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# An Effective Property of $\omega$ -Rational Functions

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**Abstract.** We prove that  $\omega$ -regular languages accepted by Büchi or Muller automata satisfy an effective automata-theoretic version of the Baire property. Then we use this result to obtain a new effective property of rational functions over infinite words which are realized by finite state Büchi transducers: for each such function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , one can construct a deterministic Büchi automaton  $\mathcal{A}$  accepting a dense  $\Pi_2^0$ -subset of  $\Sigma^\omega$  such that the restriction of  $F$  to  $L(\mathcal{A})$  is continuous.

**Keywords.** Decision problems; regular languages of infinite words; infinitary rational relations; omega rational functions; topology; automatic Baire property; points of continuity.

## 1 Introduction

Infinitary rational relations were firstly studied by Gire and Nivat, [Gir81,GN84]. The  $\omega$ -rational functions over infinite words, whose graphs are (functional) infinitary rational relations accepted by 2-tape Büchi automata, have been studied by several authors [CG99,BCPS03,Sta97,Pri00].

In this paper we are mainly interested in the question of the continuity of such  $\omega$ -rational functions. Recall that Prieur proved that one can decide whether a given  $\omega$ -rational function is continuous, [Pri01,Pri02]. On the other hand, Carton, Finkel and Simonnet proved that one cannot decide whether a given  $\omega$ -rational function  $f$  has at least one point of continuity, [CFS08]. Notice that this decision problem is actually  $\Sigma_1^1$ -complete, hence highly undecidable, [Fin12]. It was also proved in [CFS08] that one cannot decide whether the continuity set of a given  $\omega$ -rational function  $f$  (its set of continuity points) is a regular (respectively, context-free)  $\omega$ -language. Notice that the situation was shown to be quite different in the case of *synchronous* functions. It was proved in [CFS08] that if  $f : A^\omega \rightarrow B^\omega$  is an  $\omega$ -rational synchronous function, then the continuity set  $C(f)$  of  $f$  is  $\omega$ -rational. Moreover If  $X$  is an  $\omega$ -rational  $\Pi_2^0$  subset of  $A^\omega$ , then  $X$  is the continuity set  $C(f)$  of some rational synchronous function  $f$  of domain  $A^\omega$ . Notice that these previous works on the continuity of  $\omega$ -rational functions had shown that decision problems in this area may be decidable or not, (while it is well known that most problems about regular languages accepted by finite automata are decidable).

We establish in this paper a new effective property of rational functions over infinite words. We first prove that  $\omega$ -regular languages accepted by Büchi

or Muller automata satisfy an effective automata-theoretic version of the Baire property. Then we use this result to obtain a new effective property of rational functions over infinite words which are realized by finite state Büchi transducers: for each such function  $F : \Sigma^\omega \rightarrow \Gamma^\omega$ , one can construct a deterministic Büchi automaton  $\mathcal{A}$  accepting a dense  $\Pi_2^0$ -subset of  $\Sigma^\omega$  such that the restriction of  $F$  to this dense set  $L(\mathcal{A})$  is continuous.

The paper is organized as follows. We recall basic notions on automata and on the Borel hierarchy in Section 2. The automatic Baire property for regular  $\omega$ -languages is proved in Section 3. We prove our main new result on  $\omega$ -rational functions in Section 4. Some concluding remarks are given in Section 5.

## 2 Recall of basic notions

We assume the reader to be familiar with the theory of formal ( $\omega$ )-languages [Tho90,Sta97]. We recall some usual notations of formal language theory.

When  $\Sigma$  is a finite alphabet, a *non-empty finite word* over  $\Sigma$  is any sequence  $x = a_1 \dots a_k$ , where  $a_i \in \Sigma$  for  $i = 1, \dots, k$ , and  $k$  is an integer  $\geq 1$ . The length of  $x$  is  $|x| = k$ . The *set of finite words* (including the empty word whose length is zero) over  $\Sigma$  is denoted  $\Sigma^*$ .

The *first infinite ordinal* is  $\omega$ . An  $\omega$ -word over  $\Sigma$  is an  $\omega$ -sequence  $a_1 \dots a_n \dots$ , where for all integers  $i \geq 1$ ,  $a_i \in \Sigma$ . When  $\sigma$  is an  $\omega$ -word over  $\Sigma$ , we write  $\sigma = \sigma(1)\sigma(2) \dots \sigma(n) \dots$ , where for all  $i$ ,  $\sigma(i) \in \Sigma$ , and  $\sigma[n] = \sigma(1)\sigma(2) \dots \sigma(n)$ .

The usual concatenation product of two finite words  $u$  and  $v$  is denoted  $u \cdot v$  and sometimes just  $uv$ . This product is extended to the product of a finite word  $u$  and an  $\omega$ -word  $v$ : the infinite word  $u \cdot v$  is then the  $\omega$ -word such that:  $(u \cdot v)(k) = u(k)$  if  $k \leq |u|$ , and  $(u \cdot v)(k) = v(k - |u|)$  if  $k > |u|$ .

The *set of  $\omega$ -words* over the alphabet  $\Sigma$  is denoted by  $\Sigma^\omega$ . An  $\omega$ -language over an alphabet  $\Sigma$  is a subset of  $\Sigma^\omega$ .

**Definition 1.** : A *finite state machine (FSM)* is a quadruple  $\mathcal{M} = (K, \Sigma, \delta, q_0)$ , where  $K$  is a finite set of states,  $\Sigma$  is a finite input alphabet,  $q_0 \in K$  is the initial state and  $\delta$  is a mapping from  $K \times \Sigma$  into  $2^K$ . A *FSM is called deterministic (DFSM)* iff:  $\delta : K \times \Sigma \rightarrow K$ .

A *Büchi automaton (BA)* is a 5-tuple  $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$  where  $\mathcal{M} = (K, \Sigma, \delta, q_0)$  is a finite state machine and  $F \subseteq K$  is the set of final states.

A *Muller automaton (MA)* is a 5-tuple  $\mathcal{A} = (K, \Sigma, \delta, q_0, \mathcal{F})$  where  $\mathcal{M} = (K, \Sigma, \delta, q_0)$  is a FSM and  $\mathcal{F} \subseteq 2^K$  is the collection of designated state sets.

A *Büchi or Muller automaton is said deterministic* if the associated FSM is deterministic.

Let  $\sigma = a_1 a_2 \dots a_n \dots$  be an  $\omega$ -word over  $\Sigma$ .

A *sequence of states*  $r = q_1 q_2 \dots q_n \dots$  is called an *(infinite) run of  $\mathcal{M}$*   $= (K, \Sigma, \delta, q_0)$  on  $\sigma$ , starting in state  $p$ , iff: 1)  $q_1 = p$  and 2) for each  $i \geq 1$ ,  $q_{i+1} \in \delta(q_i, a_i)$ .

In case a run  $r$  of  $\mathcal{M}$  on  $\sigma$  starts in state  $q_0$ , we call it simply “a run of  $\mathcal{M}$  on  $\sigma$ ”. For every (infinite) run  $r = q_1 q_2 \dots q_n \dots$  of  $\mathcal{M}$ ,  $\text{In}(r)$  is the set of states

in  $K$  entered by  $M$  infinitely many times during run  $r$ :  $\text{In}(r) = \{q \in K \mid \exists^\infty i \geq 1 \text{ } q_i = q\}$  is infinite}.

For  $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$  a BA, the  $\omega$ -language accepted by  $\mathcal{A}$  is:

$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } \text{In}(r) \cap F \neq \emptyset\}$ .

For  $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$  a MA, the  $\omega$ -language accepted by  $\mathcal{A}$  is:

$L(\mathcal{A}) = \{\sigma \in \Sigma^\omega \mid \text{there exists a run } r \text{ of } \mathcal{A} \text{ on } \sigma \text{ such that } \text{In}(r) \in \mathcal{F}\}$ .

By R. Mc Naughton's Theorem, see [PP04], the expressive power of deterministic MA (DMA) is equal to the expressive power of non deterministic MA (NDMA) which is also equal to the expressive power of non deterministic BA (NDBA).

**Theorem 2.** For any  $\omega$ -language  $L \subseteq \Sigma^\omega$ , the following conditions are equivalent:

1. There exists a DMA that accepts  $L$ .
2. There exists a MA that accepts  $L$ .
3. There exists a BA that accepts  $L$ .

An  $\omega$ -language  $L$  satisfying one of the conditions of the above Theorem is called a regular  $\omega$ -language. The class of regular  $\omega$ -languages will be denoted by  $REG_\omega$ .

Recall that, from a Büchi (respectively, Muller) automaton  $\mathcal{A}$ , one can effectively construct a deterministic Muller (respectively, non-deterministic Büchi) automaton  $\mathcal{B}$  such that  $L(\mathcal{A}) = L(\mathcal{B})$ .

A way to study the complexity of  $\omega$ -languages accepted by various automata is to study their topological complexity.

We assume the reader to be familiar with basic notions of topology which may be found in [Mos80,LT94,Kec95,Sta97,PP04]. If  $X$  is a finite alphabet containing at least two letters, then the set  $X^\omega$  of infinite words over  $X$  may be equipped with the product topology of the discrete topology on  $X$ . This topology is induced by a natural metric which is called the *prefix metric* and defined as follows. For  $u, v \in X^\omega$  and  $u \neq v$  let  $\delta(u, v) = 2^{-l_{\text{pref}(u,v)}}$  where  $l_{\text{pref}(u,v)}$  is the first integer  $n$  such that the  $(n+1)^{\text{st}}$  letter of  $u$  is different from the  $(n+1)^{\text{st}}$  letter of  $v$ . The topological space  $X^\omega$  is a Cantor space. The open sets of  $X^\omega$  are the sets in the form  $W \cdot X^\omega$ , where  $W \subseteq X^*$ . A set  $L \subseteq X^\omega$  is a closed set iff its complement  $X^\omega - L$  is an open set. Closed sets are characterized by the following:

**Proposition 3.** A set  $L \subseteq X^\omega$  is a closed set of  $X^\omega$  iff for every  $\sigma \in X^\omega$ ,  $[\forall n \geq 1, \exists u \in X^\omega \text{ such that } \sigma(1) \dots \sigma(n).u \in L]$  implies that  $\sigma \in L$ .

Define now the next classes of the Borel Hierarchy:

**Definition 4.** The classes  $\Sigma_n^0$  and  $\Pi_n^0$  of the Borel Hierarchy on the topological space  $X^\omega$  are defined as follows:  $\Sigma_1^0$  is the class of open sets of  $X^\omega$ ,  $\Pi_1^0$  is the class of closed sets of  $X^\omega$ . And for any integer  $n \geq 1$ :  $\Sigma_{n+1}^0$  is the class of countable unions of  $\Pi_n^0$ -subsets of  $X^\omega$ , and  $\Pi_{n+1}^0$  is the class of countable intersections of  $\Sigma_n^0$ -subsets of  $X^\omega$ .

**Remark 5.** *The hierarchy defined above is the hierarchy of Borel sets of finite rank. The Borel Hierarchy is also defined for transfinite levels (see [Mos80,Kec95]) but we shall not need this in the sequel.*

It turns out that there is a characterization of  $\mathbf{\Pi}_2^0$ -subsets of  $X^\omega$ , involving the notion of  $W^\delta$  which we now recall.

**Definition 6.** *For  $W \subseteq X^*$ , we set:  $W^\delta = \{\sigma \in X^\omega \mid \exists^\infty i \text{ such that } \sigma[i] \in W\}$ . ( $\sigma \in W^\delta$  iff  $\sigma$  has infinitely many prefixes in  $W$ ).*

Then we can state the following proposition.

**Proposition 7.** *A subset  $L$  of  $X^\omega$  is a  $\mathbf{\Pi}_2^0$ -subset of  $X^\omega$  iff there exists a set  $W \subseteq X^*$  such that  $L = W^\delta$ .*

It is easy to see, using the above characterization of  $\mathbf{\Pi}_2^0$ -sets, that every  $\omega$ -language accepted by a deterministic Büchi automaton is a  $\mathbf{\Pi}_2^0$ -set. Thus every regular  $\omega$ -language is a boolean combination of  $\mathbf{\Pi}_2^0$ -sets, because it is accepted by a deterministic Muller automaton and this implies that it is a boolean combination of  $\omega$ -languages accepted by deterministic Büchi automata.

Landweber studied the topological properties of regular  $\omega$ -languages in [Lan69]. He characterized the regular  $\omega$ -languages in each of the Borel classes  $\mathbf{\Sigma}_1^0, \mathbf{\Pi}_1^0, \mathbf{\Sigma}_2^0, \mathbf{\Pi}_2^0$ , and showed that one can decide, for an effectively given regular  $\omega$ -language  $L$ , whether  $L$  is in  $\mathbf{\Sigma}_1^0, \mathbf{\Pi}_1^0, \mathbf{\Sigma}_2^0$ , or  $\mathbf{\Pi}_2^0$ . In particular, it turned out that a regular  $\omega$ -language is in the class  $\mathbf{\Pi}_2^0$  iff it is accepted by a deterministic Büchi automaton.

Recall that, from a Büchi or Muller automaton  $\mathcal{A}$ , one can construct some Büchi or Muller automata  $\mathcal{B}$  and  $\mathcal{C}$ , such that  $L(\mathcal{B})$  is equal to the topological closure of  $L(\mathcal{A})$ , and  $L(\mathcal{C})$  is equal to the topological interior of  $L(\mathcal{A})$ , see [Sta97,PP04].

### 3 The automatic Baire property

In this section we are going to prove an automatic version of the result stating that every Borel (and even every analytic) set has the Baire property.

We firstly recall some basic definitions about meager sets, see [Kec95]. In a topological space  $\mathcal{X}$ , a set  $A \subseteq \mathcal{X}$  is said to be *nowhere dense* if its closure  $\bar{A}$  has empty interior, i.e.  $\text{Int}(\bar{A}) = \emptyset$ . A set  $A \subseteq \mathcal{X}$  is said to be *meager* if it is the union of countably many nowhere dense sets, or equivalently if it is included in a countable union of closed sets with empty interiors. This means that  $A$  is meager if there exist countably many closed sets  $A_n$ ,  $n \geq 1$ , such that  $A \subseteq \bigcup_{n \geq 1} A_n$  where for every integer  $n \geq 1$ ,  $\text{Int}(A_n) = \emptyset$ . A set is *comeager* if its complement is meager, i.e. if it contains the intersection of countably many dense open sets. Notice that the notion of *meager* set is a notion of *small* set, while the notion of *comeager* set is a notion of *big* set.

Recall that a Baire space is a topological space  $\mathcal{X}$  in which every intersection of countably many dense open sets is dense, or equivalently in which every

countable union of closed sets with empty interiors has also an empty interior. It is well known that every Cantor space  $\Sigma^\omega$  is a Baire space. In the sequel we will consider only Cantor spaces.

We now recall the notion of Baire property. For any sets  $A, B \subseteq \Sigma^\omega$ , we denote  $A\Delta B$  the symmetric difference of  $A$  and  $B$ , and we write  $A =^* B$  if and only if  $A\Delta B$  is meager.

**Definition 8.** *A set  $A \subseteq \Sigma^\omega$  has the Baire property (BP) if there exists an open set  $U \subseteq \Sigma^\omega$  such that  $A =^* U$ .*

An important result of descriptive set theory is the following result, see [Kec95, page 47].

**Theorem 9.** *Every Borel set of a Cantor space has the Baire property.*

We now consider regular  $\omega$ -languages  $L \subseteq \Sigma^\omega$  for a finite alphabet  $\Sigma$ . These languages are Borel and thus have the Baire property. We are going to prove the following automatic version of the above theorem.

**Theorem 10.** *Let  $L = L(\mathcal{A}) \subseteq \Sigma^\omega$  be a regular  $\omega$ -language accepted by a Büchi or Muller automaton  $\mathcal{A}$ . Then one can construct Büchi automata  $\mathcal{B}$  and  $\mathcal{C}$  such that  $L(\mathcal{B}) \subseteq \Sigma^\omega$  is open,  $L(\mathcal{C}) \subseteq \Sigma^\omega$  is a countable union of closed sets with empty interior, and  $L(\mathcal{A})\Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ , and we shall say that the  $\omega$ -language  $L(\mathcal{A})$  has the automatic Baire property.*

**Proof.** We reason by induction on the topological complexity of the regular  $\omega$ -language  $L = L(\mathcal{A}) \subseteq \Sigma^\omega$  accepted by a Büchi automaton  $\mathcal{A}$ .

If  $L = L(\mathcal{A})$  is an open set then we immediately see that we get the result with  $\mathcal{B} = \mathcal{A}$  and  $\mathcal{C}$  is any Büchi automaton accepting the empty set.

If  $L = L(\mathcal{A})$  is a closed set then  $L \setminus \text{Int}(L)$  is a closed set with empty interior. Moreover it is known that one can construct from the Büchi automaton  $\mathcal{A}$  another Büchi automaton  $\mathcal{B}$  accepting  $\text{Int}(L)$ , and then also a Büchi automaton  $\mathcal{C}$  accepting  $L \setminus \text{Int}(L)$ . Then we have  $L(\mathcal{A})\Delta L(\mathcal{B}) = L \setminus \text{Int}(L) = L(\mathcal{C})$ , with  $L(\mathcal{B})$  open and  $L(\mathcal{C})$  is a closed set with empty interior.

We now consider the case of a regular  $\omega$ -language  $L = L(\mathcal{A})$  which is a  $\Sigma_2^0$ -set. Recall that then the  $\omega$ -language  $L$  is accepted by a deterministic finite automaton  $\mathcal{A} = (K, \Sigma, \delta, q_0, F)$  with co-Büchi acceptance condition. Moreover we may assume that the automaton  $\mathcal{A}$  is complete. An  $\omega$ -word  $x$  is accepted by  $\mathcal{A}$  with co-Büchi acceptance condition if the run of the automaton  $\mathcal{A}$  on  $x$  (which is then unique since the automaton is deterministic) goes only finitely many times through the set of states  $F$ . Let now  $n \geq 1$  and  $L_n$  be the  $\omega$ -language of  $\omega$ -words  $x \in \Sigma^\omega$  such that the run of the automaton  $\mathcal{A}$  over  $x$  goes at most  $n$  times through the set of states  $F$ . We have clearly that  $L = \bigcup_{n \geq 1} L_n$ . Moreover it is easy to see that for every  $n \geq 1$  the set  $L_n$  is closed. And the interior of  $L_n$  is the union of basic open sets  $u.\Sigma^\omega$ , for  $u \in \Sigma^\omega$ , such that the automaton  $\mathcal{A}$  enters at most  $n$  times in states from  $F$  during the reading of  $u$  and ends the reading of  $u$  in a state  $q$  such that none state of  $F$  is accessible from this

state  $q$ . We have of course that  $L_n \setminus \text{Int}(L_n)$  is a closed set with empty interior and thus  $L_n =^* \text{Int}(L_n)$ . Then  $\bigcup_{n \geq 1} L_n =^* \bigcup_{n \geq 1} \text{Int}(L_n)$  and we can easily see that  $\bigcup_{n \geq 1} \text{Int}(L_n)$  is accepted by a Büchi automaton  $\mathcal{B}$  which is essentially the automaton  $\mathcal{A}$  with a modified Büchi acceptance condition expressing “some state  $q$  has been reached from which none state of  $F$  is accessible”. Moreover  $L(\mathcal{A}) \Delta L(\mathcal{B}) = (\bigcup_{n \geq 1} L_n) \Delta (\bigcup_{n \geq 1} \text{Int}(L_n)) \subseteq \bigcup_{n \geq 1} (L_n \setminus \text{Int}(L_n))$  and the set  $\bigcup_{n \geq 1} (L_n \setminus \text{Int}(L_n))$  is a countable union of closed sets with empty interiors which is easily seen to be accepted by a co-Büchi automaton  $\mathcal{C}$  which is essentially the automaton  $\mathcal{A}$  where we have deleted every state  $q$  from which none state of  $F$  is accessible.

We now look at boolean operations. Assume that  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ , where  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  are Büchi automata,  $L(\mathcal{B})$  is open and  $L(\mathcal{C})$  is a countable union of closed sets with empty interiors. Notice that  $(\Sigma^\omega \setminus L(\mathcal{A})) \Delta (\Sigma^\omega \setminus L(\mathcal{B})) = L(\mathcal{A}) \Delta L(\mathcal{B})$ . Moreover we can construct a Büchi automaton  $\mathcal{D}$  accepting the closed set  $(\Sigma^\omega \setminus L(\mathcal{B}))$  and next also a Büchi automaton  $\mathcal{D}'$  accepting the open set  $\text{Int}((\Sigma^\omega \setminus L(\mathcal{B})))$ , and a Büchi automaton  $\mathcal{E}$  accepting the closed set with empty interior  $(L(\mathcal{D}) \setminus \text{Int}(L(\mathcal{D})))$ . It is now easy to see that  $(\Sigma^\omega \setminus L(\mathcal{A})) \Delta L(\mathcal{D}') \subseteq (\Sigma^\omega \setminus L(\mathcal{A})) \Delta (\Sigma^\omega \setminus L(\mathcal{B})) \cup L(\mathcal{E}) \subseteq L(\mathcal{C}) \cup L(\mathcal{E})$  and we can construct a Büchi automaton  $\mathcal{C}'$  accepting the  $\omega$ -language  $L(\mathcal{C}) \cup L(\mathcal{E})$  which is a countable union of closed sets with empty interiors so that  $(\Sigma^\omega \setminus L(\mathcal{A})) \Delta L(\mathcal{D}') \subseteq L(\mathcal{C}')$ . Notice that this implies that the automatic Baire property stated in the theorem is satisfied for regular  $\omega$ -languages in the Borel class  $\mathbf{\Pi}_2^0$  since we have already solved the case of the class  $\mathbf{\Sigma}_2^0$ .

We now consider the finite union operation. Assume we have  $L(\mathcal{A}) \Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ , and  $L(\mathcal{A}') \Delta L(\mathcal{B}') \subseteq L(\mathcal{C}')$  where  $\mathcal{A}$ ,  $\mathcal{A}'$ ,  $\mathcal{B}$ ,  $\mathcal{B}'$ , and  $\mathcal{C}$ ,  $\mathcal{C}'$ , are Büchi automata,  $L(\mathcal{B})$  and  $L(\mathcal{B}')$  are open and  $L(\mathcal{C})$  and  $L(\mathcal{C}')$  are countable unions of closed sets with empty interiors. Now we can see that  $(L(\mathcal{A}) \cup L(\mathcal{A}')) \Delta (L(\mathcal{B}) \cup L(\mathcal{B}')) \subseteq (L(\mathcal{A}) \Delta L(\mathcal{B})) \cup (L(\mathcal{A}') \Delta L(\mathcal{B}')) \subseteq L(\mathcal{C}) \cup L(\mathcal{C}')$ . Moreover we can construct Büchi automata  $\mathcal{B}''$  and  $\mathcal{C}''$  such that  $L(\mathcal{B}'')$  is the open set  $L(\mathcal{B}) \cup L(\mathcal{B}')$  and  $L(\mathcal{C}'') = L(\mathcal{C}) \cup L(\mathcal{C}')$  is a countable union of closed sets with empty interiors and then we have  $(L(\mathcal{A}) \cup L(\mathcal{A}')) \Delta L(\mathcal{B}'') \subseteq L(\mathcal{C}'')$ .

We now return to the general case of a regular  $\omega$ -language  $L \subseteq \Sigma^\omega$ , accepted by a Büchi or Muller automaton. We know that we can construct a deterministic Muller automaton  $\mathcal{A} = (K, \Sigma, \delta, q_0, \mathcal{F})$  accepting  $L$ . Recall that  $\mathcal{F} \subseteq 2^K$  is here the collection of designated state sets. For each state  $q \in K$ , we now denote by  $\mathcal{A}_q$  the automaton  $\mathcal{A}$  but viewed as a (deterministic) Büchi automaton with the single accepting state  $q$ , i.e.  $\mathcal{A}_q = (K, \Sigma, \delta, q_0, \{q\})$ . We know that the languages  $L(\mathcal{A}_q)$  are Borel  $\mathbf{\Pi}_2^0$ -sets and thus satisfy the automatic Baire property. Moreover we have the following equality:

$$L(\mathcal{A}) = \bigcup_{F \in \mathcal{F}} [\bigcap_{q \in F} L(\mathcal{A}_q) \setminus \bigcup_{q \notin F} L(\mathcal{A}_q)]$$

This implies, from the case of  $\mathbf{\Pi}_2^0$   $\omega$ -regular languages and from the preceding remarks about the preservation of the automatic Baire property by boolean operations, that we can construct Büchi automata  $\mathcal{B}$  and  $\mathcal{C}$ , such that  $L(\mathcal{B})$  is

open and  $L(\mathcal{C})$  is a countable union of closed sets with empty interiors, which satisfy  $L(\mathcal{A})\Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ .  $\square$

**Corollary 11.** *On can decide, for a given Büchi or Muller automaton  $\mathcal{A}$ , whether  $L(\mathcal{A})$  is meager.*

**Proof.** Let  $\mathcal{A}$  be a Büchi or Muller automaton. The  $\omega$ -language  $L(\mathcal{A})$  has the automatic Baire property and we can construct Büchi automata  $\mathcal{B}$  and  $\mathcal{C}$ , such that  $L(\mathcal{B})$  is open and  $L(\mathcal{C})$  is a countable union of closed sets with empty interiors, which satisfy  $L(\mathcal{A})\Delta L(\mathcal{B}) \subseteq L(\mathcal{C})$ . It is easy to see that  $L(\mathcal{A})$  is meager if and only if  $L(\mathcal{B})$  is empty, since any non-empty open set is non-meager, and this can be decided from the automaton  $\mathcal{B}$ .  $\square$

**Remark 12.** *The above Corollary followed already from Staiger's paper [Sta98], see also [MMS18]. So we get here another proof of this result, based on the automatic Baire property.*

## 4 An application to $\omega$ -rational functions

### 4.1 Infinitary rational relations

We now recall the definition of infinitary rational relations, via definition by Büchi transducers:

**Definition 13.** *A 2-tape Büchi automaton is a sextuple  $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$ , where  $K$  is a finite set of states,  $\Sigma$  and  $\Gamma$  are finite sets called the input and the output alphabets,  $\Delta$  is a finite subset of  $K \times \Sigma^* \times \Gamma^* \times K$  called the set of transitions,  $q_0$  is the initial state, and  $F \subseteq K$  is the set of accepting states. A computation  $\mathcal{C}$  of the automaton  $\mathcal{T}$  is an infinite sequence of consecutive transitions*

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \dots, (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \dots$$

*The computation is said to be successful iff there exists a final state  $q_f \in F$  and infinitely many integers  $i \geq 0$  such that  $q_i = q_f$ . The input word and output word of the computation are respectively  $u = u_1.u_2.u_3 \dots$  and  $v = v_1.v_2.v_3 \dots$ . The input and the output words may be finite or infinite. The infinitary rational relation  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$  accepted by the 2-tape Büchi automaton  $\mathcal{T}$  is the set of couples  $(u, v) \in \Sigma^\omega \times \Gamma^\omega$  such that  $u$  and  $v$  are the input and the output words of some successful computation  $\mathcal{C}$  of  $\mathcal{T}$ . The set of infinitary rational relations will be denoted  $RAT_2$ .*

If  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$  is an infinitary rational relation recognized by the 2-tape Büchi automaton  $\mathcal{T}$  then we denote

$$Dom(R(\mathcal{T})) = \{u \in \Sigma^\omega \mid \exists v \in \Gamma^\omega (u, v) \in R(\mathcal{T})\}$$

and

$$Im(R(\mathcal{T})) = \{v \in \Gamma^\omega \mid \exists u \in \Sigma^\omega (u, v) \in R(\mathcal{T})\}.$$



It is well known that, for each infinitary rational relation  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$ , the sets  $Dom(R(\mathcal{T}))$  and  $Im(R(\mathcal{T}))$  are regular  $\omega$ -languages and that one can construct, from the Büchi transducer  $\mathcal{T}$ , some Büchi automata  $\mathcal{A}$  and  $\mathcal{B}$  accepting the  $\omega$ -languages  $Dom(R(\mathcal{T}))$  and  $Im(R(\mathcal{T}))$ .

The 2-tape Büchi automaton  $\mathcal{T} = (K, \Sigma, \Gamma, \Delta, q_0, F)$  is said to be synchronous if the set of transitions  $\Delta$  is a finite subset of  $K \times \Sigma \times \Gamma \times K$ , i.e. if each transition is labelled with a pair  $(a, b) \in \Sigma \times \Gamma$ . An infinitary rational relation recognized by a synchronous 2-tape Büchi automaton is in fact an  $\omega$ -language over the product alphabet  $\Sigma \times \Gamma$  which is accepted by a Büchi automaton. It is called a synchronous infinitary rational relation. An infinitary rational relation is said to be asynchronous if it can not be recognized by any synchronous 2-tape Büchi automaton. Recall now the following undecidability result of C. Frougny and J. Sakarovitch.

**Theorem 14 ([FS93]).** *One cannot decide whether a given infinitary rational relation is synchronous.*

We proved in [Fin09] that many decision problems about infinitary rational relations are highly undecidable. In fact many of them, like the universality problem, the equivalence problem, the inclusion problem, the cofiniteness problem, the unambiguity problem, are  $\Pi_2^1$ -complete, hence located at the second level of the analytical hierarchy.

## 4.2 Continuity of $\omega$ -rational functions

Recall that an infinitary rational relation  $R(\mathcal{T}) \subseteq \Sigma^\omega \times \Gamma^\omega$  is said to be functional iff it is the graph of a function, i.e. iff

$$[\forall x \in Dom(R(\mathcal{T})) \exists! y \in Im(R(\mathcal{T})) (x, y) \in R(\mathcal{T})].$$

Then the functional relation  $R(\mathcal{T})$  defines an  $\omega$ -rational (partial) function  $F_{\mathcal{T}} : Dom(R(\mathcal{T})) \subseteq \Sigma^\omega \rightarrow \Gamma^\omega$  by: for each  $u \in Dom(R(\mathcal{T}))$ ,  $F_{\mathcal{T}}(u)$  is the unique  $v \in \Gamma^\omega$  such that  $(u, v) \in R(\mathcal{T})$ .

An  $\omega$ -rational (partial) function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is said to be synchronous if there is a synchronous 2-tape Büchi automaton  $\mathcal{T}$  such that  $f = f_{\mathcal{T}}$ .

An  $\omega$ -rational (partial) function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is said to be asynchronous if there is no synchronous 2-tape Büchi automaton  $\mathcal{T}$  such that  $f = f_{\mathcal{T}}$ .

Recall the following previous decidability result.

**Theorem 15 ([Gir86]).** *One can decide whether an infinitary rational relation recognized by a given 2-tape Büchi automaton  $\mathcal{T}$  is a functional infinitary rational relation (respectively, a synchronous functional infinitary rational relation).*

It is very natural to consider the notion of continuity for  $\omega$ -rational functions defined by 2-tape Büchi automata.

We recall that a function  $f : \text{Dom}(f) \subseteq \Sigma^\omega \rightarrow \Gamma^\omega$ , whose domain is  $\text{Dom}(f)$ , is said to be continuous at point  $x \in \text{Dom}(f)$  if :

$$\forall n \geq 1 \quad \exists k \geq 1 \quad \forall y \in \text{Dom}(f) \quad [ d(x, y) < 2^{-k} \Rightarrow d(f(x), f(y)) < 2^{-n} ]$$

The continuity set  $C(f)$  of the function  $f$  is the set of points of continuity of  $f$ . Notice that the continuity set  $C(f)$  of a function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is always a Borel  $\Pi_2^0$ -subset of  $\Sigma^\omega$ , see [CFS08].

The function  $f$  is said to be continuous if it is continuous at every point  $x \in \text{Dom}(f)$ , i. e. if  $C(f) = \text{Dom}(f)$ .

Priour proved the following decidability result.

**Theorem 16 (Priour [Pri01, Pri02]).** *One can decide whether a given  $\omega$ -rational function is continuous.*

On the other hand the following undecidability result was proved in [CFS08].

**Theorem 17 (see [CFS08]).** *One cannot decide whether a given  $\omega$ -rational function  $f$  has at least one point of continuity.*

The exact complexity of this undecidable problem was given in [Fin12]. It is  $\Sigma_1^1$ -complete to determine whether a given  $\omega$ -rational function  $f$  has at least one point of continuity.

We now consider the continuity set of an  $\omega$ -rational function and its possible complexity. The following undecidability result was proved in [CFS08].

**Theorem 18 (see [CFS08]).** *One cannot decide whether the continuity set of a given  $\omega$ -rational function  $f$  is a regular (respectively, context-free)  $\omega$ -language.*

The exact complexity of the first above undecidable problem, and an approximation of the complexity of the second one, were given in [Fin12]. It is  $\Pi_1^1$ -complete to determine whether the continuity set  $C(f)$  of a given  $\omega$ -rational function  $f$  is a regular  $\omega$ -language. Moreover the problem to determine whether the continuity set  $C(f)$  of a given  $\omega$ -rational function  $f$  is a context-free  $\omega$ -language is  $\Pi_1^1$ -hard and in the class  $\Pi_2^1 \setminus \Sigma_1^1$ .

The situation is quite different in the case of *synchronous* functions. The following results were proved in [CFS08].

**Theorem 19 ([CFS08]).** *Let  $f : A^\omega \rightarrow B^\omega$  be a rational synchronous function. The continuity set  $C(f)$  of  $f$  is rational.*

**Theorem 20 ([CFS08]).** *Let  $X$  be a rational  $\Pi_2^0$  subset of  $A^\omega$ . Then  $X$  is the continuity set  $C(f)$  of some rational synchronous function  $f$  of domain  $A^\omega$ .*

We are now going to prove another effective result about  $\omega$ -rational functions.

We first recall the following result of descriptive set theory, in the particular case of Cantor spaces  $\Sigma^\omega$  and  $\Gamma^\omega$ . A Borel function  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  is a function for which the inverse image of any Borel subset of  $\Gamma^\omega$ , or equivalently of any open set of  $\Gamma^\omega$ , is a Borel subset of  $\Sigma^\omega$ .

**Theorem 21** (see Theorem 8.38 of [Kec95]). *Let  $\Sigma$  and  $\Gamma$  be two finite alphabets and  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  be a Borel function. Then there is a dense  $\mathbf{\Pi}_2^0$ -subset  $G$  of  $\Sigma^\omega$  such that the restriction of  $f$  to  $G$  is continuous.*

We now state an automatic version of this theorem.

**Theorem 22.** *Let  $\Sigma$  and  $\Gamma$  be two finite alphabets and  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  be an  $\omega$ -rational function. Then there is a dense  $\omega$ -regular  $\mathbf{\Pi}_2^0$ -subset  $G$  of  $\Sigma^\omega$  such that the restriction of  $f$  to  $G$  is continuous. Moreover one can construct, from a 2-tape Büchi automaton accepting the graph of the function  $f$ , a deterministic Büchi automaton accepting a dense  $\mathbf{\Pi}_2^0$ -subset  $G$  of  $\Sigma^\omega$  such that the restriction of  $f$  to  $G$  is continuous.*

**Proof.** In the classical context of descriptive set theory, the proof of the above Theorem 21 is the following. Let  $U_n$  be a basic open subset of  $\Gamma^\omega$  (there is a countable basis for the usual Cantor topology on  $\Gamma^\omega$ ). Then  $f^{-1}(U_n) = R_n$  is a Borel subset of  $\Sigma^\omega$  since  $f$  is a Borel function. Thus  $R_n$  has the Baire property and there exists some open set  $V_n$  such that  $R_n \Delta V_n \subseteq F_n$ , where  $F_n$  is a countable union of closed sets with empty interiors. Let now  $G_n = \Sigma^\omega \setminus F_n$  and  $G = \bigcap_{n \geq 1} G_n = \Sigma^\omega \setminus \bigcup_{n \geq 1} F_n$ . Then  $G$  is a countable intersection of dense open subsets of  $\Sigma^\omega$ , hence also a dense  $\mathbf{\Pi}_2^0$ -subset of  $\Sigma^\omega$  since in the Cantor space any countable intersection of dense open sets is dense. Moreover the restriction  $f_G$  of the function  $f$  to  $G$  is continuous. Indeed for every basic open set  $U_n$  it holds that  $f_G^{-1}(U_n) = f^{-1}(U_n) \cap G = V_n \cap G$  is an open subset of  $G$ , and this implies that the inverse image of any open subset of  $\Gamma^\omega$  by  $f_G$  is also an open subset of  $G$ .

We now want to reason in the automata-theoretic framework in order to prove Theorem 22. Let then  $\Sigma$  and  $\Gamma$  be two finite alphabets and  $f : \Sigma^\omega \rightarrow \Gamma^\omega$  be an  $\omega$ -rational function whose graph is accepted by a 2-tape Büchi automaton  $\mathcal{A} = (K, \Sigma, \Gamma, \Delta, q_0, F)$ .

We assume that we have an enumeration of the finite words over the alphabet  $\Gamma$  given by  $(u_n)_{n \geq 1}$ ,  $u_n \in \Gamma^*$ . For  $q \in K$  we also denote  $\mathcal{A}_q$  the automaton  $\mathcal{A}$  in which we have changed the initial state so that the initial state of  $\mathcal{A}_q$  is  $q$  instead of  $q_0$ .

Let us now consider the basic open set of the space  $\Gamma^\omega$  given by  $U_n = u_n \cdot \Gamma^\omega$ . We first describe  $f^{-1}(U_n)$ . An  $\omega$ -word  $x \in \Sigma^\omega$  belongs to the set  $f^{-1}(U_n)$  iff  $x$  can be written in the form  $x = v \cdot y$  for some words  $v \in \Sigma^*$  and  $y \in \Sigma^\omega$ , and there is a partial run of the automaton  $\mathcal{A}$  reading  $(v, u_n)$  for which  $\mathcal{A}$  is in state  $q$  after having read the initial pair  $(v, u_n) \in \Sigma^* \times \Gamma^*$  (where the finite words  $v$  and  $u_n$  might have different lengths if the automaton  $\mathcal{A}$  is not synchronous),

and  $y \in \text{Dom}(L(\mathcal{A}_q))$ . Recall that  $L(\mathcal{A}_q) \subseteq (\Sigma \times \Gamma)^\omega$  is an infinitary rational relation and that  $\text{Dom}(L(\mathcal{A}_q))$  is then a regular  $\omega$ -language and that one can construct from  $\mathcal{A}$  a deterministic Muller automaton accepting this  $\omega$ -language  $\text{Dom}(L(\mathcal{A}_q))$  which will be denoted  $L_q$ . We also denote  $T(u_n, q)$  the set of finite words  $v$  over  $\Sigma$  such that the automaton  $\mathcal{A}$  is in state  $q$  after having read the initial pair  $(v, u_n) \in \Sigma^* \times \Gamma^*$ . Then the following equality holds:

$$f^{-1}(U_n) = \bigcup_{q \in K} T(u_n, q) \cdot L_q$$

We can now apply the automatic Baire property stated in the above Theorem 10. Then for each regular  $\omega$ -language  $L_q$ , one can construct a deterministic Muller automaton accepting an open set  $O_q$  and a deterministic Muller automaton accepting a countable union  $W_q$  of closed sets with empty interiors, such that for each  $q \in K$ ,

$$L_q \Delta O_q \subseteq W_q$$

Now we set

$$V_n = \bigcup_{q \in K} T(u_n, q) \cdot O_q$$

and

$$F_n = \bigcup_{q \in K} T(u_n, q) \cdot W_q$$

Notice that each set  $T(u_n, q)$  is countable and that for each finite word  $u \in T(u_n, q)$  it is easy to see that the set  $u \cdot O_q$  is open and that the set  $u \cdot W_q$  is a countable union of closed sets with empty interiors. Thus it is easy to see that  $V_n$  is open, and that  $F_n$  is a countable union of closed sets with empty interiors. Moreover it is easy to see that  $V_n$  and  $F_n$  are regular  $\omega$ -languages since each set  $T(u_n, q)$  is a regular language of finite words over the alphabet  $\Sigma$ . Moreover it holds that:

$$f^{-1}(U_n) \Delta V_n \subseteq F_n$$

We now prove that  $F = \bigcup_{n \geq 1} F_n$  is itself a regular  $\omega$ -language. It holds that

$$F = \bigcup_{n \geq 1} F_n = \bigcup_{n \geq 1} \bigcup_{q \in K} T(u_n, q) \cdot W_q = \bigcup_{q \in K} \bigcup_{n \geq 1} T(u_n, q) \cdot W_q$$

Consider now the 2-tape automaton  $\mathcal{B}_q$  which is like the 2-tape automaton  $\mathcal{A}$  but reads only pairs of finite words in  $\Sigma^* \times \Gamma^*$  and has the state  $q$  as unique accepting state. Let then  $\mathcal{C}_q$  be a finite automaton which reads only finite words over the alphabet  $\Sigma$  and such that  $L(\mathcal{C}_q) = \text{Proj}_{\Sigma^*}(L(\mathcal{B}_q))$  is the projection of the language  $L(\mathcal{B}_q)$  on  $\Sigma^*$ . We can construct, from the automaton  $\mathcal{A}$ , the automata  $\mathcal{B}_q$  and  $\mathcal{C}_q$  for each  $q \in K$ . Now it holds that:

$$F = \bigcup_{n \geq 1} F_n = \bigcup_{q \in K} \bigcup_{n \geq 1} T(u_n, q) \cdot W_q = \bigcup_{q \in K} L(\mathcal{C}_q) \cdot W_q$$

On the other hand, for each finite word  $u \in \Sigma^*$ , the set  $u \cdot W_q$  is a countable union of closed sets with empty interiors, since  $W_q$  is a countable union of closed sets with empty interiors. Thus the set

$$F = \bigcup_{q \in K} L(\mathcal{C}_q) \cdot W_q$$

is also a countable union of closed sets with empty interiors, since  $K$  is finite and each language  $L(\mathcal{C}_q)$  is countable. Moreover the  $\omega$ -language  $F$  is regular and we can construct, from the automata  $\mathcal{C}_q$  and from the deterministic Muller automata accepting the  $\omega$ -languages  $W_q$ , a deterministic Muller automaton accepting  $F$ .

We can now reason as in the classical case by setting  $G_n = \Sigma^\omega \setminus F_n$  and  $G = \bigcap_{n \geq 1} G_n = \Sigma^\omega \setminus \bigcup_{n \geq 1} F_n = \Sigma^\omega \setminus F$ . Then  $G$  is a countable intersection of dense open subsets of  $\Sigma^\omega$ , hence also a dense  $\mathbf{\Pi}_2^0$ -subset  $G$  of  $\Sigma^\omega$ . Moreover we can construct a deterministic Muller automaton and even a deterministic Büchi automaton (since  $G$  is a  $\mathbf{\Pi}_2^0$ -set, see [PP04, page 41]) accepting  $G$ , and the restriction  $f_G$  of the function  $f$  to  $G$  is continuous.  $\square$

**Remark 23.** *The above dense  $\mathbf{\Pi}_2^0$ -subset  $G$  of  $\Sigma^\omega$  is **comeager** and thus Theorem 22 shows that one can construct a deterministic Büchi automaton accepting a “large”  $\omega$ -rational subset of  $\Sigma^\omega$  on which the function  $f$  is continuous.*

## 5 Concluding remarks

We have proved a new effective property of  $\omega$ -rational functions. We hope this property will be useful for further studies involving  $\omega$ -rational functions. For instance an  $\omega$ -automatic structure is defined via synchronous infinitary rational relations, see [BG04, KL08]. On the other hand any synchronous infinitary rational relation is uniformizable by an  $\omega$ -rational function, see [CG99]. Thus we can expect that our result will be useful in particular in the study of  $\omega$ -automatic structures.

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