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# Freezing and low temperature entropy I : the case of mean-field Gaussian model

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## Abstract

The freezing property is studied in connection with the low temperature behaviour of the entropy of the Gibbs measure. In particular, from the study of a functional equation relating the free energies at two different temperatures, we prove, in the case of mean-field Gaussian spin glass model, that the maximum value of the inverse temperature beyond which the specific entropy of the Gibbs measure cannot be strictly positive is  $\beta_* = 4 \log 2$ . The low temperature behaviour of the REM's entropy is also addressed.

## 1 Introduction and main results

The “freezing phenomenon”, initially defined in the case of random polymers [5], has recently attracted considerable interest in the context of extreme value statistics. It is also connected to the properties of the Laplace functional of the randomly shifted decorated Poisson point processes (SDPPP)[10] and expected to occur in a wide class of logarithmically-correlated Gaussian fields.

Freezing can be characterized either from the properties of the glassy phase dominated by local extreme values of the random fields [8] or from the behaviour of the free energy at low temperatures. An interesting question is whether freezing could be related to the entropy. In [7] we prove that freezing can be equally defined by the vanishing of the entropy of the corresponding Gibbs measures. In particular, for random polymers on trees, multiplicative chaos and mean-field spin-glass models, we show that there exists a freezing temperature beyond which the specific entropy of the Gibbs measure cannot be strictly positive.

The purpose of this note is to present in detail the relation between entropy and freezing in the case of the (widely studied) mean-field Gaussian spin-glass model. Indeed, we introduce a simple, yet rigorous, method in order to estimate the value of the freezing temperature at which the mean entropy of the Gibbs measure vanishes. Our approach is totally self-contained; we solely make use of the existence of the thermodynamic limit of the quenched specific free energy [6] and its self-averaging property [9].

We first recall some basic definitions. Suppose that a finite set of  $n$  sites is given. With each site we associate the one-spin space  $\Sigma := \{1, -1\}$ . The natural configuration space is then the product space  $\Sigma_n = \{-1, 1\}^n$ , of card  $\Sigma_n = 2^n$ . For each  $\sigma \in \Sigma_n$ , the

finite volume Hamiltonian of the model is given by the following real-valued function on  $\Sigma_n$

$$H_n(\sigma, J) = -\frac{1}{\sqrt{n}} \sum_{1 \leq i < j \leq n} J_{ij} \sigma_i \sigma_j,$$

where the family of couplings  $J = (J_{ij})_{1 \leq i < j \leq n}$  are independent normal Gaussian variables.

At the inverse temperature  $\beta = \frac{1}{T} > 0$  (we take units such that the Boltzmann constant  $k = 1$ ), the disorder-dependent partition function  $Z_n(\beta, J)$ , is given by the sum of the Boltzmann factors

$$Z_n(\beta, J) = \sum_{\sigma} e^{-\beta H_n(\sigma, J)}.$$

Moreover, if  $E_J$  denotes the expectation with respect to the randomness  $J$ , it is very simple to show that  $E_J Z_n(\beta, J) = 2^n e^{\frac{\beta^2}{4}(n-1)}$ .

When the randomness  $J$  is fixed, the corresponding Gibbs probability measure is denoted by  $\mu_{n,\beta,J}(\sigma)$  and given by:

$$\mu_{n,\beta}(\sigma) = \frac{e^{-\beta H_n(\sigma, J)}}{Z_n(\beta, J)}.$$

We also recall the definition of the entropy of  $\mu_{n,\beta}(\sigma)$ :  $S(\mu_{n,\beta}) = -\sum_{\sigma} \mu_{n,\beta}(\sigma) \log \mu_{n,\beta}(\sigma)$ .

The real functions

$$f_n(\beta) = \frac{1}{n} E_J \log Z_n(\beta, J)$$

and

$$\bar{f}_n(\beta) = \frac{1}{n} \log E_J Z_n(\beta, J),$$

define the quenched average of the specific free energy and the annealed specific free energy respectively. The ground state energy density  $-\varepsilon_n(J)$  is given by

$$-\varepsilon_n(J) = \frac{1}{n} \inf_{\sigma \in \Sigma_n} H_n(\sigma, J).$$

We denote by  $\bar{f}_{\infty}(\beta)$ ,  $f_{\infty}(\beta)$ <sup>1</sup>, and  $\varepsilon_0$  the corresponding thermodynamic limits

$$\bar{f}_{\infty}(\beta) = \lim_{n \rightarrow \infty} \bar{f}_n(\beta, J), \quad f_{\infty}(\beta) = \lim_{n \rightarrow \infty} f_n(\beta, J),$$

and,

$$-\varepsilon_0 = -\lim_{n \rightarrow \infty} \varepsilon_n(J) = \lim_{\beta \rightarrow \infty} \frac{f_{\infty}(\beta)}{\beta}$$

We define moreover the critical  $\beta_c$  and the freezing  $\beta_f$  temperature by

$$\beta_c = \sup\{\beta : f_{\infty}(\beta) = \bar{f}_{\infty}(\beta)\},$$

$$\beta_f = \inf\{\beta \geq \beta_c : \lim_{n \rightarrow \infty} \frac{1}{n} S(\mu_{n,\beta_f}) = 0, a.s.\}.$$

---

<sup>1</sup>These functions are related to the usual thermodynamic specific free energies  $F_{\infty}$  and  $\bar{F}_{\infty}$  through  $f_{\infty}(\beta) = -\beta F_{\infty}(\beta)$  and  $\bar{f}_{\infty}(\beta) = -\beta \bar{F}_{\infty}(\beta)$ .

The reader can recognize in the previous setting mean field spin glass-models (as the Sherrington-Kirkpatrick (SK) model and its variants). These models have been widely studied in the mathematics and physics literature and have led the development of numerous important analytical techniques. Namely, after the first result on the self-averaging [9], the long-standing problem of the existence and uniqueness of the infinite volume limit of the free energy at low temperature ( $\beta > 1$ ) is obtained by the use of interpolation techniques. A subsequent work [2] provided with a broad variational principle over random overlap structures and various bounds on the free energy. Important mathematical arguments towards a rigorous proof of the original solution proposed by Parisi are obtained in [11]. This has been recently completed by the proof that the Parisi measure at zero temperature is supported by infinitely many points [3].

On the other hand, the behaviour of the entropy of the Gibbs measure in the low temperature region remains poorly understood. For mean-field models the specific entropy decreases with the temperature and it can easily be estimated for the high temperature region. By lowering the temperature the entropy should eventually vanish [9] and an early result, given in [1], corroborates the idea that the entropy does not vanish very fast.

The main motivation of the present work is the link between freezing and entropy and is summarized in the following

**Theorem:** *Almost surely, for the Gaussian mean-field spin-glass model, the specific entropy  $s(\mu_{\beta_*})$  of the Gibbs measure vanishes at the freezing temperature  $\beta_* = 4 \log 2 = 2.77258\dots$ :*

$$s(\mu_{\beta_*}) := \lim_{n \rightarrow \infty} \frac{1}{n} S(\mu_{n, \beta_*}) = - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n, \beta_*}(\sigma) \log \mu_{n, \beta_*}(\sigma) = 0.$$

The formulation of the above statement assumes that  $\lim_{n \rightarrow \infty} \frac{1}{n} S(\mu_{n, \beta_*, J})$  exists and is independent of  $J$ . This follows from general principles and can immediately be obtained from the existence and self-averaging of the low temperature specific free energy.

Let us emphasize that we do not consider here the Parisi measure which is defined on overlaps and has been studied in detail in [3]. Indeed, one can show that the entropy of this measure is strictly positive  $\forall \beta > 0$ .

## 2 Proof of the theorem

It should be noticed that, for all  $\beta > 0$ , the quenched limit  $f_{\infty}(\beta)$  exists and is a convex function of  $\beta$  [6]. Let  $\beta_1 \equiv 1$ . From the high temperature results [1], we have, almost surely, that

$$\begin{aligned} f_{\infty}(\beta_1) &= \lim_{n \rightarrow \infty} \frac{1}{n} E_J \log Z_n(\beta_1, J) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E_J Z_n(\beta_1, J) \\ &= \bar{f}_{\infty}(\beta_1) = \log 2 + \frac{\beta_1^2}{4} = \log 2 + \frac{1}{4}. \end{aligned}$$

Thus, for  $\beta = \beta_1$ , the quenched limit  $f_{\infty}(\beta_1)$  equals to the annealed one  $\bar{f}_{\infty}(\beta_1) = \beta_1^2/4 + \log 2$ , where the term  $\beta_1^2/4$  comes from the mean value of the Boltzmann factor, *i.e.* the

typical behaviour and the mean behaviour coincide at this temperature.

The particular value of  $\beta_*$  appearing in the theorem is defined by the intersection of the graph of  $\bar{f}_\infty(\beta)$  and the straight line  $\frac{\beta}{\beta_1} f_\infty(\beta_1)$ . As one can see in figure 1, the annealed free energy  $\bar{f}_\infty(\beta) = \log 2 + \frac{\beta^2}{4}$  is plotted as a function of  $\beta$  and the straight line is defined by  $\frac{\beta}{\beta_1} f_\infty(\beta_1) \equiv \beta f_\infty(\beta_1)$ . The two graphs intersect at  $\beta_1 \equiv 1$  and  $\beta_* \equiv 4 \log 2 = 2,77258\dots$ . We can now readily check that the annealed limit, at  $\beta = \beta_*$ , is simply related to the limit  $f_\infty(\beta_1)$  by

$$\bar{f}_\infty(\beta_*) = \frac{\beta_*^2}{4} + \log 2 = \frac{\beta_*}{\beta_1} \left( \frac{\beta_* \beta_1}{4} + \beta_1 \log 2 \right) = \frac{\beta_*}{\beta_1} \left( \log 2 + \frac{1}{4} \right) = \frac{\beta_*}{\beta_1} f_\infty(\beta_1).$$

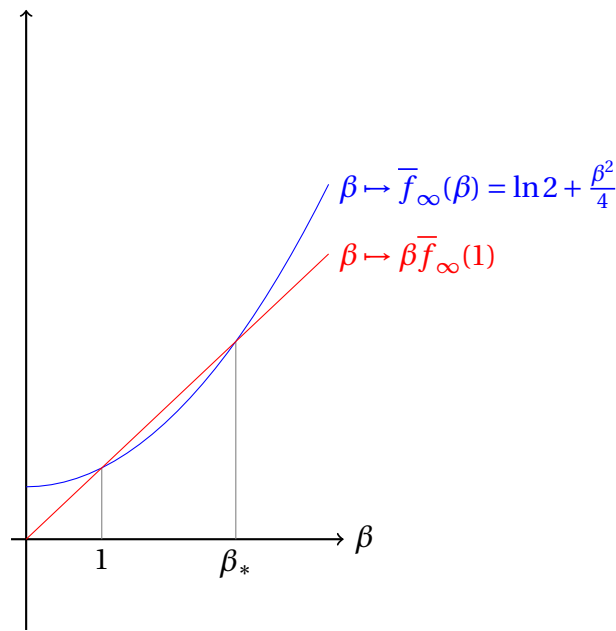


Figure 1: The value  $\beta_* = 4 \log 2$ , is given by the intersection of the graph of the annealed free energy  $\bar{f}_\infty(\beta)$  with the straight line  $\beta \bar{f}_\infty(1)$ .

With the preceding in hand, we define, for all  $\beta \geq 0$ , the affine mapping  $T_\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , by  $T_\beta x = \frac{\beta}{\beta_1} x$ ; on the values of the limits and for  $\beta = \beta_*$ , this mapping reads

$$T_{\beta_*} : f_\infty(\beta_1) \mapsto \bar{f}_\infty(\beta_*) = \frac{\beta_*}{\beta_1} f_\infty(\beta_1). \quad (1)$$

Let moreover denote by  $a$  the deviation of the limit  $f_\infty(\beta_*)$  from its mean value:

$$a := \bar{f}_\infty(\beta_*) - f_\infty(\beta_*) = \frac{\beta_*}{\beta_1} f_\infty(\beta_1) - f_\infty(\beta_*).$$

It is then intuitively appealing to expect that the previous difference depends on the behaviour of the Gibbs measures at  $\beta_1$  and  $\beta_*$ . In order to establish this dependence,

we introduce a slightly different notation that simplifies the proof. Let  $W_n(\sigma, \beta_1)$  be the random weight defined by  $W_n(\sigma, \beta_1) = e^{-\beta_1 H_n(\sigma, J)} / e^{\frac{\beta_1}{2} n}$ . For all  $\beta \geq \beta_1$ , we denote by  $\bar{g}(\beta)$  and  $g_\infty(\beta)$  the tilted annealed and quenched limits respectively,

$$\bar{g}_\infty(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E_J \sum_{\sigma} W_n^{\frac{\beta}{\beta_1}}(\sigma, \beta_1) = \bar{f}_\infty(\beta) - \frac{\beta}{2},$$

and,

$$g_\infty(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} E_J \log \sum_{\sigma} W_n^{\frac{\beta}{\beta_1}}(\sigma, \beta_1) = f_\infty(\beta) - \frac{\beta}{2}.$$

At  $\beta_1$  and  $\beta_*$  this reads

$$g_\infty(\beta_1) = f_\infty(\beta_1) - \frac{\beta_1}{2} = \log 2 + \frac{\beta_1^2}{4} - \frac{\beta_1}{4} = \log 2 - \frac{1}{4},$$

and,

$$g_\infty(\beta_*) = f_\infty(\beta_*) - \frac{\beta_*}{2}.$$

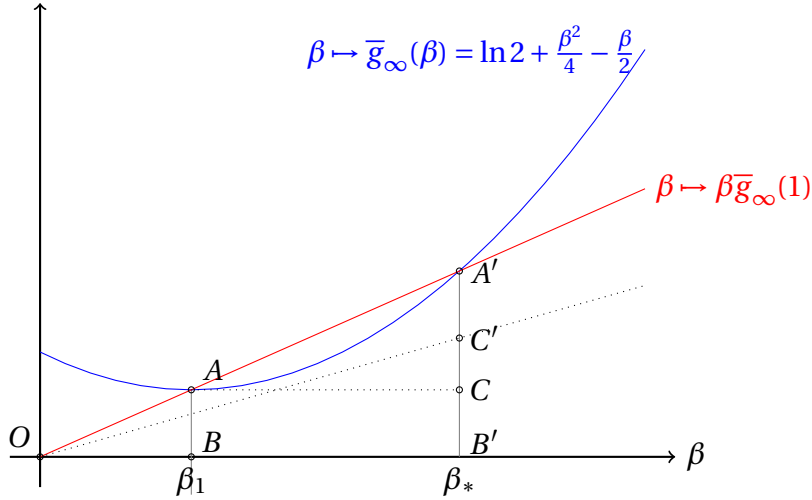


Figure 2: Mappings  $T_{\beta_*}$  for the tilted functions.

In fig. 2, the graph of the convex function  $\bar{g}_\infty(\beta)$  is plotted and it is easy to check that the minimum is reached at  $\beta = 1$ . The same holds for the minimum of the limit  $g_\infty(\beta)$  because  $\bar{g}'_\infty(\beta_1) = \bar{g}'_\infty(\beta_1 - 0) = g'_\infty(\beta_1 - 0) = 0$  and the convexity of  $g_\infty$ .

One can now make two remarks. First, we recall the definition of the relative entropy density  $s(\lambda|\lambda')$  of the probability measure  $\lambda$  w.r.t. the probability measure  $\lambda'$ , defined on an arbitrary discrete space  $X$ :

$$s(\lambda|\lambda') := \begin{cases} \sum_{x \in X} \lambda(x) \log \frac{\lambda(x)}{\lambda'(x)} & \text{if } \lambda \ll \lambda' \\ +\infty & \text{otherwise.} \end{cases}$$

The relative entropy is a non-negative function, vanishing in case the two measures are equal, and gives the extent to which the measure  $\lambda$  “differs” from the measure  $\lambda'$ . Using this definition, the limit  $g_\infty(\beta)$  can now be related to the relative entropy  $s(\mu_{\beta_1}|\mu_\beta)$  for all  $\beta \geq \beta_1$ . Namely,

$$\begin{aligned} s(\mu_{\beta_1}|\mu_\beta) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_1}(\sigma) \log \frac{\mu_{n,\beta_1}(\sigma)}{\mu_{n,\beta}(\sigma)} \\ &= -s(\mu_{\beta_1}) - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_1}(\sigma) \log \mu_{n,\beta}(\sigma) \\ &= -s(\mu_{\beta_1}) - \frac{\beta}{2} + f_\infty(\beta) = -s(\mu_{\beta_1}) + g_\infty(\beta). \end{aligned}$$

Thus, for all  $\beta \geq \beta_1$ , the limit  $g_\infty(\beta)$  is simply given by

$$g_\infty(\beta) = s(\mu_{\beta_1}) + s(\mu_{\beta_1}|\mu_\beta). \quad (2)$$

Second, since for  $\beta_1 = 1$ ,

$$\begin{aligned} s(\mu_{\beta_1}) &= \lim_{n \rightarrow \infty} \frac{1}{n} S(\mu_{n,\beta_1}) = - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_1}(\sigma) \log \mu_{n,\beta_1}(\sigma) \\ &= - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_1}(\sigma) \log e^{-\beta_1 H_n(\sigma, J)} + f_\infty(\beta_1) \\ &= -\frac{\beta_1^2}{2} + f_\infty(\beta_1) = \log 2 - \frac{\beta_1^2}{4} = g_\infty(\beta_1), \end{aligned}$$

we also obtain that  $g_\infty(\beta) = g_\infty(\beta_1) + s(\mu_{\beta_1}|\mu_\beta)$ . By making use of the mapping  $T_{\beta_*}$ , one readily obtains that

$$\bar{g}_\infty(\beta_*) = \frac{\beta_*}{\beta_1} g_\infty(\beta_1) = \frac{\beta_*}{\beta_1} s(\mu_{\beta_1}) = g_\infty(\beta_*) + a,$$

and, moreover,

$$s(\mu_{\beta_1}|\mu_{\beta_*}) + a = (\beta_* - 1)s(\mu_{\beta_1}) = (\beta_* - 1)g_\infty(\beta_1). \quad (3)$$

It should be noticed that although eq. 2 is valid  $\forall \beta \geq \beta_1$ , eq. 3 (establishing the connection between the limit  $a$  and the entropy  $s(\mu_{\beta_1})$ ) is valid only at  $\beta = \beta_*$ .

The reader can furthermore remark that, in figure 2, the segment  $CA'$  equals  $s(\mu_{\beta_1}|\mu_{\beta_*}) + a$  and the limits  $g_\infty(\beta_1)$  and  $\bar{g}_\infty(\beta_*)$  are represented by the lengths of the segments  $AB$  and  $A'B'$  respectively.

The proof of the theorem can now be accomplished in two steps. In the first one, we shall obtain a lower bound for the relative entropy  $s(\mu_{\beta_1}|\mu_{\beta_*})$  by the comparison of the limits  $a$  and  $g_\infty(\beta_1)$  ( $= s(\mu_{\beta_1})$ ) thanks to equations 2 and 3. Indeed, on one hand, one has that  $g_\infty(\beta_*) = g_\infty(\beta_1) + s(\mu_{\beta_1}|\mu_{\beta_*})$ , and, on the other hand,  $s(\mu_{\beta_1}|\mu_{\beta_*}) + a = (\beta_* - 1)s(\mu_{\beta_1})$ . Reporting now on fig. 2 the segment  $B'C'$ , of length  $(\beta_* - 1)s(\mu_{\beta_1})$ , one realizes that the relative entropy  $s(\mu_{\beta_1}|\mu_{\beta_*})$  cannot be smaller than the segment  $CC'$ .

In other words,

$$\begin{aligned}
a &= (\beta_* - 1)s(\mu_{\beta_1}) - s(\mu_{\beta_1}|\mu_{\beta_*}) = BC' - s(\mu_{\beta_1}|\mu_{\beta_*}) \\
&= s(\mu_{\beta_1}) + CC' - s(\mu_{\beta_1}|\mu_{\beta_*}) \\
&\leq s(\mu_{\beta_1}) (= \log 2 - \frac{\beta_1^2}{4}),
\end{aligned}$$

and this provides the lower bound for the the relative entropy  $s(\mu_{\beta_1}|\mu_{\beta_*}) : s(\mu_{\beta_1}|\mu_{\beta_*}) \geq (\beta_* - 2)s(\mu_{\beta_1})$ .

In the second step of the proof we consider the relative entropy  $s(\mu_{\beta_*}|\mu_{\beta_1})$ ,

$$\begin{aligned}
s(\mu_{\beta_*}|\mu_{\beta_1}) : &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_*}(\sigma) \log \frac{\mu_{n,\beta_*}(\sigma)}{\mu_{n,\beta_1}(\sigma)} \\
&= -s(\mu_{\beta_*}) - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_*}(\sigma) \log \mu_{n,\beta_1}(\sigma) \\
&= -s(\mu_{\beta_*}) + s(\mu_{\beta_1}) - A_{\infty},
\end{aligned}$$

where

$$A_{\infty} = \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{\sigma} \mu_{n,\beta_*}(\sigma) \log e^{-\beta_1 H_n(\sigma, J)} - \sum_{\sigma} \mu_{n,\beta_1}(\sigma) \log e^{-\beta_1 H_n(\sigma, J)} \right).$$

Recalling that

$$\begin{aligned}
g_{\infty}(\beta_*) &= s(\mu_{\beta_*}) + \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{\sigma} \mu_{n,\beta_*}(\sigma) \log e^{-\beta_* H_n(\sigma, J)} - \frac{\beta_*}{2} \\
&= s(\mu_{\beta_*}) + \lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{\sigma} \mu_{n,\beta_*}(\sigma) \log e^{-\beta_* H_n(\sigma, J)} - \sum_{\sigma} \mu_{n,\beta_1}(\sigma) \log e^{-\beta_* H_n(\sigma, J)} \right) \\
&= s(\mu_{\beta_*}) + \beta_* A_{\infty},
\end{aligned}$$

we have the following relation between the two relative entropies

$$s(\mu_{\beta_*}|\mu_{\beta_1}) + s(\mu_{\beta_1}|\mu_{\beta_*}) = (\beta_* - 1)A_{\infty}. \quad (4)$$

We can now complete the proof by using the following argument. Assume that the obtained bound for the limit  $a$  is saturated *i.e.*  $a = a_{\max} = s(\mu_{\beta_1})$ . In that case,  $s(\mu_{\beta_1}|\mu_{\beta_*}) = (\beta_* - 2)s(\mu_{\beta_1})$ , and  $g_{\infty}(\beta_*) = (\beta_* - 1)s(\mu_{\beta_1})$ . We notice moreover that thanks to the mapping  $T_{\beta}$ , the limit  $g_{\infty}(\beta_*)$  must be pulled backwards at  $\beta = \beta_1$  as

$$T^{-1}g_{\infty}(\beta_*) = \frac{\beta_1}{\beta_*}g_{\infty}(\beta_*) = (\beta_* - 1)\frac{s(\mu_{\beta_1})}{\beta_*}.$$



We deduce that

$$\begin{aligned}
g_\infty(\beta_*) &= s(\mu_{\beta_*}) + A_\infty + (\beta_* - 1)A_\infty \\
&= s(\mu_{\beta_*}) + A_\infty + s(\mu_{\beta_*}|\mu_{\beta_1}) + s(\mu_{\beta_1}|\mu_{\beta_*}) \\
&= T^{-1}g_\infty(\beta_*) + s(\mu_{\beta_2}|\mu_{\beta_1}) + (\beta_* - 2)s(\mu_{\beta_1}) \\
&= (\beta_* - 1)\frac{s(\mu_{\beta_1})}{\beta_*} + s(\mu_{\beta_2}|\mu_{\beta_1}) + (\beta_* - 2)s(\mu_{\beta_1}) \\
&= (\beta_* - 1)s(\mu_{\beta_1}),
\end{aligned}$$

which is consistent only with

$$s(\mu_{\beta_*}|\mu_{\beta_1}) = s(\mu_{\beta_1})/\beta_*, \quad A_\infty = (\beta_* - 1)\frac{s(\mu_{\beta_1})}{\beta_*}, \quad \text{and, } s(\mu_{\beta_*}) = 0.$$

We have thus established that, under the assumption of saturation of the lower bound for  $g_\infty(\beta_*)$ , the entropy  $s(\mu_{\beta_*})$  vanishes and, moreover,  $\bar{g}_\infty(\beta_*) - g_\infty(\beta_*) = \beta_* s(\mu_{\beta_*}|\mu_{\beta_1})$ . If now  $g_\infty(\beta_*) > (\beta_* - 1)s(\mu_{\beta_1})$ , (*i.e.* the relative entropy  $s(\mu_{\beta_1}|\mu_{\beta_*})$  is bigger than its lower bound), one can remark that, in this case, the entropy had already vanished at an inverse temperature smaller than  $\beta_*$ .<sup>2</sup> Indeed, if the relative entropy  $s(\mu_{\beta_1}|\mu_{\beta_*})$  increases, the difference between the entropies of the measures  $s(\mu_{\beta_1})$  and  $s(\mu_{\beta_*})$  cannot decrease. Since the entropy cannot be positive for all values of  $\beta > \beta_*$ , the freezing temperature of the model is given by  $\beta_f \leq \beta_*$ . Thus, the value  $\beta_*$  provides the maximum value beyond which the specific entropy of the Gibbs measure cannot be strictly positive.  $\square$

The previous result implies the following lower bound for the tilted limit  $g_\infty(\beta_*)$ ,

$$g_\infty(\beta_*) = s(\mu_{\beta_1}) + s(\mu_{\beta_1}|\mu_{\beta_*}) \geq (\beta_* - 1)s(\mu_{\beta_1}) = \frac{\beta_*^2}{4} - \frac{\beta_*}{2} + \frac{\beta_1^2}{4},$$

and, respectively, for the quenched limit  $f_\infty(\beta_*)$ ,

$$f_\infty(\beta_*) \geq \frac{\beta_*^2}{4} + \frac{1}{4} = 2.1718\dots$$

Using now the spherical (upper) bound, we have almost surely, that at  $\beta_* = 4 \log 2$ ,

$$2.1718\dots \leq f_\infty(\beta_*) \leq 2.2058\dots$$

In addition, recalling the positivity of the entropy and the geometric fact that at each point on the graph of a convex function there is a tangent line which never lies above the graph, we obtain,  $\forall \beta \geq \beta_*$ , the convexity boundary of the limit  $g_\infty(\beta)$ . This boundary is illustrated by the dotted line in fig. 2.

Although evident, the non-standard expression of the free energy in terms of the entropy  $s(\mu_{\beta_1})$  and the relative entropy  $s(\mu_{\beta_1}|\mu_\beta)$  (eq. 2),  $\forall \beta > \beta_1$ , has special importance not only for the present proof. Indeed, one can apply this expression to recover

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<sup>2</sup>This arises in particular in the case of the REM.

information about the measure at  $\beta_1$ , from information represented by  $s(\mu_{\beta_1}|\mu_\beta)$  (for instance, in the case of compressing sensing).

Let us also add that a simple calculation shows that the functional relationship (1) between the annealed limits  $\overline{f}_\infty(\beta_*)$  and  $f_\infty(\beta_1)$  is also valid for the corresponding limits of the Random Energy Model (REM). The REM, introduced in [4], is defined by  $2^n$  energy levels  $E_i (i = 1, \dots, n)$ , a family of random, independent, identically distributed Gaussian variables of variance  $N/2$ ; many results are qualitatively the same as those of the model we have studied in this work. It is well known that the REM undergoes a phase transition at the critical inverse temperature  $\beta_c = 2\sqrt{\log 2}$  and the entropy of the Gibbs measure vanishes almost surely  $\forall \beta \geq \beta_c$ . One can easily apply our approach to the REM. Indeed, it is straightforward to calculate the limit  $g_\infty^{\text{REM}}(\beta_*)$  and verify that  $\beta_f = \beta_c$ .

### 3 Concluding remarks

In this note we proved that the mean entropy of the Gibbs measure is zero at the inverse (freezing) temperature  $\beta_* = 4\log 2 = 2.7725\dots$ . Obviously, the low temperature ultrametric picture occurs in the interval  $[\beta_c, \beta_f]$  and is related to the Parisi solution. One can also readily check that  $\beta_* = \beta_c^2$ , where  $\beta_c = 2\sqrt{\log 2} = 1.6651\dots$  is the critical temperature of the Random Energy Model (REM). Both  $\beta_c$  and  $\beta_*$  are to be compared with the value at  $\beta_1 \equiv 1$ , *i.e.* the maximum value of  $\beta$  where the free energies of the two models coincide. What we learn by the comparison of the relative entropies of the two models is that the Gibbs measure of the Sherrington-Kirkpatrick has seemingly different concentration properties than for the REM. In the case of the REM, the quenched and annealed limit of the free energy are equal for all temperatures of positive entropy and consequently the critical temperature coincides with the freezing one. This does not occur in the case of SK, where the two limits are equal only for  $\beta \leq 1$ . In [7], we study the relation of the freezing property with the entropy of the Gibbs measure in case of Gaussian Models. The same relation occurs for the log-correlated gaussian models as well. Another remark concerns the ground state energy of the model and the Hausdorff dimension of the support of the Gibbs measure. One can show using the theorem of this note that this dimension vanishes at  $\beta_*$ . We also mention the remaining open question of particular interest, namely the correspondence between our result and the low temperature behaviour of the Parisi measure.

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