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The Automatic Baire Property and an Effective Property of ω -Rational Functions

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Abstract. We prove that ω -regular languages accepted by Büchi or Muller automata satisfy an effective automata-theoretic version of the Baire property. Then we use this result to obtain a new effective property of rational functions over infinite words which are realized by finite state Büchi transducers: for each such function $F : \Sigma^\omega \rightarrow \Gamma^\omega$, one can construct a deterministic Büchi automaton \mathcal{A} accepting a dense $\mathbf{\Pi}_2^0$ -subset of Σ^ω such that the restriction of F to $L(\mathcal{A})$ is continuous.

Keywords. Decision problems; regular languages of infinite words; infinitary rational relations; omega rational functions; topology; automatic Baire property; points of continuity.

1 Introduction

Infinitary rational relations were first studied by Gire and Nivat [Gir81,GN84]. The ω -rational functions over infinite words, whose graphs are (functional) infinitary rational relations accepted by 2-tape Büchi automata, have been studied by several authors [CG99,BCPS03,Sta97,Pri00].

In this paper we are mainly interested in the question of the continuity of such ω -rational functions. Recall that Prieur proved that one can decide whether a given ω -rational function is continuous [Pri01,Pri02]. On the other hand, Carton, Finkel and Simonnet proved that one cannot decide whether a given ω -rational function f has at least one point of continuity [CFS08]. Notice that this decision problem is actually Σ_1^1 -complete, hence highly undecidable [Fin12]. It was also proved in [CFS08] that one cannot decide whether the continuity set of a given ω -rational function f (its set of continuity points) is a regular (respectively, context-free) ω -language. Notice that the situation was shown to be quite different in the case of *synchronous* functions. It was proved in [CFS08] that if $f : A^\omega \rightarrow B^\omega$ is an ω -rational synchronous function, then the continuity set $C(f)$ of f is ω -rational. Moreover, if X is an ω -rational $\mathbf{\Pi}_2^0$ subset of A^ω , then X is the continuity set $C(f)$ of some rational synchronous function f of domain A^ω . Notice that these previous works on the continuity of ω -rational functions had shown that decision problems in this area may be decidable or not, while it is well known that most problems about regular languages accepted by finite automata are decidable.

We establish in this paper a new effective property of rational functions over infinite words. We first prove that ω -regular languages accepted by Büchi or Muller automata satisfy an effective automata-theoretic version of the Baire property. Then we use this result to obtain a new effective property of rational functions over infinite words which are realized by finite state Büchi transducers: for each such function $F : \Sigma^\omega \rightarrow \Gamma^\omega$, one can construct a deterministic Büchi automaton \mathcal{A} accepting a dense Π_2^0 -subset of Σ^ω such that the restriction of F to this dense set $L(\mathcal{A})$ is continuous.

The paper is organized as follows. We recall basic notions on automata and on the Borel hierarchy in Section 2. The automatic Baire property for regular ω -languages is proved in Section 3. We prove our main new result on ω -rational functions in Section 4. Some concluding remarks are given in Section 5.

2 Recall of basic notions

We assume the reader to be familiar with the theory of formal (ω)-languages [Tho90,Sta97]. We recall some usual notations of formal language theory.

When Σ is a finite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \dots a_k$, where $a_i \in \Sigma$ for $i = 1, \dots, k$, and k is an integer ≥ 1 . The length of x is $|x| = k$. The *set of finite words* (including the empty word ε whose length is zero) over Σ is denoted Σ^* .

The *first infinite ordinal* is ω . An ω -word over Σ is an ω -sequence $a_1 \dots a_n \dots$, where for all integers $i \geq 1$, $a_i \in \Sigma$. When σ is an ω -word over Σ , we write $\sigma = \sigma(1)\sigma(2) \dots \sigma(n) \dots$, where for all i , $\sigma(i) \in \Sigma$, and $\sigma[n] = \sigma(1)\sigma(2) \dots \sigma(n)$.

The usual concatenation product of two finite words u and v is denoted $u \cdot v$ and sometimes just uv . This product is extended to the product of a finite word u and an ω -word v . The infinite word $u \cdot v$ is then the ω -word such that: $(u \cdot v)(k) = u(k)$ if $k \leq |u|$, and $(u \cdot v)(k) = v(k - |u|)$ if $k > |u|$. The concatenation product can be extended in an obvious way to the concatenation of an infinite sequence of finite words. The concatenation of a set U of finite words with a set V of infinite words is the set of infinite words $U \cdot V = \{u \cdot v \mid u \in U \text{ and } v \in V\}$. If u is a finite word and V is a set of infinite words then $u \cdot V = \{u \cdot v \mid v \in V\}$.

The *set of ω -words* over the alphabet Σ is denoted by Σ^ω . An ω -language over an alphabet Σ is a subset of Σ^ω .

Definition 1. : A *finite state machine (FSM)* is a quadruple $\mathcal{M} = (K, \Sigma, \delta, q_0)$, where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state and δ is a mapping from $K \times \Sigma$ into 2^K . A FSM is called *deterministic* iff: $\delta : K \times \Sigma \rightarrow \{\{q\} \mid q \in K\}$. (As usual, by a clear identification, we might consider in that case that $\delta : K \times \Sigma \rightarrow K$).

A *Büchi automaton (BA)* is a 5-tuple $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$ where $\mathcal{M} = (K, \Sigma, \delta, q_0)$ is a finite state machine and $F \subseteq K$ is the set of final states.

A *Muller automaton (MA)* is a 5-tuple $\mathcal{A} = (K, \Sigma, \delta, q_0, \mathcal{F})$ where $\mathcal{M} = (K, \Sigma, \delta, q_0)$ is a FSM and $\mathcal{F} \subseteq 2^K$ is the collection of designated state sets.

A Büchi or Muller automaton is said to be *deterministic* if the associated FSM is deterministic.

Let $\sigma = a_1 a_2 \dots a_n \dots$ be an ω -word over Σ .

A sequence of states $r = q_1 q_2 \dots q_n \dots$ is called an (infinite) run of $\mathcal{M} = (K, \Sigma, \delta, q_0)$ on σ , starting in state p , iff: 1) $q_1 = p$ and 2) for each $i \geq 1$, $q_{i+1} \in \delta(q_i, a_i)$.

In case a run r of \mathcal{M} on σ starts in state q_0 , we call it simply “a run of \mathcal{M} on σ ”. For every (infinite) run $r = q_1 q_2 \dots q_n \dots$ of \mathcal{M} , $\text{In}(r)$ is the set of states in K entered by \mathcal{M} infinitely many times during run r : $\text{In}(r) = \{q \in K \mid \exists^\infty i \geq 1 \ q_i = q\}$ is infinite}.

For $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$ a BA, the ω -language accepted by \mathcal{A} is:

$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } \text{In}(r) \cap F \neq \emptyset\}$.

For $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$ a MA, the ω -language accepted by \mathcal{A} is:

$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } \text{In}(r) \in \mathcal{F}\}$.

By R. Mc Naughton’s Theorem, see [PP04], the expressive power of deterministic MA (DMA) is equal to the expressive power of non deterministic MA which is also equal to the expressive power of non deterministic BA.

Theorem 2. For any ω -language $L \subseteq \Sigma^\omega$, the following conditions are equivalent:

1. There exists a DMA that accepts L .
2. There exists a MA that accepts L .
3. There exists a BA that accepts L .

An ω -language L satisfying one of these conditions is called a regular ω -language.

Recall that, from a Büchi (respectively, Muller) automaton \mathcal{A} , one can effectively construct a deterministic Muller (respectively, non-deterministic Büchi) automaton \mathcal{B} such that $L(\mathcal{A}) = L(\mathcal{B})$.

A way to study the complexity of ω -languages accepted by various automata is to study their topological complexity.

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80,LT94,Kec95,Sta97,PP04]. If X is a finite alphabet containing at least two letters, then the set X^ω of infinite words over X may be equipped with the product topology of the discrete topology on X . This topology is induced by a natural metric which is called the *prefix metric* and is defined as follows. For $u, v \in X^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\text{pref}(u,v)}}$ where $l_{\text{pref}(u,v)}$ is the first integer n such that the $u(n+1)$ is different from $v(n+1)$. The topological space X^ω is a Cantor space. The open sets of X^ω are the sets of the form $W \cdot X^\omega$, where $W \subseteq X^*$. A set $L \subseteq X^\omega$ is a closed set iff its complement $X^\omega - L$ is an open set. Closed sets are characterized by the following:

Proposition 3. A set $L \subseteq X^\omega$ is a closed set of X^ω iff for every $\sigma \in X^\omega$, $[\forall n \geq 1, \exists u \in X^\omega \text{ such that } \sigma[n] \cdot u \in L]$ implies that $\sigma \in L$.

Define now the next classes of the Borel hierarchy:

Definition 4. The classes Σ_n^0 and Π_n^0 of the Borel Hierarchy on the topological space X^ω are defined as follows: Σ_1^0 is the class of open sets of X^ω , Π_1^0 is the class of closed sets of X^ω . And for any integer $n \geq 1$: Σ_{n+1}^0 is the class of countable unions of Π_n^0 -subsets of X^ω , and Π_{n+1}^0 is the class of countable intersections of Σ_n^0 -subsets of X^ω .

Remark 5. The hierarchy defined above is the hierarchy of Borel sets of finite rank. The Borel Hierarchy is also defined for transfinite levels (see [Mos80, Kec95]) but we shall not need this in the sequel. Recall that the class of Borel subsets of a Cantor space is the closure of the class of open sets under countable unions and countable intersections.

It turns out that there is a characterization of Π_2^0 -subsets of X^ω , involving the notion of W^δ which we now recall.

Definition 6. For $W \subseteq X^*$, we set: $W^\delta = \{\sigma \in X^\omega \mid \exists^\infty i \text{ such that } \sigma[i] \in W\}$. ($\sigma \in W^\delta$ iff σ has infinitely many prefixes in W .)

Then we can state the following proposition.

Proposition 7. A subset L of X^ω is a Π_2^0 -subset of X^ω iff there exists a set $W \subseteq X^*$ such that $L = W^\delta$.

It is easy to see, using the above characterization of Π_2^0 -sets, that every ω -language accepted by a deterministic Büchi automaton is a Π_2^0 -set. Thus every regular ω -language is a finite boolean combination of Π_2^0 -sets, because it is accepted by a deterministic Muller automaton and this implies that it is a finite boolean combination of ω -languages accepted by deterministic Büchi automata.

Landweber studied the topological properties of regular ω -languages in [Lan69]. He characterized the regular ω -languages in each of the Borel classes $\Sigma_1^0, \Pi_1^0, \Sigma_2^0, \Pi_2^0$, and showed that one can decide, for an effectively given regular ω -language L , whether L is in $\Sigma_1^0, \Pi_1^0, \Sigma_2^0$, or Π_2^0 . In particular, it turned out that a regular ω -language is in the class Π_2^0 iff it is accepted by a deterministic Büchi automaton.

Recall that, from a Büchi or Muller automaton \mathcal{A} , one can construct some Büchi or Muller automata \mathcal{B} and \mathcal{C} , such that $L(\mathcal{B})$ is equal to the topological closure of $L(\mathcal{A})$, and $L(\mathcal{C})$ is equal to the topological interior of $L(\mathcal{A})$, see [Sta97, PP04].

3 The automatic Baire property

In this section we are going to prove an automatic version of the result stating that every Borel (and even every analytic) set has the Baire property.

We firstly recall some basic definitions about meager sets, see [Kec95]. In a topological space \mathcal{X} , a set $A \subseteq \mathcal{X}$ is said to be *nowhere dense* if its closure \bar{A} has empty interior, i.e. $\text{Int}(\bar{A}) = \emptyset$. A set $A \subseteq \mathcal{X}$ is said to be *meager* if it is the union of countably many nowhere dense sets, or equivalently if it is included in a

countable union of closed sets with empty interiors. This means that A is meager if there exist countably many closed sets A_n , $n \geq 1$, such that $A \subseteq \bigcup_{n \geq 1} A_n$ where for every integer $n \geq 1$, $\text{Int}(A_n) = \emptyset$. A set is *comeager* if its complement is meager, i.e. if it contains the intersection of countably many dense open sets. Notice that the notion of a *meager* set is a notion of a *small* set, while the notion of a *comeager* set is a notion of a *big* set.

Recall that a Baire space is a topological space \mathcal{X} in which every intersection of countably many dense open sets is dense, or equivalently in which every countable union of closed sets with empty interiors has also an empty interior. It is well known that every Cantor space Σ^ω is a Baire space. In the sequel we will consider only Cantor spaces.

We now recall the notion of Baire property. For any sets $A, B \subseteq \Sigma^\omega$, we denote by $A \Delta B$ the symmetric difference of A and B , and we write $A =^* B$ if and only if $A \Delta B$ is meager.

Definition 8. *A set $A \subseteq \Sigma^\omega$ has the Baire property (BP) if there exists an open set $U \subseteq \Sigma^\omega$ such that $A =^* U$.*

An important result of descriptive set theory is the following result, see [Kec95, page 47].

Theorem 9. *Every Borel set of a Cantor space has the Baire property.*

We are going to prove an automatic version of the above theorem. We first give the following definition.

Definition 10. *Let $L = L(\mathcal{A}) \subseteq \Sigma^\omega$ be a regular ω -language accepted by a Büchi or Muller automaton \mathcal{A} . The ω -language L is said to have the automatic Baire property if one can construct from \mathcal{A} some Büchi automata \mathcal{B} and \mathcal{C} such that $L(\mathcal{B}) \subseteq \Sigma^\omega$ is open, $L(\mathcal{C}) \subseteq \Sigma^\omega$ is a countable union of closed sets with empty interior, i.e. a meager Σ_2^0 -set, and $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$.*

We already know that the regular ω -languages have the Baire property since they are Borel. We now state the following theorem which gives an automatic version of this result.

Theorem 11. *Let $L = L(\mathcal{A}) \subseteq \Sigma^\omega$ be a regular ω -language accepted by a Büchi or Muller automaton \mathcal{A} . Then one can construct Büchi automata \mathcal{B} and \mathcal{C} such that $L(\mathcal{B}) \subseteq \Sigma^\omega$ is open, $L(\mathcal{C}) \subseteq \Sigma^\omega$ is a meager Σ_2^0 -set, and $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$, i.e. the ω -language $L(\mathcal{A})$ has the automatic Baire property.*

In order to prove this result, we first prove the following lemmas.

Lemma 12. *Every regular ω -language which is open or closed has the automatic Baire property.*

Proof. Let $L = L(\mathcal{A}) \subseteq \Sigma^\omega$ be a regular ω -language accepted by a Büchi or Muller automaton \mathcal{A} .

If $L = L(\mathcal{A})$ is an open set then we immediately see that we get the result with $\mathcal{B} = \mathcal{A}$ and \mathcal{C} is any Büchi automaton accepting the empty set.

If $L = L(\mathcal{A})$ is a closed set then $L \setminus \text{Int}(L)$ is a closed set with empty interior. Moreover it is known that one can construct from the Büchi automaton \mathcal{A} another Büchi automaton \mathcal{B} accepting $\text{Int}(L)$, and then also a Büchi automaton \mathcal{C} accepting $L \setminus \text{Int}(L)$. Then we have $L(\mathcal{A})\Delta L(\mathcal{B}) = L \setminus \text{Int}(L) = L(\mathcal{C})$, with $L(\mathcal{B})$ open and $L(\mathcal{C})$ is a closed set with empty interior. \square

Lemma 13. *Every regular ω -language which is a Σ_2^0 -set has the automatic Baire property.*

Proof. Let L be a regular ω -language which is a Σ_2^0 -set. Recall that then the ω -language L is accepted by a deterministic finite automaton $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$ with co-Büchi acceptance condition. Moreover we may assume that the automaton \mathcal{A} is complete, i.e. for any state q and any letter $a \in \Sigma$ there is a transition from the state q and reading the current input letter a . An ω -word x is accepted by \mathcal{A} with co-Büchi acceptance condition if the run of the automaton \mathcal{A} on x (which is then unique since the automaton is deterministic) goes only finitely many times through the set of states F . Let now $n \geq 1$ and L_n be the ω -language of ω -words $x \in \Sigma^\omega$ such that the run of the automaton \mathcal{A} over x goes at most n times through the set of states F . We have clearly that $L = \bigcup_{n \geq 1} L_n$. Moreover it is easy to see that for every $n \geq 1$ the set L_n is closed. And the interior of L_n is the union of basic open sets $u \cdot \Sigma^\omega$, for $u \in \Sigma^*$, such that the automaton \mathcal{A} enters states from F at most n times during the reading of u and ends the reading of u in a state q such that no state of F is accessible from this state q . We have of course that $L_n \setminus \text{Int}(L_n)$ is a closed set with empty interior and thus $L_n = {}^* \text{Int}(L_n)$. Then $\bigcup_{n \geq 1} L_n = {}^* \bigcup_{n \geq 1} \text{Int}(L_n)$ and we can easily see that $\bigcup_{n \geq 1} \text{Int}(L_n)$ is accepted by a Büchi automaton \mathcal{B} which is essentially the automaton \mathcal{A} with a modified Büchi acceptance condition expressing “some state q has been reached from which no state of F is accessible”. Moreover $L(\mathcal{A})\Delta L(\mathcal{B}) = (\bigcup_{n \geq 1} L_n)\Delta(\bigcup_{n \geq 1} \text{Int}(L_n)) \subseteq \bigcup_{n \geq 1} (L_n \setminus \text{Int}(L_n))$ and the set $\bigcup_{n \geq 1} (L_n \setminus \text{Int}(L_n))$ is a countable union of closed sets with empty interiors which is easily seen to be accepted by a co-Büchi automaton \mathcal{C} which is essentially the automaton \mathcal{A} where we have deleted every state q from which no state of F is accessible. \square

Lemma 14. *Let $L \subseteq \Sigma^\omega$ be a regular ω -language which has the automatic Baire property. Then its complement $\Sigma^\omega \setminus L$ has also the automatic Baire property.*

Proof. Assume that the regular ω -language $L = L(\mathcal{A})$ has the automatic Baire property and that $L(\mathcal{A})\Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$, where \mathcal{A} , \mathcal{B} , and \mathcal{C} are Büchi automata, $L(\mathcal{B})$ is open and $L(\mathcal{C})$ is a meager Σ_2^0 -set. Notice that $(\Sigma^\omega \setminus L(\mathcal{A}))\Delta(\Sigma^\omega \setminus L(\mathcal{B})) = L(\mathcal{A})\Delta L(\mathcal{B})$. Moreover we can construct a Büchi automaton \mathcal{D} accepting the closed set $(\Sigma^\omega \setminus L(\mathcal{B}))$ and next also a Büchi automaton \mathcal{D}' accepting the open set $\text{Int}((\Sigma^\omega \setminus L(\mathcal{B})))$, and a Büchi automaton \mathcal{E} accepting the closed set with empty interior $(L(\mathcal{D}) \setminus \text{Int}(L(\mathcal{D})))$. It is now easy to see that $(\Sigma^\omega \setminus L(\mathcal{A}))\Delta L(\mathcal{D}') \subseteq (\Sigma^\omega \setminus L(\mathcal{A}))\Delta(\Sigma^\omega \setminus L(\mathcal{B})) \cup L(\mathcal{E}) \subseteq L(\mathcal{C}) \cup L(\mathcal{E})$ and we can construct a Büchi

automaton \mathcal{C}' accepting the ω -language $L(\mathcal{C}) \cup L(\mathcal{E})$ which is a meager Σ_2^0 -set so that $(\Sigma^\omega \setminus L(\mathcal{A})) \Delta L(\mathcal{D}') \subseteq L(\mathcal{C}')$. Thus the regular ω -language $(\Sigma^\omega \setminus L(\mathcal{A}))$ has also the automatic Baire property.

Notice that this implies that the regular ω -languages in the Borel class Π_2^0 have also the automatic Baire property since we have already solved the case of the class Σ_2^0 in the proof of Lemma 13. \square

Lemma 15. *The class of regular ω -languages having the automatic Baire property is closed under finite union and under finite intersection.*

Proof. We first consider the finite union operation. Let $L(\mathcal{A})$ and $L(\mathcal{B})$ be regular ω -languages accepted by Büchi automata \mathcal{A} and \mathcal{B} . Assume that we have $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$, and $L(\mathcal{A}') \Delta L(\mathcal{B}') \subseteq L(\mathcal{C}')$ where $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$, and $\mathcal{C}, \mathcal{C}'$, are Büchi automata, $L(\mathcal{B})$ and $L(\mathcal{B}')$ are open and $L(\mathcal{C})$ and $L(\mathcal{C}')$ are countable unions of closed sets with empty interiors. Now we can see that $(L(\mathcal{A}) \cup L(\mathcal{A}')) \Delta (L(\mathcal{B}) \cup L(\mathcal{B}')) \subseteq (L(\mathcal{A}) \Delta L(\mathcal{B})) \cup (L(\mathcal{A}') \Delta L(\mathcal{B}')) \subseteq L(\mathcal{C}) \cup L(\mathcal{C}')$. Moreover we can construct Büchi automata \mathcal{B}'' and \mathcal{C}'' such that $L(\mathcal{B}'')$ is the open set $L(\mathcal{B}) \cup L(\mathcal{B}')$ and $L(\mathcal{C}'') = L(\mathcal{C}) \cup L(\mathcal{C}')$ is a meager Σ_2^0 -set and then we have $(L(\mathcal{A}) \cup L(\mathcal{A}')) \Delta L(\mathcal{B}'') \subseteq L(\mathcal{C}'')$. This implies that the union $(L(\mathcal{A}) \cup L(\mathcal{A}'))$ has the automatic Baire property.

The case of the intersection of two regular ω -languages now follows from the case of the union and the case of the complement proved in Lemma 14. \square

End of Proof of Theorem 11. We now return to the general case of a regular ω -language $L \subseteq \Sigma^\omega$, accepted by a Büchi or Muller automaton. We know that we can construct a deterministic Muller automaton $\mathcal{A} = (K, \Sigma, \delta, q_0, \mathcal{F})$ accepting L . Recall that $\mathcal{F} \subseteq 2^K$ is here the collection of designated state sets. For each state $q \in K$, we now denote by $\mathcal{A}^{(q)}$ the automaton \mathcal{A} but viewed as a (deterministic) Büchi automaton with the single accepting state q , i.e. $\mathcal{A}^{(q)} = (K, \Sigma, \delta, q_0, \{q\})$. We know that the languages $L(\mathcal{A}^{(q)})$ are Borel Π_2^0 -sets and thus satisfy the automatic Baire property by Lemmas 13 and 14. Moreover we have the following equality:

$$L(\mathcal{A}) = \bigcup_{F \in \mathcal{F}} \left[\bigcap_{q \in F} L(\mathcal{A}^{(q)}) \setminus \bigcup_{q \notin F} L(\mathcal{A}^{(q)}) \right]$$

This implies, from the previous lemmas about the preservation of the automatic Baire property by boolean operations, that we can construct Büchi automata \mathcal{B} and \mathcal{C} , such that $L(\mathcal{B})$ is open and $L(\mathcal{C})$ is a meager Σ_2^0 -set, which satisfy $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$. Thus the ω -language L has the automatic Baire property. \square

Corollary 16. *On can decide, for a given Büchi or Muller automaton \mathcal{A} , whether $L(\mathcal{A})$ is meager.*

Proof. Let \mathcal{A} be a Büchi or Muller automaton. The ω -language $L(\mathcal{A})$ has the automatic Baire property and we can construct Büchi automata \mathcal{B} and \mathcal{C} , such that $L(\mathcal{B})$ is open and $L(\mathcal{C})$ is a countable union of closed sets with empty

interiors, which satisfy $L(\mathcal{A})\Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$. It is easy to see that $L(\mathcal{A})$ is meager if and only if $L(\mathcal{B})$ is empty, since any non-empty open set is non-meager, and this can be decided from the automaton \mathcal{B} . \square

Remark 17. *The above Corollary followed already from Staiger's paper [Sta98], see also [MMS18]. So we get here another proof of this result, based on the automatic Baire property.*

4 An application to ω -rational functions

4.1 Infinitary rational relations

We now recall the definition of infinitary rational relations, via definition by Büchi transducers:

Definition 18. *A 2-tape Büchi automaton is a sextuple $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$, where K is a finite set of states, Σ and Γ are finite sets called the input and the output alphabets, Δ is a finite subset of $K \times (\Sigma \cup \varepsilon) \times (\Gamma \cup \varepsilon) \times K$ called the set of transitions, q_0 is the initial state, and $F \subseteq K$ is the set of accepting states. A computation \mathcal{C} of the automaton \mathcal{T} is an infinite sequence of consecutive transitions*

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \dots, (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \dots$$

The computation is said to be successful iff there exists a final state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$. The input word and output word of the computation are respectively $u = u_1.u_2.u_3\dots$ and $v = v_1.v_2.v_3\dots$. The input and the output words may be finite or infinite. The infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ accepted by the 2-tape Büchi automaton \mathcal{T} is the set of pairs $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ such that u and v are the input and the output words of some successful computation \mathcal{C} of \mathcal{T} .

The 2-tape Büchi automaton $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$ is said to be synchronous if the set of transitions Δ is a finite subset of $K \times \Sigma \times \Gamma \times K$, i.e. if each transition is labelled with a pair $(a, b) \in \Sigma \times \Gamma$. An infinitary rational relation recognized by a synchronous 2-tape Büchi automaton is in fact, via the natural identification of $\Sigma^\omega \times \Gamma^\omega$ with $(\Sigma \times \Gamma)^\omega$, an ω -language over the product alphabet $\Sigma \times \Gamma$ which is accepted by a Büchi automaton. It is called a synchronous infinitary rational relation. An infinitary rational relation is said to be asynchronous if it can not be recognized by any synchronous 2-tape Büchi automaton.

Remark 19. *In the above definition, we could have defined the set of transitions Δ as a subset of $K \times \Sigma^* \times \Gamma^* \times K$. We have chosen to define Δ as a finite subset of $K \times (\Sigma \cup \varepsilon) \times (\Gamma \cup \varepsilon) \times K$ to simplify the proofs. However this is done without loss of generality because it is easy to see that this convention does not change the class of infinitary rational relations.*

If $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ is an infinitary rational relation recognized by the 2-tape Büchi automaton \mathcal{T} then we denote

$$Dom(R(\mathcal{T})) = \{u \in \Sigma^\omega \mid \exists v \in \Gamma^\omega \ (u, v) \in R(\mathcal{T})\}$$

and

$$Im(R(\mathcal{T})) = \{v \in \Gamma^\omega \mid \exists u \in \Sigma^\omega \ (u, v) \in R(\mathcal{T})\}.$$

It is well known that, for each infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$, the sets $Dom(R(\mathcal{T}))$ and $Im(R(\mathcal{T}))$ are regular ω -languages and that one can construct, from the Büchi transducer \mathcal{T} , some Büchi automata \mathcal{A} and \mathcal{B} accepting the ω -languages $Dom(R(\mathcal{T}))$ and $Im(R(\mathcal{T}))$.

Recall now the following undecidability result of Frougny and Sakarovitch.

Theorem 20 ([FS93]). *One cannot decide whether a given infinitary rational relation is synchronous.*

We proved in [Fin09] that many decision problems about infinitary rational relations are highly undecidable. In fact many of them, like the universality problem, the equivalence problem, the inclusion problem, the cofiniteness problem, the unambiguity problem, are Π_2^1 -complete, hence located at the second level of the analytical hierarchy.

4.2 Continuity of ω -rational functions

Recall that an infinitary rational relation $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ is said to be functional iff it is the graph of a function, i.e. iff

$$[\forall x \in Dom(R(\mathcal{T})) \ \exists! y \in Im(R(\mathcal{T})) \ (x, y) \in R(\mathcal{T})].$$

Then the functional relation $R(\mathcal{T})$ defines an ω -rational (partial) function $F_{\mathcal{T}} : Dom(R(\mathcal{T})) \rightarrow \Gamma^\omega$ by: for each $u \in Dom(R(\mathcal{T}))$, $F_{\mathcal{T}}(u)$ is the unique $v \in \Gamma^\omega$ such that $(u, v) \in R(\mathcal{T})$.

An ω -rational (partial) function $f : \Sigma^\omega \rightarrow \Gamma^\omega$ is said to be synchronous if there is a synchronous 2-tape Büchi automaton \mathcal{T} such that $f = F_{\mathcal{T}}$.

An ω -rational (partial) function $f : \Sigma^\omega \rightarrow \Gamma^\omega$ is said to be asynchronous if there is no synchronous 2-tape Büchi automaton \mathcal{T} such that $f = F_{\mathcal{T}}$.

Recall the following previous decidability result.

Theorem 21 (Gire [Gir86]). *One can decide whether an infinitary rational relation recognized by a given 2-tape Büchi automaton \mathcal{T} is a functional infinitary rational relation.*

It is very natural to consider the notion of continuity for ω -rational functions defined by 2-tape Büchi automata.

We recall that a function $f : \text{Dom}(f) \subseteq \Sigma^\omega \rightarrow \Gamma^\omega$, whose domain is $\text{Dom}(f)$, is said to be continuous at point $x \in \text{Dom}(f)$ if :

$$\forall n \geq 1 \quad \exists k \geq 1 \quad \forall y \in \text{Dom}(f) \quad [\delta(x, y) < 2^{-k} \Rightarrow \delta(f(x), f(y)) < 2^{-n}]$$

The continuity set $C(f)$ of the function f is the set of points of continuity of f . Notice that the continuity set $C(f)$ of a function $f : \Sigma^\omega \rightarrow \Gamma^\omega$ is always a Borel Π_2^0 -subset of Σ^ω , see [CFS08].

The function f is said to be continuous if it is continuous at every point $x \in \text{Dom}(f)$, i. e. if $C(f) = \text{Dom}(f)$.

Priour proved the following decidability result.

Theorem 22 (Priour [Pri01,Pri02]). *One can decide whether a given ω -rational function is continuous.*

On the other hand the following undecidability result was proved in [CFS08].

Theorem 23 (see [CFS08]). *One cannot decide whether a given ω -rational function f has at least one point of continuity.*

The exact complexity of this undecidable problem was given in [Fin12]. It is Σ_1^1 -complete to determine whether a given ω -rational function f has at least one point of continuity.

We now consider the continuity set of an ω -rational function and its possible complexity. The following undecidability result was proved in [CFS08].

Theorem 24 (see [CFS08]). *One cannot decide whether the continuity set of a given ω -rational function f is a regular (respectively, context-free) ω -language.*

The exact complexity of the first above undecidable problem, and an approximation of the complexity of the second one, were given in [Fin12]. It is Π_1^1 -complete to determine whether the continuity set $C(f)$ of a given ω -rational function f is a regular ω -language. Moreover the problem to determine whether the continuity set $C(f)$ of a given ω -rational function f is a context-free ω -language is Π_1^1 -hard and in the class $\Pi_2^1 \setminus \Sigma_1^1$.

The situation is quite different in the case of *synchronous* functions. The following results were proved in [CFS08].

Theorem 25 ([CFS08]). *Let $f : A^\omega \rightarrow B^\omega$ be a rational synchronous function. The continuity set $C(f)$ of f is rational.*

Theorem 26 ([CFS08]). *Let X be a rational Π_2^0 subset of A^ω . Then X is the continuity set $C(f)$ of some rational synchronous function f of domain A^ω .*

We are now going to prove another effective result about ω -rational functions.

We first recall the following result of descriptive set theory, in the particular case of Cantor spaces Σ^ω and Γ^ω . A Borel function $f : \Sigma^\omega \rightarrow \Gamma^\omega$ is a function for which the inverse image of any Borel subset of Γ^ω , or equivalently of any open set of Γ^ω , is a Borel subset of Σ^ω .

Theorem 27 (see **Theorem 8.38 of [Kec95]**). *Let Σ and Γ be two finite alphabets and $f : \Sigma^\omega \rightarrow \Gamma^\omega$ be a Borel function. Then there is a dense $\mathbf{\Pi}_2^0$ -subset G of Σ^ω such that the restriction of f to G is continuous.*

We now state an automatic version of this theorem.

Theorem 28. *Let Σ and Γ be two finite alphabets and $f : \Sigma^\omega \rightarrow \Gamma^\omega$ be an ω -rational function. Then one can construct, from a 2-tape Büchi automaton accepting the graph of the function f , a deterministic Büchi automaton accepting a dense $\mathbf{\Pi}_2^0$ -subset G of Σ^ω such that the restriction of f to G is continuous.*

Proof. Let Σ and Γ be two finite alphabets and $f : \Sigma^\omega \rightarrow \Gamma^\omega$ be an ω -rational function whose graph is accepted by a 2-tape Büchi automaton $\mathcal{A} = (K, \Sigma, \Gamma, \Delta, q_0, F)$.

Notice that one can also consider the 2-tape automaton \mathcal{A} reading pairs of finite words $(v, u) \in \Sigma^* \times \Gamma^*$. A partial computation of the 2-tape automaton \mathcal{A} reading such a pair (v, u) is simply a finite sequence of consecutive transitions

$$(q_0, a_1, b_1, q_1), (q_1, a_2, b_2, q_2), \dots, (q_{i-1}, a_i, b_i, q_i), (q_i, a_{i+1}, b_{i+1}, q_{i+1})$$

such that $v = a_1 a_2 \dots a_{i+1}$ and $u = b_1 b_2 \dots b_{i+1}$. This computation ends in state q_{i+1} .

We assume that we have an effective enumeration of the finite words over the alphabet Γ given by $(u_n)_{n \geq 1}$, $u_n \in \Gamma^*$. For $q \in K$ we also denote \mathcal{A}_q the automaton \mathcal{A} in which we have changed the initial state so that the initial state of \mathcal{A}_q is q instead of q_0 .

Let us now consider the basic open set of the space Γ^ω given by $U_n = u_n \cdot \Gamma^\omega$. We first describe $f^{-1}(U_n)$. An ω -word $x \in \Sigma^\omega$ belongs to the set $f^{-1}(U_n)$ iff x can be written in the form $x = v \cdot y$ for some words $v \in \Sigma^*$ and $y \in \Sigma^\omega$, and there is a partial computation of the automaton \mathcal{A} reading (v, u_n) for which \mathcal{A} is in state q after having read the initial pair $(v, u_n) \in \Sigma^* \times \Gamma^*$ (where the finite words v and u_n might have different lengths if the automaton \mathcal{A} is not synchronous), and $y \in \text{Dom}(R(\mathcal{A}_q))$. Recall that $R(\mathcal{A}_q) \subseteq (\Sigma \times \Gamma)^\omega$ is an infinitary rational relation and that $\text{Dom}(R(\mathcal{A}_q))$ is then a regular ω -language and that one can construct from \mathcal{A} a deterministic Muller automaton accepting this ω -language $\text{Dom}(R(\mathcal{A}_q))$ which will be denoted L_q . We also denote $T(u_n, q)$ the set of finite words v over Σ such that the automaton \mathcal{A} may be in state q after having read the initial pair $(v, u_n) \in \Sigma^* \times \Gamma^*$. Then the following equality holds:

$$f^{-1}(U_n) = \bigcup_{q \in K} T(u_n, q) \cdot L_q$$

We can now apply the automatic Baire property stated in the above Theorem 11. Then for each regular ω -language L_q , one can construct a deterministic Muller automaton accepting an open set O_q and a deterministic Muller automaton accepting a countable union W_q of closed sets with empty interiors, such that for each $q \in K$,

$$L_q \Delta O_q \subseteq W_q$$

Now we set

$$V_n = \bigcup_{q \in K} T(u_n, q) \cdot O_q$$

and

$$F_n = \bigcup_{q \in K} T(u_n, q) \cdot W_q$$

Notice that each set $T(u_n, q)$ is countable and that for each finite word $u \in T(u_n, q)$ it is easy to see that the set $u \cdot O_q$ is open and that the set $u \cdot W_q$ is a countable union of closed sets with empty interiors. Thus it is easy to see that V_n is open, and that F_n is a countable union of closed sets with empty interiors. Moreover it is easy to see that V_n and F_n are regular ω -languages since each set $T(u_n, q)$ is a regular language of finite words over the alphabet Σ . Moreover it holds that:

$$f^{-1}(U_n) \Delta V_n \subseteq F_n$$

We now prove that $F = \bigcup_{n \geq 1} F_n$ is itself a regular ω -language. It holds that

$$F = \bigcup_{n \geq 1} F_n = \bigcup_{n \geq 1} \bigcup_{q \in K} T(u_n, q) \cdot W_q = \bigcup_{q \in K} \bigcup_{n \geq 1} T(u_n, q) \cdot W_q$$

Consider now the 2-tape automaton \mathcal{B}_q which is like the 2-tape automaton \mathcal{A} but reads only pairs of finite words in $\Sigma^* \times \Gamma^*$ and has the state q as unique accepting state. Let then \mathcal{C}_q be a finite automaton which reads only finite words over the alphabet Σ and such that $L(\mathcal{C}_q) = \text{Proj}_{\Sigma^*}(L(\mathcal{B}_q))$ is the projection of the language $L(\mathcal{B}_q)$ on Σ^* . We can construct, from the automaton \mathcal{A} , the automata \mathcal{B}_q and \mathcal{C}_q for each $q \in K$. Now it holds that:

$$F = \bigcup_{n \geq 1} F_n = \bigcup_{q \in K} \bigcup_{n \geq 1} T(u_n, q) \cdot W_q = \bigcup_{q \in K} L(\mathcal{C}_q) \cdot W_q$$

On the other hand, for each finite word $u \in \Sigma^*$, the set $u \cdot W_q$ is a meager Σ_2^0 -set, since W_q is a meager Σ_2^0 -set. Thus the set

$$F = \bigcup_{q \in K} L(\mathcal{C}_q) \cdot W_q$$

is also a countable union of closed sets with empty interiors, since K is finite and each language $L(\mathcal{C}_q)$ is countable. Moreover the ω -language F is regular and we can construct, from the automata \mathcal{C}_q and from the deterministic Muller automata accepting the ω -languages W_q , a deterministic Muller automaton accepting F .

We can now set $G_n = \Sigma^\omega \setminus F_n$ and $G = \bigcap_{n \geq 1} G_n = \Sigma^\omega \setminus \bigcup_{n \geq 1} F_n = \Sigma^\omega \setminus F$. Then G is a countable intersection of dense open subsets of Σ^ω , hence also a dense $\mathbf{\Pi}_2^0$ -subset G of Σ^ω . Moreover we can construct a deterministic Muller automaton and even a deterministic Büchi automaton (since G is a $\mathbf{\Pi}_2^0$ -set, see [PP04, page 41]) accepting G . We can now see that the restriction f_G of the function f to G is continuous. This follows from the fact that the inverse image of every basic open set of Γ^ω by the function f_G is an open subset of G because for each integer $n \geq 1$, it holds that $f_G^{-1}(U_n) = f^{-1}(U_n) \cap G = V_n \cap G$. \square

Remark 29. *The above dense $\mathbf{\Pi}_2^0$ -subset G of Σ^ω is comeager and thus Theorem 28 shows that one can construct a deterministic Büchi automaton accepting a “big” ω -rational subset of Σ^ω on which the function f is continuous.*

5 Concluding remarks

We have proved a new effective property of ω -rational functions. We hope this property will be useful for further studies involving ω -rational functions. For instance an ω -automatic structure is defined via synchronous infinitary rational relations, see [BG04, KL08]. On the other hand, any (synchronous) infinitary rational relation is uniformizable by a (synchronous) ω -rational function, see [CG99]. Thus we can expect that our result will be useful in particular in the study of ω -automatic structures.

We also hope that the automatic Baire property will be useful in other studies involving regular ω -languages like the study of infinite games specified by automata.

References

- [BCPS03] M.-P. Béal, O. Carton, C. Prieur, and J. Sakarovitch. Squaring transducers: an efficient procedure for deciding functionality and sequentiality. *Theoretical Computer Science*, 292(1):45–63, 2003.
- [BG04] A. Blumensath and E. Grädel. Finite presentations of infinite structures: Automata and interpretations. *Theory of Computing Systems*, 37(6):641–674, 2004.
- [CFS08] O. Carton, O. Finkel, and P. Simonnet. On the continuity set of an omega rational function. *Theoretical Informatics and Applications*, 42(1):183–196, 2008.
- [CG99] C. Choffrut and S. Grigorieff. Uniformization of rational relations. In Juhani Karhumäki, Hermann A. Maurer, Gheorghe Paun, and Grzegorz Rozenberg, editors, *Jewels are Forever, Contributions on Theoretical Computer Science in Honor of Arto Salomaa*, pages 59–71. Springer, 1999.
- [Fin09] O. Finkel. Highly undecidable problems for infinite computations. *RAIRO-Theoretical Informatics and Applications*, 43(2):339–364, 2009.
- [Fin12] O. Finkel. Three applications to rational relations of the high undecidability of the infinite Post correspondence problem in a regular ω -language. *International Journal of Foundations of Computer Science*, 23(7):1481–1498, 2012.

- [FS93] C. Frougny and J. Sakarovitch. Synchronized rational relations of finite and infinite words. *Theoretical Computer Science*, 108(1):45–82, 1993.
- [Gir81] F. Gire. *Relations rationnelles infinitaires*. PhD thesis, Université Paris VII, 1981.
- [Gir86] F. Gire. Two decidability problems for infinite words. *Information Processing Letters*, 22(3):135–140, 1986.
- [GN84] F. Gire and M. Nivat. Relations rationnelles infinitaires. *Calcolo*, pages 91–125, 1984.
- [Kec95] A. S. Kechris. *Classical descriptive set theory*. Springer-Verlag, New York, 1995.
- [KL08] D. Kuske and M. Lohrey. First-order and counting theories of omega-automatic structures. *Journal of Symbolic Logic*, 73(1):129–150, 2008.
- [Lan69] L.H. Landweber. Decision problems for ω -automata. *Mathematical Systems Theory*, 3(4):376–384, 1969.
- [LT94] H. Lescow and W. Thomas. Logical specifications of infinite computations. In J. W. de Bakker, W. P. de Roever, and G. Rozenberg, editors, *A Decade of Concurrency*, volume 803 of *Lecture Notes in Computer Science*, pages 583–621. Springer, 1994.
- [MMS18] H. Michalewski, M. Mio, and M. Skrzypczak. Monadic second order logic with measure and category quantifiers. *Logical Methods in Computer Science*, 14(2), 2018.
- [Mos80] Y. N. Moschovakis. *Descriptive set theory*. North-Holland Publishing Co., Amsterdam, 1980.
- [PP04] D. Perrin and J.-E. Pin. *Infinite words, automata, semigroups, logic and games*, volume 141 of *Pure and Applied Mathematics*. Elsevier, 2004.
- [Pri00] C. Prieur. *Fonctions Rationnelles de Mots Infinites et Continuité*. PhD thesis, Université Paris VII, 2000.
- [Pri01] C. Prieur. How to decide continuity of rational functions on infinite words. *Theoretical Computer Science*, 250(1–2):71–82, 2001.
- [Pri02] C. Prieur. How to decide continuity of rational functions on infinite words. *Theoretical Computer Science*, 276(1–2):445–447, 2002.
- [Sta97] L. Staiger. ω -languages. In *Handbook of formal languages, Vol. 3*, pages 339–387. Springer, Berlin, 1997.
- [Sta98] L. Staiger. Rich ω -words and monadic second-order arithmetic. In *Computer science logic (Aarhus, 1997)*, pages 478–490. Springer, Berlin, 1998.
- [Tho90] W. Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B, Formal models and semantics, pages 135–191. Elsevier, 1990.