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# The Automatic Baire Property and some Effective Properties of $\omega$ -Rational Functions

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**Abstract.** We prove that  $\omega$ -regular languages accepted by Büchi or Muller automata satisfy an effective automata-theoretic version of the Baire property. Then we use this result to obtain a new effective property of rational functions over infinite words which are realized by finite state Büchi transducers: for each such function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , one can construct a deterministic Büchi automaton  $\mathcal{A}$  accepting a dense  $\mathbf{\Pi}_2^0$ -subset of  $\Sigma^\omega$  such that the restriction of  $F$  to  $L(\mathcal{A})$  is continuous. We also give a new proof of the decidability of the first Baire class for synchronous  $\omega$ -rational functions from which we get an extension of this result.

**Keywords:** Decision problems · Regular languages of infinite words · Infinitary rational relations ·  $\omega$ -rational functions · Topology · Automatic Baire property · Points of continuity · first Baire class · decidability · Wadge-Wagner class

## 1 Introduction

Infinitary rational relations were first studied by Gire and Nivat [12, 14]. The  $\omega$ -rational functions over infinite words, whose graphs are (functional) infinitary rational relations accepted by 2-tape Büchi automata, have been studied by several authors [6, 1, 26, 20].

In this paper we are mainly interested in topological properties of such  $\omega$ -rational functions, and in particular in the question of the continuity. Recall that Prieur proved that one can decide whether a given  $\omega$ -rational function is continuous [20, 21]. On the other hand, Carton, Finkel and Simonnet proved that one cannot decide whether a given  $\omega$ -rational function  $f$  has at least one point of continuity [5]. Notice that this decision problem is actually  $\Sigma_1^1$ -complete, hence highly undecidable [9]. It was also proved in [5] that one cannot decide whether the continuity set of a given  $\omega$ -rational function  $f$  (its set of continuity points) is a regular (respectively, context-free)  $\omega$ -language. Notice that the situation was shown to be quite different in the case of *synchronous* functions. It was proved in [5] that if  $f : A^\omega \rightarrow B^\omega$  is an  $\omega$ -rational synchronous function, then the continuity set  $C(f)$  of  $f$  is  $\omega$ -rational. Moreover, if  $X$  is an  $\omega$ -rational  $\mathbf{\Pi}_2^0$  subset of  $A^\omega$ , then  $X$  is the continuity set  $C(f)$  of some rational synchronous function  $f$  of domain  $A^\omega$ . Notice that these previous works on the continuity of  $\omega$ -rational

functions had shown that decision problems in this area may be decidable or not, while it is well known that most problems about regular languages accepted by finite automata are decidable.

We establish in this paper some new effective properties of rational functions over infinite words. We first prove that  $\omega$ -regular languages accepted by Büchi or Muller automata satisfy an effective automata-theoretic version of the Baire property. Then we use this result to obtain a new effective property of rational functions over infinite words which are realized by finite state Büchi transducers: for each such function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , one can construct a deterministic Büchi automaton  $\mathcal{A}$  accepting a dense  $\mathbf{\Pi}_2^0$ -subset of  $\Sigma^\omega$  such that the restriction of  $F$  to this dense set  $L(\mathcal{A})$  is continuous.

On the other hand, every  $\omega$ -rational function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  is a Borel function, i.e. the inverse image by  $F$  of every Borel subset of  $\Gamma^\omega$  is a Borel subset of  $\Sigma^\omega$ . Recall that the Borel functions  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  from a Cantor space to another Cantor space can be stratified by the Baire hierarchy of the so called Baire class  $\xi$  functions, for countable ordinals  $\xi$  [15]. However every  $\omega$ -rational function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  is either of Baire class 0 (continuous), or of Baire class 1 (the inverse image of every open subset of  $\Gamma^\omega$  is a Borel  $\Sigma_2^0$ -subset of  $\Sigma^\omega$ ) or of Baire class 2 (the inverse image of every open subset of  $\Gamma^\omega$  is a Borel  $\Sigma_3^0$ -subset of  $\Sigma^\omega$ ) see [3, 4]. It is then very natural to ask whether one can decide whether a given  $\omega$ -rational function is of Baire class 1. Cagnard and Simonnet proved in [3] that one can decide whether a given synchronous  $\omega$ -rational function is of Baire class 1. Their proof used the notion of undergraph and overgraph of a function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , when the set  $\Gamma^\omega$  is equipped with the lexicographic ordering. We give here a more direct automata theoretic proof of this result. This way one also obtains an extension of this result. We show that one can decide, for a given synchronous  $\omega$ -rational function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , whether the inverse image by  $F$  of any basic open subset of  $\Gamma^\omega$  is in a given Wadge class of the Wadge-Wagner hierarchy of  $\omega$ -regular sets (where the Wadge-Wagner hierarchy of  $\omega$ -regular sets is a great refinement of the Borel hierarchy of  $\omega$ -regular sets).

This paper is an extended version of a paper which appeared in the proceedings of the 14th International Conference on Language and Automata Theory and Applications, LATA 2020 [10]. It contains the full proofs of Lemmas 13, 14 and 15, which could not be included in the conference paper due to lack of space. Moreover an additional Section 5 has been added with more results related to the decidability of the first Baire class for  $\omega$ -rational functions and some refinements of this question.

The paper is organized as follows. We recall basic notions on automata and on the Borel hierarchy in Section 2. The automatic Baire property for regular  $\omega$ -languages is proved in Section 3. We prove our first new result on  $\omega$ -rational functions in Section 4. The decidability of the first Baire class for  $\omega$ -rational functions and further extensions are studied in Section 5. Some concluding remarks are given in Section 6.

## 2 Recall of basic notions

We assume the reader to be familiar with the theory of formal ( $\omega$ )-languages [28, 26]. We recall some usual notations of formal language theory.

When  $\Sigma$  is a finite alphabet, a *non-empty finite word* over  $\Sigma$  is any sequence  $x = a_1 \dots a_k$ , where  $a_i \in \Sigma$  for  $i = 1, \dots, k$ , and  $k$  is an integer  $\geq 1$ . The length of  $x$  is  $|x| = k$ . The *set of finite words* (including the empty word  $\varepsilon$  whose length is zero) over  $\Sigma$  is denoted  $\Sigma^*$ .

The *first infinite ordinal* is  $\omega$ . An  $\omega$ -word over  $\Sigma$  is an  $\omega$ -sequence  $a_1 \dots a_n \dots$ , where for all integers  $i \geq 1$ ,  $a_i \in \Sigma$ . When  $\sigma$  is an  $\omega$ -word over  $\Sigma$ , we write  $\sigma = \sigma(1)\sigma(2) \dots \sigma(n) \dots$ , where for all  $i$ ,  $\sigma(i) \in \Sigma$ , and  $\sigma[n] = \sigma(1)\sigma(2) \dots \sigma(n)$ .

The usual concatenation product of two finite words  $u$  and  $v$  is denoted  $u \cdot v$  and sometimes just  $uv$ . This product is extended to the product of a finite word  $u$  and an  $\omega$ -word  $v$ . The infinite word  $u \cdot v$  is then the  $\omega$ -word such that:  $(u \cdot v)(k) = u(k)$  if  $k \leq |u|$ , and  $(u \cdot v)(k) = v(k - |u|)$  if  $k > |u|$ . The concatenation product can be extended in an obvious way to the concatenation of an infinite sequence of finite words. The concatenation of a set  $U$  of finite words with a set  $V$  of infinite words is the set of infinite words  $U \cdot V = \{u \cdot v \mid u \in U \text{ and } v \in V\}$ . If  $u$  is a finite word and  $V$  is a set of infinite words then  $u \cdot V = \{u \cdot v \mid v \in V\}$ .

The *set of  $\omega$ -words* over the alphabet  $\Sigma$  is denoted by  $\Sigma^\omega$ . An  $\omega$ -language over an alphabet  $\Sigma$  is a subset of  $\Sigma^\omega$ .

**Definition 1.** : A *finite state machine (FSM)* is a quadruple  $\mathcal{M} = (K, \Sigma, \delta, q_0)$ , where  $K$  is a finite set of states,  $\Sigma$  is a finite input alphabet,  $q_0 \in K$  is the initial state and  $\delta$  is a mapping from  $K \times \Sigma$  into  $2^K$ . A FSM is called *deterministic* iff:  $\delta : K \times \Sigma \rightarrow \{\{q\} \mid q \in K\}$ . (As usual, by a clear identification, we might consider in that case that  $\delta : K \times \Sigma \rightarrow K$ ).

A *Büchi automaton (BA)* is a 5-tuple  $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$  where  $\mathcal{M} = (K, \Sigma, \delta, q_0)$  is a finite state machine and  $F \subseteq K$  is the set of final states.

A *Muller automaton (MA)* is a 5-tuple  $\mathcal{A} = (K, \Sigma, \delta, q_0, \mathcal{F})$  where  $\mathcal{M} = (K, \Sigma, \delta, q_0)$  is a FSM and  $\mathcal{F} \subseteq 2^K$  is the collection of designated state sets.

A Büchi or Muller automaton is said to be *deterministic* if the associated FSM is deterministic.

Let  $\sigma = a_1 a_2 \dots a_n \dots$  be an  $\omega$ -word over  $\Sigma$ .

A sequence of states  $r = q_1 q_2 \dots q_n \dots$  is called an (*infinite*) *run* of  $\mathcal{M} = (K, \Sigma, \delta, q_0)$  on  $\sigma$ , starting in state  $p$ , iff: 1)  $q_1 = p$  and 2) for each  $i \geq 1$ ,  $q_{i+1} \in \delta(q_i, a_i)$ .

In case a run  $r$  of  $\mathcal{M}$  on  $\sigma$  starts in state  $q_0$ , we call it simply “a run of  $\mathcal{M}$  on  $\sigma$ ”. For every (*infinite*) run  $r = q_1 q_2 \dots q_n \dots$  of  $\mathcal{M}$ ,  $\text{In}(r)$  is the set of states in  $K$  entered by  $\mathcal{M}$  infinitely many times during run  $r$ :  $\text{In}(r) = \{q \in K \mid \exists^\infty i \geq 1 \ q_i = q\}$  is infinite}.

For  $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$  a BA, the  $\omega$ -language accepted by  $\mathcal{A}$  is:

$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } \text{In}(r) \cap F \neq \emptyset\}$ .

For  $\mathcal{A} = (K, \Sigma, \delta, q_0, \mathcal{F})$  a MA, the  $\omega$ -language accepted by  $\mathcal{A}$  is:

$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } \text{In}(r) \in \mathcal{F}\}$ .

By R. Mc Naughton's Theorem, see [19], the expressive power of deterministic MA (DMA) is equal to the expressive power of non deterministic MA which is also equal to the expressive power of non deterministic BA.

**Theorem 2.** *For any  $\omega$ -language  $L \subseteq \Sigma^\omega$ , the following conditions are equivalent:*

1. *There exists a DMA that accepts  $L$ .*
2. *There exists a MA that accepts  $L$ .*
3. *There exists a BA that accepts  $L$ .*

*An  $\omega$ -language  $L$  satisfying one of these conditions is called a regular  $\omega$ -language.*

Recall that, from a Büchi (respectively, Muller) automaton  $\mathcal{A}$ , one can effectively construct a deterministic Muller (respectively, non-deterministic Büchi) automaton  $\mathcal{B}$  such that  $L(\mathcal{A}) = L(\mathcal{B})$ .

A way to study the complexity of  $\omega$ -languages accepted by various automata is to study their topological complexity.

We assume the reader to be familiar with basic notions of topology which may be found in [15, 26, 19]. If  $X$  is a finite alphabet containing at least two letters, then the set  $X^\omega$  of infinite words over  $X$  may be equipped with the product topology of the discrete topology on  $X$ . This topology is induced by a natural metric which is called the *prefix metric* and is defined as follows. For  $u, v \in X^\omega$  and  $u \neq v$  let  $\delta(u, v) = 2^{-l_{\text{pref}(u,v)}}$  where  $l_{\text{pref}(u,v)}$  is the first integer  $n$  such that the  $u(n+1)$  is different from  $v(n+1)$ . The topological space  $X^\omega$  is a Cantor space. The open sets of  $X^\omega$  are the sets of the form  $W \cdot X^\omega$ , where  $W \subseteq X^*$ . A set  $L \subseteq X^\omega$  is a closed set iff its complement  $X^\omega - L$  is an open set. Closed sets are characterized by the following:

**Proposition 3.** *A set  $L \subseteq X^\omega$  is a closed set of  $X^\omega$  iff for every  $\sigma \in X^\omega$ ,  $\forall n \geq 1, \exists u \in X^\omega$  such that  $\sigma[n] \cdot u \in L$  implies that  $\sigma \in L$ .*

Define now the next classes of the Borel hierarchy:

**Definition 4.** *The classes  $\Sigma_n^0$  and  $\Pi_n^0$  of the Borel Hierarchy on the topological space  $X^\omega$  are defined as follows:  $\Sigma_1^0$  is the class of open sets of  $X^\omega$ ,  $\Pi_1^0$  is the class of closed sets of  $X^\omega$ . And for any integer  $n \geq 1$ :  $\Sigma_{n+1}^0$  is the class of countable unions of  $\Pi_n^0$ -subsets of  $X^\omega$ , and  $\Pi_{n+1}^0$  is the class of countable intersections of  $\Sigma_n^0$ -subsets of  $X^\omega$ .*

**Remark 5.** *The hierarchy defined above is the hierarchy of Borel sets of finite rank. The Borel Hierarchy is also defined for transfinite levels (see [15]) but we shall not need this in the sequel. Recall that the class of Borel subsets of a Cantor space is the closure of the class of open sets under countable unions and countable intersections.*

It turns out that there is a characterization of  $\Pi_2^0$ -subsets of  $X^\omega$ , involving the notion of  $W^\delta$  which we now recall, see [26, 19].

**Definition 6.** For  $W \subseteq X^*$ , we set:  $W^\delta = \{\sigma \in X^\omega \mid \exists^\infty i \text{ such that } \sigma[i] \in W\}$ . ( $\sigma \in W^\delta$  iff  $\sigma$  has infinitely many prefixes in  $W$ .)

Then we can state the following proposition.

**Proposition 7.** A subset  $L$  of  $X^\omega$  is a  $\mathbf{\Pi}_2^0$ -subset of  $X^\omega$  iff there exists a set  $W \subseteq X^*$  such that  $L = W^\delta$ .

It is easy to see, using the above characterization of  $\mathbf{\Pi}_2^0$ -sets, that every  $\omega$ -language accepted by a deterministic Büchi automaton is a  $\mathbf{\Pi}_2^0$ -set. Thus every regular  $\omega$ -language is a finite Boolean combination of  $\mathbf{\Pi}_2^0$ -sets, because it is accepted by a deterministic Muller automaton and this implies that it is a finite boolean combination of  $\omega$ -languages accepted by deterministic Büchi automata.

Landweber studied the topological properties of regular  $\omega$ -languages in [17]. He characterized the regular  $\omega$ -languages in each of the Borel classes  $\mathbf{\Sigma}_1^0, \mathbf{\Pi}_1^0, \mathbf{\Sigma}_2^0, \mathbf{\Pi}_2^0$ , and showed that one can decide, for an effectively given regular  $\omega$ -language  $L$ , whether  $L$  is in  $\mathbf{\Sigma}_1^0, \mathbf{\Pi}_1^0, \mathbf{\Sigma}_2^0$ , or  $\mathbf{\Pi}_2^0$ . In particular, it turned out that a regular  $\omega$ -language is in the class  $\mathbf{\Pi}_2^0$  iff it is accepted by a deterministic Büchi automaton.

Recall that, from a Büchi or Muller automaton  $\mathcal{A}$ , one can construct some Büchi or Muller automata  $\mathcal{B}$  and  $\mathcal{C}$ , such that  $L(\mathcal{B})$  is equal to the topological closure of  $L(\mathcal{A})$ , and  $L(\mathcal{C})$  is equal to the topological interior of  $L(\mathcal{A})$ , see [26, 19].

### 3 The automatic Baire property

In this section we are going to prove an automatic version of the result stating that every Borel (and even every analytic) set has the Baire property.

We firstly recall some basic definitions about meager sets, see [15]. In a topological space  $\mathcal{X}$ , a set  $A \subseteq \mathcal{X}$  is said to be *nowhere dense* if its closure  $\bar{A}$  has empty interior, i.e.  $\text{Int}(\bar{A}) = \emptyset$ . A set  $A \subseteq \mathcal{X}$  is said to be *meager* if it is the union of countably many nowhere dense sets, or equivalently if it is included in a countable union of closed sets with empty interiors. This means that  $A$  is meager if there exist countably many closed sets  $A_n, n \geq 1$ , such that  $A \subseteq \bigcup_{n \geq 1} A_n$  where for every integer  $n \geq 1$ ,  $\text{Int}(A_n) = \emptyset$ . A set is *comeager* if its complement is meager, i.e. if it contains the intersection of countably many dense open sets. Notice that the notion of a *meager* set is a notion of a *small* set, while the notion of a *comeager* set is a notion of a *big* set.

Recall that a Baire space is a topological space  $\mathcal{X}$  in which every intersection of countably many dense open sets is dense, or equivalently in which every countable union of closed sets with empty interiors has also an empty interior. It is well known that every Cantor space  $\Sigma^\omega$  is a Baire space. In the sequel we will consider only Cantor spaces.

We now recall the notion of Baire property. For any sets  $A, B \subseteq \Sigma^\omega$ , we denote by  $A \Delta B$  the symmetric difference of  $A$  and  $B$ , and we write  $A =^* B$  if and only if  $A \Delta B$  is meager.

**Definition 8.** A set  $A \subseteq \Sigma^\omega$  has the Baire property (BP) if there exists an open set  $U \subseteq \Sigma^\omega$  such that  $A =^* U$ .

An important result of descriptive set theory is the following result, see [15, page 47].

**Theorem 9.** Every Borel set of a Cantor space has the Baire property.

We are going to prove an automatic version of the above theorem. We first give the following definition.

**Definition 10.** Let  $L = L(\mathcal{A}) \subseteq \Sigma^\omega$  be a regular  $\omega$ -language accepted by a Büchi or Muller automaton  $\mathcal{A}$ . The  $\omega$ -language  $L$  is said to have the automatic Baire property if one can construct from  $\mathcal{A}$  some Büchi automata  $\mathcal{B}$  and  $\mathcal{C}$  such that  $L(\mathcal{B}) \subseteq \Sigma^\omega$  is open,  $L(\mathcal{C}) \subseteq \Sigma^\omega$  is a countable union of closed sets with empty interior, i.e. a meager  $\Sigma_2^0$ -set, and  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ .

We already know that the regular  $\omega$ -languages have the Baire property since they are Borel. We now state the following theorem which gives an automatic version of this result.

**Theorem 11.** Let  $L = L(\mathcal{A}) \subseteq \Sigma^\omega$  be a regular  $\omega$ -language accepted by a Büchi or Muller automaton  $\mathcal{A}$ . Then one can construct Büchi automata  $\mathcal{B}$  and  $\mathcal{C}$  such that  $L(\mathcal{B}) \subseteq \Sigma^\omega$  is open,  $L(\mathcal{C}) \subseteq \Sigma^\omega$  is a meager  $\Sigma_2^0$ -set, and  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ , i.e. the  $\omega$ -language  $L(\mathcal{A})$  has the automatic Baire property.

In order to prove this result, we first prove the following lemmas.

**Lemma 12.** Every regular  $\omega$ -language which is open or closed has the automatic Baire property.

**Proof.** Let  $L = L(\mathcal{A}) \subseteq \Sigma^\omega$  be a regular  $\omega$ -language accepted by a Büchi or Muller automaton  $\mathcal{A}$ .

If  $L = L(\mathcal{A})$  is an open set then we immediately see that we get the result with  $\mathcal{B} = \mathcal{A}$  and  $\mathcal{C}$  is any Büchi automaton accepting the empty set.

If  $L = L(\mathcal{A})$  is a closed set then  $L \setminus \text{Int}(L)$  is a closed set with empty interior. Moreover it is known that one can construct from the Büchi automaton  $\mathcal{A}$  another Büchi automaton  $\mathcal{B}$  accepting  $\text{Int}(L)$ , and then also a Büchi automaton  $\mathcal{C}$  accepting  $L \setminus \text{Int}(L)$ . Then we have  $L(\mathcal{A}) \Delta L(\mathcal{B}) = L \setminus \text{Int}(L) = L(\mathcal{C})$ , with  $L(\mathcal{B})$  open and  $L(\mathcal{C})$  is a closed set with empty interior.  $\square$

**Lemma 13.** Every regular  $\omega$ -language which is a  $\Sigma_2^0$ -set has the automatic Baire property.

**Proof.** Let  $L$  be a regular  $\omega$ -language which is a  $\Sigma_2^0$ -set. Recall that then the  $\omega$ -language  $L$  is accepted by a deterministic finite automaton  $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$  with co-Büchi acceptance condition. Moreover we may assume that the automaton  $\mathcal{A}$  is complete, i.e. for any state  $q$  and any letter  $a \in \Sigma$  there is a transition

from the state  $q$  and reading the current input letter  $a$ . An  $\omega$ -word  $x$  is accepted by  $\mathcal{A}$  with co-Büchi acceptance condition if the run of the automaton  $\mathcal{A}$  on  $x$  (which is then unique since the automaton is deterministic) goes only finitely many times through the set of states  $F$ . Let now  $n \geq 1$  and  $L_n$  be the  $\omega$ -language of  $\omega$ -words  $x \in \Sigma^\omega$  such that the run of the automaton  $\mathcal{A}$  over  $x$  goes at most  $n$  times through the set of states  $F$ . We have clearly that  $L = \bigcup_{n \geq 1} L_n$ . Moreover it is easy to see that for every  $n \geq 1$  the set  $L_n$  is closed. And the interior of  $L_n$  is the union of basic open sets  $u \cdot \Sigma^\omega$ , for  $u \in \Sigma^*$ , such that the automaton  $\mathcal{A}$  enters states from  $F$  at most  $n$  times during the reading of  $u$  and ends the reading of  $u$  in a state  $q$  such that no state of  $F$  is accessible from this state  $q$ . We have of course that  $L_n \setminus \text{Int}(L_n)$  is a closed set with empty interior and thus  $L_n =^* \text{Int}(L_n)$ . Then  $\bigcup_{n \geq 1} L_n =^* \bigcup_{n \geq 1} \text{Int}(L_n)$  and we can easily see that  $\bigcup_{n \geq 1} \text{Int}(L_n)$  is accepted by a Büchi automaton  $\mathcal{B}$  which is essentially the automaton  $\mathcal{A}$  with a modified Büchi acceptance condition expressing “some state  $q$  has been reached from which no state of  $F$  is accessible”. Moreover  $L(\mathcal{A}) \Delta L(\mathcal{B}) = (\bigcup_{n \geq 1} L_n) \Delta (\bigcup_{n \geq 1} \text{Int}(L_n)) \subseteq \bigcup_{n \geq 1} (L_n \setminus \text{Int}(L_n))$  and the set  $\bigcup_{n \geq 1} (L_n \setminus \text{Int}(L_n))$  is a countable union of closed sets with empty interiors which is easily seen to be accepted by a co-Büchi automaton  $\mathcal{C}$  which is essentially the automaton  $\mathcal{A}$  where we have deleted every state  $q$  from which no state of  $F$  is accessible.  $\square$

**Lemma 14.** *Let  $L \subseteq \Sigma^\omega$  be a regular  $\omega$ -language which has the automatic Baire property. Then its complement  $\Sigma^\omega \setminus L$  has also the automatic Baire property.*

**Proof.** Assume that the regular  $\omega$ -language  $L = L(\mathcal{A})$  has the automatic Baire property and that  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ , where  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  are Büchi automata,  $L(\mathcal{B})$  is open and  $L(\mathcal{C})$  is a meager  $\Sigma_2^0$ -set. Notice that  $(\Sigma^\omega \setminus L(\mathcal{A})) \Delta (\Sigma^\omega \setminus L(\mathcal{B})) = L(\mathcal{A}) \Delta L(\mathcal{B})$ . Moreover we can construct a Büchi automaton  $\mathcal{D}$  accepting the closed set  $(\Sigma^\omega \setminus L(\mathcal{B}))$  and next also a Büchi automaton  $\mathcal{D}'$  accepting the open set  $\text{Int}(\Sigma^\omega \setminus L(\mathcal{B}))$ , and a Büchi automaton  $\mathcal{E}$  accepting the closed set with empty interior  $(L(\mathcal{D}) \setminus \text{Int}(L(\mathcal{D})))$ . It is now easy to see that  $(\Sigma^\omega \setminus L(\mathcal{A})) \Delta L(\mathcal{D}') \subseteq (\Sigma^\omega \setminus L(\mathcal{A})) \Delta (\Sigma^\omega \setminus L(\mathcal{B})) \cup L(\mathcal{E}) \subseteq L(\mathcal{C}) \cup L(\mathcal{E})$  and we can construct a Büchi automaton  $\mathcal{C}'$  accepting the  $\omega$ -language  $L(\mathcal{C}) \cup L(\mathcal{E})$  which is a meager  $\Sigma_2^0$ -set so that  $(\Sigma^\omega \setminus L(\mathcal{A})) \Delta L(\mathcal{D}') \subseteq L(\mathcal{C}')$ . Thus the regular  $\omega$ -language  $(\Sigma^\omega \setminus L(\mathcal{A}))$  has also the automatic Baire property.

Notice that this implies that the regular  $\omega$ -languages in the Borel class  $\Pi_2^0$  have also the automatic Baire property since we have already solved the case of the class  $\Sigma_2^0$  in the proof of Lemma 13.  $\square$

**Lemma 15.** *The class of regular  $\omega$ -languages having the automatic Baire property is closed under finite union and under finite intersection.*

**Proof.** We first consider the finite union operation. Let  $L(\mathcal{A})$  and  $L(\mathcal{A}')$  be regular  $\omega$ -languages accepted by Büchi automata  $\mathcal{A}$  and  $\mathcal{A}'$ . Assume that we have  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ , and  $L(\mathcal{A}') \Delta L(\mathcal{B}') \subseteq L(\mathcal{C}')$  where  $\mathcal{A}$ ,  $\mathcal{A}'$ ,  $\mathcal{B}$ ,  $\mathcal{B}'$ , and  $\mathcal{C}$ ,  $\mathcal{C}'$ , are Büchi automata,  $L(\mathcal{B})$  and  $L(\mathcal{B}')$  are open and  $L(\mathcal{C})$  and  $L(\mathcal{C}')$  are countable unions of closed sets with empty interiors. Now we can see that



$(L(\mathcal{A}) \cup L(\mathcal{A}')) \Delta (L(\mathcal{B}) \cup L(\mathcal{B}')) \subseteq (L(\mathcal{A}) \Delta L(\mathcal{B})) \cup (L(\mathcal{A}') \Delta L(\mathcal{B}')) \subseteq L(\mathcal{C}) \cup L(\mathcal{C}')$ .  
Moreover we can construct Büchi automata  $\mathcal{B}''$  and  $\mathcal{C}''$  such that  $L(\mathcal{B}'')$  is the open set  $L(\mathcal{B}) \cup L(\mathcal{B}')$  and  $L(\mathcal{C}'') = L(\mathcal{C}) \cup L(\mathcal{C}')$  is a meager  $\Sigma_2^0$ -set and then we have  $(L(\mathcal{A}) \cup L(\mathcal{A}')) \Delta L(\mathcal{B}'') \subseteq L(\mathcal{C}'')$ . This implies that the union  $(L(\mathcal{A}) \cup L(\mathcal{A}'))$  has the automatic Baire property.

The case of the intersection of two regular  $\omega$ -languages now follows from the case of the union and the case of the complement proved in Lemma 14.  $\square$

**End of Proof of Theorem 11.** We now return to the general case of a regular  $\omega$ -language  $L \subseteq \Sigma^\omega$ , accepted by a Büchi or Muller automaton. We know that we can construct a deterministic Muller automaton  $\mathcal{A} = (K, \Sigma, \delta, q_0, \mathcal{F})$  accepting  $L$ . Recall that  $\mathcal{F} \subseteq 2^K$  is here the collection of designated state sets. For each state  $q \in K$ , we now denote by  $\mathcal{A}^{(q)}$  the automaton  $\mathcal{A}$  but viewed as a (deterministic) Büchi automaton with the single accepting state  $q$ , i.e.  $\mathcal{A}^{(q)} = (K, \Sigma, \delta, q_0, \{q\})$ . We know that the languages  $L(\mathcal{A}^{(q)})$  are Borel  $\Pi_2^0$ -sets and thus satisfy the automatic Baire property by Lemmas 13 and 14. Moreover we have the following equality:

$$L(\mathcal{A}) = \bigcup_{F \in \mathcal{F}} \left[ \bigcap_{q \in F} L(\mathcal{A}^{(q)}) \setminus \bigcup_{q \notin F} L(\mathcal{A}^{(q)}) \right]$$

This implies, from the previous lemmas about the preservation of the automatic Baire property by Boolean operations, that we can construct Büchi automata  $\mathcal{B}$  and  $\mathcal{C}$ , such that  $L(\mathcal{B})$  is open and  $L(\mathcal{C})$  is a meager  $\Sigma_2^0$ -set, which satisfy  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ . Thus the  $\omega$ -language  $L$  has the automatic Baire property.  $\square$

**Corollary 16.** *On can decide, for a given Büchi or Muller automaton  $\mathcal{A}$ , whether  $L(\mathcal{A})$  is meager.*

**Proof.** Let  $\mathcal{A}$  be a Büchi or Muller automaton. The  $\omega$ -language  $L(\mathcal{A})$  has the automatic Baire property and we can construct Büchi automata  $\mathcal{B}$  and  $\mathcal{C}$ , such that  $L(\mathcal{B})$  is open and  $L(\mathcal{C})$  is a countable union of closed sets with empty interiors, which satisfy  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ . It is easy to see that  $L(\mathcal{A})$  is meager if and only if  $L(\mathcal{B})$  is empty, since any non-empty open set is non-meager, and it can be decided from the automaton  $\mathcal{B}$  whether  $L(\mathcal{B})$  is empty.  $\square$

**Remark 17.** *The above Corollary followed already from Staiger's paper [27], see also [18]. So we get here another proof of this result, based on the automatic Baire property.*

## 4 An application to $\omega$ -rational functions

### 4.1 Infinitary rational relations

We now recall the definition of infinitary rational relations, via definition by Büchi transducers:

**Definition 18.** A 2-tape Büchi automaton is a 6-tuple  $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$ , where  $K$  is a finite set of states,  $\Sigma$  and  $\Gamma$  are finite sets called the input and the output alphabets,  $\Delta$  is a finite subset of  $K \times (\Sigma \cup \varepsilon) \times (\Gamma \cup \varepsilon) \times K$  called the set of transitions,  $q_0$  is the initial state, and  $F \subseteq K$  is the set of accepting states. A computation  $\mathcal{C}$  of the automaton  $\mathcal{T}$  is an infinite sequence of consecutive transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \dots, (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \dots$$

The computation is said to be successful iff there exists a final state  $q_f \in F$  and infinitely many integers  $i \geq 0$  such that  $q_i = q_f$ . The input word and output word of the computation are respectively  $u = u_1.u_2.u_3 \dots$  and  $v = v_1.v_2.v_3 \dots$ . The input and the output words may be finite or infinite. The infinitary rational relation  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$  accepted by the 2-tape Büchi automaton  $\mathcal{T}$  is the set of pairs  $(u, v) \in \Sigma^\omega \times \Gamma^\omega$  such that  $u$  and  $v$  are the input and the output words of some successful computation  $\mathcal{C}$  of  $\mathcal{T}$ .

The 2-tape Büchi automaton  $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$  is said to be synchronous if the set of transitions  $\Delta$  is a finite subset of  $K \times \Sigma \times \Gamma \times K$ , i.e. if each transition is labelled with a pair  $(a, b) \in \Sigma \times \Gamma$ . An infinitary rational relation recognized by a synchronous 2-tape Büchi automaton is in fact, via the natural identification of  $\Sigma^\omega \times \Gamma^\omega$  with  $(\Sigma \times \Gamma)^\omega$ , an  $\omega$ -language over the product alphabet  $\Sigma \times \Gamma$  which is accepted by a Büchi automaton. It is called a synchronous infinitary rational relation. An infinitary rational relation is said to be asynchronous if it can not be recognized by any synchronous 2-tape Büchi automaton.

**Remark 19.** In the above definition, we could have defined the set of transitions  $\Delta$  as a subset of  $K \times \Sigma^* \times \Gamma^* \times K$ . We have chosen to define  $\Delta$  as a finite subset of  $K \times (\Sigma \cup \varepsilon) \times (\Gamma \cup \varepsilon) \times K$  to simplify the proofs. However this is done without loss of generality because it is easy to see that this convention does not change the class of infinitary rational relations.

If  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$  is an infinitary rational relation recognized by the 2-tape Büchi automaton  $\mathcal{T}$  then we denote

$$Dom(R(\mathcal{T})) = \{u \in \Sigma^\omega \mid \exists v \in \Gamma^\omega \ (u, v) \in R(\mathcal{T})\}$$

and

$$Im(R(\mathcal{T})) = \{v \in \Gamma^\omega \mid \exists u \in \Sigma^\omega \ (u, v) \in R(\mathcal{T})\}.$$

It is well known that, for each infinitary rational relation  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ , the sets  $Dom(R(\mathcal{T}))$  and  $Im(R(\mathcal{T}))$  are regular  $\omega$ -languages and that one can construct, from the Büchi transducer  $\mathcal{T}$ , some (non-deterministic) Büchi automata  $\mathcal{A}$  and  $\mathcal{B}$  accepting the  $\omega$ -languages  $Dom(R(\mathcal{T}))$  and  $Im(R(\mathcal{T}))$ .

Recall now the following undecidability result of Frougny and Sakarovitch.

**Theorem 20 ([11]).** *One cannot decide whether a given infinitary rational relation is synchronous.*

We proved in [8] that many decision problems about infinitary rational relations are highly undecidable. In fact many of them, like the universality problem, the equivalence problem, the inclusion problem, the cofiniteness problem, the unambiguity problem, are  $\Pi_2^1$ -complete, hence located at the second level of the analytical hierarchy.

## 4.2 Continuity of $\omega$ -rational functions

Recall that an infinitary rational relation  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$  is said to be functional iff it is the graph of a function, i.e. iff

$$[\forall x \in \text{Dom}(R(\mathcal{T})) \ \exists! y \in \text{Im}(R(\mathcal{T})) \ (x, y) \in R(\mathcal{T})].$$

Then the functional relation  $R(\mathcal{T})$  defines an  $\omega$ -rational (partial) function  $F_{\mathcal{T}} : \text{Dom}(R(\mathcal{T})) \rightarrow \Gamma^\omega$  by: for each  $u \in \text{Dom}(R(\mathcal{T}))$ ,  $F_{\mathcal{T}}(u)$  is the unique  $v \in \Gamma^\omega$  such that  $(u, v) \in R(\mathcal{T})$ .

An  $\omega$ -rational (partial) function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is said to be synchronous if there is a synchronous 2-tape Büchi automaton  $\mathcal{T}$  such that  $f = F_{\mathcal{T}}$ .

An  $\omega$ -rational (partial) function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is said to be asynchronous if there is no synchronous 2-tape Büchi automaton  $\mathcal{T}$  such that  $f = F_{\mathcal{T}}$ .

Recall the following previous decidability result.

**Theorem 21 (Gire [13]).** *One can decide whether an infinitary rational relation recognized by a given 2-tape Büchi automaton  $\mathcal{T}$  is a functional infinitary rational relation.*

It is very natural to consider the notion of continuity for  $\omega$ -rational functions defined by 2-tape Büchi automata.

We recall that a function  $f : \text{Dom}(f) \subseteq \Sigma^\omega \rightarrow \Gamma^\omega$ , whose domain is  $\text{Dom}(f)$ , is said to be continuous at point  $x \in \text{Dom}(f)$  if :

$$\forall n \geq 1 \ \exists k \geq 1 \ \forall y \in \text{Dom}(f) \ [ \delta(x, y) < 2^{-k} \Rightarrow \delta(f(x), f(y)) < 2^{-n} ]$$

The continuity set  $C(f)$  of the function  $f$  is the set of points of continuity of  $f$ . Notice that the continuity set  $C(f)$  of a function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is always a Borel  $\Pi_2^0$ -subset of  $\Sigma^\omega$ , see [5].

The function  $f$  is said to be continuous if it is continuous at every point  $x \in \text{Dom}(f)$ , i. e. if  $C(f) = \text{Dom}(f)$ .

Prieur proved the following decidability result.

**Theorem 22 (Prieur [20, 21]).** *One can decide whether a given  $\omega$ -rational function is continuous.*

On the other hand the following undecidability result was proved in [5].

**Theorem 23 (see [5]).** *One cannot decide whether a given  $\omega$ -rational function  $f$  has at least one point of continuity.*

The exact complexity of this undecidable problem was given in [9]. It is  $\Sigma_1^1$ -complete to determine whether a given  $\omega$ -rational function  $f$  has at least one point of continuity.

We now consider the continuity set of an  $\omega$ -rational function and its possible complexity. The following undecidability result was proved in [5].

**Theorem 24 (see [5]).** *One cannot decide whether the continuity set of a given  $\omega$ -rational function  $f$  is a regular (respectively, context-free)  $\omega$ -language.*

The situation is quite different in the case of *synchronous* functions. The following results were proved in [5].

**Theorem 25 ([5]).** *Let  $f : A^\omega \rightarrow B^\omega$  be a rational synchronous function. The continuity set  $C(f)$  of  $f$  is rational.*

**Theorem 26 ([5]).** *Let  $X$  be a rational  $\Pi_2^0$  subset of  $A^\omega$ . Then  $X$  is the continuity set  $C(f)$  of some rational synchronous function  $f$  of domain  $A^\omega$ .*

We are now going to prove another effective result about  $\omega$ -rational functions.

We first recall the following result of descriptive set theory, in the particular case of Cantor spaces  $\Sigma^\omega$  and  $\Gamma^\omega$ . A Borel function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is a function for which the inverse image of any Borel subset of  $\Gamma^\omega$ , or equivalently of any open set of  $\Gamma^\omega$ , is a Borel subset of  $\Sigma^\omega$ .

**Theorem 27 (see Theorem 8.38 of [15]).** *Let  $\Sigma$  and  $\Gamma$  be two finite alphabets and  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  be a Borel function. Then there is a dense  $\Pi_2^0$ -subset  $G$  of  $\Sigma^\omega$  such that the restriction of  $f$  to  $G$  is continuous.*

We now state an automatic version of this theorem.

**Theorem 28.** *Let  $\Sigma$  and  $\Gamma$  be two finite alphabets and  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  be an  $\omega$ -rational function. Then one can construct, from a 2-tape Büchi automaton accepting the graph of the function  $f$ , a deterministic Büchi automaton accepting a dense  $\Pi_2^0$ -subset  $G$  of  $\Sigma^\omega$  such that the restriction of  $f$  to  $G$  is continuous.*

**Proof.** Let  $\Sigma$  and  $\Gamma$  be two finite alphabets and  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  be an  $\omega$ -rational function whose graph is accepted by a 2-tape Büchi automaton  $\mathcal{A} = (K, \Sigma, \Gamma, \Delta, q_0, F)$ .

Notice that one can also consider the 2-tape automaton  $\mathcal{A}$  reading pairs of finite words  $(v, u) \in \Sigma^* \times \Gamma^*$ . A partial computation of the 2-tape automaton  $\mathcal{A}$  reading such a pair  $(v, u)$  is simply a finite sequence of consecutive transitions

$$(q_0, a_1, b_1, q_1), (q_1, a_2, b_2, q_2), \dots, (q_{i-1}, a_i, b_i, q_i), (q_i, a_{i+1}, b_{i+1}, q_{i+1})$$

such that  $v = a_1 a_2 \dots a_{i+1}$  and  $u = b_1 b_2 \dots b_{i+1}$ . This computation ends in state  $q_{i+1}$ .

We assume that we have an effective enumeration of the finite words over the alphabet  $\Gamma$  given by  $(u_n)_{n \geq 1}$ ,  $u_n \in \Gamma^*$ . For  $q \in K$  we also denote  $\mathcal{A}_q$  the

automaton  $\mathcal{A}$  in which we have changed the initial state so that the initial state of  $\mathcal{A}_q$  is  $q$  instead of  $q_0$ .

Let us now consider the basic open set of the space  $\Gamma^\omega$  given by  $U_n = u_n \cdot \Gamma^\omega$ . We first describe  $f^{-1}(U_n)$ . An  $\omega$ -word  $x \in \Sigma^\omega$  belongs to the set  $f^{-1}(U_n)$  iff  $x$  can be written in the form  $x = v \cdot y$  for some words  $v \in \Sigma^*$  and  $y \in \Sigma^\omega$ , and there is a partial computation of the automaton  $\mathcal{A}$  reading  $(v, u_n)$  for which  $\mathcal{A}$  is in state  $q$  after having read the initial pair  $(v, u_n) \in \Sigma^* \times \Gamma^*$  (where the finite words  $v$  and  $u_n$  might have different lengths if the automaton  $\mathcal{A}$  is not synchronous), and  $y \in \text{Dom}(R(\mathcal{A}_q))$ . Recall that  $R(\mathcal{A}_q) \subseteq (\Sigma \times \Gamma)^\omega$  is an infinitary rational relation and that  $\text{Dom}(R(\mathcal{A}_q))$  is then a regular  $\omega$ -language and that one can construct from  $\mathcal{A}$  a deterministic Muller automaton accepting this  $\omega$ -language  $\text{Dom}(R(\mathcal{A}_q))$  which will be denoted  $L_q$ . We also denote  $T(u_n, q)$  the set of finite words  $v$  over  $\Sigma$  such that the automaton  $\mathcal{A}$  may be in state  $q$  after having read the initial pair  $(v, u_n) \in \Sigma^* \times \Gamma^*$ . Then the following equality holds:

$$f^{-1}(U_n) = \bigcup_{q \in K} T(u_n, q) \cdot L_q$$

We can now apply the automatic Baire property stated in the above Theorem 11. Then for each regular  $\omega$ -language  $L_q$ , one can construct a deterministic Muller automaton accepting an open set  $O_q$  and a deterministic Muller automaton accepting a countable union  $W_q$  of closed sets with empty interiors, such that for each  $q \in K$ ,

$$L_q \Delta O_q \subseteq W_q$$

Now we set

$$V_n = \bigcup_{q \in K} T(u_n, q) \cdot O_q \quad \text{and} \quad F_n = \bigcup_{q \in K} T(u_n, q) \cdot W_q$$

Notice that each set  $T(u_n, q)$  is countable and that for each finite word  $u \in T(u_n, q)$  it is easy to see that the set  $u \cdot O_q$  is open and that the set  $u \cdot W_q$  is a countable union of closed sets with empty interiors. Thus it is easy to see that  $V_n$  is open, and that  $F_n$  is a countable union of closed sets with empty interiors. Moreover it is easy to see that  $V_n$  and  $F_n$  are regular  $\omega$ -languages since each set  $T(u_n, q)$  is a regular language of finite words over the alphabet  $\Sigma$ . Moreover it holds that:

$$f^{-1}(U_n) \Delta V_n \subseteq F_n$$

We now prove that  $F = \bigcup_{n \geq 1} F_n$  is itself a regular  $\omega$ -language. It holds that

$$F = \bigcup_{n \geq 1} F_n = \bigcup_{n \geq 1} \bigcup_{q \in K} T(u_n, q) \cdot W_q = \bigcup_{q \in K} \bigcup_{n \geq 1} T(u_n, q) \cdot W_q$$

Consider now the 2-tape automaton  $\mathcal{B}_q$  which is like the 2-tape automaton  $\mathcal{A}$  but reads only pairs of finite words in  $\Sigma^* \times \Gamma^*$  and has the state  $q$  as unique accepting state. Let then  $\mathcal{C}_q$  be a finite automaton which reads only finite words over the alphabet  $\Sigma$  and such that  $L(\mathcal{C}_q) = \text{Proj}_{\Sigma^*}(L(\mathcal{B}_q))$  is the projection of the

language  $L(\mathcal{B}_q)$  on  $\Sigma^*$ . We can construct, from the automaton  $\mathcal{A}$ , the automata  $\mathcal{B}_q$  and  $\mathcal{C}_q$  for each  $q \in K$ . Now it holds that:

$$F = \bigcup_{n \geq 1} F_n = \bigcup_{q \in K} \bigcup_{n \geq 1} T(u_n, q) \cdot W_q = \bigcup_{q \in K} L(\mathcal{C}_q) \cdot W_q$$

On the other hand, for each finite word  $u \in \Sigma^*$ , the set  $u \cdot W_q$  is a meager  $\Sigma_2^0$ -set, since  $W_q$  is a meager  $\Sigma_2^0$ -set. Thus the set

$$F = \bigcup_{q \in K} L(\mathcal{C}_q) \cdot W_q$$

is also a countable union of closed sets with empty interiors, since  $K$  is finite and each language  $L(\mathcal{C}_q)$  is countable. Moreover the  $\omega$ -language  $F$  is regular and we can construct, from the automata  $\mathcal{C}_q$  and from the deterministic Muller automata accepting the  $\omega$ -languages  $W_q$ , a deterministic Muller automaton accepting  $F$ .

We can now set  $G_n = \Sigma^\omega \setminus F_n$  and  $G = \bigcap_{n \geq 1} G_n = \Sigma^\omega \setminus \bigcup_{n \geq 1} F_n = \Sigma^\omega \setminus F$ . Then  $G$  is a countable intersection of dense open subsets of  $\Sigma^\omega$ , hence also a dense  $\Pi_2^0$ -subset  $G$  of  $\Sigma^\omega$ . Moreover we can construct a deterministic Muller automaton and even a deterministic Büchi automaton (since  $G$  is a  $\Pi_2^0$ -set, see [19, page 41]) accepting  $G$ . We can now see that the restriction  $f_G$  of the function  $f$  to  $G$  is continuous. This follows from the fact that the inverse image of every basic open set of  $\Gamma^\omega$  by the function  $f_G$  is an open subset of  $G$  because for each integer  $n \geq 1$ , it holds that  $f_G^{-1}(U_n) = f^{-1}(U_n) \cap G = V_n \cap G$ .  $\square$

**Remark 29.** *The above dense  $\Pi_2^0$ -subset  $G$  of  $\Sigma^\omega$  is comeager and thus Theorem 28 shows that one can construct a deterministic Büchi automaton accepting a “big”  $\omega$ -rational subset of  $\Sigma^\omega$  on which the function  $f$  is continuous.*

## 5 Deciding the first Baire class and more . . .

Notice that in this section we consider total functions. We first give a definition of Baire class  $n$  functions.

**Definition 30** (see [15]). *Let  $X$  and  $Y$  be metrizable spaces and  $F : X \rightarrow Y$  be a function. Then  $F$  is said to be of Baire class 0 if  $F$  is continuous. The function  $F$  is said to be of Baire class 1 if for every open subset  $V$  of  $Y$  the inverse image of  $V$  by  $F$  is a  $\Sigma_2^0$ -subset of  $X$ . Then for every integer  $n > 1$  the function  $F$  is said to be of Baire class  $n$  if it is the pointwise limit of a sequence  $(F_k)_{k \geq 0}$  of functions of Baire class  $n - 1$ .*

**Remark 31.** *This definition can be extended to a transfinite hierarchy of Baire class  $\xi$  functions, for countable ordinals  $\xi$ . However we just mention this and refer the interested reader to [15] for more on this subject since we shall not need this extension in the sequel.*

If we only consider functions  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  from a Cantor space to another Cantor space we have another characterization of Baire class  $n$  functions which is given by the following:

**Theorem 32.** *Let  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  be a function from a Cantor space to another Cantor space. Then, for every integer  $n \geq 0$ , the function  $F$  is of Baire class  $n$  if and only if for every open subset  $V \subseteq \Gamma^\omega$  the inverse image of  $V$  by  $F$  is in the Borel class  $\Sigma_{n+1}^0$ .*

On the other hand, a Cantor space  $\Gamma^\omega$  has a countable basis of clopen (i.e. closed and open) sets formed by the collection  $(u \cdot \Gamma^\omega)$  for  $u \in \Gamma^*$ . Therefore we easily get the following equivalent characterization.

**Corollary 33.** *Let  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  be a function from a Cantor space to another Cantor space. Then, for every integer  $n \geq 0$ , the function  $F$  is of Baire class  $n$  if and only if for every  $u \in \Gamma^*$  the inverse image of the clopen set  $(u \cdot \Gamma^\omega)$  by  $F$  is in the Borel class  $\Delta_{n+1}^0$ .*

Notice that we also have the following interesting link between Baire class 1 functions and the notion of continuity set of a function.

**Theorem 34 (Baire, see [15]).** *Let  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  be a function. If the function  $F$  is of Baire class 1 then the continuity set  $C(F)$  of  $F$  is a dense  $\Pi_2^0$ -set.*

If we now consider an  $\omega$ -rational function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , then every basic open set  $(u \cdot \Gamma^\omega)$  of  $\Gamma^\omega$  is  $\omega$ -regular and this implies that  $F^{-1}(u \cdot \Gamma^\omega)$  is also  $\omega$ -regular and hence  $\Delta_3^0$ . Then we have the following proposition.

**Proposition 35 ([3, 4]).** *Every  $\omega$ -rational function is of Baire class 2.*

It is then very natural to ask whether one can decide whether a given  $\omega$ -rational function is of Baire class 1. Cagnard and Simonnet proved in [3] that one can decide whether a given synchronous  $\omega$ -rational function is of Baire class 1. Their proof used the notion of undergraph and overgraph of a function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , when the set  $\Gamma^\omega$  is equipped with the lexicographic ordering.

**Theorem 36 (Cagnard-Simonnet [3]).** *One can decide whether a given synchronous  $\omega$ -rational function, whose graph is an  $\omega$ -regular language accepted by a Büchi or Muller automaton, is of Baire class 1.*

We first give here a more direct automata theoretic proof of this result, which do not use the notions of undergraph and overgraph of a function.

**Proof.** Let  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  be a synchronous  $\omega$ -rational function whose graph  $G$  is accepted by a Büchi or Muller automaton over the product alphabet  $\Sigma \times \Gamma$ . We know that we can construct a deterministic Muller automaton  $\mathcal{A} = (K, \Sigma \times \Gamma, \delta, q_0, \mathcal{F})$  accepting  $G$ , where  $\mathcal{F} \subseteq 2^K$  is here the collection of designated state sets.

As in the preceding section, for  $q \in K$  we also denote  $\mathcal{A}_q$  the automaton  $\mathcal{A}$  in which we have changed the initial state so that the initial state of  $\mathcal{A}_q$  is  $q$  instead of  $q_0$ .

Let then  $u \in \Gamma^*$ . We are going to describe the set  $F^{-1}(u \cdot \Gamma^\omega)$ . We denote

$$A_u = \{v \in \Sigma^* \mid |v| = |u| \text{ and } (v, u) \cdot (\Sigma^\omega \times \Gamma^\omega) \cap G \neq \emptyset\}.$$

Then for every  $v \in A_u$ , we denote  $q_{(v,u)}$  the state in which the automaton  $\mathcal{A}$  is after the reading of the pair  $(v, u)$ . The rational relation  $R(\mathcal{A}_{q_{(v,u)}}) \subseteq \Sigma^\omega \times \Gamma^\omega$  is functional since the relation  $R(\mathcal{A})$  is functional. Then  $\text{Dom}(R(\mathcal{A}_{q_{(v,u)}})) \subseteq \Sigma^\omega$  is a regular  $\omega$ -language and one can construct a deterministic Muller automaton accepting it. It is now easy to see that

$$F^{-1}(u \cdot \Gamma^\omega) = \bigcup_{v \in A_u} v \cdot \text{Dom}(R(\mathcal{A}_{q_{(v,u)}}))$$

We can now notice that  $F^{-1}(u \cdot \Gamma^\omega)$  is a  $\Delta_2^0$ -set iff for all  $v \in A_u$  it holds that  $\text{Dom}(R(\mathcal{A}_{q_{(v,u)}}))$  is a  $\Delta_2^0$ -set:

Firstly, if for every  $v \in A_u$  it holds that  $\text{Dom}(R(\mathcal{A}_{q_{(v,u)}}))$  is a  $\Delta_2^0$ -set then  $v \cdot \text{Dom}(R(\mathcal{A}_{q_{(v,u)}}))$  is also a  $\Delta_2^0$ -set and therefore by finite union  $F^{-1}(u \cdot \Gamma^\omega)$  is a  $\Delta_2^0$ -set. (Notice that this includes the particular case where  $A_u = \emptyset$  and then  $F^{-1}(u \cdot \Gamma^\omega) = \emptyset$  is a  $\Delta_2^0$ -set.)

Secondly, if  $F^{-1}(u \cdot \Gamma^\omega)$  is a  $\Delta_2^0$ -set then for every  $v \in A_u$  the set  $F^{-1}(u \cdot \Gamma^\omega) \cap v \cdot \Gamma^\omega = v \cdot \text{Dom}(R(\mathcal{A}_{q_{(v,u)}}))$  is the intersection of a  $\Delta_2^0$ -set and of an open set hence also a  $\Delta_2^0$ -set. Moreover “ $v \cdot \text{Dom}(R(\mathcal{A}_{q_{(v,u)}}))$  is a  $\Delta_2^0$ -set” easily implies that  $\text{Dom}(R(\mathcal{A}_{q_{(v,u)}}))$  is a  $\Delta_2^0$ -set.

To conclude it suffices now to remark that the function  $F$  is of Baire class 1 iff for all states  $q$  of  $\mathcal{A}$  which are accessible from  $q_0$  (when reading some pair  $(v, u)$ ) it holds that  $\text{Dom}(R(\mathcal{A}_q))$  is a  $\Delta_2^0$ -set. This is clearly decidable since one can decide whether a regular  $\omega$ -language is a  $\Delta_2^0$ -set.  $\square$

From the above proof we are now going to show that one can also easily obtain an extension of Theorem 36 based on the notion of the Wadge hierarchy which is a great refinement of the Borel hierarchy, firstly defined by Wadge via reductions by continuous functions [29].

We first now recall the notion of Wadge reduction and Wadge hierarchy on a Cantor space.

**Definition 37 (Wadge [29]).** *Let  $X, Y$  be two finite alphabets. For  $L \subseteq X^\omega$  and  $L' \subseteq Y^\omega$ ,  $L$  is said to be Wadge reducible to  $L'$  ( $L \leq_W L'$ ) iff there exists a continuous function  $f : X^\omega \rightarrow Y^\omega$ , such that  $L = f^{-1}(L')$ . The sets  $L$  and  $L'$  are Wadge equivalent iff  $L \leq_W L'$  and  $L' \leq_W L$ . This will be denoted by  $L \equiv_W L'$ . And we shall say that  $L <_W L'$  iff  $L \leq_W L'$  but not  $L' \leq_W L$ . A set  $L \subseteq X^\omega$  is said to be self dual iff  $L \equiv_W L^-$ , and otherwise it is said to be non self dual.*



The relation  $\leq_W$  is reflexive and transitive, and  $\equiv_W$  is an equivalence relation. The *equivalence classes* of  $\equiv_W$  are called *Wadge degrees*. The Wadge hierarchy  $WH$  is the class of Borel subsets of a set  $X^\omega$ , where  $X$  is a finite set, equipped with  $\leq_W$  and with  $\equiv_W$ .

For  $L \subseteq X^\omega$  and  $L' \subseteq Y^\omega$ , if  $L \leq_W L'$  and  $L = f^{-1}(L')$  where  $f$  is a continuous function from  $X^\omega$  into  $Y^\omega$ , then  $f$  is called a continuous reduction of  $L$  to  $L'$ . Intuitively it means that  $L$  is less complicated than  $L'$  because to check whether  $x \in L$  it suffices to check whether  $f(x) \in L'$  where  $f$  is a continuous function. Hence the Wadge degree of an  $\omega$ -language is a measure of its topological complexity. The Wadge class of a set  $L \subseteq X^\omega$  is equal to  $[L] = \{L' \subseteq Y^\omega \mid Y \text{ is a finite alphabet and } L' \leq_W L\}$ .

Recall that each Borel class  $\Sigma_n^0$  and  $\Pi_n^0$  is a *Wadge class* (and a similar result holds for Borel classes of transfinite ranks).

Notice that in the above definition, we consider that a subset  $L \subseteq X^\omega$  is given together with the alphabet  $X$ .

There is a close relationship between Wadge reducibility and the notion of a particular class of infinite 2-player games now called Wadge games.

**Definition 38.** *Let  $L \subseteq X^\omega$  and  $L' \subseteq Y^\omega$ . The Wadge game  $W(L, L')$  is a game with perfect information between two players, player 1 who is in charge of  $L$  and player 2 who is in charge of  $L'$ . Player 1 first writes a letter  $a_1 \in X$ , then player 2 writes a letter  $b_1 \in Y$ , then player 1 writes a letter  $a_2 \in X$ , and so on. The two players alternatively write letters  $a_n$  of  $X$  for player 1 and  $b_n$  of  $Y$  for player 2. After  $\omega$  steps, the player 1 has written an  $\omega$ -word  $a \in X^\omega$  and the player 2 has written an  $\omega$ -word  $b \in Y^\omega$ . The player 2 is allowed to skip, even infinitely often, provided he really writes an  $\omega$ -word in  $\omega$  steps. The player 2 wins the play iff  $[a \in L \leftrightarrow b \in L']$ , i.e. iff :*

$$[(a \in L \text{ and } b \in L') \text{ or } (a \notin L \text{ and } b \notin L' \text{ and } b \text{ is infinite})].$$

Recall that a strategy for player 1 is a function  $\sigma : (Y \cup \{s\})^* \rightarrow X$ . And a strategy for player 2 is a function  $f : X^+ \rightarrow Y \cup \{s\}$ . The strategy  $\sigma$  is a winning strategy for player 1 iff he always wins a play when he uses the strategy  $\sigma$ , i.e. when the  $n^{\text{th}}$  letter he writes is given by  $a_n = \sigma(b_1 \cdots b_{n-1})$ , where  $b_i$  is the letter written by player 2 at step  $i$  and  $b_i = s$  if player 2 skips at step  $i$ . A winning strategy for player 2 is defined in a similar manner.

Martin's Theorem states that every Gale-Stewart game  $G(X)$  (see [15]), with  $X$  a Borel set, is determined and this implies the following :

**Theorem 39 (Wadge).** *Let  $L \subseteq X^\omega$  and  $L' \subseteq Y^\omega$  be two Borel sets, where  $X$  and  $Y$  are finite alphabets. Then the Wadge game  $W(L, L')$  is determined: one of the two players has a winning strategy. And  $L \leq_W L'$  iff the player 2 has a winning strategy in the game  $W(L, L')$ .*

**Theorem 40 (Wadge).** *Up to the complement and  $\equiv_W$ , the class of Borel subsets of  $X^\omega$ , for a finite alphabet  $X$  having at least two letters, is a well ordered hierarchy. There is an ordinal  $|WH|$ , called the length of the hierarchy, and a*

map  $d_W^0$  from  $WH$  onto  $|WH| - \{0\}$ , such that for all  $L, L' \subseteq X^\omega$ :  
 $d_W^0 L < d_W^0 L' \leftrightarrow L <_W L'$  and  
 $d_W^0 L = d_W^0 L' \leftrightarrow [L \equiv_W L' \text{ or } L \equiv_W L'^-]$ .

The Wadge hierarchy of Borel sets of **finite rank** has length  ${}^1\varepsilon_0$  where  ${}^1\varepsilon_0$  is the limit of the ordinals  $\alpha_n$  defined by  $\alpha_1 = \omega_1$  and  $\alpha_{n+1} = \omega_1^{\alpha_n}$  for  $n$  a non negative integer,  $\omega_1$  being the first non countable ordinal. Then  ${}^1\varepsilon_0$  is the first fixed point of the ordinal exponentiation of base  $\omega_1$ .

The Wadge hierarchy of  $\mathbf{\Delta}_2^0$ -sets has length  $\omega_1$  and the length of the Wadge hierarchy of boolean combinations of  $\mathbf{\Pi}_2^0$ -sets is equal to the ordinal  $\omega_1^\omega$ .

The length of the whole Wadge hierarchy of Borel sets is a huge ordinal, with regard to the ordinal  ${}^1\varepsilon_0$ . We refer the interested reader to [29, 7] for a description of this ordinal, using the Veblen functions.

On the other hand, the trace of the Wadge hierarchy on the  $\omega$ -regular languages is called the Wagner hierarchy. It has been completely described by Wagner in [30]. Its length is the (countable) ordinal  $\omega^\omega$ . Wagner gave an automaton-like characterization of this hierarchy, based on the notions of chain and super-chain, together with an algorithm to compute the Wadge (Wagner) degree of any given  $\omega$ -regular language, see also [22, 24, 23, 25].

We now see that we can get a refinement of the Baire hierarchy of functions, using the notion of the Wadge hierarchy. Recall that a function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  is of Baire class  $n$  if and only if for every  $u \in \Gamma^*$  the inverse image of the clopen set  $(u \cdot \Gamma^\omega)$  by  $F$  is in the Borel class  $\mathbf{\Delta}_{n+1}^0$ . One can refine this notion by considering, for each Wadge class  $[L] \subseteq \mathbf{\Delta}_{n+1}^0$ , the class  $\mathcal{C}_{[L]}$  of functions  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  such that for every  $u \in \Gamma^*$  the inverse image of the clopen set  $(u \cdot \Gamma^\omega)$  by  $F$  is in the Wadge class  $[L]$ .

We can now state the following refinement of Theorem 36.

**Theorem 41.** *For a given regular  $\omega$ -language  $L$  accepted by a Büchi or Muller automaton and a given synchronous  $\omega$ -rational function  $F$ , whose graph is an  $\omega$ -regular language accepted by a Büchi or Muller automaton, one can decide whether  $F$  belongs to the class  $\mathcal{C}_{[L]}$ .*

**Proof.** Let, as in the above proof of Theorem 36,  $F : \Sigma^\omega \rightarrow \Gamma^\omega$  be a synchronous  $\omega$ -rational function whose graph  $G$  is accepted by a deterministic Muller automaton over the product alphabet  $\Sigma \times \Gamma$ . Recall that, for  $u \in \Gamma^*$ , we got the following description of the set  $F^{-1}(u \cdot \Gamma^\omega)$ .

$$F^{-1}(u \cdot \Gamma^\omega) = \bigcup_{v \in A_u} v \cdot \text{Dom}(R(\mathcal{A}_{q(v,u)}))$$

We can now determine the Wadge degree of  $F^{-1}(u \cdot \Gamma^\omega)$ . We first notice that we can determine the Wadge-Wagner degrees of every  $\omega$ -language  $\text{Dom}(R(\mathcal{A}_{q(v,u)}))$  since for every state  $q \in K$  we can construct a deterministic Muller automaton  $\mathcal{B}_q$  accepting the  $\omega$ -language  $\text{Dom}(R(\mathcal{A}_q))$ . We now consider the following cases.

**Case 1.**  $u = \varepsilon$ . In that case it holds that  $F^{-1}(u \cdot \Gamma^\omega) = F^{-1}(\Gamma^\omega) = \Sigma^\omega$ .

**Case 2.**  $u \neq \varepsilon$  and  $A_u = \emptyset$ . In that case  $F^{-1}(u \cdot \Gamma^\omega) = \emptyset$ .

**Case 3.**  $u \neq \varepsilon$  and  $F^{-1}(u \cdot \Gamma^\omega) = \Sigma^\omega$ .

**Case 4.**  $u \neq \varepsilon$  and there is some  $v_0 \in A_u$  such that for every  $v \in A_u$  it holds that  $Dom(R(\mathcal{A}_{q(v,u)})) \leq_W Dom(R(\mathcal{A}_{q(v_0,u)})) \neq \Sigma^\omega$ . (Notice that this implies in particular that  $F^{-1}(u \cdot \Gamma^\omega) \neq \Sigma^\omega$ .)

In that case we prove that  $F^{-1}(u \cdot \Gamma^\omega) \equiv_W \cdot Dom(R(\mathcal{A}_{q(v_0,u)}))$ .

Consider firstly the Wadge game  $W(F^{-1}(u \cdot \Gamma^\omega), Dom(R(\mathcal{A}_{q(v_0,u)})))$ . We claim that Player 2 has a winning strategy in this game. Player 2, following this strategy, begins by skipping his turn until Player 1 has written a word  $v$  of length  $|u|$ . Then if  $v \notin A_u$  then Player 1 is now like a player in charge of the emptyset therefore at the end of the play the infinite word written by Player 1 will be surely outside her (empty) set. Thus Player 2 can write in  $\omega$  steps an infinite word which is in  $\Sigma^\omega \setminus Dom(R(\mathcal{A}_{q(v_0,u)}))$  so that he wins the play. And if  $v \in A_u$  then Player 1 is now like a player in charge of the set  $Dom(R(\mathcal{A}_{q(v,u)}))$  and Player 2 can win the play since by hypothesis  $Dom(R(\mathcal{A}_{q(v,u)})) \leq_W Dom(R(\mathcal{A}_{q(v_0,u)}))$ . This shows that  $F^{-1}(u \cdot \Gamma^\omega) \leq_W Dom(R(\mathcal{A}_{q(v_0,u)}))$ .

Secondly consider the Wadge game  $W(Dom(R(\mathcal{A}_{q(v_0,u)})), F^{-1}(u \cdot \Gamma^\omega))$ . We claim that Player 2 has a winning strategy in this game. Indeed Player 2 can first write  $v_0$  in the first  $|u|$  steps of the play. Then Player 2 just follows what has written Player 1 so that at the end of the play Player 1 has written an  $\omega$ -word  $x$  and Player 2 has written an  $\omega$ -word  $v_0 \cdot x$  and Player 2 wins the play.

Finally  $F^{-1}(u \cdot \Gamma^\omega) \equiv_W Dom(R(\mathcal{A}_{q(v_0,u)}))$ .

**Case 5.**  $u \neq \varepsilon$  and  $F^{-1}(u \cdot \Gamma^\omega) \neq \Sigma^\omega$  and there is some  $v_0 \in A_u$  such that for every  $v \in A_u$  it holds that  $Dom(R(\mathcal{A}_{q(v,u)})) \leq_W Dom(R(\mathcal{A}_{q(v_0,u)})) = \Sigma^\omega$ .

Then for every  $v \in A_u$  it holds that  $Dom(R(\mathcal{A}_{q(v,u)})) = \Sigma^\omega$  since the set  $\Sigma^\omega$  is at the bottom of the Wadge hierarchy and there are no other sets below it. Thus  $F^{-1}(u \cdot \Gamma^\omega) = \bigcup_{v \in A_u} v \cdot \Sigma^\omega$  and  $A_u \subsetneq \{v \in \Sigma^* \mid |v| = |u|\}$  since  $F^{-1}(u \cdot \Gamma^\omega) \neq \Sigma^\omega$ . In that case  $F^{-1}(u \cdot \Gamma^\omega)$  is a clopen set (different from the emptyset and from the whole set  $\Sigma^\omega$  and just above them in the Wadge hierarchy on the space  $\Sigma^\omega$ ) of Wadge degree 2.

**Case 6.**  $u \neq \varepsilon$  and there exist  $v_0, v'_0 \in A_u$  such that for every  $v \in A_u$  it holds that  $Dom(R(\mathcal{A}_{q(v,u)})) \leq_W Dom(R(\mathcal{A}_{q(v_0,u)}))$  or  $Dom(R(\mathcal{A}_{q(v,u)})) \leq_W Dom(R(\mathcal{A}_{q(v'_0,u)}))$ . We also assume that  $Dom(R(\mathcal{A}_{q(v_0,u)}))$  is non self-dual and  $Dom(R(\mathcal{A}_{q(v_0,u)})) \equiv_W Dom(R(\mathcal{A}_{q(v'_0,u)}))^-$ . (Notice that this implies in particular that  $F^{-1}(u \cdot \Gamma^\omega) \neq \Sigma^\omega$  since  $Dom(R(\mathcal{A}_{q(v_0,u)})) \neq \Sigma^\omega$  and  $Dom(R(\mathcal{A}_{q(v'_0,u)})) \neq \Sigma^\omega$ .)

In that case

$$F^{-1}(u \cdot \Gamma^\omega) \equiv_W a \cdot Dom(R(\mathcal{A}_{q(v_0,u)})) \cup b \cdot Dom(R(\mathcal{A}_{q(v'_0,u)})),$$

where  $a$  and  $b$  are two distinct letters of  $\Sigma$ . Then  $F^{-1}(u \cdot \Gamma^\omega)$  belongs to the first self-dual degree just above the non self-dual degrees of  $Dom(R(\mathcal{A}_{q(v_0,u)}))$  and

$Dom(R(\mathcal{A}_{q(v'_0, u)}))$ . This can be proved using Wadge games in a very similar way as in the above case 4. Details are here left to the reader.

Notice that these cases 1–6 are exhaustive since the Wadge-Wagner hierarchy is, up to complement and  $\equiv_W$ , a well ordered hierarchy.

We are now going to determine the smaller Wadge class  $[T]$  such that  $F$  belongs to the class  $\mathcal{C}_{[T]}$ .

Case 1 shows that there is some  $u$  ( $u = \varepsilon$ ) such that  $F^{-1}(u \cdot \Gamma^\omega) = \Sigma^\omega$ . Moreover it is not possible that Case 2 occurs for all  $u \neq \varepsilon$  since there is at least one  $\omega$ -word in the range of  $F$ . It is also not possible that Case 3 occurs for all  $u \neq \varepsilon$  because if for some  $u \neq \varepsilon$  it holds that  $F^{-1}(u \cdot \Gamma^\omega) = \Sigma^\omega$  then for  $v \in \Gamma^*$ ,  $|v| = |u|$ , it will hold that  $F^{-1}(v \cdot \Gamma^\omega) = \emptyset$ .

If only Cases 1 – 3 occur it is easy to see that the function  $F$  is actually constant (and in that case its range contains only an ultimately periodic word since the range is a regular  $\omega$ -language). Indeed if the image of  $F$  contains (at least) two  $\omega$ -words  $x$  and  $y$  then there is some integer  $n \geq 1$  such that  $x[n] \neq y[n]$ . Then  $F^{-1}(x[n] \cdot \Gamma^\omega)$  and  $F^{-1}(y[n] \cdot \Gamma^\omega)$  are both different from the empty set and from the whole set  $\Sigma^\omega$ . Notice that we can construct a Büchi or Muller automaton accepting the range of  $F$  and decide whether it is a singleton. If this is the case then for  $u \in \Gamma^*$ , the set  $F^{-1}(u \cdot \Gamma^\omega)$  can be either empty or equal to  $\Sigma^\omega$ . And the least Wadge class above the two non self dual degrees of  $\emptyset$  and of  $\Sigma^\omega$  is the class of clopen sets which can be written  $[a \cdot \Sigma^\omega]$  for some letter  $a \in \Sigma$ .

If on the contrary the function  $F$  is not constant then cases 4 or 5 or 6 may appear. Then we begin by determining the Wadge-Wagner degrees of the regular  $\omega$ -languages  $Dom(R(\mathcal{A}_q))$ , for all states  $q$  of  $\mathcal{A}$  which are accessible from  $q_0$  (when reading some pair  $(v, u)$ ) and such that  $R(\mathcal{A}_q) \neq \emptyset$ .

If all these sets  $Dom(R(\mathcal{A}_q))$  are equal to  $\Sigma^\omega$  then only case 5 may appear. In that case we have seen that  $F^{-1}(u \cdot \Gamma^\omega)$  is a clopen set (different from the empty set and from the whole set  $\Sigma^\omega$ ). And since cases 4 and 6 cannot appear, then the function  $F$  belongs to the class  $\mathcal{C}_{[a \cdot \Sigma^\omega]}$  for any letter  $a \in \Sigma$ .

Assume now that there is a state  $q_1 \in K$  (accessible from  $q_0$ ) with  $R(\mathcal{A}_{q_1}) \neq \emptyset$  and such that for all other states  $q \in K$  which are accessible from  $q_0$  with  $R(\mathcal{A}_q) \neq \emptyset$  it holds that  $Dom(R(\mathcal{A}_q)) \leq_W Dom(R(\mathcal{A}_{q_1})) \neq \Sigma^\omega$ . Then we can deduce from the above cases 4 – 6 that there is some  $u \in \Gamma^*$  such that  $F^{-1}(u \cdot \Gamma^\omega) \equiv_W Dom(R(\mathcal{A}_{q_1}))$ . Moreover this implies that  $[Dom(R(\mathcal{A}_{q_1}))]$  is the least Wadge class  $[T]$  such that  $F$  belongs to the class  $\mathcal{C}_{[T]}$ .

Assume now that there are some states  $q_1 \in K$  and  $q_2 \in K$  (accessible from  $q_0$ ) with  $R(\mathcal{A}_{q_1}) \neq \emptyset$ ,  $R(\mathcal{A}_{q_2}) \neq \emptyset$  and such that for all other states  $q \in K$  which are accessible from  $q_0$  with  $R(\mathcal{A}_q) \neq \emptyset$  it holds that  $Dom(R(\mathcal{A}_q)) \leq_W Dom(R(\mathcal{A}_{q_1})) \neq \Sigma^\omega$  or  $Dom(R(\mathcal{A}_q)) \leq_W Dom(R(\mathcal{A}_{q_2})) \neq \Sigma^\omega$ . Moreover we assume that  $Dom(R(\mathcal{A}_{q_1}))$  and  $Dom(R(\mathcal{A}_{q_2}))$  are non self-dual and that  $Dom(R(\mathcal{A}_{q_1})) \equiv Dom(R(\mathcal{A}_{q_2}))^-$ . Then we can deduce from the above cases 4 and 6 that either there are some  $u_1, u_2 \in \Gamma^*$  such that  $F^{-1}(u_1 \cdot \Gamma^\omega) \equiv_W Dom(R(\mathcal{A}_{q_1}))$  and  $F^{-1}(u_2 \cdot \Gamma^\omega) \equiv_W Dom(R(\mathcal{A}_{q_2}))$  or there is some  $u \in \Sigma^\omega$  such that  $F^{-1}(u \cdot \Gamma^\omega) \equiv a \cdot Dom(R(\mathcal{A}_{q_1})) \cup b \cdot Dom(R(\mathcal{A}_{q_2}))$ , where  $a$  and  $b$  are two dis-

tinct letters of  $\Sigma$ . In both cases we see that  $[a \cdot \text{Dom}(R(\mathcal{A}_{q_1})) \cup b \cdot \text{Dom}(R(\mathcal{A}_{q_2}))]$  is the least Wadge class  $[T]$  such that  $F$  belongs to the class  $\mathcal{C}_{[T]}$ .

Finally we have determined the least Wadge class  $[T]$  such that  $F$  belongs to the class  $\mathcal{C}_{[T]}$ . Moreover we have found a regular  $\omega$ -language  $R$  such that  $[T] = [R]$ . On the other hand, we know that the Wadge-Wagner hierarchy is effective therefore we can decide, for a given regular  $\omega$ -language  $L$ , whether the function  $F$  belongs to the class  $\mathcal{C}_{[L]}$ .  $\square$

## 6 Concluding remarks

We have proved some new effective properties, related to the points of continuity, of  $\omega$ -rational functions. We hope these properties will be useful for further studies involving  $\omega$ -rational functions. For instance an  $\omega$ -automatic structure is defined via synchronous infinitary rational relations, see [2, 16]. On the other hand, any (synchronous) infinitary rational relation is uniformizable by a (synchronous)  $\omega$ -rational function, see [6]. Thus we can expect that our results will be useful in particular in the study of  $\omega$ -automatic structures.

We also hope that the automatic Baire property will be useful in other studies involving regular  $\omega$ -languages like the study of infinite games specified by automata.

We have got a new simple proof of the decidability of the Baire class 1 synchronous  $\omega$ -rational functions, and showed that one can decide, for a given synchronous  $\omega$ -rational function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , whether the inverse image by  $F$  of any basic open subset of  $\Gamma^\omega$  is in a given Wadge class of the Wadge-Wagner hierarchy of  $\omega$ -regular sets. This provides a refinement of Cagnard and Simonnet's result. The case of asynchronous  $\omega$ -rational functions is left for further study.

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