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An Identity for Horadam Sequences

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Abstract

We derive an identity connecting any two Horadam sequences having the same recurrence relation but whose initial terms may be different. Binomial and ordinary summation identities arising from the identity are developed. Illustrative examples are drawn from Fibonacci, Fibonacci-Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences and their generalizations. Our new results subsume previously known identities.

1 Introduction

This paper is concerned with establishing an identity connecting any two Horadam sequences, \((G_n)_{n \in \mathbb{Z}}\) and \((H_n)_{n \in \mathbb{Z}}\), having the same recurrence relation but whose initial terms may be different. Thus, for \(n \geq 2\) and with \(p\) and \(q\) arbitrary fixed non-zero complex constants, we define

\[ G_n = pG_{n-1} + qG_{n-2}, \] (1.1)

where the initial terms \(G_0\) and \(G_1\) are given arbitrary integers, not both zero; and

\[ H_n = pH_{n-1} + qH_{n-2}, \] (1.2)

with initial terms \(H_0\) and \(H_1\) given arbitrary integers, not both zero.

Extension of the definition of \(G_n\) and \(H_n\) to negative subscripts is provided by writing the recurrence relation as

\[ G_{-n} = (G_{-n+2} - pG_{-n+1})/q \] (1.3)

and

\[ H_{-n} = (H_{-n+2} - pH_{-n+1})/q. \] (1.4)

In section 2, we will derive an identity connecting \((G_n)\) and \((H_n)\), for arbitrary integers. We will illustrate the results by deriving identities for six well-known Horadam sequences, namely, Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences.
1.1 Notation and definitions

The Fibonacci numbers, \(F_n\), and the Lucas numbers, \(L_n\), are defined, for \(n \in \mathbb{Z}\), as usual, through the recurrence relations \(F_n = F_{n-1} + F_{n-2}\) \((n \geq 2)\), \(F_0 = 0\), \(F_1 = 1\) and \(L_n = L_{n-1} + L_{n-2}\) \((n \geq 2)\), \(L_0 = 2\), \(L_1 = 1\), with \(F_{-n} = (-1)^{n-1}F_n\) and \(L_{-n} = (-1)^nL_n\). Exhaustive discussion of the properties of Fibonacci and Lucas numbers can be found in Vajda [9] and in Koshy [6]. Generalized Fibonacci numbers having the same recurrence as the Fibonacci and Lucas numbers but with arbitrary initial values will be denoted \(F_n\).

The Jacobsthal numbers, \(J_n\), and the Jacobsthal-Lucas numbers, \(j_n\), are defined, for \(n \in \mathbb{Z}\), through the recurrence relations \(J_n = J_{n-1} + 2J_{n-2}\) \((n \geq 2)\), \(J_0 = 0\), \(J_1 = 1\) and \(j_n = j_{n-1} + 2j_{n-2}\) \((n \geq 2)\), \(j_0 = 2\), \(j_1 = 1\), with \(J_{-n} = (-1)^{n-1}J_n\) and \(j_{-n} = (-1)^n2^{-n}j_n\). Horadam [5] and Aydin [2] are good reference materials on the Jacobsthal and associated sequences. Generalized Jacobsthal numbers having the same recurrence as the Jacobsthal and Jacobsthal-Lucas numbers but with arbitrary initial values will be denoted \(J_n\).

The Pell numbers, \(P_n\), and Pell-Lucas numbers, \(Q_n\), are defined, for \(n \in \mathbb{Z}\), through the recurrence relations \(P_n = 2P_{n-1} + P_{n-2}\) \((n \geq 2)\), \(P_0 = 0\), \(P_1 = 1\) and \(Q_n = 2Q_{n-1} + Q_{n-2}\) \((n \geq 2)\), \(Q_0 = 2\), \(Q_1 = 2\), with \(P_{-n} = (-1)^{n-1}P_n\) and \(Q_{-n} = (-1)^nQ_n\). Koshy [7], Horadam [4] and Patel and Shrivastava [8] are useful source materials on Pell and Pell-Lucas numbers. Generalized Pell numbers having the same recurrence as the Pell and Pell-Lucas numbers but with arbitrary initial values will be denoted \(P_n\).

For reference, the first few values of the six sequences are given below:

<table>
<thead>
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<th>(n)</th>
<th>(-5)</th>
<th>(-4)</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>(0)</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
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<tr>
<td>(F_n)</td>
<td>5</td>
<td>-3</td>
<td>2</td>
<td>-1</td>
<td>1</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
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<tr>
<td>(L_n)</td>
<td>-11</td>
<td>7</td>
<td>-4</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>18</td>
<td>29</td>
<td>47</td>
</tr>
<tr>
<td>(P_n)</td>
<td>29</td>
<td>-12</td>
<td>5</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>29</td>
<td>70</td>
<td>169</td>
<td>408</td>
</tr>
<tr>
<td>(Q_n)</td>
<td>-82</td>
<td>34</td>
<td>-14</td>
<td>6</td>
<td>-2</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>14</td>
<td>34</td>
<td>82</td>
<td>198</td>
<td>476</td>
<td>1154</td>
</tr>
<tr>
<td>(J_n)</td>
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<td>-5/16</td>
<td>3/8</td>
<td>-1/4</td>
<td>1/2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>11</td>
<td>21</td>
<td>43</td>
<td>85</td>
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<tr>
<td>(j_n)</td>
<td>-31/32</td>
<td>17/16</td>
<td>-7/8</td>
<td>5/4</td>
<td>-1/2</td>
<td>2</td>
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<td>7</td>
<td>17</td>
<td>31</td>
<td>65</td>
<td>127</td>
<td>257</td>
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2 Main results

2.1 Recurrence relations and related identities

**Theorem 1.** Let \((G_n)_{n \in \mathbb{Z}}\) and \((H_n)_{n \in \mathbb{Z}}\) be any two Horadam sequences having the same recurrence relation. Then, the following identity holds for arbitrary integers \(n\), \(m\), \(a\), \(b\), \(c\) and \(d\):

\[
(G_{d-b}G_{c-a} - G_{d-a}G_{c-b})H_{n+m} = (G_{d-b}G_{m-a} - G_{d-a}G_{m-b})H_{n+c} + (G_{c-a}G_{m-b} - G_{c-b}G_{m-a})H_{n+d}.
\]

In particular, we have

\[
(G_{d-b}G_{c-a} - G_{d-a}G_{c-b})G_{n+m} = (G_{d-b}G_{m-a} - G_{d-a}G_{m-b})G_{n+c} + (G_{c-a}G_{m-b} - G_{c-b}G_{m-a})G_{n+d},
\]

for any Horadam sequence, \((G_n)_{n \in \mathbb{Z}}\).
Proof. Since both sequences \((G_n)\) and \((H_n)\) have the same recurrence relation we choose a basis set in one and express the other in this basis. We write
\[
H_{n+m} = \lambda_1 G_{m-a} + \lambda_2 G_{m-b},
\]
where \(a, b, n\) and \(m\) are arbitrary integers and the coefficients \(\lambda_1\) and \(\lambda_2\) are to be determined. Setting \(m = c\) and \(m = d\), in turn, produces two simultaneous equations:
\[
H_{n+c} = \lambda_1 G_{c-a} + \lambda_2 G_{c-b}, \quad H_{n+d} = \lambda_1 G_{d-a} + \lambda_2 G_{d-b}.
\]
The identity of Theorem 1 is established by solving these equations for \(\lambda_1\) and \(\lambda_2\) and substituting the solutions into identity (2.2). \(\square\)

Corollary 2. The following identity holds for integers \(a, b, n\) and \(m\):
\[
(G_{a-b}G_{b-a} - G_0^2)H_{n+m} = (G_{b-a}G_{m-b} - G_0 G_{m-a})H_{n+a} + (G_{a-b}G_{m-a} - G_0 G_{m-b})H_{n+b}.
\]

In particular,
\[
(G_{a-b}G_{b-a} - G_0^2)G_{n+m} = (G_{b-a}G_{m-b} - G_0 G_{m-a})G_{n+a} + (G_{a-b}G_{m-a} - G_0 G_{m-b})G_{n+b}.
\]

2.2 Summation identities

2.2.1 Summation identities not involving binomial coefficients

Lemma 1 ([1, Lemma 1]). Let \(\{X_n\}\) and \(\{Y_n\}\) be any two sequences such that \(X_n\) and \(Y_n\), \(n \in \mathbb{Z}\), are connected by a three-term recurrence relation \(X_n = f_1 X_{n-a} + f_2 Y_{n-b}\), where \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(r\), and \(a\) and \(b\) are integers. Then,
\[
f_2 \sum_{j=0}^{k} \frac{Y_{n-ka+b+j}}{f_1^j} = \frac{X_n}{f_1^k} - f_1 X_{n-(k+1)a},
\]
for \(k\) a non-negative integer.

Lemma 2 ([1, Lemma 2]). Let \(\{X_n\}\) be any arbitrary sequence, where \(X_n\), \(n \in \mathbb{Z}\), satisfies a three-term recurrence relation \(X_n = f_1 X_{n-a} + f_2 X_{n-b}\), where \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(r\), and \(a\) and \(b\) are integers. Then, the following identities hold for integer \(k\):
\[
f_2 \sum_{j=0}^{k} \frac{X_{n-ka+b+j}}{f_1^j} = \frac{X_n}{f_1^k} - f_1 X_{n-(k+1)a},
\]
\[
f_1 \sum_{j=0}^{k} \frac{X_{n-kb+a+bj}}{f_2^j} = \frac{X_n}{f_2^k} - f_2 X_{n-(k+1)b}
\]
and
\[
\sum_{j=0}^{k} \frac{X_{n-(a-b)k+b+(a-b)j}}{(-f_1/f_2)^j} = \frac{f_2X_n}{(-f_1/f_2)^k} + f_1X_{n-(k+1)(a-b)}.
\]
In order to state the next two theorems in a compact form, we introduce the following notation:

\[ f_G(u, v; s, t) = G_{u-s}G_{v-t} - G_{u-t}G_{v-s}, \]

(2.7)

with the following symmetry properties:

\[ f_G(u, v; t, s) = -f_G(u, v; s, t), \quad f_G(v, u; s, t) = -f_G(u, v; s, t), \]

(2.8)

\[ f_G(v, u; t, s) = f_G(u, v; s, t), \]

(2.9)

and

\[ f_G(u, u; s, t) = 0, \quad f_G(u, v; s, s) = 0. \]

(2.10)

In this notation, the identity of Theorem 1 becomes

\[ f_G(d, c; b, a)H_{n+m} = f_G(d, m; b, a)H_{n+c} + f_G(c, m; a, b)H_{n+d}. \]

(2.11)

The results in the next theorem follow from direct substitutions from identity (2.11) into Lemma 2.

**Theorem 3.** The following identities hold for arbitrary integers \( a, b, c, d \) and \( m \) for which \( f_G(d, m; b, a) \neq 0 \) and \( f_G(c, m; b, a) \neq 0 \):

\[
\begin{align*}
  f_G(c, m; a, b) & \sum_{j=0}^{k} \left( \frac{f_G(d, c; b, a)}{f_G(d, m; b, a)} \right)^j H_{n-(m-c)j} (m-d) + (m-c)j \\
  & = \frac{f_G(d, c; b, a)^{k+1}}{f_G(d, m; b, a)^k} H_n - f_G(d, m; b, a)H_{n-(m-c)(k+1)},
\end{align*}
\]

(2.12)

\[
\begin{align*}
  f_G(d, m; b, a) & \sum_{j=0}^{k} \left( \frac{f_G(d, c; b, a)}{f_G(c, m; a, b)} \right)^j H_{n-(m-d)j} (m-c) + (m-d)j \\
  & = \frac{f_G(d, c; b, a)^{k+1}}{f_G(c, m; a, b)^k} H_n - f_G(c, m; b, a)H_{n-(m-d)(k+1)}
\end{align*}
\]

(2.13)

and

\[
\begin{align*}
  f_G(d, c; b, a) & \sum_{j=0}^{k} \left( -\frac{f_G(d, m; b, a)}{f_G(c, m; a, b)} \right)^j H_{n-(c-d)j} (c+d) + (c-d)j \\
  & = (-1)^k \frac{f_G(d, m; b, a)^{k+1}}{f_G(c, m; a, b)^k} H_n + f_G(c, m; b, a)H_{n-(c-d)(k+1)}. \]
\]

(2.14)

### 2.2.2 Binomial summation identities

**Lemma 3** ([1, Lemma 3]). Let \( \{X_n\} \) be any arbitrary sequence. Let \( X_n, n \in \mathbb{Z}, \) satisfy a three-term recurrence relation \( X_n = f_1X_{n-a} + f_2X_{n-b}, \) where \( f_1 \) and \( f_2 \) are non-vanishing complex functions, not dependent on \( n, \) and \( a \) and \( b \) are integers. Then,

\[
\sum_{j=0}^{k} \binom{k}{j} \left( \frac{f_1}{f_2} \right)^j X_{n-bk+(a-b)j} = \frac{X_n}{f_2^k}, \]

(2.15)
\[
\sum_{j=0}^{k} \binom{k}{j} \frac{X_{n+(a-b)k+bj}}{(-f_2)^j} = \left( -\frac{f_1}{f_2} \right)^k X_n
\] (2.16)

and
\[
\sum_{j=0}^{k} \binom{k}{j} \frac{X_{n+(b-a)k+a_j}}{(-f_1)^j} = \left( -\frac{f_2}{f_1} \right)^k X_n,
\] (2.17)

for \( k \) a non-negative integer.

Substituting from identity (2.11) into Lemma 3, we have the results stated in the next theorem.

**Theorem 4.** The following identities hold for positive integer \( k \) and arbitrary integers \( a, b, c, d \), and \( m \) for which \( f_G(d, m; b, a) \neq 0 \) and \( f_G(c, m; b, a) \neq 0 \):

\[
\sum_{j=0}^{k} \binom{k}{j} \left( \frac{f_G(d, m; b, a)}{f_G(c, m; a, b)} \right)^j H_{n-(m-d)k+(c-d)j} = \left( \frac{f_G(d, c; b, a)}{f_G(c, m; a, b)} \right)^k H_n,
\] (2.18)

\[
\sum_{j=0}^{k} \binom{k}{j} \left( -\frac{f_G(d, c; b, a)}{f_G(d, m; b, a)} \right)^j H_{n-(c-d)k+(m-d)j} = \left( \frac{f_G(c, m; a, b)}{f_G(d, m; b, a)} \right)^k H_n
\] (2.19)

and
\[
\sum_{j=0}^{k} \binom{k}{j} \left( -\frac{f_G(d, c; b, a)}{f_G(d, m; b, a)} \right)^i H_{n+(c-d)k+(m-c)j} = \left( \frac{f_G(c, m; a, b)}{f_G(d, m; b, a)} \right)^k H_n.
\] (2.20)

### 3 Application

#### 3.1 Identities involving Fibonacci, Lucas and generalized Fibonacci numbers

**3.1.1 Recurrence relations and related identities**

In Corollary 2, let \((G_n) \equiv (F_n)\) be the Fibonacci sequence and let \((H_n) \equiv (\mathcal{F}_n)\) be the generalized Fibonacci sequence. Then, the identity of Corollary 2 reduces to

\[
F_{a-b}\mathcal{F}_{n+m} = F_{m-b}\mathcal{F}_{n+a} - (-1)^{a-b} F_{m-a}\mathcal{F}_{n+b}.
\] (3.1)

The presumably new identity (3.1) subsumes most known three term recurrence relations involving Fibonacci numbers, Lucas numbers and the generalized Fibonacci numbers. We will give a couple of examples to illustrate this point.

Incidentally, identity (3.1) can also be written as

\[
F_{a-b}\mathcal{F}_{n+m} = \mathcal{F}_{m-b}\mathcal{F}_{n+a} - (-1)^{a-b} \mathcal{F}_{m-a}\mathcal{F}_{n+b}.
\] (3.2)

Setting \( a = 0, b = m - n \) in identity (3.1) gives

\[
F_{n-m}\mathcal{F}_{n+m} = F_n\mathcal{F}_n - (-1)^{n-m} F_m\mathcal{F}_m,
\] (3.3)
which is a generalization of Catalan’s identity:

\[ F_{n-m}F_{n+m} = F_n^2 + (-1)^{n+m+1}F_m^2. \]  

(3.4)

Setting \( b = -a \) in identity (3.1) gives

\[ F_{2a}F_{n+m} = F_{m+a}F_{n+a} - F_{m-a}F_{n-a}, \]

(3.5)

with the special case

\[ F_{n+m} = F_{n+1}F_{n+1} - F_{n-1}F_{n-1}, \]

(3.6)

which is a generalization of the following known identity (Halton [3, Identity (63)]):

\[ F_{n+m} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}. \]

(3.7)

Setting \( b = 2k, a = 1 \) and \( b = 2k, a = 0 \), in turn, in identity (3.1) produces

\[ F_{2k-1}F_{n+m} = F_{m-2k}F_{n+1} + F_{m-1}F_{n+2k}, \]

(3.8)

and

\[ F_{2k}F_{n+m} = F_{m-2k}F_{n+2k} - F_{m-2k}F_{n}. \]

(3.9)

Identity (3.9) is a generalization of the known identity (Vajda [9, Formula (8)]):

\[ F_{n+m} = F_{m-1}F_n + F_mF_{n+1}. \]

(3.10)

Setting \( a = n \) and \( b = -m \) in (3.1) produces

\[ F_{2m}F_{2n} = F_{n+m}F_{n+m} - F_{n-m}F_{n-m}. \]

(3.11)

### 3.1.2 Summation identities

Summation identities given in Theorems 5 and 6 are derived by making appropriate substitutions from identity (3.1) into Lemmata 2 and 3.

**Theorem 5.** The following identities hold for arbitrary integers \( a, b, n, m \) and \( k \):

\[
(-1)^{a+b+1}F_{m-a} \sum_{j=0}^{k} F_{m-b}^{k-j} F_{a-b}^j F_{n-(m-a)(k-(m-b)+(m-a))j} = F_{a-b}^{k+1} F_n - F_{m-b}^{k+1} F_{n-(m-a)(k+1)}.
\]

(3.12)

\[
F_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m-a}^{k-j} F_{a-b}^j F_{n-(m-b)(k-(m-a)+(m-b))j} = F_{a-b}^{k+1} F_n - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} F_{n-(m-b)(k+1)}.
\]

(3.13)

and

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m-a}^{k-j} F_{m-b}^j F_{n-(a-b)(k+(m-a)+(a-b))j} = (-1)^{(a+b)k} F_{m-b}^{k+1} F_n + (-1)^{a+b+1} F_{m-a}^{k+1} F_{n-(a-b)(k+1)}.
\]

(3.14)
In particular, we have

\[
(-1)^{a+b+1} F_{m-a} \sum_{j=0}^{k} F_{m-b}^{k-j} F_{a-b}^{j} F_{n-((m-a)k-(m-b)+(m-a)j} = F_{a-b}^{k+1} F_{n} - F_{m-b}^{k+1} F_{n-(m-a)(k+1)} ;
\]

(3.15)

\[
F_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m-a}^{k-j} F_{a-b}^{j} F_{n-((m-b)k-(m-a)+(m-b)j} = F_{a-b}^{k+1} F_{n} - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} F_{n-(m-b)(k+1)} ;
\]

(3.16)

and

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m-a}^{k-j} F_{a-b}^{j} F_{n-((a-b)k+(m-a)+(a-b)j} = (-1)^{(a+b)k} F_{m-b}^{k+1} F_{n} + (-1)^{a+b+1} F_{m-a}^{k+1} F_{n-(a-b)(k+1)} ;
\]

(3.17)

and the corresponding results involving Lucas numbers:

\[
(-1)^{a+b+1} F_{m-a} \sum_{j=0}^{k} F_{m-b}^{k-j} F_{a-b}^{j} L_{n-((m-a)k-(m-b)+(m-a)j} = F_{a-b}^{k+1} L_{n} - F_{m-b}^{k+1} L_{n-(m-a)(k+1)} ;
\]

(3.18)

\[
F_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m-a}^{k-j} F_{a-b}^{j} L_{n-((m-b)k-(m-a)+(m-b)j} = F_{a-b}^{k+1} L_{n} - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} L_{n-(m-b)(k+1)} ;
\]

(3.19)

and

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m-a}^{k-j} F_{a-b}^{j} L_{n-((a-b)k+(m-a)+(a-b)j} = (-1)^{(a+b)k} F_{m-b}^{k+1} L_{n} + (-1)^{a+b+1} F_{m-a}^{k+1} L_{n-(a-b)(k+1)} .
\]

(3.20)

**Theorem 6.** The following identities hold for positive integer \( k \) and arbitrary integers \( a, b, n, m \):

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m-b}^{j} F_{a-b}^{k-j} F_{n-((m-b)k+(a-b)j} = F_{a-b}^{k} F_{n} ,
\]

(3.21)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} F_{a-b}^{j} F_{m-b}^{k-j} F_{n-((a-b)k+(m-b)j} = (-1)^{a+b} F_{m-b}^{k} F_{n} .
\]

(3.22)

and

\[
\sum_{j=0}^{k} (-1)^{j} \binom{k}{j} F_{a-b}^{j} F_{m-b}^{k-j} F_{n+(a-b)k+(m-a)j} = (-1)^{a+b} F_{m-a}^{k} F_{n} .
\]

(3.23)
In particular we have
\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m-b}^j F_{m-a}^{k-j} F_{n-(m-b)k+(a-b)j} = F_{a-b}^k F_n,
\] (3.24)
\[
\sum_{j=0}^{k} (-1)^{a+bj} \binom{k}{j} F_{m-b}^j F_{n-(a-b)k+(m-b)j} = (-1)^{a+b} F_{m-b}^k F_n
\] (3.25)
and
\[
\sum_{j=0}^{k} (-1)^{a+b} \binom{k}{j} F_{m-b}^j F_{n+(a-b)k+(m-a)j} = (-1)^{a+b} F_{m-a}^k F_n;
\] (3.26)
and the corresponding Lucas versions:
\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m-b}^j F_{m-a}^{k-j} L_{n-(m-b)k+(a-b)j} = F_{a-b}^k L_n,
\] (3.27)
\[
\sum_{j=0}^{k} (-1)^{a+bj} \binom{k}{j} F_{m-b}^j F_{m-a}^{k-j} L_{n-(a-b)k+(m-b)j} = (-1)^{a+b} F_{m-b}^k L_n
\] (3.28)
and
\[
\sum_{j=0}^{k} (-1)^{a+b} \binom{k}{j} F_{m-b}^j F_{m-a}^{k-j} L_{n+(a-b)k+(m-a)j} = (-1)^{a+b} F_{m-a}^k L_n.
\] (3.29)

### 3.2 Identities involving Pell, Pell-Lucas and generalized Pell numbers

#### 3.2.1 Recurrence relations and related identities

Since \( P_0 = 0 \) and \( P_{b-a} = (-1)^{a-b} \) for all \( a, b \in \mathbb{Z} \) just like in the Fibonacci case; we find that the Pell relations derived from the identity of Corollary 2, (with \( (G_n) \equiv (P_n) \), the Pell sequence, and \( (H_n) \equiv (P_n) \), the generalized Pell sequence), are identical to those derived in section 3.1. Thus, we have
\[
P_{a-b} P_{n+m} = P_{m-b} P_{n+a} - (-1)^{a-b} P_{m-a} P_{n+b},
\] (3.30)
and a couple of special instances:
\[
P_{n-m} P_{n+m} = P_n P_n - (-1)^{n-m} P_m P_m,
\] (3.31)
\[
P_{2a} P_{n+m} = P_{m+a} P_{n+a} - P_{m-a} P_{n-a},
\] (3.32)
\[
P_{2k-1} P_{n+m} = P_{m-2k} P_{n+1} + P_{m-1} P_{n+2k};
\] (3.33)
\[
P_{2k} P_{n+m} = P_m P_{n+2k} - P_{m-2k} P_n
\] (3.34)
and
\[
P_{2m} P_{2n} = P_{n+m} P_{n+m} - P_{n-m} P_{n-m}.
\] (3.35)

From identity (3.31), we see that Pell numbers also obey Catalan’s identity:
\[
P_{n-m} P_{n+m} = P_n^2 + (-1)^{n+m+1} P_m^2.
\] (3.36)
We have the following particular cases of identity (3.32):

\[ P_{2a}P_{n+m} = P_{m+a}P_{n+a} - P_{m-a}P_{n-a} \]  
\[ (3.37) \]

and

\[ P_{2a}Q_{n+m} = P_{m+a}Q_{n+a} - P_{m-a}Q_{n-a} , \]  
\[ (3.38) \]

with the special evaluations:

\[ 2P_{n+m} = P_{m+1}P_{n+1} - P_{m-1}P_{n-1} \]  
\[ (3.39) \]

and

\[ 2Q_{n+m} = P_{m+1}Q_{n+1} - P_{m-1}Q_{n-1} . \]  
\[ (3.40) \]

### 3.2.2 Summation identities

Summation identities given in Theorems 7 and 8 are derived by making appropriate substitutions from identity (3.30) into Lemmata 2 and 3.

**Theorem 7.** The following identities hold for arbitrary integers \( a, b, n, m \) and \( k \):

\[ (-1)^{a+b+1} P_{m-a} \sum_{j=0}^{k} P_{m-b}^{k-j} P_{a-b}^{j} P_{n-(m-a)k-(m-b)+(m-a)j} \]
\[ = P_{a-b}^{k+1} P_{n} - P_{m-b}^{k+1} P_{n-(m-a)(k+1)} , \]
\[ (3.41) \]

\[ P_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} P_{m-a}^{k-j} P_{a-b}^{j} P_{n-(a-b)k+(m-a)+(a-b)j} \]
\[ = P_{a-b}^{k+1} P_{n} - (-1)^{(a+b+1)(k+1)} P_{m-a}^{k+1} P_{n-(a-b)(k+1)} , \]
\[ (3.42) \]

and

\[ P_{a-b} \sum_{j=0}^{k} (-1)^{a+b} P_{m-a}^{k-j} P_{a-b}^{j} P_{n-(a-b)k+(m-a)+(a-b)j} \]
\[ = (-1)^a P_{m-a}^{k+1} P_{n} + (-1)^{a+b+1} P_{m-a}^{k+1} P_{n-(a-b)(k+1)} . \]
\[ (3.43) \]

In particular, we have

\[ (-1)^{a+b+1} P_{m-a} \sum_{j=0}^{k} P_{m-b}^{k-j} P_{a-b}^{j} P_{n-(m-a)k-(m-b)+(m-a)j} \]
\[ = P_{a-b}^{k+1} P_{n} - P_{m-b}^{k+1} P_{n-(m-a)(k+1)} , \]
\[ (3.44) \]

\[ P_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} P_{m-a}^{k-j} P_{a-b}^{j} P_{n-(a-b)k-(m-a)+(m-b)j} \]
\[ = P_{a-b}^{k+1} P_{n} - (-1)^{(a+b+1)(k+1)} P_{m-a}^{k+1} P_{n-(m-b)(k+1)} , \]
\[ (3.45) \]

and

\[ P_{a-b} \sum_{j=0}^{k} (-1)^{a+b} P_{m-a}^{k-j} P_{a-b}^{j} P_{n-(a-b)k+(m-a)+(a-b)j} \]
\[ = (-1)^a P_{m-a}^{k+1} P_{n} + (-1)^{a+b+1} P_{m-a}^{k+1} P_{n-(a-b)(k+1)} ; \]
\[ (3.46) \]
and the corresponding results involving Pell-Lucas numbers:

\[
(-1)^{a+b+1} P_{m-a} \sum_{j=0}^{k} P_{m-b}^j P_{a-b}^j Q_{n-(m-a)k-(m-b)+(m-a)j} = P_{a-b}^{k+1} Q_n - P_{m-b}^{k+1} Q_{n-(m-a)(k+1)},
\]

(3.47)

\[
P_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} P_{m-a}^{k-j} P_{a-b}^j Q_{n-(m-b)k-(m-a)+(m-b)j} = P_{a-b}^{k+1} Q_n - (-1)^{(a+b+1)(k+1)} P_{m-a}^{k+1} Q_{n-(m-b)(k+1)}
\]

(3.48)

\[
P_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} P_{m-a}^{k-j} P_{m-b}^j Q_{n-(a-b)k+(m-a)+(a-b)j} = (-1)^{a+b} P_{m-b}^{k+1} Q_n + (-1)^{a+b+1} P_{m-a}^{k+1} Q_{n-(a-b)(k+1)}.
\]

(3.49)

**Theorem 8.** The following identities hold for positive integer \(k\) and arbitrary integers \(a, b, n, m:\)

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} P_{m-b}^j P_{m-a}^{k-j} P_{n-(m-b)k+(a-b)j} = P_{a-b}^k P_n,
\]

(3.50)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} P_{a-b}^j P_{m-a}^{k-j} P_{n-(a-b)k+(m-a)j} = (-1)^{a+b} P_{m-b}^k P_n
\]

(3.51)

and

\[
\sum_{j=0}^{k} (-1)^{j} \binom{k}{j} P_{a-b}^j P_{m-b}^{k-j} P_{n+(a-b)k+(m-a)j} = (-1)^{a+b} P_{m-a}^k P_n.
\]

(3.52)

In particular we have

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} P_{m-b}^j P_{m-a}^{k-j} P_{n-(m-b)k+(a-b)j} = P_{a-b}^k P_n,
\]

(3.53)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} P_{a-b}^j P_{m-a}^{k-j} P_{n-(a-b)k+(m-a)j} = (-1)^{a+b} P_{m-b}^k P_n
\]

(3.54)

and

\[
\sum_{j=0}^{k} (-1)^{j} \binom{k}{j} P_{a-b}^j P_{m-b}^{k-j} P_{n+(a-b)k+(m-a)j} = (-1)^{a+b} P_{m-a}^k P_n;
\]

(3.55)

and the corresponding Pell-Lucas versions:

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} P_{m-b}^j P_{m-a}^{k-j} Q_{n-(m-b)k+(a-b)j} = P_{a-b}^k Q_n,
\]

(3.56)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} P_{a-b}^j P_{m-a}^{k-j} Q_{n-(a-b)k+(m-a)j} = (-1)^{a+b} P_{m-b}^k Q_n
\]

(3.57)

and

\[
\sum_{j=0}^{k} (-1)^{j} \binom{k}{j} P_{a-b}^j P_{m-b}^{k-j} Q_{n+(a-b)k+(m-a)j} = (-1)^{a+b} P_{m-a}^k Q_n.
\]

(3.58)
3.3 Identities involving Jacobsthal, Jacobsthal-Lucas and generalized Jacobsthal numbers

3.3.1 Recurrence relations and related identities

With \((G_n) \equiv (J_n)\), the Jacobsthal sequence, and \((H_n) \equiv (J_n)\), the generalized Jacobsthal sequence, the identity of Corollary 2 reduces to

\[ J_{a-b}J_{n+m} = J_{m-b}J_{n+a} - (-1)^{a-b}2^{a-b}J_{m-a}J_{n+b}. \]  \hspace{1cm} (3.59)

Proceeding as in section 3.1, we have the following particular instances of identity (3.59):

\[ J_{n-m}J_{n+m} = J_nJ_n - (-1)^{n-m}2^{n-m}J_mJ_m, \]  \hspace{1cm} (3.60)

\[ J_{2aJ_{n+m}} = J_{m+a}J_{n+a} - J_{m-a}J_{n-a}, \]  \hspace{1cm} (3.61)

\[ J_{2k-1J_{n+m}} = J_{m-2k}J_{n+1} + J_{m-1}J_{n+2k}, \]  \hspace{1cm} (3.62)

\[ J_{2k}J_{n+m} = J_{m}J_{n+2k} - J_{m-2k}J_{n}, \]  \hspace{1cm} (3.63)

and

\[ J_{2m}J_2n = J_{n+m}J_{n+m} - J_{n-m}J_{n-m}. \]  \hspace{1cm} (3.64)

Identity (3.60) is a generalization of

\[ J_{n-m}J_{n+m} = J_n^2 + (-1)^{n+m+1}2^{n-m}J_m^2, \]  \hspace{1cm} (3.65)

which is the Jacobsthal version of Catalan’s identity.

3.3.2 Summation identities

Summation identities given in Theorems 9 and 10 are derived by making appropriate substitutions from identity (3.59) into Lemmata 2 and 3.

**Theorem 9.** The following identities hold for arbitrary integers \(a, b, n, m\) and \(k\):

\[ (-1)^{a+b+1}2^{a-b}J_{m-a} \sum_{i=0}^{k} J_{m-b}^{k-i}J_{n-(m-a)k-(m-b)+(m-a)i} = J_{a-b}^{k+1}J_n - J_{m-b}^{k+1}J_{n-(m-a)(k+1)}, \]  \hspace{1cm} (3.66)

\[ J_{m-b} \sum_{i=0}^{k} (-1)^{(a+b+1)(k-i)}2^{(a-b)(k-i)}J_{m-a}^{k-i}J_{n-(m-b)k-(m-a)+(m-b)i} = J_{a-b}^{k+1}J_n - (-1)^{(a+b+1)(k+1)}2^{(a-b)(k+1)}J_{m-a}^{k+1}J_{n-(m-b)(k+1)} \]  \hspace{1cm} (3.67)

and

\[ J_{a-b} \sum_{i=0}^{k} (-1)^{(a+b)i}2^{(a-b)(k-i)}J_{m-a}^{k-i}J_{n-(a-b)k+(m-a)+(a-b)i} = (-1)^{(a+b)k}J_{m-a}^{k+1}J_n + (-1)^{a+b+1}2^{(a-b)(k+1)}J_{m-a}^{k+1}J_{n-(a-b)(k+1)}. \]  \hspace{1cm} (3.68)
In particular, we have

\[(−1)^{a+b+1}2^{a−b}J_{m−a}\sum_{i=0}^{k}J_{m−b}^iJ_{a−b}^iJ_{n−(m−a)k−(m−b)+(m−a)i}^{k−i}\]
\[= J_{a−b}^{k+1}J_n − J_{m−b}^{k+1}J_{n−(m−a)(k+1)} \tag{3.69}\]

\[J_{m−b}\sum_{i=0}^{k} (−1)^{(a+b+1)(k−i)}2^{(a−b)(k−i)}J_{m−a}^iJ_{a−b}^iJ_{n−(m−b)k−(m−a)+(m−b)i}\]
\[= J_{a−b}^{k+1}J_n − (−1)^{(a+b+1)(k+1)}2^{(a−b)(k+1)}J_{m−a}^{k+1}J_{n−(m−b)(k+1)} \tag{3.70}\]

and

\[J_{a−b}\sum_{i=0}^{k} (−1)^{(a+b)i}2^{(a−b)(k−i)}J_{m−a}^iJ_{a−b}^iJ_{n−(a−b)k+(m−a)+(a−b)i}\]
\[= (−1)^{(a+b)k}J_{n}^{k+1} + (−1)^{(a+b+1)(k+1)}2^{(a−b)(k+1)}J_{m−a}^{k+1}J_{n−(a−b)(k+1)} \tag{3.71}\]

and the corresponding results involving Jacobsthal-Lucas numbers:

\[(−1)^{a+b+1}2^{a−b}J_{m−a}\sum_{i=0}^{k}J_{m−b}^iJ_{a−b}^iJ_{n−(m−a)k−(m−b)+(m−a)i}^{k−i}\]
\[= J_{a−b}^{k+1}J_n − J_{m−b}^{k+1}J_{n−(m−a)(k+1)} \tag{3.72}\]

\[J_{m−b}\sum_{i=0}^{k} (−1)^{(a+b+1)(k−i)}2^{(a−b)(k−i)}J_{m−a}^iJ_{a−b}^iJ_{n−(m−b)k−(m−a)+(m−b)i}\]
\[= J_{a−b}^{k+1}J_n − (−1)^{(a+b+1)(k+1)}2^{(a−b)(k+1)}J_{m−a}^{k+1}J_{n−(m−b)(k+1)} \tag{3.73}\]

and

\[J_{a−b}\sum_{i=0}^{k} (−1)^{(a+b)i}2^{(a−b)(k−i)}J_{m−a}^iJ_{a−b}^iJ_{n−(a−b)k+(m−a)+(a−b)i}\]
\[= (−1)^{(a+b)k}J_{n}^{k+1} + (−1)^{(a+b+1)(k+1)}2^{(a−b)(k+1)}J_{m−a}^{k+1}J_{n−(a−b)(k+1)} \tag{3.74}\]

**Theorem 10.** The following identities hold for positive integer $k$ and arbitrary integers $a$, $b$, $n$, $m$:

\[\sum_{i=0}^{k} (−1)^{(a+b+1)(k−i)}\binom{k}{i}2^{(a−b)(k−i)}J_{m−a}^iJ_{n−(m−b)k+(a−b)i} = J_{a−b}^{k}J_{n} \tag{3.75}\]

\[\sum_{i=0}^{k} (−1)^{(a+b)i}\binom{k}{i}2^{(a−b)(k−i)}J_{m−a}^iJ_{n−(a−b)k+(m−b)i} = (−1)^{a+b}J_{m−b}^{k}J_{n} \tag{3.76}\]

and

\[\sum_{i=0}^{k} (−1)^i\binom{k}{i}J_{a−b}^{k−i}J_{m−b}^{i}J_{n+(a−b)k+(m−a)i} = (−1)^{a+b}2^{(a−b)k}J_{m−a}^{k}J_{n} \tag{3.77}\]
In particular we have
\[
\sum_{i=0}^{k} (-1)^{(a+b+1)(k-i)} \binom{k}{i} J_{m-b}^i 2^{(a-b)(k-i)} J_{m-a}^{k-i} J_{n-(m-b)k+(a-b)i} = J_{m-b}^k J_n, \tag{3.78}
\]
\[
\sum_{i=0}^{k} (-1)^{(a+b)i} \binom{k}{i} J_{m-b}^i 2^{(a-b)(k-i)} J_{m-a}^{k-i} J_{n-(a-b)k+(m-b)i} = (-1)^{a+b} J_{m-b}^k J_n \tag{3.79}
\]
and
\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} J_{m-b}^i J_{m-a}^{k-i} J_{n+(a-b)k+(m-a)i} = (-1)^{a+b} 2^{(a-b)k} J_{m-a}^k J_n, \tag{3.80}
\]
and the corresponding Jacobsthal-Lucas versions:
\[
\sum_{i=0}^{k} (-1)^{(a+b+1)(k-i)} \binom{k}{i} J_{m-b}^i 2^{(a-b)(k-i)} J_{m-a}^{k-i} J_{n-(m-b)k+(a-b)i} = J_{m-b}^k J_n, \tag{3.81}
\]
\[
\sum_{i=0}^{k} (-1)^{(a+b)i} \binom{k}{i} J_{m-b}^i 2^{(a-b)(k-i)} J_{m-a}^{k-i} J_{n-(a-b)k+(m-b)i} = (-1)^{a+b} J_{m-b}^k J_n \tag{3.82}
\]
and
\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} J_{m-b}^i J_{m-a}^{k-i} J_{n+(a-b)k+(m-a)i} = (-1)^{a+b} 2^{(a-b)k} J_{m-a}^k J_n. \tag{3.83}
\]

References


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