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24th International Congress of Theoretical and Applied Mechanics  
**2D versus 1D models for Shallow Water Equations**

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**Abstract**

In this paper we present a general framework to construct 1D width averaged models when the flow is constrained -e.g. by topography- to be almost 1D. We start from two dimensional shallow water equations, perform an asymptotic expansion of the fluid elevation and velocity field in the spirit of wave diffusive equations and establish a set of 1D equations made of a mass, momentum and energy equations which are close to the one usually used in hydraulic engineering. We show that in some special cases, like the U-shaped river bed, that our set of equations reduces to the classical 1d shallow water equations. Out of these configurations, there is an  $O(1)$  deviation of our model from the classical one.

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**Keywords:** Energy Balance ; Asymptotic expansion; Saint Venant equations

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**1. Introduction**

The so called shallow water equations (Saint Venant equations in the French community) are widely used in hydraulic engineering to compute stationary or transient flows in rivers. In the case of a varying river section, it is written as

$$\begin{cases} S_t + Q_x = 0, \\ Q_t + \left(\frac{Q^2}{S}\right)_x + gH_x S = gS(I - J), \end{cases} \quad (1)$$

where  $S = \int_{y_-}^{y_+} h(x, y) dy = \int_0^H L(s, x) ds$  is the wetted section (see Figure 1),  $Q$  the total discharge rate,  $I$  the bottom slope whereas  $J$  is the friction term usually given by  $J = \frac{U^2}{C_h^2 R_h}$  with  $U = Q/S$  the average velocity,  $R_h = \frac{S}{P}$  the hydraulic radius,  $P = L(x, 0) + 2 \int_0^H \sqrt{1 + \frac{(L_x(x, z))^2}{4}} dz$  the wetted perimeter and  $C_h$  is the Chezy coefficient.

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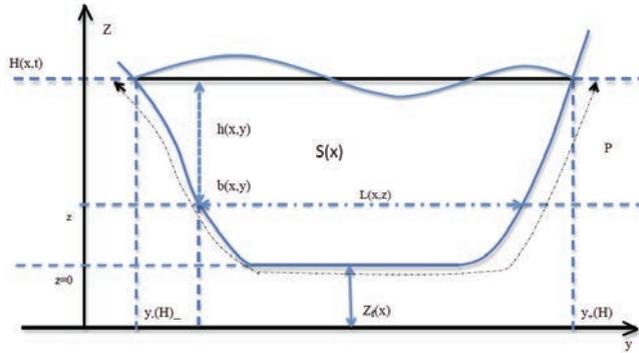


Fig. 1. Wetted Section

Note that equation (1) admits a kinetic energy balance law in the form:

$$\left(\frac{1}{2}S U^2\right)_t + \frac{1}{2}(S U^3)_x = gQ \left( \Lambda - \frac{Q|Q|}{C_h^2 R_h S^2} \right), \quad \text{with } \Lambda = I - gH_x. \tag{2}$$

The potential gravity energy is defined as  $\mathcal{E} = g \int_0^H L(x, z) z dz$  so that (2) is also written as

$$\left(\frac{1}{2}S U^2 + \mathcal{E}\right)_t + \left(U \left(\frac{1}{2}S U^2 + gS H\right)\right)_x = gQ \left( I - \frac{Q|Q|}{C_h^2 R_h S^2} \right)$$

or, equivalently

$$\left(\frac{1}{2}S U^2 + \mathcal{E} + g b_0 S\right)_t + \left(U \left(\frac{1}{2}S U^2 + gS (H + b_0)\right)\right)_x = -gQ \frac{Q|Q|}{C_h^2 R_h S^2}.$$

In order to take into account the fact that the cross-stream flow is not uniform, various modifications of (1) have been proposed (see eg Chow<sup>5</sup>). For that purpose, one introduces the Boussinesq coefficient  $\beta$  in the momentum equation and the Coriolis coefficient  $\alpha$  in the definition of the kinetic energy :

$$\beta S U^2 = \left(\int_{y_-}^{y_+} h u^2 dy\right), \quad \alpha S U^3 = \left(\int_{y_-}^{y_+} h u^3 dy\right). \tag{3}$$

In practice (see e.g. Liggett<sup>4</sup>) the momentum equation is transformed into

$$Q_t + \left(\beta \frac{Q^2}{S}\right)_x + gH_x S = gS (I - J). \tag{4}$$

Note that no practical rules have been proposed in order to compute the Boussinesq coefficient (except that  $\beta \geq 1$  and close to 1). It is also well known that the Coriolis coefficient may correct the energy balance laws. There remains an inconsistency since once one has modified the momentum equation, one cannot get a correct balance energy equation. We shall prove below that the exact energy balance should be taken as :

$$\left(\frac{1}{2}\beta S V^2\right)_t + \frac{1}{2}(\alpha S V^3)_x = gQ \left( \Lambda - \frac{Q|Q|}{C_h^2 R_h S^2} \right). \tag{5}$$

Though, equation (5) is not compatible with (4) unless  $\beta = \alpha = 1$  (which is generally untrue - see eg Table p.28 in Chow<sup>5</sup>).

In the following, we proceed to a direct width averaging of 2D Saint Venant equations. In order to withdraw the ad hoc hypothesis made on the velocity field ( see e.g. Szymkiewicz<sup>3</sup>, Wu<sup>2</sup>) to close the resulting set of equations, we compute an asymptotic expansion in regimes close to the kinematic or diffusive/kinematic waves (see eg Sing<sup>8</sup>), quite representative of flood propagation in rivers. We start from the 2D shallow water equations written as  $(\mathbf{u} = (u, v)^T)$ :

$$\begin{cases} h_t + \text{div}(h\mathbf{u}) = 0 \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla h = g \left( -\nabla Z - \frac{\mathbf{u} \|\mathbf{u}\|}{C_h^2 h^m} \right). \end{cases}$$

The shallow water system admits an additional energy conservation laws which reads, by denoting  $E = \frac{1}{2}h \|\mathbf{u}\|^2 + \frac{1}{2}gh^2$

$$(E + ghZ)_t + \text{div} \left( \mathbf{u} \left( E + ghZ + \frac{1}{2}gh^2 \right) \right) = -gh \frac{\|\mathbf{u}\|^3}{C_h^2 h^m}.$$

Here, the Chezy friction coefficient can be chosen non uniform:  $C_h = C_h(x, y)$ .

The aim of this paper is to provide a better understanding of the averaging process. In order to make the flow almost one-dimensional, we assume that the bottom topography  $b(x, y)$  is given by an equation of the form  $Z = B_0 b_0 \left( \frac{x}{L} \right) + h_0 \varphi \left( \frac{x}{L_x}, \frac{y}{L_y} \right)$  where  $h_0$  is a typical length characteristic of the height of the flow,  $L$  a typical longitudinal length,  $\varepsilon = \frac{h_0}{L}$  a small parameter and  $L_x, L_y$  two other length scale (to be determined later).

In this paper, we propose a methodology to build asymptotic expansions of 2D shallow water flows along the xline, where the small parameter will be  $\frac{\varepsilon F^2}{I_0}$  ( see exact definition below) and we will prove that in the non dimensional framework the width averaging process leads to

$$\begin{cases} S_t + Q_x = 0 \\ \left( \frac{1}{2} \beta S V^2 \right)_t + \frac{1}{2} (\alpha S V^3)_x = \frac{I_0}{\varepsilon F^2} Q \left( \Lambda - \frac{Q|Q|}{C_h^2 R_h S^2} \right). \end{cases}$$

In the spirit of the extended versions of the shallow water equations proposed recently by (Richard and Gavryliuk<sup>6</sup>) and (Richard Ruyer Quil and Vila<sup>7</sup>), we propose a class of four equations models which are consistent both with the width averaged momentum and the energy equations. This system is written in non dimensional form as

$$\begin{cases} (i) & S_t + Q_x = 0, \\ (ii) & Q_t + \left( \frac{Q^2}{S} + P^{tot} \right)_x = \frac{I_0 S}{\varepsilon F^2} \left( I - \frac{Q|Q|}{C_h^2 R_h S^2} + A_1 (\Pi^{(0)} - \Pi) \right) + B(H, x), \\ (iii) & E_t^{2D} + \left( \frac{Q}{S} (E^{2D} + P^{tot}) \right)_x = \frac{I_0}{\varepsilon F^2} Q \left( I - \frac{Q|Q|}{C_h^2 R_h S^2} \right), \\ (iv) & (S^3 \Pi)_t + V (S^3 \Pi)_x = \frac{I_0}{\varepsilon F^2} A_2 (\Psi^{(0)} - \Psi), \end{cases} \tag{6}$$

where  $B(H, x) = \frac{1}{F^2} \int_0^H L_x(x, z) (H - z) dz$  represents the lateral pressure effects,  $P^{hydro} = \frac{1}{F^2} \int_0^H L(x, z) (H - z) dz$  the hydrostatic pressure effect,  $P^{tot} = \frac{1}{2} S^3 (\Pi - \Psi) + P^{hydro}$  an effective total pressure and  $E^{2D} = \frac{1}{2} \frac{Q^2}{S} + \frac{1}{2} S^3 \Psi + \mathcal{E}$  a total energy. The function  $\Psi$  is called the enstrophy (following the terminology found in (Richard & all)<sup>6,7</sup>): it is related to the Boussinesq coefficient through  $\Psi = (\beta - 1) \frac{U^2}{S}$ , whereas the potential  $\Pi$  is related to the Coriolis coefficient through the relation  $\Pi = (\alpha - 1) \frac{U^2}{S}$ .

The function  $\Psi^{(0)}$  (resp.  $\Pi^{(0)}$ ) are two equilibrium distribution functions depending on  $S$  and given below by (17). Finally,  $A_1$  and  $A_2$  are two free parameters. The above system is close to the Saint Venant standard system (1). More precisely if  $\Pi^{(0)} = \Psi^{(0)} = 0$  and  $\Pi = \Psi = 0$  initially, then  $\Pi = \Psi = 0$  for all  $t > 0$  and the system reduces exactly to the standard system. The condition  $\Pi^{(0)} = \Psi^{(0)} = 0$  is true if the channel is U-shaped ( $L_z(x, z) = 0$ ) and the Chezy coefficient homogeneous in the transverse direction: in the 2D case, this means  $C_h = C_h(x)$ . Note also that by taking  $P^{tot} = P^{hydro}$  and  $A_1 = 0$ , we find

$$Q_t + \left( \frac{Q^2}{S} + P^{hydro} \right)_x = \frac{I_0 S}{\varepsilon F^2} \left( I - \frac{Q|Q|}{C_h^2 R_h S^2} \right) + B(H, x),$$

which is the conservative form (necessary to compute correctly discontinuous solution such as hydraulic jumps) of Saint Venant equations which have been proposed in (Vila<sup>9</sup>). This equation is equivalent to

$$Q_t + \left(\frac{Q^2}{S}\right)_x + \frac{1}{F^2} S H_x = \frac{I_0 S}{\varepsilon F^2} \left( I - \frac{Q|Q|}{C_h^2 R_h S^2} \right)$$

which is the non dimensional form of the momentum equation (1).

### 2. Non dimensional form of Equations

We choose reference velocity, length, and time scale according to  $u = u_0 u'$ ,  $v = v_0 v'$ ,  $x = D_x x'$ ,  $y = D_y y'$ ,  $t = \frac{D_x}{u_0} t'$  with

$$\frac{g h_0}{u_0^2} = \frac{1}{F^2}, R_v = \frac{v_0}{u_0}, R_L = \frac{D_y}{D_x}, I_0 = \frac{B_0}{D_x}, J_0 = \frac{u_0^2}{(C_h^0)^2 h_0^m}, \varepsilon = \frac{h_0}{D_x}.$$

We expect  $R_v \ll 1$  and  $R_L \ll 1$ . The longitudinal slope is  $I_0 = \frac{B_0}{D_x}$  whereas the transverse slope is  $I_T = \frac{h_0}{D_y}$ . We take  $Z = \mathcal{Z}\left(\frac{x}{D_x}, \frac{y}{D_y}, \frac{D_x}{D_x}\right)$  with

$$\mathcal{Z} = B_0 b_0 \left(\frac{x}{D_x}\right) + h_0 \varphi\left(\frac{x}{D_x}, \frac{y}{D_y}\right) = B_0 \left(b_0 \left(\frac{x}{D_x}\right) + \frac{h_0}{B_0} \varphi\left(\frac{x}{D_x}, \frac{y}{D_y}\right)\right) = B_0 b \left(\frac{x}{D_x}, \frac{y}{D_y}, \frac{D_x}{D_x}\right)$$

In order to simplify the analysis, we assume  $R_X = \frac{D_x}{D_x} = 1$ ,  $R_v = R_L = \varepsilon$  and  $J_0 = I_0$ . We get  $b(x, y, R_X) = b_0(x) + \frac{\varepsilon}{I_0} \varphi(x, y)$ . We take  $C_h = C_h^0 C(x, y, R_X)$  which turns to  $C^2(x, y, R_X) = (1 + c(x, y))^2$ . Omitting the ' and denoting  $E = \frac{1}{2} h (u^2 + (\varepsilon)^2 v^2) + \frac{1}{2F^2} h^2$ , the 2D shallow water system in non dimensional form reads :

$$\begin{aligned} (i) \quad & h_t + (hu)_x + (hv)_y = 0 \\ (ii) \quad & u_t + uu_x + vu_y + \frac{1}{F^2} h_x = \frac{1}{\varepsilon F^2} \left( -I_0 b_{0,x} - \varepsilon R_X \varphi_{,1} - I_0 \frac{u(1+c)^{-2}}{h^m} \sqrt{u^2 + \varepsilon^2 v^2} \right) \\ (iii) \quad & (h_y + \varphi_y) = \varepsilon^2 \left( -\frac{I_0 v(1+c)^{-2}}{\varepsilon h^m} \sqrt{u^2 + \varepsilon^2 v^2} \right) - F^2 (R_v)^2 (v_t + uv_x + vv_y) \end{aligned} \tag{7}$$

$$\left( E + \frac{I_0}{F^2 \varepsilon} hb \right)_t + \left( u \left( E + \frac{I_0}{F^2 \varepsilon} hb + \frac{1}{2F^2} h^2 \right) \right)_x + \left( v \left( E + \frac{I_0}{F^2 \varepsilon} hb + \frac{1}{2F^2} h^2 \right) \right)_y = -\frac{I_0}{\varepsilon F^2} h (1+c)^{-2} \frac{(u^2 + \varepsilon^2 v^2)^{\frac{3}{2}}}{h^m} \tag{8}$$

whereas the 1D standard system in non dimensional form is obtained by setting with  $u = u_0 u'$ ,  $S = h_0 L_y S'$ ,  $x = D_x x'$ ,  $Q = h_0 L_y u_0 Q'$ ,  $t = \frac{D_x}{u_0} t'$ ,  $I = I_0 I'$ ,  $C_h^2 = C_{0h}^2 C_{0h}^2$ ,  $R_h = h_0 R'_h$ , taking  $\frac{u_0^2}{H_0 (C_{0h})^2} = J_0$  :

$$\begin{cases} S_t + Q_x = 0 \\ Q_t + \left(\frac{Q^2}{S}\right)_x + \frac{1}{F^2} H_x S = \frac{I_0}{\varepsilon F^2} S \left( I - \frac{Q|Q|}{C_h^2 R_h S^2} \right). \end{cases} \tag{9}$$

Note that the above 1D systems admit an energy balance equation which can also be written as a kinetic energy balance law

$$\left(\frac{1}{2} S U^2\right)_t + \frac{1}{2} (S U^3)_x = \frac{I_0}{\varepsilon F^2} Q \left( \Lambda - \frac{Q|Q|}{C_h^2 R_h S^2} \right) \tag{10}$$

where  $\Lambda = I - \frac{\varepsilon}{I_0} H_x$ . By introducing the potential gravity energy  $\mathcal{E} = \frac{1}{F^2} \int_0^H L(x, z) z dz$ , one also finds

$$\left(\frac{1}{2} S U^2 + \mathcal{E} + \frac{I_0}{\varepsilon F^2} b_0 S\right)_t + \left(U \left(\frac{1}{2} S U^2 + \frac{1}{F^2} S \left(H + \frac{I_0}{\varepsilon} b_0\right)\right)\right)_x = -\frac{I_0}{\varepsilon F^2} Q \frac{Q|Q|}{C_h^2 R_h S^2}$$

which is close to energy equation (8).

### 3. Two Asymptotic Regime

We focus on two asymptotic regimes connected with kinematic waves and wave diffusive approximation of river flood (see eg (Singh<sup>8</sup>) or the seminal work of (Lighthill and Whitham<sup>9</sup>)). We consider:

*Regime 1.* Small parameter :  $\frac{\varepsilon}{I_0} F^2 = \varepsilon^q$  with  $\frac{\varepsilon}{I_0} = O(1)$ , it copes in particularly with flows in large estuaries where mascaret (or tidal bore) may occurs in the  $x$  direction, the main equilibrium is

$$\frac{1}{F^2} \left( h_x + \frac{I_0}{\varepsilon} b_{0,x} + R_X \varphi_{,1} \right) \approx \frac{1}{F^2} \left( -\frac{J_0}{\varepsilon} \frac{u(1+c)^{-2}}{h^m} \sqrt{u^2 + \varepsilon^2 v^2} \right)$$

and we can refer it as the diffusive wave approximation .

*Regime 2.* Small parameter :  $\varepsilon \frac{F^2}{I_0} = \varepsilon^s$  with  $\frac{1}{F^2} \ll \frac{I_0}{F^2 \varepsilon}$  or equivalently with  $\frac{\varepsilon}{I_0} = o(1)$ . It copes with standard flood flow. In the  $x$  direction, the main equilibrium is.

$$\frac{1}{F^2} \left( \frac{I_0}{\varepsilon} b_{0,x} \right) \approx \frac{1}{F^2} \left( -\frac{J_0}{\varepsilon} \frac{u}{h^m} \sqrt{u^2 + \varepsilon^2 v^2} \right)$$

and is closer to the kinematic wave approximation of the literature. In our framework, results on regime 2 are deduced from the results on regime 1.

#### 3.1. Width integrated equations

By setting  $I = -b_{0,x}$  and recalling that  $E = \frac{1}{2} h (u^2 + \varepsilon^2 v^2) + \frac{1}{2F^2} h^2$ , the mass,  $x$ -momentum and energy equations averaged over the width of the channel are:

$$\left\{ \begin{array}{l} (i) \quad S_t + Q_x = 0 \\ (ii) \quad Q_t + \left( \int_{y_-}^{y_+} (hu^2) dy \right)_x + \left( \int_{y_-}^{y_+} \frac{h^2}{2F^2} dy \right)_x = \\ \quad \frac{I_0}{\varepsilon F^2} \left( \int_{y_-}^{y_+} h \left( I - \frac{R_X}{I_0} \varphi_{,1} \right) dy - \int_{y_-}^{y_+} h^{1-m} (1+c)^{-2} u \sqrt{u^2 + \varepsilon^2 v^2} dy \right) \\ (iii) \quad \left( \int_{y_-}^{y_+} \left( E + \frac{I_0}{\varepsilon F^2} hb \right) dy \right)_t + \left( \int_{y_-}^{y_+} u \left( E + \frac{I_0}{\varepsilon F^2} hb + \frac{1}{2F^2} h^2 \right) dy \right)_x \\ \quad = -\frac{I_0}{\varepsilon F^2} \int_{y_-}^{y_+} h^{1-m} (1+c)^{-2} (u^2 + \varepsilon^2 v^2)^{\frac{3}{2}} dy \end{array} \right. \quad (11)$$

These equations have a structure which is similar to the one of standard Saint Venant system (1).

#### 3.2. Transverse water level and preliminary computations

We first use the  $y$ -momentum equation (ii) (7) to expand the fluid elevation. By assuming that  $\int_{y_-}^{y_+} \frac{v}{h^m} \sqrt{u^2 + \varepsilon^2 v^2} dy = O(1)$ , one finds that  $(h + \varphi)_y = O(\varepsilon I_0 + F^2 \varepsilon^2)$  which in turns yields:

$$h = H(x, t) - \varphi(x, y) + O(\varepsilon^2) \quad (12)$$

where  $H(x, t)$  is a new unknown ( the local level of the water ). Thus the free surface is nearly horizontal in the cross-stream direction. For later use, we introduce some tools to compute integrals of the type  $\int_{y_-}^{y_+} (1 + c(x, y))^p h^q dy$  for  $q > 0$  which comes later in the derivation of the asymptotic expansion. We compute them by introducing the surface width function :  $L(x, z) = y_+(z) - y_-(z)$ . Note that  $L(x, z)$  may possesses a finite number of discontinuities in  $z$ , located at  $\{z_i\}$ . Thus  $L_z(x, z) = L_z^{smooth}(x, z) + \sum_i [L_i](x) \delta(z - z_i)$ . We introduce a ‘‘Chezy’’ weighted surface width function  $\mathcal{L}_p$  such that  $(1 + c(x, y(z)))^p L_z(x, z) := \mathcal{L}_{p,z}(x, z)$ . Finally, by taking

$$M_{q, \mathcal{L}_p}(H, x) = \int_0^H (H - z)^q \mathcal{L}_{p,z}(x, z) dz$$

one finds  $\int_{y_-}^{y_+} (1 + c(R_X x, y))^p h^q dy = \int_0^H (H - z)^q \mathcal{L}_{p,z}(x, z) dz + o(\varepsilon)$ .

### 3.3. Asymptotic Expansion of the Velocity field

With these tools at hand, we can deduce an asymptotic expansion of the velocity field from the  $x$ -momentum equation and relation (12). We start from

$$(1 + c)^{-2} u |u| = \left( \mathcal{I}(x) - \frac{\varepsilon}{I_0} H_x \right) h^m - \frac{\varepsilon F^2 h^m}{I_0} (u_t + uu_x + vu_y) + O(\varepsilon^2)$$

and get successively, by introducing  $\Lambda(x, t) = \mathcal{I}(x) - \frac{\varepsilon}{I_0} H_x$

$$u = u^{(0)} + \frac{\varepsilon F^2}{I_0} u^{(1)} + O\left(\frac{\varepsilon^{t+1}}{I_0}\right)$$

with

$$u^{(0)} = (1 + c) \operatorname{sgn}(\Lambda(x, t)) \sqrt{h^m |\Lambda(x, t)|}, \quad u^{(1)} = -\frac{u^{(0)}}{2\Lambda} \left( u_t^{(0)} + u^{(0)} u_x^{(0)} + v^{(0)} u_y^{(0)} \right)$$

We also get

$$Q = \int_{y_-}^{y_+} h u dy = Q^{(0)}(H, x) + O\left(\frac{\varepsilon}{I_0} F^2\right), \quad Q^{(0)}(H, x) = \operatorname{sgn}(\Lambda) \sqrt{|\Lambda|} M_{1+m/2, \mathcal{L}_1}$$

and

$$h v^{(0)}(x, y) = -(y - y_-) H_t - \int_{y_-}^y \left( (1 + c) h^{1+\frac{m}{2}} \operatorname{sgn}(\Lambda) \sqrt{|\Lambda|} \right)_x dy.$$

The term  $Q^{(1)} = \int_{y_-}^{y_+} h u^{(1)} dy$  can be computed by using mass conservation (i) (7) to get

$$Q^{(1)} = -\frac{1}{4\Lambda} \left( \left( \int_{y_-}^{y_+} h (u^{(0)})^2 dy \right)_t + \left( \int_{y_-}^{y_+} h (u^{(0)})^3 dy \right)_x \right) \tag{13}$$

or, equivalently,

$$Q^{(1)} = -\frac{1}{4\Lambda} \left( |\Lambda| M_{1+m, \mathcal{L}_2} \right)_t - \frac{1}{4\Lambda} \left( \operatorname{sgn}(\Lambda) |\Lambda|^{\frac{3}{2}} M_{1+\frac{3m}{2}, \mathcal{L}_3} \right)_x.$$

Note here that (13) can be interpreted as a kinetic energy averaged balance equation. As a byproduct of this analysis, we have established some diffusive wave equation which are consistent with our asymptotic regime. It is a direct consequence of (averaged) mass conservation law.  $S_t + Q_x = 0$  together with

$$Q = Q^{(0)} + \frac{\varepsilon F^2}{I_0} Q^{(1)} + o\left(\frac{\varepsilon F^2}{I_0}\right)$$

## 4. Consistent 1D Shallow Water type models

### 4.1. Momentum balance and Friction model

The mass conservation equation is exact and given by (11)(i)

$$S_t + Q_x = 0.$$

Then we look carefully at momentum and energy conservation laws. By considering averaged momentum equation (11)(ii) together with (12) we get (recall  $\Lambda(x, t) = \mathcal{I}(x) - \frac{\varepsilon}{I_0} H_x$ ):

$$Q_t + \left( \int_{y_-}^{y_+} (h u^2) dy \right)_x = \frac{I_0}{\varepsilon F^2} \left( \Lambda S - \int_{y_-}^{y_+} h^{1-m} (1 + c)^{-2} u \sqrt{u^2 + \varepsilon^2 v^2} dy \right) + O\left(\frac{\varepsilon^t}{F^2}\right). \tag{14}$$

The friction term is given by  $T = \int_{y_-}^{y_+} h^{1-m} (1 + c)^{-2} u \sqrt{u^2 + \varepsilon^2 v^2} dy$ . Recall that uniform stationary flows satisfy

$$S \mathcal{I} = \int_{y_-}^{y_+} h^{1-m} (1 + c)^{-2} u \sqrt{u^2 + \varepsilon^2 v^2} dy.$$

We search for a friction model similar to standard engineering ones :  $\int_{y_-}^{y_+} h^{1-m} (1+c)^{-2} u^2 dy = \frac{Q^2}{C_h^2 R_h S} + O\left(\frac{\varepsilon F^2}{I_0}\right)$ , we thus need  $C_h^2 R_h = \frac{(Q_{unif}^{(0)})^2}{IS^2} = \frac{(M_{1+m/2, L_1})^2}{(M_{1,L})^3}$  and take in the following

$$C_h^2 = \frac{\mathcal{P} (M_{1+m/2, L_1})^2}{(M_{1,L})^3}. \quad (15)$$

#### 4.2. Energy Balance

As a consequence of the choice (15), we find that  $\frac{I_0}{\varepsilon F^2} Q \left( \Lambda - \frac{Q|Q|}{C_h^2 R_h S^2} \right) = -2Q^{(1)} + O\left(\frac{\varepsilon F^2}{I_0}\right)$ . We thus get with (13)

$$\left( \frac{1}{2} \beta S U^2 \right)_t + \frac{1}{2} (\alpha S U^3)_x = \frac{I_0}{\varepsilon F^2} Q \left( \Lambda - \frac{Q|Q|}{C_h^2 R_h S^2} \right) + O\left(\frac{\varepsilon F^2}{I_0}\right) \quad (16)$$

which is exactly the expected result. In view of providing a complete model similar to those proposed in (6) and (7), we introduce instead of Boussinesq and Coriolis coefficient the enstrophy  $\Psi$  defined as  $S^3 \Psi = \int_{y_-}^{y_+} (hu^2) dy - \frac{Q^2}{S}$ , together with the potential  $\Pi$  defined as  $\int_{y_-}^{y_+} hu^3 dy = \left( \frac{Q^2}{S} + S^3 \Pi \right) \frac{Q}{S}$ . Note that  $\Psi = (\beta - 1) \frac{U^2}{S}$  and  $\Pi = (\alpha - 1) \frac{U^2}{S}$ . We easily obtain  $\Psi = \Psi^{(0)} + O\left(\frac{\varepsilon F^2}{I_0}\right)$  and  $\Pi = \Pi^{(0)} + O\left(\frac{\varepsilon F^2}{I_0}\right)$  with

$$S^3 \Psi^{(0)} = |\Lambda| \left( M_{1+m, L_2} - \frac{(M_{1+m/2, L_1})^2}{M_{1,L}} \right), \quad S^3 \Pi^{(0)} = |\Lambda|^{3/2} \operatorname{sgn}(\Lambda) \left( \frac{M_{1,L} M_{1+3m/2, L_3}}{M_{1+m/2, L_1}} - \frac{(M_{1+m/2, L_1})^2}{M_{1,L}} \right) \quad (17)$$

Introducing the enstrophy  $\Psi$  and the potential  $\Pi$  in ((16)) together with the potential gravity energy  $\mathcal{E}$ , we get

$$\left( \frac{1}{2} S U^2 + \mathcal{E} + S^3 \Psi + \frac{I_0}{\varepsilon F^2} b_0 S \right)_t + \left( U \left( \frac{1}{2} S U^2 + S^3 \Pi + \frac{1}{F^2} S \left( H + \frac{I_0}{\varepsilon} b_0 \right) \right) \right)_x = -\frac{I_0}{\varepsilon F^2} Q \frac{Q|Q|}{C_h^2 R_h S^2} + O\left(\frac{\varepsilon F^2}{I_0}\right) \quad (18)$$

It can be proved that ((18)) is exactly the total energy width averaged ((11)) (iii) equation up to  $O\left(\frac{\varepsilon F^2}{I_0}\right)$ . Note also that we can also writes

$$\left( \frac{1}{2} S U^2 + \mathcal{E} + S^3 \Psi \right)_t + \left( U \left( \frac{1}{2} S U^2 + S^3 \Pi + \frac{1}{F^2} S H \right) \right)_x = \frac{I_0}{\varepsilon F^2} Q \left( I - \frac{Q|Q|}{C_h^2 R_h S^2} \right) + O\left(\frac{\varepsilon F^2}{I_0}\right) \quad (19)$$

#### 4.3. Computation of Backwater curves

Taking  $\Psi = \Psi^{(0)}$  and  $\Pi = \Pi^{(0)}$  in ((18)) together with mass conservation equation, we get a closed system of equation consistent up to with our asymptotic ansatz. However this system as a very poor mathematical structure ( unless  $0 = \Psi^{(0)} = \Pi^{(0)}$ , where it becomes the standard Saint Venant model). Though, it provides an efficient way to compute backwater curves (i.e. stationary solution of the system), which are thus characterized by  $Q = cst$  and

$$\left( \frac{1}{S} \left( \frac{1}{2} \frac{Q^2}{S} + S^3 \Pi^{(0)} + \frac{1}{F^2} S \left( H + \frac{I_0}{\varepsilon} b_0 \right) \right) \right)_x = -\frac{I_0}{\varepsilon F^2} \frac{Q|Q|}{C_h^2 R_h S^2}$$

which is a first order ordinary differential equation if we take  $\Lambda = I$  in the definition of  $\Pi^{(0)}$  ( which is correct if we take asymptotic regime 2 instead of regime 1). Note that it coincides with the usual backwater curve equation iff  $\Pi^{(0)} = 0$ . An easy inspection of (17) leads easily to  $0 = \Psi^{(0)} = \Pi^{(0)}$  in the case of square shaped channel with  $c(R_X x, y) = c(x)$ . We thus have in this situation coincidence of our model with standard Saint Venant system. However the term  $S^3 \Pi^{(0)}$  can be  $O(1)$  and leads to  $O(1)$  deviation from the standard Saint Venant backwater curves.

#### 4.4. Complete 4 equation model

Fortunately, following ideas developed in (Richard Gavriluk<sup>6</sup>) and (Richard Ruyer-Quil Vila<sup>7</sup>) we can propose the following class of model

$$\begin{cases} (i) & S_t + Q_x = 0 \\ (ii) & Q_t + \left( \frac{Q^2}{S} + P^{tot} \right)_x = \frac{I_0 S}{\varepsilon F^2} \left( \mathcal{I} - \frac{Q|Q|}{C_h^2 R_h S^2} + A_1 (\Pi^{(0)} - \Pi) \right) + B(H, x) \\ (iii) & E_t^{2D} + \left( \frac{Q}{S} (E^{2D} + P^{tot}) \right)_x = \frac{I_0}{\varepsilon F^2} Q \left( \mathcal{I} - \frac{Q|Q|}{C_h^2 R_h S^2} \right) \\ (iv) & (S^3 \Pi)_t + V (S^3 \Pi)_x = \frac{I_0}{\varepsilon F^2} A_2 (\Psi^{(0)} - \Psi) \end{cases}$$

where  $B(H, x) = \frac{1}{F^2} \int_0^H L_x(x, z)(H - z) dz$  represent the lateral pressure effects,  $P^{hydro} = \frac{1}{F^2} \int_0^H L(x, z)(H - z) dz$  the hydrostatic pressure effect,  $P^{tot} = \frac{1}{2} S^3 (\Pi - \Psi) + P^{hydro}$  an effective total pressure and  $E^{2D} = \frac{1}{2} \frac{Q^2}{S} + \frac{1}{2} S^3 \Psi + \mathcal{E}$  a total energy. Equation (iii) is just (with adhoc notation) the Energy Balance equation (19). After an inspection of momentum equation and the additional transport equation of potential  $\Pi$  we observe that in our asymptotic regime we obtain, taking first the form (16) of energy balance, that  $Q = Q^{(0)} + O\left(\frac{\varepsilon F^2}{I_0}\right)$ , then (iv) gives  $\Psi = \Psi^{(0)} + O\left(\frac{\varepsilon F^2}{I_0}\right)$  and (ii) gives  $\Pi = \Pi^{(0)} + O\left(\frac{\varepsilon F^2}{I_0}\right)$ , turning back to ((16)) we get  $Q = Q^{(0)} + \frac{\varepsilon F^2}{I_0} Q^{(1)} + o\left(\frac{\varepsilon F^2}{I_0}\right)$  and complete consistency of energy and momentum conservation equations with the original ones.

For completeness we give the analytic formulae for  $\Psi^{(0)}$  and  $\Pi^{(0)}$  in the case of a Trapezoidal channel such that  $L = l(x) \left(1 + 2z \frac{\xi}{H}\right)$  with  $\xi = \frac{H}{l} = \frac{L(x, H) - l}{2l}$ . We thus get

$$\Psi^{(0)} = \frac{H^m \xi m^2 (m + 2\xi + 4) |\Lambda|}{l^2 H^2 (\xi + 1)^4 (2 + m) (4 + m)^2}$$

and

$$\Pi^{(0)} = \frac{\Lambda H^m \xi m^2 (3\xi m^2 + 4\xi^2 m + 6m^2 + 48\xi m + 48\xi^2 + 48m + 144\xi + 96) \sqrt{|\Lambda|}}{l^2 H^2 (4\xi + 4 + m) (4 + 3m) (4 + m)^2 (\xi + 1)^4}$$

Such a model has a structure closely related to Euler compressible equations, and is fully compatible with the diffusive wave approximation of 2D Saint Venant equations. It reduces exactly to standard Saint Venant model in the case of U-shaped model. We expect that future study may bring some confidence in this new model.

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