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Approximability and inapproximability of the star p -hub center problem with parameterized triangle inequality^{☆,☆☆}

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Abstract

A complete weighted graph $G = (V, E, w)$ is called Δ_β -metric, for some $\beta \geq 1/2$, if G satisfies the β -triangle inequality, *i.e.*, $w(u, v) \leq \beta \cdot (w(u, x) + w(x, v))$ for all vertices $u, v, x \in V$. Given a Δ_β -metric graph $G = (V, E, w)$ and a center $c \in V$, and an integer p , the Δ_β -STAR p -HUB CENTER PROBLEM (Δ_β -SpHCP) is to find a depth-2 spanning tree T of G rooted at c such that c has exactly p children (also called hubs) and the diameter of T is minimized. In this paper, we study Δ_β -SpHCP for all $\beta \geq \frac{1}{2}$. We show that for any $\epsilon > 0$, to approximate the Δ_β -SpHCP to a ratio $g(\beta) - \epsilon$ is NP-hard and give $r(\beta)$ -approximation algorithms for the same problem where $g(\beta)$ and $r(\beta)$ are functions of β . A subclass of metric graphs is identified that Δ_β -SpHCP is polynomial-time solvable. Moreover, some $r(\beta)$ -approximation algorithms given in this paper meet approximation lower bounds.

1. Introduction

The *hub location problems* have various applications in transportation and telecommunication systems. Variants of hub location problems have been defined and well-studied in

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the literatures (see the two survey papers [15, 1]). Suppose that we have a set of demand nodes that want to communicate with each other through some hubs in a network. A *single allocation hub location problem* requests each demand node can only be served by exactly one hub. Conversely, if a demand node can be served by several hubs, then this kind of hub location problem is called *multi-allocation*. Classical hub location problems ask to minimize the total cost of all origin-destination pairs (see *e.g.*, [32]). However, minimizing the total routing cost would lead to the result that the poorest service quality is extremely bad. In this paper, we consider a single hub location problem with min-max criterion, called Δ_β -STAR p -HUB CENTER PROBLEM which is different from the classic hub location problems. The min-max criterion is able to avoid the drawback of minimizing the total cost.

A complete weighted graph $G = (V, E, w)$ is called Δ_β -metric, for some $\beta \geq 1/2$, if the distance function $w(\cdot, \cdot)$ satisfies $w(v, v) = 0$, $w(u, v) = w(v, u)$, and the β -triangle inequality, *i.e.*, $w(u, v) \leq \beta \cdot (w(u, x) + w(x, v))$ for all vertices $u, v, x \in V$. If $\beta > 1$ then we speak about *relaxed triangle inequality*, and if $\beta < 1$ we speak about *sharpened triangle inequality*. Let u, v be two vertices in a tree T . Use $d_T(u, v)$ to denote the distance between u, v in T . Define $D(T) = \max_{u, v \in T} d_T(u, v)$ called the diameter of T .

We give the definition of the Δ_β -STAR p -HUB CENTER PROBLEM as follows.

Δ_β -STAR p -HUB CENTER PROBLEM (Δ_β -SpHCP)

Input: A Δ_β -metric graph $G = (V, E, w)$, a center vertex $c \in V$, and a positive integer p , $|V| \geq 2p + 1$.

Output: A depth-2 spanning tree T^* rooted at c (called the central hub) such that c has exactly p children (called hubs) and the diameter of T^* , $D(T^*)$, is minimized.

Here, we assume that the number of non-hubs is at least as many as the number of hubs, *i.e.*, $|V| \geq 2p + 1$. The assumption $|V| \geq 2p + 1$ is reasonable because in real applications, a hub could be a post office or an airport, and a non-hub could be a mail post, a customer, or a passenger.

The Δ_β -SpHCP problem is a general version of the original STAR p -HUB CENTER PROBLEM (SpHCP) since the original problem assumes the input graph to be a metric graph, *i.e.*, $\beta = 1$. We use SpHCP to denote the Δ_β -SpHCP for $\beta = 1$. Yaman and Elloumi [33] showed that SpHCP is NP-hard and gave two integer programming formulations for the same problem. Liang [28] showed that SpHCP does not admit a $(1.25 - \epsilon)$ -approximation algorithm for any $\epsilon > 0$ unless $P = NP$ and gave a 3.5-approximation algorithm.

The SINGLE ALLOCATION p -HUB CENTER PROBLEM was introduced in [31, 14] which is similar to SpHCP with min-max criterion and well-studied in [27, 20, 29, 16]. The difference between the two problems is that the SINGLE ALLOCATION p -HUB CENTER PROBLEM assumes that hubs are fully interconnected. Thus, for the SINGLE ALLOCATION p -HUB CENTER PROBLEM, the communication between hubs is not necessary to go through a specified central hub c .

For $\beta = 1$, Δ_β -SpHCP is NP-hard and even NP-hard to have a $(1.25 - \epsilon)$ -approximation algorithm for any $\epsilon > 0$. In this paper, we investigate the complexity of Δ_β -SpHCP parameterized by β -triangle inequality. The motivation of this research for $\beta < 1$ is to investigate whether there exists a large subclasses of input instances of Δ_β -SpHCP that can be solved in polynomial time or admit polynomial-time approximation algorithms

Table 1: The main results where Δ_β -SpHCP cannot be approximated within $g(\beta)$ and has an $r(\beta)$ -approximation algorithm.

β	lower bound $g(\beta)$	upper bound $r(\beta)$
$[\frac{1}{2}, \frac{3-\sqrt{3}}{2}]$	1	1
$(\frac{3-\sqrt{3}}{2}, \frac{2}{3}]$	$\frac{1+2\beta-2\beta^2}{4(1-\beta)}$	$\frac{1+2\beta-2\beta^2}{4(1-\beta)}$
$[\frac{2}{3}, 0.7737\dots]$	$\frac{5\beta+1}{4}$	$\frac{1+2\beta-2\beta^2}{4(1-\beta)}$
$[0.7737\dots, 1]$	$\frac{5\beta+1}{4}$	$1 + \frac{4\beta^2}{5\beta+1}$
$[1, 2]$	$\beta + \frac{1}{2}$	$\beta + \frac{4\beta^2-2\beta}{2+\beta}$
$[2, \infty)$	$\beta + \frac{1}{2}$	$2\beta + 1$

with a reasonable approximation ratio. For $\beta \geq 1$, it is an interesting issue to see whether there exists a polynomial-time approximation algorithm with an approximation ratio linear in β .

Our study uses the well-known concept of *stability of approximation* [10, 12, 25, 26]. The idea behind this concept is to find a parameter (characteristic) of the input instances that captures the hardness of particular inputs. An approximation algorithm is called *stable* with respect to this parameter, if its approximation ratio grows with this parameter but not with the size of the input instances. A nice example is the Traveling Salesman Problem (TSP) that does not admit any polynomial-time approximation algorithm with an approximation ratio bounded by a polynomial in the size of the input instance, but is $\frac{3}{2}$ -approximable for metric input instances. Here, one can characterize the input instances by their “distance” to metric instances. This can be expressed by the β -triangle inequality for any $\beta \geq \frac{1}{2}$.

In a sequence of papers [2, 3, 5, 9, 10, 11, 13, 30], it was shown that one can partition the set of all input instances of TSP into infinitely many subclasses according to the degree of violation of the triangle inequality, and for each subclass one can guarantee upper and lower bounds on the approximation ratio. Similar studies were performed for the problem of constructing 2-connected spanning subgraphs of a given complete graph whose edge weights obey the β -triangle inequality [6], and for the problem of finding, for a given positive integer $k \geq 2$ and an edge-weighted graph G , a minimum k -edge- or k -vertex-connected spanning subgraph [7, 8], demonstrating that for these problems the β -triangle inequality can serve as a measure of hardness of the input instances.

In Table 1, we list the main results of this paper. The rest of the paper is organized as follows. In Section 2, for $\beta \geq \frac{3-\sqrt{3}}{2}$, we show that for any $\epsilon > 0$, to approximate Δ_β -SpHCP to a ratio $g(\beta) - \epsilon$ is NP-hard. In Section 3, we give $r(\beta)$ -approximation algorithms for Δ_β -SpHCP. If $\beta \leq \frac{3-\sqrt{3}}{2}$, we have $r(\beta) = g(\beta) = 1$, *i.e.*, Δ_β -SpHCP is polynomial time solvable. If $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$, we have $r(\beta) = g(\beta)$. For $\frac{2}{3} \leq \beta \leq 1$, $r(\beta) = \min\{\frac{1+2\beta-2\beta^2}{4(1-\beta)}, 1 + \frac{4\beta^2}{5\beta+1}\}$ and $g(\beta) = \frac{5\beta+1}{4}$. Moreover, for $\beta \geq 1$, we have $r(\beta) = \min\{\beta + \frac{4\beta^2-2\beta}{2+\beta}, 2\beta + 1\}$ and $g(\beta) = \beta + \frac{1}{2}$. For $\beta \geq 2$, the approximability of the problem (*i.e.*, upper and lower bound) is linear in β . For $\beta = 1$, we reduce the gap between the

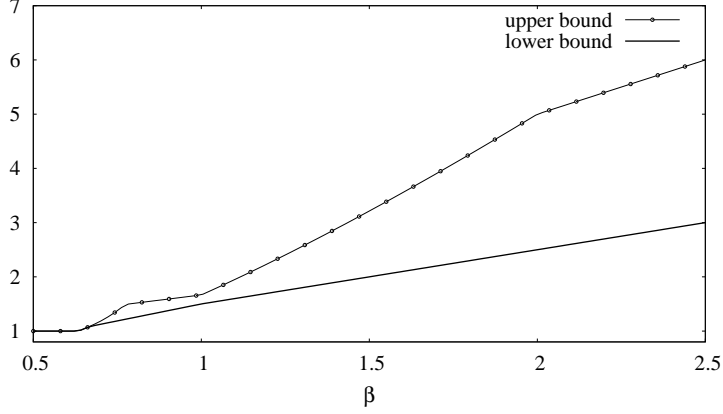


Figure 1: The curves depict the functions in Table 1.

upper and lower bounds of approximability of $SpHCP$. We obtain that for any $\epsilon > 0$, to approximate $SpHCP$ to a ratio $1.5 - \epsilon$ is NP-hard and a $\frac{5}{3}$ -approximation algorithms for $SpHCP$.

We close this section with some notation and definitions, and a property of Δ_β -metric. For a vertex v in a tree T , we use $N_T(v)$ to denote the set of vertices adjacent to v in T and $N_T[v] = N_T(v) \cup \{v\}$. Let $f(v)$ be the parent of v in T and $f(v) = v$ if v is the root of T . Let T^* be an optimal solution of Δ_β - $SpHCP$ in a given β -metric graph $G = (V, E, w)$. For a non-hub x in T^* , we use $f^*(x)$ to denote the hub in T^* that is adjacent to x . We use \tilde{T} to denote the best solution among all solutions in \mathcal{T} where \mathcal{T} is the collection of all solutions satisfying that all non-hubs are adjacent to the same hub for Δ_β - $SpHCP$ in a given β -metric graph $G = (V, E, w)$.

Lemma 1 ([9]). *Let $G = (V, E)$ be a Δ_β -metric graph for $\frac{1}{2} \leq \beta < 1$. For any two edges $(u, x), (v, x)$ with a common endvertex x in G , $w(u, x) \leq \frac{\beta}{1-\beta} \cdot w(v, x)$.*

2. Inapproximability results

In this section, we show that for $\beta > \frac{3-\sqrt{3}}{2}$, to approximate Δ_β - $SpHCP$ to a factor $g(\beta) - \epsilon$ is NP-hard where $g(\beta) = \frac{1+2\beta-2\beta^2}{4(1-\beta)}$ for $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$, $g(\beta) = \frac{5\beta+1}{4}$ for $\frac{2}{3} \leq \beta \leq 1$, and $g(\beta) = \beta + \frac{1}{2}$ for $\beta \geq 1$.

In the following lemma, we first show that for $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$ and for any $\epsilon > 0$, it is NP-hard to approximate Δ_β - $SpHCP$ to a factor $\frac{1+2\beta-2\beta^2}{4(1-\beta)} - \epsilon$.

Lemma 2. *Let $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$. For any $\epsilon > 0$, to approximate Δ_β - $SpHCP$ to a factor $\frac{1+2\beta-2\beta^2}{4(1-\beta)} - \epsilon$ is NP-hard.*

Proof. We will prove that, if Δ_β - $SpHCP$ can be approximated to within a factor $\frac{1+2\beta-2\beta^2}{4(1-\beta)} - \epsilon$ in polynomial time, for some $\epsilon > 0$, then SET COVER can be solved in polynomial time. This will complete the proof, since SET COVER is well-known to be NP-hard [21].

Let (\mathcal{S}, U) be an instance of SET COVER where U is the universal set, $|U| = n$, and \mathcal{S} is a collection of subsets of U , $|\mathcal{S}| = m$. The goal is to decide whether \mathcal{S} has a subset \mathcal{S}' of size k such that $\bigcup_{s_i \in \mathcal{S}'} s_i = U$. In the following, we construct a β -metric graph $G = (V \cup S \cup Y \cup \{c\}, E, w)$ according to (\mathcal{S}, U) . For each element $v \in U$, construct a

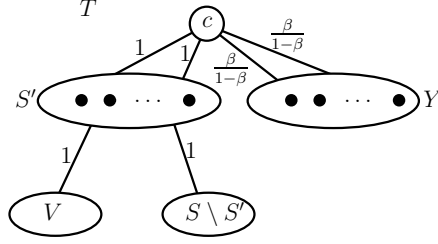


Figure 2: A solution of Δ_β -SHCP in G

vertex $v \in V$, *i.e.*, $V = U$. For each set $s \in \mathcal{S}$, construct a vertex $s \in S$, $|S| = |\mathcal{S}|$. We add $m - k + 2$ vertices in Y and a center c in G . The reason to create $(m - k + 2)$ many vertices in Y is to avoid all elements in U to be selected as hubs in some solutions of Δ_β -SpHCP. The edge cost of G is defined in Table 2.

Table 2: The costs of edges (a, b) in G

$w(a, b)$	$b \in S$	$b \in V$	$b \in Y$
$a = c$	1	2β	$\frac{\beta}{1-\beta}$
$a \in S$	1	1 if $b \in a$ 2β otherwise	$\frac{\beta}{1-\beta}$
$a \in V$	1 if $a \in b$ 2β otherwise	1	$\frac{\beta}{1-\beta}$
$a \in Y$	$\frac{\beta}{1-\beta}$	$\frac{\beta}{1-\beta}$	$\frac{\beta}{1-\beta}$

Clearly, G can be constructed in polynomial time. It is easy to verify that G is a β -metric graph. Let (G, c) be the input of Δ_β -SpHCP constructed according to (\mathcal{S}, U) where c is the specified center and $p = m + 2$.

Let $\mathcal{S}' \subset \mathcal{S}$ be a set cover of (\mathcal{S}, U) of size k . We then construct a solution T of Δ_β -SpHCP according to \mathcal{S}' . For each set $s \in \mathcal{S}'$, collect the vertex $s \in S'$ in G . Let $S' \cup Y$ be hubs adjacent to c where $|S'| = |\mathcal{S}'|$. Note that \mathcal{S}' is a set cover. For each $v \in V$, connect v to exactly one set $s \in S'$ satisfying that $v \in s$. Connect all vertices in $S \setminus S'$ to a vertex $s \in S'$. Since each $v \in V$ is connected to a set $s \in S'$ satisfying that $v \in s$, we see that $w(v, s) = 1$. Hence $D(T) = 4$ (see Fig. 2). Let T^* denote an optimal solution of Δ_β -SpHCP in G . We have $D(T^*) \leq 4$.

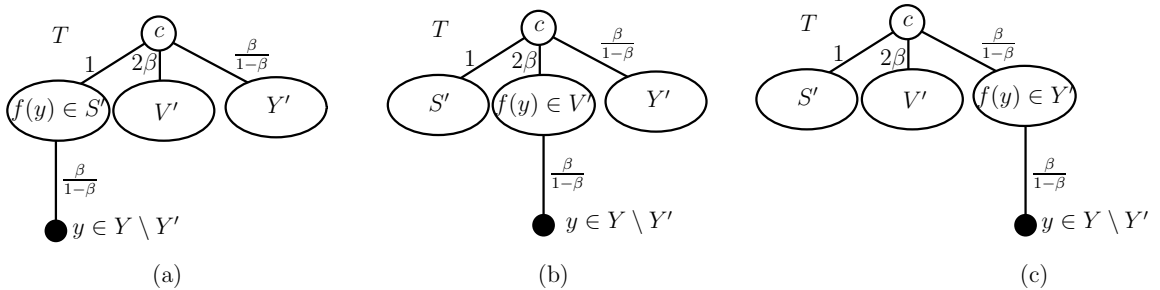


Figure 3: The three cases that there exists $y \in Y \setminus Y'$ not adjacent to c .

Assume that there exists a polynomial time algorithm that finds a solution T of Δ_β -SpHCP in G with $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$. W.l.o.g., assume that $N_T(c) = S' \cup V' \cup Y'$ where

$S' \subseteq S$, $V' \subseteq V$, and $Y' \subseteq Y$.

CLAIM 1. All $y \in Y$ must be hubs, *i.e.*, $Y' = Y$.

Proof. Suppose there exists $y \in Y$ not a hub in T . Since $p = m + 2$ and $|S| = m$, there exists $x \in V' \cup Y'$. There are three cases.

- If $f(y) \in S'$ as Fig. 3(a), then there exists $x \in V' \cup Y'$ with

$$\begin{aligned} d_T(x, y) &= w(x, c) + w(c, f(y)) + w(f(y), y) \\ &\geq \min\{2\beta, \frac{\beta}{1-\beta}\} + 1 + \frac{\beta}{1-\beta} \\ &= 2\beta + 1 + \frac{\beta}{1-\beta} \quad (\text{since } \frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}) \\ &= \frac{1+2\beta-2\beta^2}{1-\beta}, \end{aligned}$$

a contradiction to the assumption that $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$.

- If $f(y) \in V'$ Fig. 3(b), then there exists $x \in V' \cup Y'$ with

$$\begin{aligned} d_T(x, y) &= w(x, c) + w(c, f(y)) + w(f(y), y) \\ &\geq \min\{2\beta, \frac{\beta}{1-\beta}\} + 2\beta + \frac{\beta}{1-\beta} \\ &= 2\beta + 2\beta + \frac{\beta}{1-\beta} \quad (\text{since } \frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}) \\ &\geq \frac{1+2\beta-2\beta^2}{1-\beta}, \end{aligned}$$

a contradiction to the assumption that $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$.

- If $f(y) \in Y'$ Fig. 3(c), then there exists $x \in V' \cup Y'$ with

$$\begin{aligned} d_T(x, y) &= w(x, c) + w(c, f(y)) + w(f(y), y) \\ &\geq \min\{2\beta, \frac{\beta}{1-\beta}\} + \frac{\beta}{1-\beta} + \frac{\beta}{1-\beta} \\ &= 2\beta + \frac{\beta}{1-\beta} + \frac{\beta}{1-\beta} \quad (\text{since } \frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}) \\ &\geq \frac{1+2\beta-2\beta^2}{1-\beta}, \end{aligned}$$

a contradiction to the assumption that $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$.

Thus, all $y \in Y$ must be hubs, *i.e.*, $Y' = Y$. □

CLAIM 2. Each $y \in Y$ does not have a child in T .

Proof. Suppose that there exists $y \in Y$ has a child. There are two cases.

- If y has a child $s \in S \setminus S'$, then there exists $x \in N_T(c)$ with

$$\begin{aligned} d_T(x, s) &= w(x, c) + w(c, y) + w(y, s) \\ &\geq \min\{1, 2\beta, \frac{\beta}{1-\beta}\} + \frac{\beta}{1-\beta} + \frac{\beta}{1-\beta} \\ &= 1 + 2 \cdot \frac{\beta}{1-\beta} \quad (\text{since } \frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}) \\ &\geq \frac{1+2\beta-2\beta^2}{1-\beta}, \end{aligned}$$

a contradiction to the assumption that $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$.

- If y has a child $v \in V \setminus V'$, then there exist $x \in N_T(c)$ with

$$\begin{aligned}
d_T(x, v) &= w(x, c) + w(c, y) + w(y, v) \\
&\geq \min\{1, 2\beta, \frac{\beta}{1-\beta}\} + \frac{\beta}{1-\beta} + \frac{\beta}{1-\beta} \\
&= 1 + 2 \cdot \frac{\beta}{1-\beta} \quad (\text{since } \frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}) \\
&\geq \frac{1+2\beta-2\beta^2}{1-\beta},
\end{aligned}$$

a contradiction to the assumption that $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$.

Thus, each $y \in Y$ has no child in T . □

CLAIM 3. No $v \in V \setminus V'$ is adjacent to any $u \in V'$.

Proof. Suppose that there exists $v \in V \setminus V'$ is adjacent to $u \in V'$ in T . We see that for any $y \in Y$

$$d_T(v, y) = w(v, u) + w(u, c) + w(c, y) = 1 + 2\beta + \frac{\beta}{1-\beta} \geq \frac{1+2\beta-2\beta^2}{1-\beta},$$

a contradiction to the assumption that $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$. Thus, no $v \in V \setminus V'$ is adjacent to any $u \in V'$. □

According to Claims 1, 2, and 3, in T all vertices $V \setminus V'$ must be adjacent to vertices in S' . If there exists $v \in V \setminus V'$ satisfying that $w(v, f(v)) = 2\beta$, then

$$d_T(v, y) = w(v, f(v)) + w(f(v), c) + w(c, y) = 2\beta + 1 + \frac{\beta}{1-\beta} \geq \frac{1+2\beta-2\beta^2}{1-\beta},$$

a contradiction to the assumption that $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$. Thus, each $v \in V \setminus V'$ satisfies that $w(v, f(v)) = 1$. We see that S' forms a set cover of $V \setminus V'$. For each $u \in V'$, pick a set $s \in \mathcal{S}$ satisfying $u \in s$, call the collection of sets S'' . It is easy to see that $|S''| = |V'|$ and $S' \cup S''$ forms a set cover of $V = U$ of size at most k . This shows that if Δ_β -SpHCP has a solution T with $D(T) < \frac{1+2\beta-2\beta^2}{1-\beta}$ that can be found in polynomial time, then SET COVER can be solved in polynomial time. However, SET COVER is a well-known NP-hard problem [21]. By the fact that SET COVER is NP-hard and $D(T^*) \leq 4$, this implies that for any $\epsilon > 0$, to approximate Δ_β -SpHCP to a factor $\frac{1+2\beta-2\beta^2}{4(1-\beta)} - \epsilon$ is NP-hard. □

In the following lemma, we show that for $\frac{2}{3} \leq \beta \leq 1$ and for any $\epsilon > 0$, it is NP-hard to approximate Δ_β -SpHCP to a factor $\frac{5\beta+1}{4} - \epsilon$.

Lemma 3. Let $\frac{2}{3} \leq \beta \leq 1$. For any $\epsilon > 0$, to approximate Δ_β -SpHCP to a factor $\frac{5\beta+1}{4} - \epsilon$ is NP-hard.

Proof. We will prove that, if Δ_β -SpHCP can be approximated within a factor $\frac{5\beta+1}{4} - \epsilon$ in polynomial time for some $\epsilon > 0$, then a 2-approximate solution of set cover problem can be found in polynomial time. This will complete the proof, since for any $\epsilon > 0$, to approximate SET COVER to within a factor $(1 - \epsilon) \ln n$ is NP-hard [19].

Let (S, U) be an instance of the set cover problem, where U , $|U| = n$, is the universal set and S , $|S| = m$, is a collection of subsets of U . The goal of the problem is to decide whether there exists a subset $S' \subseteq S$ of size k such that $\bigcup_{s_i \in S'} s_i = U$.

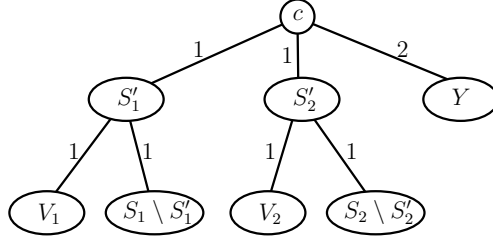


Figure 4: $D(T) = 4$

Construct a β -metric graph $G = (V_1 \cup V_2 \cup S_1 \cup S_2 \cup Y \cup \{c\}, E, w)$ of Δ_β -SpHCP as follows. For each element $v \in U$, construct a copy of v in V_1 and another copy of v in V_2 , i.e., $V_1 = V_2 = U$. For each set $s \in S$, construct a vertex in S_1 and a vertex in S_2 , $|S_1| = |S_2| = |S|$. Construct $(2m - 2k + 2)$ vertices in Y . The reason to create $(2m - 2k + 2)$ many vertices in Y is to avoid all elements in U to be selected as hubs in some solutions Δ_β -SpHCP. Add a vertex c in G as the specified center. Let $p = 2m + 2$. The edge cost of G is defined in Table 3. It is not hard to see that any three vertices in G satisfying β -triangle inequality.

Table 3: The cost of edges (a, b) in G

$w(a, b)$	$b \in S_1$	$b \in S_2$	$b \in V_1$	$b \in V_2$	$b \in Y$
$a = c$	1	1	2β	2β	2
$a \in S_1$	1	2β	$\frac{1 \text{ if } b \in a}{2\beta \text{ otherwise}}$	$\beta + 2\beta^2$	3β
$a \in S_2$	2β	1	$\beta + 2\beta^2$	$\frac{1 \text{ if } b \in a}{2\beta \text{ otherwise}}$	3β
$a \in V_1$	$\frac{1 \text{ if } a \in b}{2\beta \text{ otherwise}}$	$\beta + 2\beta^2$	2β	$\beta + \beta^2 + 2\beta^3$	$\beta + 3\beta^2$
$a \in V_2$	$\beta + 2\beta^2$	$\frac{1 \text{ if } a \in b}{2\beta \text{ otherwise}}$	$\beta + \beta^2 + 2\beta^3$	2β	$\beta + 3\beta^2$
$a \in Y$	3β	3β	$\beta + 3\beta^2$	$\beta + 3\beta^2$	2

Suppose (S, U) has a set cover $S' \subset S$ of size k . We then construct a solution of Δ_β -SpHCP according to S' as Fig. 4. Let $S'_1 \subset S_1$ be the copy of S' and $S'_2 \subset S_2$ be the copy of S' . Let all vertices in $S'_1 \cup S'_2 \cup Y$ be hubs connected to c . For each $v \in V_1$, connect v to exactly one set $s \in S'_1$ such that $v \in s$. For each $v' \in V_2$, connect v' to exactly one set $s' \in S'_2$ such that $v' \in s'$. Connect all vertices in $S_1 \setminus S'_1$ to any vertex in S'_1 and connect all vertices in $S_2 \setminus S'_2$ to any vertex in S'_2 . Notice that S' is a set cover of U . We see that for $v \in V_1$, $w(v, f(v)) = 1$ where $f(v) \in S'_1$ is the hub adjacent to v satisfying that $v \in f(v)$ and for $v' \in V_2$, $w(v', f(v')) = 1$ where $f(v') \in S'_2$ satisfying that $v' \in f(v')$. We see that $D(T) = 4$. Let T^* denote an optimal solution of Δ_β -SpHCP. Then $D(T^*) \leq 4$.

Suppose that there exists a polynomial time algorithm for Δ_β -SpHCP that computes a solution T such that $D(T) < 5\beta + 1$.

Let $S'_1 \cup S'_2 \cup V'_1 \cup V'_2 \cup Y'$ be the set of children of c , i.e., $N_T(c) = S'_1 \cup S'_2 \cup V'_1 \cup V'_2 \cup Y'$

CLAIM 1. All $y \in Y$ must be hubs, i.e., $Y' = Y$.

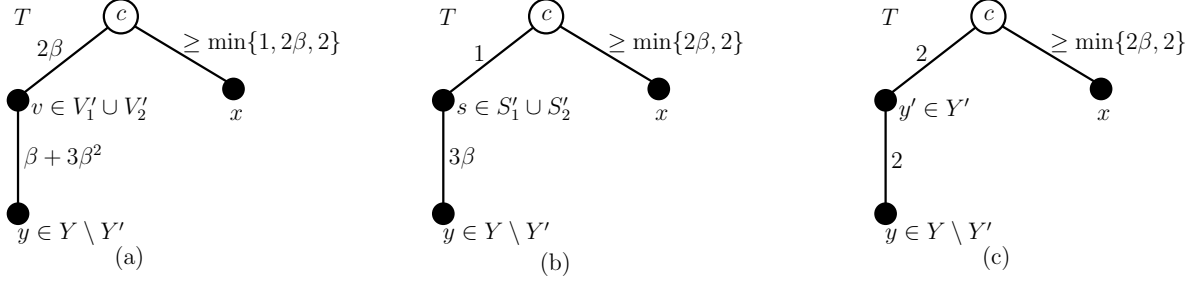


Figure 5: The three cases that there exists $y \in Y$ not adjacent to c where (a) $x \in N_T(c)$; (b) $x \in V'_1 \cup V'_2 \cup Y'$; (c) $x \in V'_1 \cup V'_2 \cup Y'$.

Proof. Suppose there exists $y \in Y$ not a hub. There are three cases.

- y is a child of $v \in V'_1 \cup V'_2$ as Fig. 5(a). For $x \in N(c) \setminus \{v\}$, we see that

$$\begin{aligned}
 d_T(y, x) &= w(y, v) + w(v, c) + w(c, x) \\
 &\geq (\beta + 3\beta^2) + 2\beta + \min\{1, 2\beta, 2\} \\
 &= 3\beta^2 + 3\beta + 1 \\
 &\geq 5\beta + 1 \quad (\text{since } \beta \geq \frac{2}{3}),
 \end{aligned}$$

a contradiction to the assumption that $D(T) < 5\beta + 1$.

- y is a child of $s \in S'_1 \cup S'_2$ as Fig. 5(b). Since $|N_T(c)| = p = 2m + 2$ and $|S_1| = |S_2| = m$, $N_T(c) \setminus (S'_1 \cup S'_2) \neq \emptyset$. There exists $x \in V'_1 \cup V'_2 \cup Y'$ such that

$$d_T(y, x) = w(y, s) + w(s, c) + w(c, x) \geq 3\beta + 1 + \min\{2\beta, 2\} = 5\beta + 1,$$

a contradiction to the assumption that $D(T) < 5\beta + 1$.

- y is a child of $y' \in Y$ as Fig. 5(c). Since $|N_T(c)| = p = 2m + 2$ and $|S_1| = |S_2| = m$, we see $N_T(c) \setminus (S'_1 \cup S'_2) \neq \emptyset$. There exists $x \in V'_1 \cup V'_2 \cup Y'$ such that

$$d_T(y, x) = w(y, y') + w(y', c) + w(c, x) \geq 2 + 2 + \min\{2\beta, 2\} = 4 + 2\beta \geq 5\beta + 1,$$

a contradiction to the assumption that $D(T) < 5\beta + 1$.

This shows that all $y \in Y$ must be hubs in T . □

CLAIM 2. Each $y \in Y$ has no children in T .

Proof. Suppose that there exists $y \in Y$ having a child in T . Notice that by Claim 1, all $y \in Y$ are hubs. There are two cases.

- y has a child $v \in V_1 \cup V_2$. Then there exists a hub $x \in N_T(c)$ such that

$$\begin{aligned}
 d_T(x, v) &= w(x, c) + w(c, y) + w(y, v) \\
 &\geq \min\{1, 2\beta, 2\} + 2 + (\beta + 3\beta^2) \\
 &= 3\beta^2 + \beta + 3 \\
 &> 5\beta + 1 \quad (\text{since } \beta \geq \frac{2}{3})
 \end{aligned}$$

a contradiction to the assumption $D(T) < 5\beta + 1$.

- y has a child $s \in S_1 \cup S_2$. Then there exists a hub $x \in N_T(c)$ and

$$\begin{aligned}
d_T(x, s) &= w(x, c) + w(c, y) + w(y, s) \\
&\geq \min\{1, 2\beta, 2\} + 2 + 3\beta \\
&= 3\beta + 3 \\
&\geq 5\beta + 1 \quad (\text{since } \beta \leq 1)
\end{aligned}$$

a contradiction to the assumption $D(T) < 5\beta + 1$.

This shows that each $y \in Y$ has no children in T . \square

According to Claims 1 and 2, if $D(T) < 5\beta + 1$, all vertices in Y must be hubs adjacent to c and each $y \in Y$ has no children in T .

CLAIM 3. There is no $v \in V'_1 \cup V'_2$ having a child in $(V_1 \cup V_2) \setminus N_T(c)$.

Proof. Suppose that there exists $v \in V'_1 \cup V'_2$ having a child $u \in (V_1 \cup V_2) \setminus N_T(c)$.

If $v \in V'_i$ and $u \in V_i \setminus V'_i$, $i \in \{1, 2\}$, then for any $y \in Y$

$$\begin{aligned}
d_T(u, y) &= w(u, v) + w(v, c) + w(c, y) \\
&= 2\beta + 2\beta + 2 \\
&\geq 5\beta + 1 \quad (\text{since } \beta \leq 1)
\end{aligned}$$

a contradiction to the assumption $D(T) < 5\beta + 1$.

If $v \in V'_1$ and $u \in V_2 \setminus V'_2$, then for any $y \in Y$

$$\begin{aligned}
d_T(u, y) &= w(u, v) + w(v, c) + w(c, y) \\
&= (\beta + \beta^2 + 2\beta^3) + 2\beta + 2 \\
&= 2\beta^3 + \beta^2 + 3\beta + 2 \\
&> 5\beta + 1 \quad (\text{since } \beta \geq \frac{2}{3})
\end{aligned}$$

a contradiction to the assumption $D(T) < 5\beta + 1$.

If $v \in V'_2$ and $u \in V_1 \setminus V'_1$, then for any $y \in Y$

$$\begin{aligned}
d_T(u, y) &= w(u, v) + w(v, c) + w(c, y) \\
&= (\beta + \beta^2 + 2\beta^3) + 2\beta + 2 \\
&= 2\beta^3 + \beta^2 + 3\beta + 2 \\
&> 5\beta + 1 \quad (\text{since } \beta \geq \frac{2}{3})
\end{aligned}$$

a contradiction to the assumption the $D(T) < 5\beta + 1$.

Thus, no vertex in $V'_1 \cup V'_2$ is adjacent to any vertex in $(V_1 \cup V_2) \setminus N_T(c)$ in T . This completes the proof. \square

CLAIM 4. Let S''_1 be a subset of S satisfying that for each $u \in V'_1$ there is exactly one set in S''_1 containing u . If for all $v \in V_1 \setminus V'_1$, $d_T(v, c) \leq 2$, then $S'_1 \cup S''_1$ is a set cover of U , $|S'_1 \cup S''_1| \leq 2k$.

Proof. Since for all $v \in V_1 \setminus V'_1$, $d_T(v, c) = 2$, the element v must be a child of $s \in S'_1$ satisfying that $v \in s$. Thus S'_1 is a set cover of $V_1 \setminus V'_1$. For each element in $u \in V'_1$, we select exactly one set in S that contains u , call the collection of sets S''_1 . It is not hard to see that $|S''_1| = |V'_1|$ and S''_1 forms a set cover of V'_1 . Moreover, $S'_1 \cup S''_1$ is a set cover of $V_1 = U$ satisfying that $|S'_1 \cup S''_1| \leq 2k$ and can be found in polynomial time. This completes the proof. \square

CLAIM 5. If there exists $v \in V_1 \setminus V_1'$ such that $d_T(v, c) > 2$, then $S_2' \cup S_2''$ is a set cover of U , $|S_2' \cup S_2''| \leq 2k$, where $S_2'' \subset S$ satisfying that for each $u \in V_2'$ there is exactly one set in S_2'' containing u .

Proof. Suppose that condition of Claim 4 is not satisfied, there exists $v \in V_1 \setminus V_1'$ such that $d_T(v, c) > 2$. According to Claims 1, 2, and 3, v is not adjacent to any vertex in $V_1' \cup V_2' \cup Y$. If the parent of v is in S_2' , then for any $y \in Y$

$$\begin{aligned} d_T(v, y) &= w(v, f(v)) + w(f(v), c) + w(c, y) \\ &= (\beta + 2\beta^2) + 1 + 2 \\ &= 2\beta^2 + \beta + 3 \\ &\geq 5\beta + 1, \end{aligned}$$

a contradiction to the assumption $D(T) < 5\beta + 1$. Thus, the parent of v is in S_1' . We see that

$$d_T(v, c) = w(v, f(v)) + w(f(v), c) = 2\beta + 1$$

where $f(v)$ is the parent of v in T and $f(v) \in S_1'$.

Notice that no $u \in V_2 \setminus V_2'$ is adjacent to $V_1' \cup V_2' \cup Y$ according to Claims 1, 2, and 3. Suppose that there exists $u \in V_2 \setminus V_2'$ adjacent to a hub in S_1' . We see that for any $y \in Y$

$$\begin{aligned} d_T(u, y) &= w(u, f(u)) + w(f(u), c) + w(c, y) \\ &= (\beta + 2\beta^2) + 1 + 2 \\ &= 2\beta^2 + \beta + 3 \\ &\geq 5\beta + 1, \end{aligned}$$

a contradiction to the assumption $D(T) < 5\beta + 1$. Thus, all vertices $u \in V_2 \setminus V_2'$ must be adjacent to vertices in S_2' in T . If there exists $u \in V_2 \setminus V_2'$ adjacent to a hub $f(u) \in S_2'$ such that $w(u, f(u)) = 2\beta$, then

$$\begin{aligned} d_T(u, v) &= w(u, f(u)) + w(f(u), c) + w(c, f(v)) + w(f(v), v) \\ &= 2\beta + 1 + 1 + 2\beta \\ &\geq 5\beta + 1 \quad (\text{since } \beta \leq 1) \end{aligned}$$

a contradiction to the assumption $D(T) < 5\beta + 1$. Thus, in T each $u \in V_2 \setminus V_2'$ must be adjacent to a hub $f(u) \in S_2'$ satisfying that $w(u, f(u)) = 1$. This implies that S_2' is a set cover of $V_2 \setminus V_2'$. For each element $x \in V_2'$, we pick a set $s \in S$ satisfying that $x \in s$, call the collection of sets S_2'' . It is easy to see that $|S_2''| = |V_2'|$. Moreover, $S_2' \cup S_2''$, $|S_2' \cup S_2''| \leq 2k$, is a set cover of U and can be found in polynomial time. \square

By Claims 4 and 5, if $D(T) < 5\beta + 1$, then SET COVER has a 2-approximation algorithm running in polynomial time. Notice that $D(T^*) \leq 4$. Thus, for any $\varepsilon > 0$, if $\Delta_\beta\text{-SpHCP}$ can be approximated to a factor $(\frac{5\beta+1}{4} - \varepsilon)$ in polynomial time, then SET COVER has a 2-approximation algorithm running in polynomial time. However, for any $\varepsilon > 0$, to approximate SET COVER to a factor $(1 - \varepsilon) \ln n$ is NP-hard [19]. Therefore, for any $\varepsilon > 0$, to approximate $\Delta_\beta\text{-SpHCP}$ to a factor $\frac{5\beta+1}{4} - \varepsilon$ is NP-hard. This completes the proof. \square

Next, we show that for any $\beta \geq 1$ and for any $\varepsilon > 0$, it is NP-hard to approximate $\Delta_\beta\text{-SpHCP}$ to a factor $\beta + \frac{1}{2} - \varepsilon$.

Lemma 4. *Let $\beta \geq 1$. For any $\varepsilon > 0$, to approximate $\Delta_\beta\text{-SpHCP}$ to a factor $\beta + \frac{1}{2} - \varepsilon$ is NP-hard.*

Proof. We will prove that, if Δ_β -SpHCP can be approximated within a factor $\beta + \frac{1}{2} - \varepsilon$ in polynomial time for some $\varepsilon > 0$, then a 3-approximate solution of set cover problem can be found in polynomial time. This will complete the proof of the lemma, since for any $\varepsilon > 0$, to approximate SET COVER to within a factor $(1 - \varepsilon) \ln n$ is NP-hard [19].

Let (S, U) be an instance of the set cover problem, where U , $|U| = n$, is the universal set and S , $|S| = m$, is a collection of subsets of U . The goal of the problem is to decide whether there exists a subset $S' \subseteq S$ of size k such that $\bigcup_{s_i \in S'} s_i = U$.

Construct a β -metric graph $G = (V_1 \cup V_2 \cup S_1 \cup S_2 \cup Y \cup \{c\}, E, w)$ of Δ_β -SpHCP as follows. For each element $v \in U$, construct a copy of v in V_1 and another copy of v in V_2 , i.e., $V_1 = V_2 = U$. For each set $s \in S$, construct a vertex in S_1 and a vertex in S_2 , $|S_1| = |S_2| = |S|$. Construct two vertices in Y , $Y = \{y_1, y_2\}$. Add a vertex c in G as the specified center. Let $p = 2k + 2$. The edge cost of G is defined in Table 4. It is not hard to see that any three vertices in G satisfying β -triangle inequality.

Table 4: The cost of edges (a, b) in G

$w(a, b)$	$b \in S_1$	$b \in S_2$	$b \in V_1$	$b \in V_2$	$b \in Y$
$a = c$	1	1	2β	2β	2
$a \in S_1$	1	2β	$\frac{1 \text{ if } b \in a}{2\beta \text{ otherwise}}$	$\beta + 2\beta^2$	3β
$a \in S_2$	2β	1	$\beta + 2\beta^2$	$\frac{1 \text{ if } b \in a}{2\beta \text{ otherwise}}$	3β
$a \in V_1$	$\frac{1 \text{ if } a \in b}{2\beta \text{ otherwise}}$	$\beta + 2\beta^2$	2β	$4\beta^2$	$2\beta + 2\beta^2$
$a \in V_2$	$\beta + 2\beta^2$	$\frac{1 \text{ if } a \in b}{2\beta \text{ otherwise}}$	$4\beta^2$	2β	$2\beta + 2\beta^2$
$a \in Y$	3β	3β	$2\beta + 2\beta^2$	$2\beta + 2\beta^2$	4β

Suppose (S, U) has a set cover $S' \subseteq S$ of size k . We then construct a solution of Δ_β -SpHCP according to S' as Fig. 4. Let $S'_1 \subseteq S_1$ be the copy of S' and $S'_2 \subseteq S_2$ be the copy of S' . Let all vertices in $S'_1 \cup S'_2 \cup Y$ be hubs connected to c . For each $v \in V_1$, connect v to exactly one set $s \in S'_1$ such that $v \in s$. For each $v' \in V_2$, connect v' to exactly one set $s' \in S'_2$ such that $v' \in s'$. Connect all vertices in $S_1 \setminus S'_1$ to any vertex in S'_1 and connect all vertices in $S_2 \setminus S'_2$ to any vertex in S'_2 . Since S' is a set cover of U , we see that for $v \in V_1$, $w(v, f(v)) = 1$ where $f(v) \in S'_1$ is the hub adjacent to v satisfying that $v \in f(v)$ and for $v' \in V_2$, $w(v', f(v')) = 1$ where $f(v') \in S'_2$ satisfying that $v' \in f(v')$. We see that $D(T) = 4$. Let T^* denote an optimal solution of Δ_β -SpHCP. Then $D(T^*) \leq 4$.

Suppose that there exists a polynomial time algorithm for Δ_β -SpHCP that computes a solution T such that $D(T) < 4\beta + 2$. Let $S'_1 \cup S'_2 \cup V'_1 \cup V'_2 \cup Y'$ be the set of children of c , i.e., $N_T(c) = S'_1 \cup S'_2 \cup V'_1 \cup V'_2 \cup Y'$ where $S'_1 \subseteq S_1$, $S'_2 \subseteq S_2$, $V'_1 \subseteq V_1$, $V'_2 \subseteq V_2$, and $Y' \subseteq Y$.

CLAIM 1. Either y_1 or y_2 is a hub.

Proof. Suppose that both y_1 and y_2 are non-hubs. Let $f(y_1)$ (resp. $f(y_2)$) be the hub adjacent to y_1 (resp. y_2) in T . Note that $f(y_1), f(y_2) \in S'_1 \cup S'_2 \cup V'_1 \cup V'_2$. There are two cases.

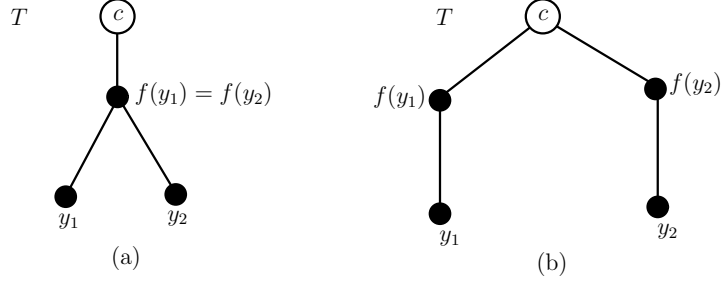


Figure 6: The two cases that both y_1 and y_2 are not hubs in T .

- $f(y_1) = f(y_2)$ as Fig. 6(a). Then

$$\begin{aligned}
 d_T(y_1, y_2) &= w(y_1, f(y_1)) + w(y_2, f(y_2)) \\
 &\geq \min\{3\beta, 2\beta + 2\beta^2\} + \min\{3\beta, 2\beta + 2\beta^2\} \\
 &= 3\beta + 3\beta \\
 &\geq 4\beta + 2 \quad (\text{since } \beta \geq 1)
 \end{aligned}$$

a contradiction to the assumption $D(T) < 4\beta + 2$.

- $f(y_1) \neq f(y_2)$ as Fig. 6(b). Then

$$\begin{aligned}
 d_T(y_1, y_2) &= w(y_1, f(y_1)) + w(f(y_1), c) + w(c, f(y_2)) + w(f(y_2), y_2) \\
 &> w(y_1, f(y_1)) + w(y_2, f(y_2)) \\
 &\geq \min\{3\beta, 2\beta + 2\beta^2\} + \min\{3\beta, 2\beta + 2\beta^2\} \\
 &= 3\beta + 3\beta \\
 &\geq 4\beta + 2 \quad (\text{since } \beta \geq 1)
 \end{aligned}$$

a contradiction to the assumption $D(T) < 4\beta + 2$.

Thus, either y_1 or y_2 is a hub. □

In the rest of the proof, we assume that y_1 is a hub in T .

CLAIM 2. For each $y \in Y'$, y has no child in $(V_1 \cup V_2 \cup Y) \setminus N_T(c)$.

Proof. Suppose $y_1 \in Y'$ has a child x . There are two cases.

- If $x = y_2$, then

$$d_T(y_2, c) = w(y_2, y_1) + w(y_1, c) = 4\beta + 2,$$

a contradiction to the assumption $D(T) < 4\beta + 2$.

- If $x \in (V_1 \cup V_2) \setminus N_T(c)$, then

$$d_T(x, c) = w(x, y_1) + w(y_1, c) = (2\beta + 2\beta^2) + 2 \geq 4\beta + 2,$$

a contradiction to the assumption $D(T) < 4\beta + 2$.

This completes the proof of the claim. □

CLAIM 3. All $u \in V_1 \setminus V_1'$ (resp. $u' \in V_2 \setminus V_2'$) are adjacent to S_1' (resp. S_2') in T .

Proof. By Claim 2, $y_1 \in Y'$ has no child in $(V_1 \cup V_2 \cup Y) \setminus N_T(c)$. Thus, each $u \in V_1 \setminus V_1'$ must be adjacent to some vertex in $S_1' \cup S_2' \cup V_1' \cup V_2'$ in T .

Suppose that there exists $u \in V_1 \setminus V_1'$ adjacent to $s' \in S_2'$. We see that

$$d_T(u, y_1) = w(u, s') + w(s', c) + w(c, y_1) = (\beta + 2\beta^2) + 1 + 2 \geq 4\beta + 2,$$

a contradiction to the assumption $D(T) < 4\beta + 2$.

Suppose that there exists $u \in V_1 \setminus V_1'$ adjacent to $x \in V_1' \cup V_2'$. We see that

$$\begin{aligned} d_T(u, y_1) &= w(u, x) + w(x, c) + w(c, y_1) \\ &\geq \min\{2\beta, 4\beta^2\} + 2\beta + 2 \\ &= 2\beta + 2\beta + 2 \quad (\text{since } \beta \geq 1) \\ &= 4\beta + 2 \end{aligned}$$

a contradiction to the assumption $D(T) < 4\beta + 2$.

This shows that all $u \in V_1 \setminus V_1'$ are adjacent to vertices in S_1' in T . Similarly, we can show that all $u' \in V_2 \setminus V_2'$ are adjacent to vertices in S_2' in T . \square

CLAIM 4. Let S_1'' be a subset of S satisfying that for each $u \in V_1'$ there is exactly one set in S_1'' containing u . If for all $v \in V_1 \setminus V_1'$, $d_T(v, c) \leq 2$, then $S_1' \cup S_1''$ is a set cover of U , $|S_1' \cup S_1''| \leq 2k + 1$.

Proof. Since for all $v \in V_1 \setminus V_1'$, $d_T(v, c) = 2$, the element v must be a child of $s \in S_1'$ satisfying that $v \in s$. Thus S_1' is a set cover of $V_1 \setminus V_1'$. For each element in $u \in V_1'$, we select exactly one set in S that contains u , call the collection of sets S_1'' . It is not hard to see that $|S_1''| = |V_1'|$ and S_1'' forms a set cover of V_1' . Moreover, $S_1' \cup S_1''$ is a set cover of $V_1 = U$ satisfying that $|S_1' \cup S_1''| \leq 2k + 1$ and can be found in polynomial time. This completes the proof of the claim. \square

CLAIM 5. If there exists $v \in V_1 \setminus V_1'$ such that $d_T(v, c) > 2$, then $S_2' \cup S_2''$ is a set cover of U , $|S_2' \cup S_2''| \leq 2k + 1$, where $S_2'' \subset S$ satisfying that for each $u \in V_2'$ there is exactly one set in S_2'' containing u .

Proof. Suppose that the condition of Claim 4 is not satisfied, there exists $v \in V_1 \setminus V_1'$ such that $d_T(v, c) > 2$. According to Claims 1, 2, and 3, v is not adjacent to any vertex in $S_2' \cup V_1' \cup V_2' \cup Y'$. Thus, the parent of v is in S_1' . We see that

$$d_T(v, c) = w(v, f(v)) + w(f(v), c) = 2\beta + 1$$

where $f(v)$ is the parent of v in T and $f(v) \in S_1'$.

Notice that no $u \in V_2 \setminus V_2'$ is adjacent to $S_1' \cup V_1' \cup V_2' \cup Y$ according to Claims 1, 2, and 3. If there exists $u \in V_2 \setminus V_2'$ adjacent to a hub $f(u) \in S_2'$ such that $w(u, f(u)) = 2\beta$, then

$$\begin{aligned} d_T(u, v) &= w(u, f(u)) + w(f(u), c) + w(c, f(v)) + w(f(v), v) \\ &= 2\beta + 1 + 1 + 2\beta \\ &= 4\beta + 2, \end{aligned}$$

a contradiction to the assumption $D(T) < 4\beta + 2$. Thus, in T each $u \in V_2 \setminus V_2'$ must be adjacent to a hub $f(u) \in S_2'$ satisfying that $w(u, f(u)) = 1$. This implies that S_2' is a set cover of $V_2 \setminus V_2'$. For each element $x \in V_2'$, we pick a set $s \in S$ satisfying that $x \in s$, call the collection of sets S_2'' . It is easy to see that $|S_2''| = |V_2'|$. Moreover, $S_2' \cup S_2''$, $|S_2' \cup S_2''| \leq 2k + 1$, is a set cover of U and can be found in polynomial time. \square

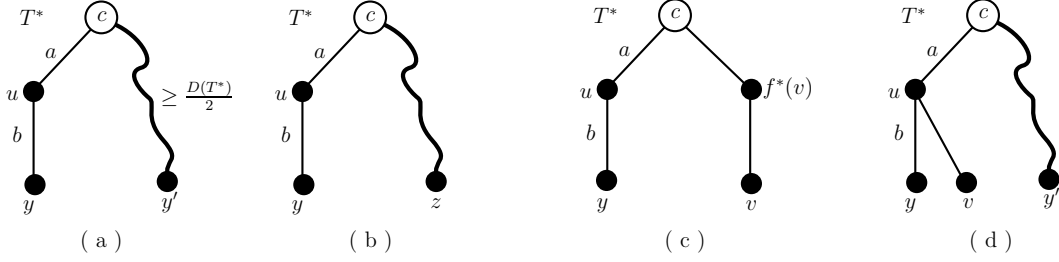


Figure 7: In T^* (a) $d_{T^*}(y', c) \geq \frac{D(T^*)}{2}$; (b) z is not adjacent to u in T^* ; (c) z is adjacent a hub $f^*(z) \neq u$; (d) z is adjacent to u

By Claims 4 and 5, if $D(T) < 4\beta + 2$, then SET COVER has a 3-approximation algorithm running in polynomial time. Notice that $D(T^*) \leq 4$. Thus, for any $\varepsilon > 0$, if Δ_β -SpHCP can be approximated to within a factor $(\beta + \frac{1}{2} - \varepsilon)$ in polynomial time, then SET COVER has a 3-approximation algorithm running in polynomial time. However, for any $\varepsilon > 0$, to approximate SET COVER to within a factor $(1 - \varepsilon) \ln n$ is NP-hard [19]. Therefore, for any $\varepsilon > 0$, to approximate Δ_β -SpHCP to within a factor $\beta + \frac{1}{2} - \varepsilon$ is NP-hard. This completes the proof. \square

We close this section with the following theorem.

Theorem 1. *Let $\beta > \frac{3-\sqrt{3}}{2}$. For any $\varepsilon > 0$, to approximate Δ_β -SpHCP to a factor $g(\beta) - \varepsilon$ is NP-hard where*

1. $g(\beta) = \frac{1+2\beta-2\beta^2}{4(1-\beta)}$ if $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$;
2. $g(\beta) = \frac{5\beta+1}{4}$ if $\frac{2}{3} \leq \beta \leq 1$;
3. $g(\beta) = \beta + \frac{1}{2}$ if $\beta \geq 1$.

Corollary 1. *For any $\varepsilon > 0$, to approximate the SpHCP to a factor $\frac{3}{2} - \varepsilon$ is NP-hard.*

Proof. Since the STAR p -HUB CENTER PROBLEM (SpHCP) assumes the input graph to be a metric graph, *i.e.*, $\beta = 1$, we see $g(1) = \frac{3}{2}$. This completes the proof. \square

3. Polynomial time algorithms

In this section, we show that for $\frac{1}{2} \leq \beta \leq \frac{3-\sqrt{3}}{2}$, Δ_β -SpHCP can be solved in polynomial time. Besides, we give polynomial time approximation algorithms for Δ_β -SpHCP for $\beta > \frac{3-\sqrt{3}}{2}$. For $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$, our approximation algorithm achieves the factor that closes the gap between the upper and lower bounds of approximability for Δ_β -SpHCP.

Lemma 5. *Let $\frac{1}{2} \leq \beta < 1$. Then the following statements hold.*

1. *There exists a solution \tilde{T} satisfying that all non-hubs are adjacent to the same hub and $D(\tilde{T}) \leq \max\{1, \frac{1+2\beta-2\beta^2}{4(1-\beta)}\} \cdot D(T^*)$.*
2. *There exists a polynomial time algorithm to compute a solution T such that $D(T) = D(\tilde{T})$.*

Proof. If T^* is an optimal solution that all non-hubs are adjacent to the same hub, then we are done. If not, let y' be the vertex in T^* satisfying $d_{T^*}(y', c) \geq \frac{D(T^*)}{2}$. Since there exists no optimal solution satisfying that all non-hubs are adjacent to the same hub, there exist a hub u and a non-hub y , $u \neq f^*(y')$, $u \neq y'$, and $y \neq y'$ such that $f^*(y) = u$ and $d_{T^*}(y, c) \leq \frac{D(T^*)}{2}$ (see Fig. 7(a)). Let $w(u, c) = a$ and $w(u, y) = b$. We see that $\min\{a, b\} \leq \frac{D(T^*)}{4}$.

We then construct a new tree T' by the following steps.

Step 1. Guess the hub u adjacent to c in T^* .

Step 2. Select $(p-1)$ vertices from $V \setminus \{c, u\}$ closest to c as hub set S adjacent to c in T' .

Step 3. Let all the remaining vertices as non-hubs adjacent to u in T' .

Now we prove that for x, z in T' , $d_{T'}(x, z) \leq \max\{1, \frac{1+2\beta-2\beta^2}{4(1-\beta)}\} \cdot D(T^*)$.

- In T' , both x and z are non-hubs.

If both x and z are adjacent to u in T^* , then

$$d_{T'}(x, z) = w(x, u) + w(z, u) = d_{T^*}(x, z) \leq D(T^*).$$

Otherwise, one of x, z is not adjacent to u in T^* , say z , as Fig. 7(b). We see that

$$w(u, z) \leq \beta \cdot (d_{T^*}(z, y) - w(u, y)) \leq \beta \cdot (D(T^*) - b).$$

By Lemma 1, we see that

$$w(x, u) \leq \frac{\beta}{1-\beta} \cdot w(u, c) = \frac{\beta}{1-\beta} \cdot a$$

and

$$w(x, u) \leq \frac{\beta}{1-\beta} \cdot w(u, y) = \frac{\beta}{1-\beta} \cdot b.$$

Thus, $w(u, x) \leq \frac{\beta}{1-\beta} \cdot \min\{a, b\}$. Next, we obtain that

$$\begin{aligned} d_{T'}(x, z) &= w(x, u) + w(z, u) \\ &\leq \beta \cdot (D(T^*) - b) + \frac{\beta}{1-\beta} \cdot \min\{a, b\} \\ &\leq \beta \cdot D(T^*) + \left(\frac{\beta}{1-\beta} - \beta\right) \cdot \min\{a, b\} \\ &\leq \beta \cdot D(T^*) + \left(\frac{\beta}{1-\beta} - \beta\right) \cdot \frac{D(T^*)}{4} \\ &= \frac{4\beta-3\beta^2}{4(1-\beta)} \cdot D(T^*) \\ &\leq \frac{1+2\beta-2\beta^2}{4(1-\beta)} \cdot D(T^*). \end{aligned}$$

- In T' , x is a hub and z is a non-hub as Fig. 8.

Notice that in T^* , any non-hub v not adjacent to u (see Fig. 7(c)), *i.e.*, $f^*(v) \neq u$, must satisfy

$$\begin{aligned} w(v, c) &\leq \beta \cdot (w(v, f^*(v)) + w(f^*(v), c)) \\ &= \beta(d_{T^*}(v, z) - a - b) \\ &\leq \beta \cdot (D(T^*) - a - b). \end{aligned}$$

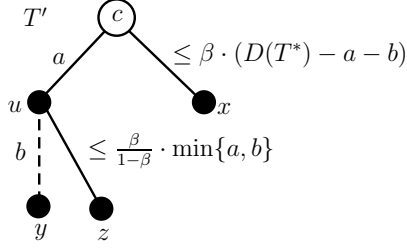


Figure 8: In T' , z is a non-hub adjacent to u and x is a hub. The dashline is an edge in T^* .

If v is a non-hub adjacent to u (see Fig. 7(d)), $f^*(v) = u$, in T^* , then

$$\begin{aligned}
w(v, c) &\leq \beta \cdot (w(v, u) + w(u, c)) \\
&= \beta \cdot (d_{T^*}(v, y') - d_{T^*}(y', c)) \\
&\leq \beta \cdot \frac{D(T^*)}{2} \quad (\text{using } d_{T^*}(v, y') \leq D(T^*) \text{ and } d_{T^*}(y', c) \geq \frac{D(T^*)}{2}) \\
&\leq \beta \cdot (D(T^*) - a - b) \quad (\text{using } a + b \leq \frac{D(T^*)}{2})
\end{aligned}$$

Note that $|V| \geq 2p + 1$. We see that in T^* , there are at least p non-hubs v satisfying that

$$w(v, c) \leq \beta \cdot (D(T^*) - a - b).$$

Since the algorithm chooses the $(p - 1)$ vertices closest to c as hubs in T' and u is a hub in T^* , we see that for all hubs s in T' , $w(s, c) \leq \beta \cdot (D(T^*) - a - b)$. Thus, $w(x, c) \leq \beta \cdot (D(T^*) - a - b)$. By definition $w(u, c) = a$. We have

$$\begin{aligned}
d_{T'}(x, z) &= w(x, c) + w(u, c) + w(z, u) \\
&\leq \beta \cdot (D(T^*) - a - b) + a + \frac{\beta}{1-\beta} \cdot \min\{w(u, c), w(u, y)\} \\
&= a \cdot (1 - \beta) - b \cdot \beta + \beta \cdot D(T^*) + \frac{\beta}{1-\beta} \cdot \min\{a, b\} \\
&= a \cdot (1 - \beta) + b \cdot (1 - \beta) - b + \beta \cdot D(T^*) + \frac{\beta}{1-\beta} \cdot \min\{a, b\} \\
&\leq (a + b) \cdot (1 - \beta) + \beta \cdot D(T^*) + \left(\frac{\beta}{1-\beta} - 1\right) \cdot \min\{a, b\} \\
&\leq \frac{D(T^*)}{2} \cdot (1 - \beta) + \beta \cdot D(T^*) + \left(\frac{\beta}{1-\beta} - 1\right) \cdot \frac{D(T^*)}{4} \\
&\quad (\text{using } a + b \leq \frac{D(T^*)}{2}) \\
&= \frac{D(T^*)}{4} \cdot \left(2 - 2\beta + 4\beta + \frac{\beta}{1-\beta} - 1\right) \\
&= \frac{D(T^*)}{4} \cdot \left(2\beta + \frac{1}{1-\beta}\right) \\
&= \frac{D(T^*) \cdot (1 + 2\beta - 2\beta^2)}{4(1-\beta)}.
\end{aligned}$$

- In T' , both x and z are hubs. Notice that u is also a hub in T^* and all the other $(p - 1)$ hubs in T' are selected as hubs since they are closer to c than other vertices. Thus,

$$d_{T'}(x, z) = w(x, c) + w(z, c) \leq D(T^*).$$

This completes the proof of (i).

Let \mathcal{T} be the collection of solutions of Δ_β -SpHCP satisfying that all non-hubs are adjacent to the same hub. Suppose that \tilde{T} is of minimum diameter among all trees in \mathcal{T} . We give the following algorithm to find a solution T and show that $D(T) = D(\tilde{T})$.

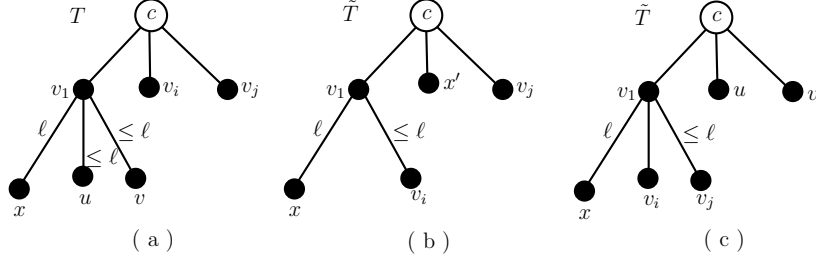


Figure 9: (a) In T , u, v are non-hubs and v_i, v_j are hubs; in \tilde{T} (b) v_i is a non-hub and v_j is a hub; (c) both v_i and v_j are non-hubs.

Step 1. Guess the unique hub that is adjacent to non-hubs in \tilde{T} , call the hub v_1 . Let v_1 be a hub adjacent to c in T .

Step 2. Guess a non-hub x that $w(x, v_1) = \ell = \max_{v \in N_{\tilde{T}}(v_1) \setminus \{c\}} w(v, v_1)$, i.e., (x, v_1) is the longest edge between non-hubs and v_1 in \tilde{T} . Let x be a non-hub adjacent to v_1 in T .

Step 3. For the remaining vertices $y \in V \setminus \{c, v_1, x\}$, if $w(y, v_1) \leq \ell$, let y be a non-hub adjacent to v_1 in T ; otherwise let y be a hub adjacent to c in T .

Step 4. If all vertices are assigned as hubs and non-hubs but the number of hubs $j < p$, select $(p - j)$ non-hubs that are closest to c as hubs.

It is easy to see that the above algorithm runs in $O(n^3)$ time. It is easy to see that $D(T) \geq D(\tilde{T})$. Now we show that $D(T) \leq D(\tilde{T})$.

Suppose that ℓ is the longest edge between hubs and non-hubs in \tilde{T} . Let x be the non-hub adjacent to v_1 in \tilde{T} satisfying that $w(x, v_1) = \ell$. For any non-hub u , $u \neq x$, in T , $d_T(u, v_1) = w(u, v_1) \leq D(\tilde{T}) - \ell$. For any hub v_i , $v_i \neq v_1$, $d_T(v_i, v_1) \leq D(\tilde{T}) - \ell$ since v_i is either a hub in \tilde{T} or v_i is selected as a hub in T in Step 4 since it is closer to c than some hub in \tilde{T} . For any two hubs v_i, v_j , $i \neq 1$ and $j \neq 1$, we see $d_T(v_i, v_j) \leq D(\tilde{T})$. Thus, for any two vertices u, v in T , $d_T(u, v) \leq \max\{D(\tilde{T}), 2D(\tilde{T}) - 2\ell\}$. If $\ell \geq \frac{D(\tilde{T})}{2}$, then $D(T) \leq D(\tilde{T})$.

Suppose that $\ell < \frac{D(\tilde{T})}{2}$. We have the following three cases.

- For any two non-hubs u, v in T (see Fig. 9(a)),

$$d_T(u, v) = w(u, v_1) + w(v, v_1) \leq 2\ell < D(\tilde{T}).$$

- For any non-hub u and any hub v_i in T (see Fig. 9(a)), there are two cases.

– If v_i is a hub in \tilde{T} , then

$$\begin{aligned} d_T(u, v_i) &= w(u, v_1) + w(v_1, c) + w(c, v_i) \\ &= w(u, v_1) + d_{\tilde{T}}(v_i, v_1) \\ &\leq \ell + d_{\tilde{T}}(v_i, v_1) \\ &= \ell + (d_{\tilde{T}}(v_i, x) - \ell) \\ &= d_{\tilde{T}}(v_i, x) \\ &\leq D(\tilde{T}) \end{aligned}$$

where x is the non-hub adjacent to v_1 in \tilde{T} satisfying $w(x, v_1) = \ell$.

- The vertex v_i is a non-hub in \tilde{T} as Fig. 9(b). Since $w(v_1, v_i) \leq \ell$, v_i must be selected as a hub in T in Step 4. The reason that v_i is selected as a hub in T rather than the other vertex x' which is a hub in \tilde{T} and a non-hub in T is because of $w(v_i, c) \leq w(x', c)$. Thus,

$$\begin{aligned}
d_T(u, v_i) &= w(u, v_1) + w(v_1, c) + w(c, v_i) \\
&\leq \ell + w(v_1, c) + w(c, x') \\
&= \ell + d_{\tilde{T}}(v_1, x') \\
&= \ell + (d_{\tilde{T}}(x, x') - \ell) \\
&= d_{\tilde{T}}(x, x') \\
&\leq D(\tilde{T})
\end{aligned}$$

where x is the non-hub adjacent to v_1 in \tilde{T} satisfying $w(x, v_1) = \ell$.

- Both v_i, v_j are hubs in T . In the following, we show that for any two hubs v_i, v_j , $d_T(v_i, v_j) \leq D(\tilde{T})$. There are three cases.

- v_i and v_j are both hubs in \tilde{T} . We have $d_T(v_i, v_j) = d_{\tilde{T}}(v_i, v_j) \leq D(\tilde{T})$.
- One of v_i and v_j is a non-hub in \tilde{T} , say v_i (see Fig. 9(b)). Since v_i is a non-hub in \tilde{T} , $w(v_1, v_i) \leq \ell$. Notice that if $w(v_1, v_i) > \ell$, v_i must be a hub in \tilde{T} . According to the fact that $w(v_1, v_i) \leq \ell$, v_i must be selected as a hub of T in Step 4. The reason that v_i is selected as a hub in T rather than the other vertex x' which is a hub in \tilde{T} and a non-hub in T is because of $w(v_i, c) \leq w(x', c)$. Thus

$$\begin{aligned}
d_T(v_i, v_j) &= w(v_i, c) + w(v_j, c) \\
&\leq w(x', c) + w(v_j, c) \\
&= d_{\tilde{T}}(x', v_j) \leq D(\tilde{T}).
\end{aligned}$$

- Both v_i and v_j are non-hubs in \tilde{T} as Fig. 9(c). Note that $w(v_i, v_1) \leq \ell$ and $w(v_j, v_1) \leq \ell$. We see that v_i and v_j must be selected as hubs in Step 4. It means there exist two hubs u, v in \tilde{T} which are non-hubs in T satisfying that $w(u, c) \geq w(v_i, c)$ and $w(v, c) \geq w(v_j, c)$. Thus,

$$\begin{aligned}
d_T(v_i, v_j) &= w(v_i, c) + w(v_j, c) \\
&\leq w(u, c) + w(v, c) \\
&= d_{\tilde{T}}(u, v) \leq D(\tilde{T}).
\end{aligned}$$

Since for all $u, v \in T$, we have shown that $d_T(u, v) \leq D(\tilde{T})$, together with the fact $D(T) \geq D(\tilde{T})$, we obtain that $D(T) = D(\tilde{T})$. This completes the proof of (ii). \square

Lemma 6. *Let $\frac{1}{2} \leq \beta \leq 0.7737\dots$. Then the following statements hold.*

1. *If $\beta \leq \frac{3-\sqrt{3}}{2}$, then Δ_β -SpHCP can be solved in polynomial time.*
2. *If $\frac{3-\sqrt{3}}{2} < \beta \leq 0.7737\dots$, there is a $\frac{1+2\beta-2\beta^2}{4(1-\beta)}$ -approximation algorithm for Δ_β -SpHCP.*

Proof. Let T^* denote an optimal solution of the Δ_β -SpHCP problem. By Lemma 5, there is a polynomial time algorithm for Δ_β -SpHCP to compute a solution T such that $D(T) \leq \max\{1, \frac{1+2\beta-2\beta^2}{4(1-\beta)}\} \cdot D(T^*)$.

For $\beta \leq \frac{3-\sqrt{3}}{2}$, $D(T) \leq \max\{1, \frac{1+2\beta-2\beta^2}{4(1-\beta)}\} \cdot D(T^*) = D(T^*)$.

For $\frac{3-\sqrt{3}}{2} < \beta \leq 0.7737\dots$, $D(T) \leq \max\{1, \frac{1+2\beta-2\beta^2}{4(1-\beta)}\} \cdot D(T^*) = \frac{1+2\beta-2\beta^2}{4(1-\beta)} \cdot D(T^*)$.

This completes the proof. \square

Algorithm 1 Approximation algorithm for Δ_β -SpHCP (G, c) if $\beta \in [0.7737\dots, 2]$

- 1: Run Algorithm APX1.
 - 2: Run Algorithm APX2.
 - 3: Return the best solution found by Algorithms APX1 and APX2.
-

Algorithm APX1

- 1: Guess the correct edge (y, z) where $w(y, z) = \ell$ is the largest edge cost in an optimal solution T^* with y as a hub and z as a non-hub. Let $U := V \setminus \{c\}$ and $h_1 = y$. Let T_1 be the tree found by the following steps and H be the children of c in T_1 . Initialize $H = \emptyset$.
 - 2: Add edge (h_1, c) in the tree T , let $H := H \cup \{h_1\}$, and let $U := U \setminus \{h_1\}$.
 - 3: For $x \in U$, if $w(h_1, x) \leq \ell$, add edges (x, h_1) in T and let $U := U \setminus \{x\}$.
 - 4: **while** $i = |H| + 1 \leq p$ and $U \neq \emptyset$ **do**
 - 5: choose $v \in U$, let $h_i = v$, add edge (h_i, c) in T , let $U := U \setminus \{v\}$, and let $H := H \cup \{h_i\}$;
 - 6: for $x \in U$, if $w(x, h_i) \leq 2\beta\ell$, then add edge (x, h_i) in T and $U := U \setminus \{x\}$.
 - 7: **end while**
 - 8: If $|H| < p$ and $U = \emptyset$, we change the shape of T by selecting $p - |H|$ vertices closest to c from the second layer to be the children of c , call the new tree T_1 ; otherwise let $T_1 := T$.
 - 9: **return** T_1 .
-

Algorithm APX2

- 1: Guess the correct edge (y, z) where $w(y, z) = \ell$ is the largest edge cost in an optimal solution T^* with y as a hub and z as a non-hub. Let T_2 be the tree found by the following steps.
 - 2: Let y be the child of c in T^* .
 - 3: Pick $(p - 1)$ vertices $\{v_1, v_2, \dots, v_{p-1}\}$ closest to c from $U \setminus \{y, z\}$. Let $N_{T_2}(c) = \{y, v_1, v_2, \dots, v_{p-1}\}$.
 - 4: Let all vertices in $U \setminus \{v_1, v_2, \dots, v_{p-1}, y\}$ be the children of y .
-

Lemma 7. *Let $0.7737\dots \leq \beta \leq 1$. Then, there is a $(1 + \frac{4\beta^2}{5\beta+1})$ -approximation algorithm for Δ_β -SpHCP.*

Proof. It is not hard to see that Algorithm 1 runs in polynomial time. Let T^* be an optimal solution of Δ_β -SpHCP and ℓ be the largest edge cost in T^* with one end vertex

as a hub and the other end vertex as a non-hub. Note that both Algorithm APX1 and Algorithm APX2 guess all possible edges (y, z) to be the longest edge in T^* with y is a hub and z is a non-hub. Let T_1 and T_2 be the best solutions returned by Algorithm APX1 and Algorithm APX2, respectively. Next, we show that for $0.7737... \leq \beta \leq 1$, Algorithm 1 returns a solution T such that $D(T) \leq (1 + \frac{4\beta^2}{5\beta+1}) \cdot D(T^*)$ where T^* is an optimal solution.

Let T_1 and T_2 be the best solution returned by Algorithm APX1 and Algorithm APX2, respectively.

CLAIM 1. $D(T_1) \leq D(T^*) + 4\beta\ell$

Proof. We first show that for any two hubs u, v in T_1 , $d_{T_1}(u, v) = w(u, c) + w(v, c) \leq D(T^*)$. Let T^* be an optimal solution of Δ_β -SpHCP. Let $f^*(u)$ and $f^*(v)$ be the parents of u and v in T^* , respectively.

If $f^*(u) \neq f^*(v)$, there are three cases.

- Suppose that $f^*(u) = c$ and $f^*(v) \neq u$. Then

$$d_{T_1}(u, v) = w(u, c) + w(v, c) \leq w(u, c) + w(c, f^*(v)) + w(f^*(v), v) = d_{T^*}(u, v) \leq D(T^*).$$

- Suppose that $f^*(u) = c$ and $f^*(v) = u$. Since $w(u, v) \leq 2\beta\ell$, v is selected as a hub in Step (iv) of Algorithm APX1. Since in Step (iv), the algorithm select $(p - |H|)$ vertices closest to c from the second layer as hubs, there exists y' which is a hub in T^* and a non-hub in T_1 satisfying $w(y', c) \geq w(v, c)$. Thus,

$$d_{T_1}(u, v) = w(u, c) + w(v, c) \leq w(u, c) + w(y', c) = d_{T^*}(u, y') \leq D(T^*).$$

- Suppose that $f^*(u) \neq c$. Then

$$\begin{aligned} d_{T_1}(u, v) &= w(u, c) + w(v, c) \\ &\leq w(u, f^*(u)) + w(f^*(u), c) + w(c, f^*(v)) + w(f^*(v), v) \\ &= d_{T^*}(u, v) \leq D(T^*). \end{aligned}$$

If $f^*(u) = f^*(v) = c$, $d_{T_1}(u, v) = d_{T^*}(u, v) \leq D(T^*)$.

If $f^*(u) = f^*(v) \neq c$, then at most one of u, v is selected as a hub in Step (iii) of Algorithm APX1 since $w(u, v) \leq 2\beta\ell$, or both u and v are selected as hubs in Step (iv).

Suppose that u is selected as a hub in Step (iii) and v is selected as a hub in Step (iv). We see that in Step (iv), the algorithms select $(p - |H|)$ vertices closest to c from the second layer as hubs. Thus, there exists y' which is a hub in T^* and a non-hub in T_1 satisfying $w(y', c) \geq w(v, c)$. We obtain that

$$d_{T_1}(u, v) = w(u, c) + w(v, c) \leq d_{T^*}(u, c) + w(y', c) = d_{T^*}(u, y') \leq D(T^*).$$

Suppose that both u, v are selected as hubs in Step (iv). We see that in Step (iv), the algorithm selects $(p - |H|)$ vertices closest to c from the second layer as hubs. Thus, there exist y_1, y_2 which are hubs in T^* and non-hubs in T_1 satisfying $w(y_1, c) \geq w(u, c)$ and $w(y_2, c) \geq w(v, c)$. We obtain that

$$d_{T_1}(u, v) = w(u, c) + w(v, c) \leq w(y_1, c) + w(y_2, c) = d_{T^*}(y_1, y_2) \leq D(T^*).$$

Notice that each non-hub v in T_1 is adjacent to a hub $f(v)$ in T_1 if $w(v, f(v)) \leq 2\beta\ell$.

Thus, for u, v in T_1 , $d_{T_1}(u, v) \leq D(T^*) + 4\beta\ell$ and $D(T_1) \leq D(T^*) + 4\beta\ell$. This completes the proof of the claim. \square

CLAIM 2. $D(T_2) \leq \max\{D(T^*), (D(T^*) - \ell) + \beta(D(T^*) - \ell)\}$.

Proof. Let T^* be an optimal solution. For a vertex v , use $f^*(v)$ to denote the parent of v in T^* . Notice that Algorithm APX2 guesses all possible edges (y, z) to be a longest hub-nonhub edge in T^* . In the following we assume that $w(y, z) = \ell$ is the cost of a longest hub-nonhub edge in T^* with y as a hub and z is a non-hub in T^* . Since Algorithm APX2 picks $(p - 1)$ vertices closest to c , y is a hub in both T^* and T_2 , and (y, z) is the longest hub-nonhub edge in T^* , we see that for any hub v in T_2 , $d_{T_2}(v, y) \leq D(T^*) - \ell$.

For two non-hubs u, v in T_2 , we have the following three cases.

- $f^*(u) = f^*(v) = y$, we see that $d_{T_2}(u, v) = d_{T^*}(u, v) \leq D(T^*)$.
- $f^*(u) = y$ and $f^*(v) \neq y$, we see that

$$d_{T_2}(u, v) = w(u, y) + w(v, y) \leq \ell + \beta \cdot d_{T^*}(v, y) \leq \ell + \beta \cdot (D(T^*) - \ell) \leq D(T^*).$$

- $f^*(u) \neq y$ and $f^*(v) \neq y$, we see that

$$\begin{aligned} d_{T_2}(u, v) &= w(u, y) + w(v, y) \\ &\leq \beta \cdot d_{T^*}(u, y) + \beta \cdot d_{T^*}(v, y) \\ &\leq 2\beta(D(T^*) - \ell) \\ &\leq (D(T^*) - \ell) + \beta(D(T^*) - \ell) \end{aligned}$$

For a non-hub u and a hub v in T_2 , there are two cases.

- If $f^*(u) = y$, we see that $d_{T_2}(u, v) = w(u, y) + d_{T_2}(v, y) \leq \ell + D(T^*) - \ell = D(T^*)$.
- If $f^*(u) \neq y$, we see that $d_{T_2}(u, v) = w(u, y) + d_{T_2}(v, y) \leq \beta \cdot (D(T^*) - \ell) + (D(T^*) - \ell)$.

For two hubs u, v in T_2 , $u \neq y$ and $v \neq y$, we see that $d_{T_2}(u, v) \leq D(T^*)$ since y is a hub in T^* and Algorithm APX2 picks the other $(p - 1)$ vertices closest to c as hubs.

Thus, $D(T_2) \leq \max\{D(T^*), (D(T^*) - \ell) + \beta(D(T^*) - \ell)\}$. This completes the proof of the claim. \square

Notice that if $\frac{\ell}{D(T^*)} \geq \frac{\beta}{1+\beta}$, $D(T_2) = D(T^*)$. Thus, the worst case approximation ratio happens when $\frac{\ell}{D(T^*)} < \frac{\beta}{1+\beta}$.

If $\frac{\ell}{D(T^*)} < \frac{\beta}{1+\beta}$, $D(T_2) \leq D(T^*) - \ell + \beta(D(T^*) - \ell)$. We see that the approximation ratio of Algorithm 1 is $r(\beta) = \min\{\frac{D(T_1)}{D(T^*)}, \frac{D(T_2)}{D(T^*)}\}$. The worst case approximation ratio of Algorithm 1 happens when $D(T_1) = D(T_2)$, *i.e.*,

$$D(T^*) + 4\beta\ell = (D(T^*) - \ell) + \beta \cdot (D(T^*) - \ell)$$

We obtain that $\frac{\ell}{D(T^*)} = \frac{\beta}{5\beta+1}$. Thus,

$$\begin{aligned} r(\beta) &= \min\left\{\frac{D(T_1)}{D(T^*)}, \frac{D(T_2)}{D(T^*)}\right\} \\ &\leq \min\left\{1 + \frac{4\beta^2}{5\beta+1}, 1 - \frac{\beta}{5\beta+1} + \beta\left(1 - \frac{\beta}{5\beta+1}\right)\right\} \\ &= 1 + \frac{4\beta^2}{5\beta+1}. \end{aligned}$$

This completes the proof. \square

Lemma 8. *Let $1 \leq \beta \leq 2$. Then, there is a $(\beta + \frac{4\beta^2-2\beta}{2+\beta})$ -approximation algorithm for Δ_β -SpHCP.*

Proof. It is not hard to see that Algorithm 1 runs in polynomial time. Let T^* be an optimal solution of Δ_β -SpHCP and ℓ be the largest edge cost in T^* with one end vertex as a hub and the other end vertex as a non-hub. Note that both Algorithm APX1 and Algorithm APX2 guess all possible edges (y, z) to be the longest edge in T^* with y is a hub and z is a non-hub. Let T_1 and T_2 be the best solutions returned by Algorithm APX1 and Algorithm APX2, respectively. In this lemma, we show that for $1 \leq \beta \leq 2$, Algorithm 1 returns a solution T such that $D(T) \leq (\beta + \frac{4\beta^2-2\beta}{2+\beta}) \cdot D(T^*)$ where T^* is an optimal solution.

CLAIM 1. $D(T_1) \leq \beta \cdot D(T^*) + 4\beta\ell$.

Proof. We first show that for any two hubs u, v in T_1 , $d_{T_1}(u, v) = w(u, c) + w(v, c) \leq \beta \cdot D(T^*)$. Let T^* be an optimal solution of Δ_β -SpHCP. Let $f^*(u)$ and $f^*(v)$ be the parents of u and v in T^* respectively.

If $f^*(u) \neq f^*(v)$, then there are three cases.

- Suppose that $f^*(u) = c$ and $f^*(v) \neq u$. Then

$$\begin{aligned} d_{T_1}(u, v) &= w(u, c) + w(v, c) \\ &\leq w(u, c) + \beta \cdot (w(c, f^*(v)) + w(f^*(v), v)) \\ &\leq \beta \cdot d_{T^*}(u, v) \leq \beta \cdot D(T^*). \end{aligned}$$

- Suppose that $f^*(u) = c$ and $f^*(v) = u$. Since $w(u, v) \leq 2\beta\ell$, v is selected as a hub in Step (iv) of Algorithm APX1. Since in Step (iv), the algorithm select $(p - |H|)$ vertices closest to c from the second layer as hubs, there exists y' which is a hub in T^* and a non-hub in T_1 satisfying $w(y', c) \geq w(v, c)$. Thus,

$$d_{T_1}(u, v) = w(u, c) + w(v, c) \leq w(u, c) + w(y', c) = d_{T^*}(u, y') \leq D(T^*) \leq \beta \cdot D(T^*).$$

- Suppose that $f^*(u) \neq c$. Then

$$\begin{aligned} d_{T_1}(u, v) &= w(u, c) + w(v, c) \\ &\leq \beta \cdot (w(u, f^*(u)) + w(f^*(u), c)) + \beta \cdot (w(c, f^*(v)) + w(f^*(v), v)) \\ &= \beta \cdot d_{T^*}(u, v) \\ &\leq \beta \cdot D(T^*). \end{aligned}$$

If $f^*(u) = f^*(v) = c$, $d_{T_1}(u, v) = d_{T^*}(u, v) \leq D(T^*) \leq \beta \cdot D(T^*)$.

If $f^*(u) = f^*(v) \neq c$, then at most one of u, v is selected as a hub in Step (iii) of Algorithm APX1 since $w(u, v) \leq 2\beta\ell$, or both u and v are selected as hubs in Step (iv). Suppose that u is selected as a hub in Step (iii) of Algorithm APX1 and v is selected as a hub in Step (iv) of Algorithm APX1. We see that in Step (iv), the algorithm selects $(p - |H|)$ vertices closest to c from the second layer as hubs. Thus, there exists y' which is a hub in T^* and a non-hub in T_1 satisfying $w(y', c) \geq w(v, c)$. We obtain that

$$d_{T_1}(u, v) = w(u, c) + w(v, c) \leq \beta \cdot d_{T^*}(u, c) + w(y', c) \leq \beta \cdot d_{T^*}(u, y') \leq \beta \cdot D(T^*).$$

Suppose that both u, v are selected as hubs in Step (iv). We see that in Step (iv), the algorithms select $(p - |H|)$ vertices closest to c from the second layer as hubs. Thus,

there exist y_1, y_2 which are hubs in T^* and non-hubs in T_1 satisfying $w(y_1, c) \geq w(u, c)$ and $w(y_2, c) \geq w(v, c)$. We obtain that

$$d_{T_1}(u, v) = w(u, c) + w(v, c) \leq w(y_1, c) + w(y_2, c) = d_{T^*}(y_1, y_2) \leq D(T^*) \leq \beta \cdot D(T^*).$$

Notice that each non-hub v in T_1 is adjacent to a hub $f(v)$ in T_1 if $w(v, f(v)) \leq 2\beta\ell$. Thus, for $u, v \in T_1$, $d_{T_1}(u, v) \leq \beta \cdot D(T^*) + 4\beta\ell$ and $D(T_1) \leq \beta \cdot D(T^*) + 4\beta\ell$. This completes the proof of the claim. \square

CLAIM 2. $D(T_2) \leq \max\{\ell + \beta^2 \cdot (D(T^*) - \ell), 2\beta^2 \cdot (D(T^*) - \ell)\}$.

Proof. Let T^* be an optimal solution. For a vertex v , use $f^*(v)$ to denote the parent of v in T^* . Since Algorithm APX2 picks $(p-1)$ vertices closest to c , y is a hub in both T^* and T_2 , and (y, z) is the longest hub-nonhub edge in T^* , we see that for any hub v in T_2 , $d_{T_2}(v, y) \leq D(T^*) - \ell$.

For two non-hubs u, v in T_2 , we have the following three cases.

- $f^*(u) = f^*(v) = y$, we see that $d_{T_2}(u, v) = d_{T^*}(u, v) \leq D(T^*) \leq \ell + \beta^2 \cdot (D(T^*) - \ell)$.
- $f^*(u) = y$ and $f^*(v) \neq y$, we see that

$$d_{T_2}(u, v) = w(u, y) + w(v, y) \leq \ell + \beta^2 \cdot d_{T^*}(v, y) \leq \ell + \beta^2 \cdot (D(T^*) - \ell).$$

- $f^*(u) \neq y$ and $f^*(v) \neq y$, we see that

$$\begin{aligned} d_{T_2}(u, v) &= w(u, v) + w(v, y) \\ &\leq \beta^2 \cdot d_{T^*}(u, y) + \beta^2 \cdot d_{T^*}(v, y) \\ &\leq 2\beta^2 \cdot (D(T^*) - \ell) \end{aligned}$$

For a non-hub u and a hub v in T_2 , there are two cases.

- If $f^*(u) = y$, we see that

$$\begin{aligned} d_{T_2}(u, v) &= w(u, y) + d_{T_2}(v, y) \\ &\leq \ell + D(T^*) - \ell \\ &\leq \ell + \beta^2(D(T^*) - \ell). \end{aligned}$$

- If $f^*(u) \neq y$, we see that

$$\begin{aligned} d_{T_2}(u, v) &= w(u, y) + d_{T_2}(v, y) \\ &\leq \beta^2 \cdot (D(T^*) - \ell) + (D(T^*) - \ell) \\ &\leq 2\beta^2 \cdot (D(T^*) - \ell). \end{aligned}$$

Thus, $D(T_2) \leq \max\{\ell + \beta^2 \cdot (D(T^*) - \ell), 2\beta^2 \cdot (D(T^*) - \ell)\}$. This completes the proof of the claim. \square

The worst case approximation ratio is

$$r(\beta) = \min\left\{\beta + 4\beta \cdot \frac{\ell}{D(T^*)}, \max\left\{\frac{\ell}{D(T^*)} + \beta^2 \cdot \left(1 - \frac{\ell}{D(T^*)}\right), 2\beta^2 \cdot \left(1 - \frac{\ell}{D(T^*)}\right)\right\}\right\}.$$

Algorithm 2 Approximation algorithm for Δ_β -SpHCP (G, c) if $\beta \geq 2$

- 1: Guess the correct values of ℓ_0, ℓ_1 and ℓ_2 (ℓ_1 and ℓ_2 will be used in Algorithm 3). Their meanings are provided in the proof.
 - 2: $H \leftarrow \{v \in V \setminus \{c\} \mid w(v, c) \leq \ell_0\}$.
 - 3: Create an instance \mathcal{J} of the k -center problem with forbidden centers, in which $V \setminus \{c\}$ is the set of input vertices, H is the set of allowed centers, $k = p$, and the distance function (satisfying the β -triangle inequality) is the restriction of w to $V \setminus \{c\}$.
 - 4: Apply the greedy approximation algorithm for the k -center problem with forbidden centers (Algorithm 3), to obtain an approximate solution of \mathcal{J} . Assume that $H^* \subseteq H$ is the set of centers opened in the solution.
 - 5: **return** the solution that opens H^* as the set of p hubs and assigns each vertex in $V \setminus \{c\}$ to its nearest hub in H^* .
-

If $\frac{\ell}{D(T^*)} > \frac{1}{2}$, then

$$\begin{aligned}
r(\beta) &\leq \frac{D(T_2)}{D(T^*)} \\
&\leq \max\left\{\frac{\ell}{D(T^*)} + \beta^2 \cdot \left(1 - \frac{\ell}{D(T^*)}\right), 2\beta^2 \cdot \left(1 - \frac{\ell}{D(T^*)}\right)\right\} \\
&\leq \max\left\{1 + \frac{\beta^2}{2}, \beta^2\right\} \\
&\leq \beta + \frac{4\beta^2 - 2\beta}{2 + \beta} \text{ for } 1 \leq \beta \leq 2.
\end{aligned}$$

If $\frac{\ell}{D(T^*)} \leq \frac{1}{2}$, we have

$$\begin{aligned}
D(T_2) &\leq \max\{\ell + \beta^2 \cdot (D(T^*) - \ell), 2\beta^2 \cdot (D(T^*) - \ell)\} \\
&= 2\beta^2 \cdot (D(T^*) - \ell).
\end{aligned}$$

Thus, $r(\beta) = \min\{\beta + 4\beta \cdot \frac{\ell}{D(T^*)}, 2\beta^2 \cdot (1 - \frac{\ell}{D(T^*)})\}$.

Let $\beta + 4\beta \cdot \frac{\ell}{D(T^*)} = 2\beta^2 \cdot (1 - \frac{\ell}{D(T^*)})$. Then, we obtain that $\frac{\ell}{D(T^*)} = \frac{2\beta-1}{2\beta+4}$. Hence

$$\begin{aligned}
r(\beta) &\leq \min\left\{\beta + 4\beta \cdot \frac{2\beta-1}{2\beta+4}, 2\beta^2 \cdot \left(1 - \frac{2\beta-1}{2\beta+4}\right)\right\} \\
&= \beta + \frac{4\beta^2 - 2\beta}{2 + \beta}.
\end{aligned}$$

This completes the proof. □

Lemma 9. *Let $\beta \geq 2$. Then, there is a $(2\beta+1)$ -approximation algorithm for Δ_β -SpHCP.*

Proof. Let T^* be an optimal solution of Δ_β -SpHCP. Let (c, q) be the longest edge incident to c in T^* , $w(c, q) = \ell_0$, i.e., $\ell_0 = \max_{v \in N_{T^*}(c)} \{w(v, c)\}$. Let ℓ_1 and ℓ_2 be the largest and second largest edge costs in T^* with one end vertex as a hub and the other end vertex as a non-hub. Note that it is possible that $\ell_1 = \ell_2$. Our algorithm is presented as Algorithm 2. Line 1 of Algorithm 2 guesses the values of ℓ_0, ℓ_1 , and ℓ_2 . There are only polynomial many possible values of ℓ_0, ℓ_2, ℓ_3 . In the following we assume that we know ℓ_0, ℓ_1 and ℓ_2 . It is easy to see that $D(T^*) \geq \ell_0 + \ell_1$ and $D(T^*) \geq \ell_1 + \ell_2$.

Let T denote the solution returned by Algorithm 2. We next prove that Algorithm 2 is indeed a $(2\beta+1)$ -approximation algorithm for Δ_β -SpHCP by establishing an upper bound of $D(T)$. According to our choice of ℓ_0 , the set H defined in line 2 contains all hub nodes in the optimal solution $N_{T^*}(c)$, i.e., $N_{T^*}(c) \subseteq H$. In Line 3, we create an instance \mathcal{J} of the k -center problem with forbidden centers. This problem is defined as follows: The

Algorithm 3 Approximation algorithm for k -center with forbidden centers.

```

1: // Let  $C$  be the input vertex set,  $C' \subseteq C$  be the set of allowed centers, and  $w$  be
   the distance function on  $C$  satisfying the  $\beta$ -triangle inequality. Assume w.l.o.g. that
    $k \leq |C'|$ .
2:  $R \leftarrow C$ ;  $S \leftarrow \emptyset$ .
3: while  $R \neq \emptyset$  and  $|S| < k$  do
4:   Choose an arbitrary vertex  $v \in C' \cap R$ .
5:    $B(v) \leftarrow \{u \in R \mid w(u, v) \leq \beta(\ell_1 + \ell_2)\}$ .
6:    $R \leftarrow R \setminus B(v)$ ;  $S \leftarrow S \cup \{v\}$ .
7: end while
8: if  $|S| < k$  and  $R = \emptyset$  then
9:   select an arbitrary vertex set  $S' \subseteq (C' \setminus S)$  of size  $k - |S|$ ;  $S \leftarrow S \cup S'$ .
10: end if
11: return  $S$ 

```

input consists of a set C of demand points in a space satisfying the β -triangle inequality, a set $C' \subseteq C$ of allowed centers, and an integer k . The goal is to open k centers in C' such that the maximum distance between any vertex in C and its nearest center among the k opened centers is minimized. This problem is a generalization of the ordinary k -center problem (in which $C' = C$), and is a special case of the k -supplier problem (in which C' may not be a subset of C) [22, 23, 24]. There is a simple greedy approximation algorithm for this problem, which is presented in Algorithm 3. Its analysis is standard and is similar to that of the traditional k -center problem (see [22, 23, 24]), and thus is omitted here.

Hence, by applying the greedy approximation algorithm (Algorithm 3) to implement line 4 of Algorithm 2, we obtain a solution H^* of \mathcal{J} with objective value at most $\beta(\ell_1 + \ell_2)$, that is,

$$\max_{v \in V \setminus \{c\}} \min_{h \in H^*} w(v, h) \leq \beta(\ell_1 + \ell_2). \quad (1)$$

Line 5 returns a solution that opens H^* as the set of p hubs. For each $v \in V \setminus (H^* \cup \{c\})$, let $f'(v) := \arg \min_{h \in H^*} w(v, h)$; *i.e.*, $f'(v)$ is the hub in H^* assigned to v in the solution returned by the algorithm. Let ℓ'_1 and ℓ'_2 be the largest value and second-largest value in the multiset $\{w(v, f'(v)) \mid v \in V \setminus \{c\}\}$. By inequality (1), we have

$$\ell'_1 + \ell'_2 \leq 2\beta(\ell_1 + \ell_2).$$

Let $x, y \in V \setminus \{c\}$ be the nodes achieving the maximum path length in T , *i.e.*, $d_T(x, y) = D(T)$. It suffices to show that $D(T) \leq (2\beta + 1) \cdot D(T^*)$.

If $f'(x) = f'(y)$, then

$$D(T) = w(x, f'(x)) + w(y, f'(y)) \leq \ell'_1 + \ell'_2.$$

If $f'(x) \neq f'(y)$, then

$$\begin{aligned} D(T) &= w(x, f'(x)) + w(f'(x), c) + w(f'(y), c) + w(y, f'(y)) \\ &\leq \ell'_1 + 2\ell_0 + \ell'_2 \end{aligned}$$

where we use $w(h, c) \leq \ell_0$ for all $h \in H$ by our choice of H . Combine with the fact that

$D(T^*) \geq \ell_0 + \ell_1$ and $D(T^*) \geq \ell_1 + \ell_2$, we always have

$$\begin{aligned}
D(T) &\leq 2\ell_0 + \ell'_1 + \ell'_2 \\
&\leq 2\ell_0 + 2\beta(\ell_1 + \ell_2) \\
&\leq 2(\ell_0 + \ell_1) + (2\beta - 1)(\ell_1 + \ell_2) \quad (\text{using } \ell_2 \leq \ell_1) \\
&\leq 2 \cdot D(T^*) + (2\beta - 1) \cdot D(T^*) \\
&= (2\beta + 1) \cdot D(T^*),
\end{aligned}$$

which indicates that Algorithm 2 is a $(2\beta + 1)$ -approximation algorithm for Δ_β -SpHCP. This completes the proof. \square

We close this section with the following theorem.

Theorem 2. *Let $\beta \geq \frac{1}{2}$. There exists a polynomial time $r(\beta)$ -approximation algorithm for Δ_β -SpHCP where*

1. $r(\beta) = 1$ if $\beta \leq \frac{3-\sqrt{3}}{2}$;
2. $r(\beta) = \frac{1+2\beta-2\beta^2}{4(1-\beta)}$ if $\frac{3-\sqrt{3}}{2} < \beta \leq 0.7737\dots$;
3. $r(\beta) = 1 + \frac{4\beta^2}{5\beta+1}$ if $0.7737\dots \leq \beta \leq 1$;
4. $r(\beta) = \beta + \frac{4\beta^2-2\beta}{2+\beta}$ if $1 \leq \beta \leq 2$;
5. $r(\beta) = 2\beta + 1$ if $\beta \geq 2$.

Corollary 2. *There exists a polynomial time $\frac{5}{3}$ -approximation algorithm for SpHCP.*

Proof. Since the STAR p -HUB CENTER PROBLEM (SpHCP) assumes the input graph to be a metric graph, *i.e.*, $\beta = 1$, we see $r(1) = \frac{5}{3}$. This completes the proof. \square

4. Conclusion

In this paper, we have studied Δ_β -SpHCP for all $\beta \geq \frac{1}{2}$. We showed that for any $\epsilon > 0$, to approximate Δ_β -SpHCP to a ratio $g(\beta) - \epsilon$ is NP-hard where $g(\beta) = \frac{1+2\beta-2\beta^2}{4(1-\beta)}$ if $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$; $g(\beta) = \frac{5\beta+1}{4}$ if $\frac{2}{3} < \beta \leq 1$; $g(\beta) = \beta + \frac{1}{2}$ if $\beta \geq 1$. Moreover, we gave $r(\beta)$ -approximation algorithms for the same problem. If $\beta \leq \frac{3-\sqrt{3}}{2}$, we have $r(\beta) = g(\beta) = 1$, *i.e.*, Δ_β -SpHCP is polynomial time solvable for $\beta \leq \frac{3-\sqrt{3}}{2}$. If $\frac{3-\sqrt{3}}{2} < \beta \leq \frac{2}{3}$, we have $r(\beta) = g(\beta) = \frac{1+2\beta-2\beta^2}{4(1-\beta)}$. For $\frac{2}{3} \leq \beta \leq 1$, $r(\beta) = \min\{\frac{1+2\beta-2\beta^2}{4(1-\beta)}, 1 + \frac{4\beta^2}{5\beta+1}\}$. For $\beta \geq 1$, we have $r(\beta) = \min\{\beta + \frac{4\beta^2-2\beta}{2+\beta}, 2\beta + 1\}$. In the future work, it is of interest to extend the range of β for Δ_β -SpHCP such that the gap between the upper and lower bounds of approximability can be reduced.

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