, \gamma_n}$. The mapping \mathcal{L} is C^∞ because \mathcal{L} is linear and continuous. The mapping \mathcal{B} is C^∞ because \mathcal{B} is i -linear and continuous. Indeed, by using (61), we have

$$\begin{aligned} \|\mathcal{B}(a_{\gamma_1, \dots, \gamma_n})\|_{\mathcal{S}_\infty} &= \left\| \frac{\mathcal{B}(a_{\gamma_1, \dots, \gamma_n})}{\rho_S} \right\|_{L^\infty(Q_T)^n} = \left\| \frac{\rho_0^{i-2} \rho_0^2 \mathcal{B}(a_{\gamma_1, \dots, \gamma_n})}{\rho_S \rho_0^i} \right\|_{L^\infty(Q_T)^n} \\ &\leq C \prod_{i=1}^n \prod_{j=1}^{\gamma_i} \|a_{i,j}\|_{Z_\infty}. \end{aligned}$$

Moreover, the differential of \mathcal{A} at the point $(0, 0)$ in the direction (Z, H^J) is

$$(103) \quad D\mathcal{A}(0, 0).(Z, H^J) = (\partial_t Z - D_J \Delta Z - A_J Z - H^J 1_\omega, Z(0, \cdot)),$$

which is onto by using Proposition 7.5. Then, by using the inverse mapping theorem (see [10, Theorem 2]), we deduce that there exists $r > 0$, such that for every $(S, Z_0) \in F$ satisfying $\|(S, Z_0)\|_F \leq r$, there exists $(Z, H^J) \in E$ such that $\mathcal{A}(Z, H^J) = (S, Z_0)$. By taking $S = 0$ and $Z_0 \in L_{inv}^\infty$ such that $\|Z_0\|_{L^\infty(\Omega)^n} \leq r$, we get the existence of $(Z, H^J) \in Z_\infty \times \mathcal{H}_\infty$ such that

$$\begin{cases} \partial_t Z - D_J \Delta Z = \underbrace{A_J Z + Q(Z)}_{G(Z) \text{ by (100)}} + H^J 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial Z}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ (Z(0, \cdot), Z(T, \cdot)) = (Z_0, 0) & \text{in } \Omega. \end{cases}$$

This concludes the proof of Theorem 4.1. \square

9. COMMENTS

9.1. More general semilinearities. In this paper, we have only considered particular semilinearities of the form:

$$(104) \quad \forall 1 \leq i \leq n, f_i(u_1, \dots, u_n) = (\beta_i - \alpha_i) \left(\prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k} \right).$$

But the main result of the article, i.e., Theorem 4.2 holds true with more general polynomial semilinearities satisfying

$$\exists R \in \mathbb{R}[X_1, \dots, X_n], \forall 1 \leq i \leq n, \exists a_i \in \mathbb{R}^*, f_i = a_i R,$$

where $\mathbb{R}[X_1, \dots, X_n]$ denotes the space of multivariate polynomials with coefficients in \mathbb{R} . In this case, $(u_i^*)_{1 \leq i \leq n}$ is a constant nonnegative stationary state if

$$(u_i^*)_{1 \leq i \leq n} \in [0, +\infty)^n \text{ and } R(u_1^*, \dots, u_n^*) = 0.$$

For example, (104) rewrites as follows

$$\forall 1 \leq i \leq n, f_i(X_1, \dots, X_n) = (\beta_i - \alpha_i) \left(\prod_{k=1}^n X_k^{\alpha_k} - \prod_{k=1}^n X_k^{\beta_k} \right).$$

9.2. Degenerate cases. In this part, we assume that Assumption 3.2 is not satisfied. Then, the usual strategy is to perform the return method, introduced by Jean-Michel Coron in [8] (see also [9, Chapter 6]). This method consists in finding a reference trajectory (\bar{U}, \bar{H}^J) verifying $U(0, \cdot) = U(T, \cdot) = U^*$ of (NL-U) (see Section 1.2) such that the linearized system of (NL-U) around (\bar{U}, \bar{H}^J) is null-controllable.

Example 9.1. For $n = 2$, we take $\alpha_1 = 3$, $\beta_1 = 0$, $\alpha_2 = 0$, $\beta_2 = 1$ and $J = \{1\}$. We get the following control reaction-diffusion system:

$$(105) \quad \begin{cases} \partial_t u_1 - d_1 \Delta u_1 = -u_1^3 + u_2 + h_1 1_\omega & \text{in } (0, T) \times \Omega, \\ \partial_t u_2 - d_2 \Delta u_2 = u_1^3 - u_2 & \text{in } (0, T) \times \Omega, \\ \frac{\partial U}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ U(0, \cdot) = U_0 & \text{in } \Omega. \end{cases}$$

In this case, Assumption 3.2 is not satisfied if and only if $(u_1^*, u_2^*) = (0, 0)$. By using the return method, Jean-Michel Coron, Sergio Guerrero and Lionel Rosier prove the local null-controllability around $(0, 0)$ of (105) (see [11]).

Example 9.2. For $n = 4$, we take $\alpha_1 = \alpha_3 = \beta_2 = \beta_4 = 1$, $\alpha_2 = \alpha_4 = \beta_1 = \beta_3 = 0$ and $J = \{1, 2, 3\}$. Then, we get the controlled reaction-diffusion system as in (40). In this case, Assumption 3.2 is not satisfied if and only if $(u_1^*, u_3^*, u_4^*) = (0, 0, 0)$. More precisely, the linearized system around $\left((0, u_2^*, 0, 0), (0, 0, 0)\right)$ is not null-controllable because the fourth equation is decoupled from the others:

$$\partial_t u_4 - d_4 \Delta u_4 = u_2^* u_4.$$

By using the return method, the author proves the local null-controllability around $(0, u_2^*, 0, 0)$ of (40) (see [28]). More precisely, for this system, a reference trajectory is not difficult to construct. Indeed, one can take $\left((0, u_2^*, g, 0), (0, 0, \partial_t g - d_3 \Delta g)\right)$ where g satisfies

$$(106) \quad g \in C^\infty(\overline{Q}), \quad g \geq 0, \quad g \neq 0, \quad \text{supp}(g) \subset (0, T) \times \omega.$$

Thus, the fourth equation of the linearized system around this trajectory is:

$$\partial_t u_4 - d_4 \Delta u_4 = g(t, x) u_1 - u_2^* u_4.$$

Roughly speaking, the component u_4 can be controlled throughout the coupling term $g(t, x) u_1$ which lives in the control zone (see for instance [11, Section 3.1, Lemma 3]).

Example 9.3. For $n = 10$, we take $\alpha_1 = \alpha_3 = \alpha_5 = \alpha_7 = \alpha_9 = \beta_2 = \beta_4 = \beta_6 = \beta_8 = \beta_{10} = 1$, $\alpha_2 = \alpha_4 = \alpha_6 = \alpha_8 = \alpha_{10} = \beta_1 = \beta_3 = \beta_5 = \beta_7 = \beta_9 = 0$ and $J = \{1, 2, 3, 4, 5\}$. The control system is the following one:

$$(107) \quad \forall 1 \leq i \leq 10, \quad \begin{cases} \partial_t u_i - d_i \Delta u_i = \\ (-1)^i (u_1 u_3 u_5 u_7 u_9 - u_2 u_4 u_6 u_8 u_{10}) + h_i 1_\omega 1_{i \in J} & \text{in } (0, T) \times \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ u_i(0, \cdot) = u_{i,0} & \text{in } \Omega. \end{cases}$$

The stationary state

$$(0, 0, 0, 0, u_5^*, u_6^*, u_7^*, u_8^*, u_9^*, u_{10}^*),$$

where $(u_5^*, u_7^*, u_9^*) \in (0, +\infty)^3$ and $(u_6^*, u_8^*, u_{10}^*) \in [0, +\infty)^3$ does not satisfy Assumption 3.2. In this case, the linearized system around

$\left((0, 0, 0, 0, u_5^*, u_6^*, u_7^*, u_8^*, u_9^*, u_{10}^*), (0, 0, 0, 0, 0)\right)$ is not null-controllable because the sixth equation of this system is:

$$\partial_t u_6 - d_6 \Delta u_6 = 0.$$

As for Example 9.2, we can easily construct a reference trajectory:

$$\left((0, 0, g, 0, u_5^*, u_6^*, u_7^*, u_8^*, u_9^*, u_{10}^*), (0, 0, \partial_t g - d_3 \Delta g, 0, 0)\right),$$

where g satisfies (106). By performing the same change of variables as in Section 3.1 and by linearizing around the reference trajectory, we find the same system as in (L-Z) (see Section 3.2 with $n = 10$, $m = 5$) where the coefficients of A are allowed

to depend on (t, x) and $a_{61}(t, x) \geq \varepsilon > 0$ on $(t_1, t_2) \times \omega_0 \subset (0, T) \times \omega$. This linear system seems to be null-controllable according to the following heuristic diagram:

$$\begin{array}{ccccccccc} h_1 & \xrightarrow{\text{controls}} & z_1, & h_2 & \xrightarrow{\text{controls}} & z_2, & h_3 & \xrightarrow{\text{controls}} & z_3, & h_4 & \xrightarrow{\text{controls}} & z_4, & h_5 & \xrightarrow{\text{controls}} & z_5, \\ z_1 & \xrightarrow[\substack{\text{controls} \\ a_{61}(t,x)z_1}]{} & z_6 & \xrightarrow[\Delta z_6]{\text{controls}} & z_7 & \xrightarrow[\Delta z_7]{\text{controls}} & z_8 & \xrightarrow[\Delta z_8]{\text{controls}} & z_9 & \xrightarrow[\Delta z_9]{\text{controls}} & z_{10}. \end{array}$$

Unfortunately, we do not know how to prove that the linearized system around this trajectory is null-controllable for technical reasons maybe. It comes from the fact that in this case $m = 5 < n - 4 = 6$. Intuitively, with the proof strategy performed in [28], we have to benefit from one coupling term of order 0 (in L^∞) and four coupling terms of order 2. This leads to the following open problem.

We introduce the following notation: $C_b^\infty(\overline{Q_T})$ is the set of functions A defined on $\overline{Q_T}$ of class C^∞ and such that all the derivatives of A are bounded. We choose to state the following open problem for Dirichlet conditions instead of Neumann conditions to avoid the constraints on the initial data.

Open problem 9.4. Let n, m be two integers such that $n \geq 6$, $m < n - 4$ and $(d_i)_{1 \leq i \leq n} \in (0, +\infty)^n$. Let $A \in C_b^\infty(\overline{Q_T})^{m \times n}$. We assume that there exist $(t_1, t_2) \subset (0, T)$, a nonempty open subset ω_0 such that $\omega_0 \subset\subset \omega$ and $\varepsilon > 0$ such that $A_{m+1,m}(t, x) \geq \varepsilon$ on $(t_1, t_2) \times \omega_0$. For $y_0 \in L^2(\Omega)^n$, $(h_i)_{1 \leq i \leq m} \in L^2((0, T) \times \Omega)^m$, we consider the linear control system:

$$(108) \quad \begin{cases} \partial_t y_i - d_i \Delta y_i = \sum_{j=1}^n A_{i,j}(t, x) y_j + h_i 1_\omega 1_{1 \leq i \leq m} & \text{in } (0, T) \times \Omega, & 1 \leq i \leq m+1 \\ \partial_t y_i - d_i \Delta y_i = \Delta y_{i-1} & \text{in } (0, T) \times \Omega, & m+2 \leq i \leq n \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0, \cdot) = y_0 & \text{in } \Omega. \end{cases}$$

Is the system (108) null-controllable in $L^2(\Omega)^n$?

Remark 9.5. Open problem 9.4 is closely related to the generalization of [15, Theorem 1.1] to linear parabolic systems with diffusion matrices that contain Jordan blocks of dimension more than 5. Indeed, the diffusion matrix of (108) is D_J defined in (32). The submatrix D_\sharp (see again (32)) looks like a Jordan block of dimension more than 5 if $m < n - 4$. Consequently, the strategy of Carleman inequalities applied to each equation of the adjoint system of (108) yields some global terms in the right hand side of the inequality that cannot be absorbed by the left hand side (see [15, Section 2]).

APPENDIX A.

A.1. Stationary states. We only have considered nonnegative stationary constant solutions of (2). It is not restrictive because of the following proposition.

Proposition A.1. *Let $(u_i)_{1 \leq i \leq n} \in C^2(\overline{\Omega})^n$ be a nonnegative solution of*

$$(109) \quad \forall 1 \leq i \leq n, \quad \begin{cases} -d_i \Delta u_i = f_i(U) & \text{in } \Omega, \\ \frac{\partial u_i}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_i(U)$ ($1 \leq i \leq n$) is defined in (6). Then, for every $1 \leq i \leq n$, u_i is constant.

The proof relies on an entropy inequality: $-\sum_{i=1}^n \log(u_i) f_i(U) \leq 0$.

Proof. Let $\varepsilon > 0$ be a small parameter. For every $1 \leq i \leq n$, we introduce

$$u_{i,\varepsilon} = u_i + \varepsilon, \quad w_{i,\varepsilon} = u_{i,\varepsilon} (\log u_{i,\varepsilon} - 1) + 1 \geq 0.$$

We have

$$(110) \quad \forall 1 \leq i \leq n, \quad \nabla w_{i,\varepsilon} = \log(u_{i,\varepsilon}) \nabla u_{i,\varepsilon}, \quad \Delta w_{i,\varepsilon} = \log(u_{i,\varepsilon}) \Delta u_{i,\varepsilon} + \frac{|\nabla u_{i,\varepsilon}|^2}{u_{i,\varepsilon}}.$$

Then, from (109) and (110), we have

$$(111) \quad \forall 1 \leq i \leq n, \begin{cases} -d_i \Delta w_{i,\varepsilon} + d_i \frac{|\nabla u_{i,\varepsilon}|^2}{u_{i,\varepsilon}} = -\log(u_{i,\varepsilon}) f_i(U) & \text{in } \Omega, \\ \frac{\partial w_{i,\varepsilon}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

We sum the n equations of (111), we integrate on Ω and we use the increasing of the function \log :

$$(112) \quad \begin{aligned} & 0 + \int_{\Omega} \sum_{i=1}^n d_i \frac{|\nabla u_{i,\varepsilon}|^2}{u_{i,\varepsilon}} \\ &= - \left(\int_{\Omega} \left\{ \log \left(\prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i} \right) - \log \left(\prod_{i=1}^n u_{i,\varepsilon}^{\beta_i} \right) \right\} \left\{ \prod_{i=1}^n u_i^{\alpha_i} - \prod_{i=1}^n u_i^{\beta_i} \right\} \right) \\ &= - \left(\int_{\Omega} \left\{ \log \left(\prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i} \right) - \log \left(\prod_{i=1}^n u_{i,\varepsilon}^{\beta_i} \right) \right\} \left\{ \prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i} - \prod_{i=1}^n u_{i,\varepsilon}^{\beta_i} + \mathcal{O}(\varepsilon) \right\} \right) \\ &\leq \int_{\Omega} \left| \log \left(\prod_{i=1}^n u_{i,\varepsilon}^{\alpha_i} \right) - \log \left(\prod_{i=1}^n u_{i,\varepsilon}^{\beta_i} \right) \right| \mathcal{O}(\varepsilon) \leq \left(\sum_{i=1}^n (\alpha_i + \beta_i) \right) |\log(\varepsilon)| \mathcal{O}(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

Moreover,

$$(113) \quad \forall 1 \leq i \leq n, \int_{\Omega} d_i \frac{|\nabla u_i^{\varepsilon}|^2}{u_i^{\varepsilon}} = \int_{\Omega} 4d_i |\nabla \sqrt{u_i^{\varepsilon}}|^2.$$

Consequently, from (112), (113), we get that

$$\forall 1 \leq i \leq n, \int_{\Omega} 4d_i |\nabla \sqrt{u_i}|^2 = 0.$$

Consequently, for every $1 \leq i \leq n$, u_i is constant. \square

Our proof of Theorem 4.2 does not treat the case of stationary states which can change of sign, contrary to the proof of [28, Theorem 3.2] (see [28, Section 6.2]). As in the previous part (see Example 9.3), the proof of [28, Theorem 3.2] can be adapted to local controllability around stationary states of (2) which can change of sign if $m \geq n - 4$ (for technical reasons maybe, see Open problem 9.4).

A.2. Proof of the existence of invariant quantities in the system. The goal of this section is to prove Proposition 2.3.

Proof. We introduce the notation $R := \prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k}$ and we take $m+1 \leq i \leq n$.

By using the fact that $u_i \in W_T$ and from [14, Lemma 3], we obtain that the mapping $t \mapsto \int_{\Omega} u_i(t, x) dx$ is absolutely continuous and for a.e. $0 \leq t \leq T$,

$$(114) \quad \frac{d}{dt} \int_{\Omega} u_i(t, x) dx = (\partial_t u_i(t, \cdot), 1)_{(H^1(\Omega))', H^1(\Omega)}.$$

Then, by using that $((u_i)_{1 \leq i \leq n}, (h_i)_{1 \leq i \leq m})$ is a trajectory of (NL-U) and by taking $w = 1$ in (18), we find that for a.e. $0 \leq t \leq T$,

$$(115) \quad \begin{aligned} (\partial_t u_i(t, \cdot), 1)_{(H^1(\Omega))', H^1(\Omega)} &= d_i (\nabla u_i(t, \cdot), \nabla 1)_{L^2(\Omega), L^2(\Omega)} + \int_{\Omega} (\beta_i - \alpha_i) R \\ &= \int_{\Omega} (\beta_i - \alpha_i) R. \end{aligned}$$

Then, by using (114) and (115), we get for a.e. $0 \leq t \leq T$,

$$(116) \quad \frac{d}{dt} \int_{\Omega} \frac{u_i(t, \cdot)}{\beta_i - \alpha_i} = \int_{\Omega} R.$$

Now, let $m + 1 \leq k \neq l \leq n$. By (116) for $i = k$ and (116) for $i = l$, we deduce that for a.e. $0 \leq t \leq T$,

$$(117) \quad \frac{d}{dt} \int_{\Omega} \left(\frac{u_k(t, \cdot)}{\beta_k - \alpha_k} - \frac{u_l(t, \cdot)}{\beta_l - \alpha_l} \right) = 0.$$

Therefore, from (117), we have for every $t \in [0, T]$,

$$\frac{1}{|\Omega|} \int_{\Omega} \left(\frac{u_k(t, x)}{\beta_k - \alpha_k} - \frac{u_l(t, x)}{\beta_l - \alpha_l} \right) dx = \frac{u_k^*}{\beta_k - \alpha_k} - \frac{u_l^*}{\beta_l - \alpha_l}.$$

If we assume that $d := d_k = d_l$, then the equation satisfied by $v := (\beta_l - \alpha_l)u_k - (\beta_k - \alpha_k)u_l$ is

$$(118) \quad \begin{cases} \partial_t v - d\Delta v = 0 & \text{in } (0, T) \times \Omega, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } (0, T) \times \partial\Omega, \\ v(T, \cdot) = (\beta_l - \alpha_l)u_k^* - (\beta_k - \alpha_k)u_l^* & \text{in } \Omega. \end{cases}$$

The backward uniqueness of the heat equation (see for instance [5, Théorème II.1]) applied to (118) leads to

$$\forall t \in [0, T], (\beta_l - \alpha_l)u_k(t, \cdot) - (\beta_k - \alpha_k)u_l(t, \cdot) = (\beta_l - \alpha_l)u_k^* - (\beta_k - \alpha_k)u_l^*.$$

This yields (21). \square

A.3. Proofs concerning the change of variables.

A.3.1. *Proof of the equivalence of the two systems.* In this section, we prove Proposition 3.1. It is based on the following algebraic lemma.

Lemma A.2. *Let s be an integer such that $s \geq 2$. Let $(a_1, \dots, a_s) \in \mathbb{C}^s$ be such that $a_i \neq a_j$ for $i \neq j$. Then, we have*

$$(119) \quad \sum_{i=1}^s \prod_{\substack{j=1 \\ j \neq i}}^s \frac{1}{a_i - a_j} = 0.$$

Proof. Let $\mathbb{C}(X)$ be the field of fractional functions with coefficients in \mathbb{C} and $F \in \mathbb{C}(X)$ be defined by

$$(120) \quad F(X) := \left(\sum_{i=1}^{s-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{s-1} \frac{1}{a_i - a_j} \right) \frac{1}{a_i - X} \right) + \prod_{j=1}^{s-1} \frac{1}{X - a_j}.$$

The partial fractional decomposition of F is the following one:

$$F(X) = \sum_{i=1}^{s-1} \frac{b_i}{X - a_i}, \quad \text{where } b_i \in \mathbb{C}.$$

For $1 \leq i \leq s-1$, we compute each b_i by multiplying (120) by $(X - a_i)$ and by evaluating $X = a_i$:

$$b_i = - \prod_{\substack{j=1 \\ j \neq i}}^{s-1} \frac{1}{a_i - a_j} + \prod_{\substack{j=1 \\ j \neq i}}^{s-1} \frac{1}{a_i - a_j} = 0.$$

We deduce that $F = 0$. By remarking that

$$F(a_s) = \sum_{i=1}^s \prod_{\substack{j=1 \\ j \neq i}}^s \frac{1}{a_i - a_j} = 0,$$

we conclude the proof of (119). \square

The following result is an easy consequence of Lemma A.2.

Corollary A.3. For every $m + 2 \leq k \leq n$, we have

$$(121) \quad \sum_{l=m+1}^k P_{kl}(\beta_l - \alpha_l) = 0.$$

Proof. By (27), we have by taking $s = k - m$ and $a_i = d_{i+m}$ for $1 \leq i \leq k - m$ in Lemma A.2

$$\sum_{l=m+1}^k P_{kl}(\beta_l - \alpha_l) = \sum_{i=1}^{k-m} P_{k,i+m}(\beta_{i+m} - \alpha_{i+m}) = \sum_{i=1}^{k-m} \prod_{\substack{j=1 \\ j \neq i}}^{i-m} \frac{1}{d_{i+m} - d_{j+m}} = 0.$$

This ends the proof of Corollary A.3. \square

Now, we turn to the proof of Proposition 3.1.

Proof. We introduce the following notation: $R := \prod_{k=1}^n u_k^{\alpha_k} - \prod_{k=1}^n u_k^{\beta_k}$.

We assume that (U, H^J) is a trajectory of (NL-U). The equations $1 \leq i \leq m + 1$ of (28) are clearly satisfied. Let $m + 2 \leq i \leq n$. We have:

$$(122) \quad \begin{aligned} \partial_t z_i - d_i \Delta z_i &= \partial_t \left(\sum_{j=m+1}^i P_{ij} u_j \right) - d_i \Delta \left(\sum_{j=m+1}^i P_{ij} u_j \right) \\ &= \sum_{j=m+1}^i P_{ij} (\partial_t u_j - d_j \Delta u_j + (d_j - d_i) \Delta u_j) \\ &= \sum_{j=m+1}^i \left(P_{ij} ((\beta_j - \alpha_j) R) + P_{ij} \underbrace{(d_j - d_i)}_{0 \text{ if } j=i} \Delta u_j \right) \\ &= R \underbrace{\sum_{j=m+1}^i P_{ij} (\beta_j - \alpha_j)}_{0 \text{ by Corollary A.3}} + \sum_{j=m+1}^{i-1} \underbrace{P_{ij} (d_j - d_i)}_{P_{i-1,j} \text{ by (27)}} \Delta u_j \\ &= \Delta z_{i-1}. \end{aligned}$$

This ends the proof of “ \Rightarrow ”.

We assume that (Z, H^J) satisfies (28). Then, the equations

$$(123) \quad \partial_t u_i - d_i \Delta u_i = (\beta_i - \alpha_i) R,$$

are clearly satisfied for $1 \leq i \leq m + 1$. We prove (123) by strong induction on $i \in \{m + 2, \dots, n\}$. By using (122) for $i = m + 2$ and (123) for $i = m + 1$, we obtain

$$\begin{aligned} \sum_{j=m+1}^{m+2} P_{m+2,j} (\partial_t u_j - d_j \Delta u_j) &= 0 \\ \Leftrightarrow P_{m+2,m+2} (\partial_t u_{m+2} - d_{m+2} \Delta u_{m+2}) &= -R P_{m+2,m+1} (\beta_{m+1} - \alpha_{m+1}). \end{aligned}$$

This leads to (123) for $i = m + 2$ by using $P_{m+2,m+1}/P_{m+2,m+2} = -(\beta_{m+2} - \alpha_{m+2})/(\beta_{m+1} - \alpha_{m+1})$ by (27). For $i > m + 2$, by induction, we have $P_{ii} (\partial_t u_i - d_i \Delta u_i) + \sum_{j=m+1}^{i-1} P_{ij} (\beta_j - \alpha_j) R = 0$ by (122). Then, from Corollary A.3, we have

$$\sum_{j=m+1}^{i-1} P_{ij} (\beta_j - \alpha_j) = -P_{ii} (\beta_i - \alpha_i). \text{ This yields (123) and ends the proof of “}\Leftarrow\text{”}.$$

This concludes the proof of Proposition 3.1. \square

A.3.2. *Proof of the equivalence concerning the mass condition.* In this section, we prove the equivalence (39) which leads to the equivalence between Theorem 4.1 and Theorem 4.2.

Proof. Assume that $Z_0 \in L_{inv}^\infty$. Then, we have

$$(124) \quad \forall m+2 \leq i \leq n, \quad \int_{\Omega} \sum_{k=m+1}^i P_{ik}(u_{k,0}(x) - u_k^*) dx = 0.$$

We prove (24) by strong induction on $k \geq m+2$. The case $k = m+2$ comes from (124) for $i = m+2$ and $P_{m+2,m+1}/P_{m+2,m+2} = -(\beta_{m+2} - \alpha_{m+2})/(\beta_{m+1} - \alpha_{m+1})$ by (27). For $i > m+2$ in (124), by induction, we have

$$\int_{\Omega} \left\{ P_{ii}(u_{i,0}(x) - u_i^*) + \sum_{k=m+1}^{i-1} P_{ik} \frac{(\beta_k - \alpha_k)(u_{m+1,0}(x) - u_{m+1}^*)}{\beta_{m+1} - \alpha_{m+1}} \right\} dx = 0.$$

Then, from Corollary A.3, we have $\sum_{k=m+1}^{i-1} P_{ik}(\beta_k - \alpha_k) = -P_{ii}(\beta_i - \alpha_i)$. This yields

(24) for $k = i$.

Assume (24) holds. From Corollary A.3, we have that for every $m+2 \leq i \leq n$,

$$\int_{\Omega} \sum_{k=m+1}^i P_{ik}(u_{k,0}(\cdot) - u_k^*) = \int_{\Omega} \sum_{k=m+1}^i P_{ik} \frac{\beta_k - \alpha_k}{\beta_{m+1} - \alpha_{m+1}} (u_{m+1,0}(\cdot) - u_{m+1}^*) = 0.$$

This ends the proof of (39). \square

A.4. Proof of an observability estimate for linear finite dimensional systems. The goal of this section is to give a self-contained proof of Lemma 5.5. By the Hilbert Uniqueness Method (see [9, Theorem 2.44]), it suffices to show the following null-controllability result for finite dimensional systems.

Proposition A.4. *There exist $C > 0$, $p_1, p_2 \in \mathbb{N}$ such that for every $\tau \in (0, 1)$, $\lambda \geq \lambda_1$ with λ_1 the first positive eigenvalue of $(-\Delta, H_{Ne}^2(\Omega))$, $y_0 \in \mathbb{R}^n$, there exists a control $h \in L^2(0, \tau; \mathbb{R}^m)$ verifying*

$$(125) \quad \|h\|_{L^2(0, \tau; \mathbb{R}^m)}^2 \leq C \left(1 + \frac{1}{\tau^{p_1}} + \lambda^{p_2} \right) \|y_0\|_{\mathbb{R}^n}^2$$

such that the solution $y \in L^2(0, \tau; \mathbb{R}^n)$ of

$$(126) \quad \begin{cases} y' = Ay + Bh, & \text{in } (0, \tau), \\ y(0) = y_0 & \text{in } \mathbb{R}^n, \end{cases}$$

where $A = -\lambda D_J + A_J$ (see (32), (36) and (37)) and $B = \begin{pmatrix} I_m \\ 0 \end{pmatrix} \in \mathcal{M}_{n,m}(\mathbb{R})$, satisfies $y(\tau) = 0$.

Remark A.5. We do not treat the case $\lambda_0 = 0$ with initial data $y_0 \in \mathbb{R}^{m+1} \times \{0\}^{n-m-1}$ because it is a simple adaptation of the following proof.

Proof. Let $\tau \in (0, 1)$, $\lambda \geq \lambda_1$, $y_0 \in \mathbb{R}^n$.

Step 1: Construction of the control h by a Brunovsky approach. We start by defining \bar{y} to be the free solution of the system (126) (take $h = 0$). We have $\bar{y}(t) = e^{tA} y_0 = e^{t(-\lambda D_J + A_J)} y_0$. We easily have that for any $l \geq 0$,

$$(127) \quad \left\| \bar{y}^{(l)} \right\|_{L^2(0, \tau; \mathbb{R}^n)} \leq C(1 + \lambda^{l-1/2}) \|y_0\|_{\mathbb{R}^n}.$$

We choose a cut-off function $\eta \in C^\infty([0, \tau]; \mathbb{R})$ such that $\eta = 1$ on $[0, \tau/3]$ and $\eta = 0$ on $[2\tau/3, \tau]$ verifying:

$$(128) \quad \forall p \in \mathbb{N}, \quad \forall t \in [0, \tau], \quad |\eta^{(p)}(t)| \leq \frac{C_p}{\tau^p}.$$

We start by choosing for every $i \in \{1, \dots, m-1, n\}$,

$$(129) \quad y_i(t) := \eta(t)\bar{y}_i(t).$$

Then, by using the cascade form of (126), we define by reverse induction on $i \in \{n-1, n-2, \dots, m+1\}$,

$$(130) \quad y_i(t) := -\frac{1}{\lambda} (y'_{i+1}(t) + \lambda d_{i+1} y_{i+1}(t)).$$

Then, y_m is defined by the equation number $(m+1)$ by

$$(131) \quad y_m(t) := \frac{1}{a_{m+1,m}} \left(y'_{m+1}(t) + \lambda d_{m+1} y_{m+1}(t) - \sum_{\substack{s=1 \\ s \neq m}}^n a_{m+1,s} y_s(t) \right).$$

Finally, we set for the control

$$(132) \quad h := y' - Ay.$$

By (131) and (132), h is of the form $h = (h_1, \dots, h_m, 0, \dots, 0)$.

Step 2: Properties of the solution y and estimate of the control h . First, we remark that,

$$(133) \quad \forall 1 \leq i \leq n, \quad \begin{cases} y_i = \bar{y}_i, & \text{in } [0, \tau/3], \\ y_i = 0, & \text{in } [2\tau/3, \tau]. \end{cases}$$

Indeed, the property (133) is clear for $i \in \{1, \dots, m-1, n\}$ by definition (129). Then, we prove (133) by reverse induction on $m \leq i \leq n$ by using (130), (131) and the definition of \bar{y} , for instance, for $t \in [0, \tau/3]$:

$$y_{n-1}(t) = -\frac{1}{\lambda} (y'_n(t) + \lambda d_n y_n(t)) = -\frac{1}{\lambda} (\bar{y}'_n(t) + \lambda d_n \bar{y}_n(t)) = \bar{y}_{n-1}(t).$$

Now, we have by (129), (128) and (127) that for every $i \in \{1, \dots, m-1\}$,

$$(134) \quad \sum_{l=0}^1 \|y_i^{(l)}\|_{L^2(0,\tau;\mathbb{R}^n)} \leq C \left(1 + \frac{1}{\tau^{1/2}} + \lambda^{1/2} \right) \|y_0\|_{\mathbb{R}^n}.$$

Then, we easily prove by reverse induction on $m \leq i \leq n$ by using (127), (128), (129), (130), (131) and (134)

$$(135) \quad \sum_{l=0}^{i+1-m} \|y_i^{(l)}\|_{L^2(0,\tau;\mathbb{R}^n)} \leq C \left(1 + \frac{1}{\tau^{n-m+1/2}} + \lambda^{n-m+1/2} \right) \|y_0\|_{L^2(0,\tau;\mathbb{R}^n)}.$$

Hence, the control h and the state y satisfy (125), (126) with $p_1 = p_2 = 2(n - m + 1/2)$ and $y(\tau) = 0$. \square

A.5. Source term method in L^r for $r \in \{2, +\infty\}$. We use the same notations as in the beginning of Section 6. The goal of this section is to prove Proposition 6.3 and Proposition 7.5. We have the following result.

Proposition A.6. *For every $S \in \mathcal{S}_r$ and $Z_0 \in L^r_{inv}$, there exists $H^J \in \mathcal{H}_r$, such that the solution Z of (L+S-Z) satisfies $Z \in \mathcal{Z}_r$. Furthermore, there exists $C > 0$, not depending on S and Z_0 , such that*

$$(136) \quad \|Z/\rho_0\|_{L^\infty([0,T];L^r(\Omega)^n)} + \|H^J\|_{\mathcal{H}_r} \leq C_T \left(\|Z_0\|_{L^r(\Omega)^n} + \|S\|_{\mathcal{S}_r} \right),$$

where $C_T = Ce^{C/T}$.

The proof is inspired by [6, Proposition 2.6].

Proof. For $k \geq 0$, we define $T_k = T(1 - q^{-k})$. On the one hand, let $a_0 = Z_0$ and, for $k \geq 0$, we define $a_{k+1} = Z_S(T_{k+1}^-, \cdot)$ where Z_S is the solution to

$$(137) \quad \begin{cases} \partial_t Z_S - D_J \Delta Z_S = A_J Z_S + S & \text{in } (0, T) \times \Omega, \\ \frac{\partial Z_S}{\partial \nu} = 0 & \text{on } (0, T) \times \partial \Omega, \\ Z_S(T_k^+, \cdot) = 0 & \text{in } \Omega. \end{cases}$$

From Definition-Proposition 2.1, using the estimates (16) and (12) for $r = 2$ or (17) and (12) for $r = +\infty$, we have

$$(138) \quad \|a_{k+1}\|_{L^r(\Omega)^n} \leq \|Z_S\|_{L^\infty([T_k, T_{k+1}]; L^r(\Omega)^n)} \leq C \|S\|_{L^r((T_k, T_{k+1}); L^r(\Omega)^n)}.$$

On the other hand, for $k \geq 0$, we also consider the control systems

$$(139) \quad \begin{cases} \partial_t Z_H - D_J \Delta Z_H = A_J Z_H + H^J 1_\omega & \text{in } (0, T) \times \Omega, \\ \frac{\partial Z_H}{\partial \nu} = 0 & \text{on } (0, T) \times \partial \Omega, \\ Z_H(T_k^+, \cdot) = a_k & \text{in } \Omega. \end{cases}$$

Using Theorem 5.1 for $r = 2$ or Theorem 7.1 for $r = +\infty$, we can define $H_k^J \in L^r((T_k, T_{k+1}) \times \Omega)^m$ such that $Z_H(T_{k+1}^-, \cdot) = 0$ and, thanks to the cost estimate (41) for $r = 2$ or (68) for $r = +\infty$ (recalling that $C_T \leq M e^{M/T}$),

$$(140) \quad \|H_k^J\|_{L^r((T_k, T_{k+1}) \times \Omega)^m} \leq M e^{\frac{M}{T_{k+1} - T_k}} \|a_k\|_{L^2(\Omega)^n}.$$

In particular, for $k = 0$, we have

$$(141) \quad \|H_0^J\|_{L^r((T_0, T_1) \times \Omega)^m} \leq M e^{\frac{qM}{T(q-1)}} \|Z_0\|_{L^2(\Omega)^n}.$$

And, since ρ_0 is decreasing

$$(142) \quad \|H_0^J / \rho_0\|_{L^r((T_0, T_1) \times \Omega)^m} \leq \rho_0^{-1}(T_1) M e^{\frac{qM}{T(q-1)}} \|Z_0\|_{L^2(\Omega)^n}.$$

For $k \geq 0$, since ρ_S is decreasing, combining (138) and (140) yields

$$(143) \quad \|H_{k+1}^J\|_{L^r((T_{k+1}, T_{k+2}) \times \Omega)^m} \leq C M e^{\frac{M}{T_{k+2} - T_{k+1}}} \rho_S(T_k) \|S / \rho_S\|_{L^r((T_k, T_{k+1}) \times \Omega)^n}.$$

In particular, by using $M e^{\frac{M}{T_{k+2} - T_{k+1}}} \rho_S(T_k) = \rho_0(T_{k+2})$ (see (59) and (60)), we have

$$(144) \quad \|H_{k+1}^J\|_{L^r((T_{k+1}, T_{k+2}) \times \Omega)^m} \leq C \rho_0(T_{k+2}) \|S / \rho_S\|_{L^r((T_k, T_{k+1}) \times \Omega)^n}.$$

Then, from (144), by using the fact that ρ_0 is decreasing,

$$(145) \quad \|H_{k+1}^J / \rho_0\|_{L^r((T_{k+1}, T_{k+2}) \times \Omega)^m} \leq C \|S / \rho_S\|_{L^r((T_k, T_{k+1}) \times \Omega)^n}.$$

As in the original proof, we can paste the controls H_k^J for $k \geq 0$ together by defining

$$(146) \quad H^J := \sum_{k \geq 0} H_k^J.$$

We have the estimate from (142) and (145)

$$(147) \quad \|H^J\|_{\mathcal{H}_r} \leq C \|S\|_{\mathcal{S}_r} + C \rho_0^{-1}(T_1) M e^{\frac{qM}{T(q-1)}} \|Z_0\|_{L^2(\Omega)^n}.$$

The state Z can also be reconstructed by concatenation of $Z_S + Z_H$, which are continuous at each junction T_k thanks to the construction. Then, we estimate the state. We use the energy estimate (16) for $r = 2$ or (17) for $r = +\infty$ from Definition-Proposition 2.1 on each time interval (T_k, T_{k+1}) :

$$(148) \quad \|Z_S\|_{L^\infty(T_k, T_{k+1}; L^r(\Omega)^n)} \leq C \|S\|_{L^r((T_k, T_{k+1}) \times \Omega)^n},$$

and

$$(149) \quad \|Z_H\|_{L^\infty(T_k, T_{k+1}; L^r(\Omega)^n)} \leq C \left(\|a_k\|_{L^r(\Omega)^n} + \|H_k^J\|_{L^r((T_k, T_{k+1}) \times \Omega)^m} \right).$$

Proceeding similarly as for the estimate on the control, we obtain respectively

$$(150) \quad \|Z_S/\rho_0\|_{L^\infty(T_k, T_{k+1}; L^r(\Omega)^n)} \leq CM^{-1} \|S\|_{\mathcal{S}_r},$$

and

$$(151) \quad \|Z_H/\rho_0\|_{L^\infty(T_k, T_{k+1}; L^r(\Omega)^n)} \leq CM^{-1} \|S\|_{\mathcal{S}_r} + C\rho_0^{-1}(T_1)Me^{\frac{qM}{T(q-1)}} \|Z_0\|_{L^\infty(\Omega)^n}.$$

Therefore, for an appropriate choice of constant $C > 0$, Z and H^J satisfy (136). This concludes the proof of Proposition A.6. \square

A.6. Proof of a strong observability inequality. We take the same notations as in the beginning of Section 6. The goal of this section is to prove Corollary 6.4.

Proof. We define $\mathcal{F}_1 : (Z_0, S) \in L_{inv}^2 \times \mathcal{S}_2 \mapsto Z(T, \cdot) \in L_{inv}^2$, where Z is the solution of (L+S-Z) with $H^J = 0$ and $\mathcal{F}_2 : H^J \in \mathcal{H}_2 \mapsto Z(T, \cdot) \in L_{inv}^2$ is the solution of (L+S-Z) with $(Z_0, S) = (0, 0)$. It is easy to see that the null-controllability of (L+S-Z) is equivalent to $\text{Range}(\mathcal{F}_1) \subset \text{Range}(\mathcal{F}_2)$.

From [9, Lemma 2.48], we have that $\text{Range}(\mathcal{F}_1) \subset \text{Range}(\mathcal{F}_2)$ is equivalent to the observability inequality

$$(152) \quad \exists C_T > 0, \forall \varphi_T \in L_{inv}^2, \|\mathcal{F}_1^*(\varphi_T)\|_{L_{inv}^2 \times \mathcal{S}_2} \leq C_T \|\mathcal{F}_2^*(\varphi_T)\|_{\mathcal{H}_2}.$$

Consequently, by using the null-controllability result for (L+S-Z): Proposition 6.3, we have that (152) holds true. Moreover, the constant C_T in (152) can be chosen such that $C_T \leq Ce^{C/T}$ by using the cost estimate (65) (see the proof of [9, Theorem 2.44] for more details between the constant of cost estimate and the constant of observability inequality).

Duality arguments between Z , the solution of (L+S-Z), and φ , the solution of (66), lead to:

$$\begin{aligned} \int_{\Omega} \mathcal{F}_1(Z_0, S)(x) \cdot \varphi_T(x) dx &= \int_{\Omega} Z_0(x) \cdot \varphi(0, x) dx + \int \int_{(0, T) \times \Omega} S \cdot \varphi, \\ ((Z_0, S), \mathcal{F}_1^*(\varphi_T))_{L^2(\Omega)^n \times \mathcal{S}_2} &= \int_{\Omega} Z_0(x) \cdot \varphi(0, x) dx + \int \int_{(0, T) \times \Omega} S \cdot \varphi \rho_S^2 \rho_S^{-2}, \\ \int_{\Omega} \mathcal{F}_2(H^J)(x) \cdot \varphi_T(x) dx &= \int \int_{(0, T) \times \omega} H^J \cdot \varphi, \\ (H^J, \mathcal{F}_2^*(\varphi_T))_{\mathcal{H}_2} &= \sum_{i=1}^m \int \int_{(0, T) \times \Omega} h_i \cdot \varphi_i \rho_0^2 1_{\omega} \rho_0^{-2}. \end{aligned}$$

Consequently, by identification, we find

$$(153) \quad \mathcal{F}_1^*(\varphi_T) = (\varphi(0, \cdot), \varphi \rho_S^2) \in L^2(\Omega)^n \times \mathcal{S}_2, \quad \mathcal{F}_2^*(\varphi_T) = (\varphi_i \rho_0^2 1_{\omega})_{1 \leq i \leq m} \in \mathcal{H}_2.$$

Finally, by putting (153) in (152), we exactly obtain (67) with $C_T = Ce^{C/T}$. This ends the proof of Corollary 6.4. \square

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