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# Shape Preserving Flows and the $p$ -Laplacian Spectra

Ido Cohen and Guy Gilboa

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## Abstract

We examine nonlinear scale-spaces in the general form  $u_t = P(u(t))$ , where  $P$  is a bounded nonlinear operator. We seek solutions with separation of variables in space and time  $u(x, t) = a(t)f(x)$ , where  $f$  is the initial condition. We term these as *shape-preserving flows* and provide necessary and sufficient conditions for their existence. We show that homogeneous operators admit the above conditions. It turns out that the initial condition must admit a nonlinear eigenvalue problem, with respect to the operator  $P$ ,  $P(f) = \lambda f$ , where  $\lambda$  is the eigenvalue. In this case we can formulate a closed form solution for any  $P$  which is homogeneous of positive degree. Consequently, we can determine if a finite extinction time exists. We show that in all cases the extinction time is inversely proportional to the eigenvalue  $\lambda$ .

Following the above analysis, we generalize the total-variation and one-homogeneous transforms to a homogeneous spectral representation. The notions of spectrum, generalized Parseval's theorem and filtering are defined. We apply these formulations to the  $p$ -Laplace operator for  $1 < p < 2$ .

## 1 Introduction and main results

A main principle in image and signal processing is to represent the data in a meaningful manner, which allows effective processing for the type of signals and application at hand. Usually, a compact and sparse representation is desired. In filtering applications, we would often prefer a representation which well distinguishes between signal and noise. Linear transforms were extensively employed for this purpose. For example, Fourier transform represents harmonic functions in the time domain by delta functions in the frequency domain.

Fourier developed his theory while investigating the solution of the heat equation (or homogeneous linear diffusion), [10]

$$u_t = \Delta u, \quad u(t=0) = f.$$

It is well known that the solution  $u(t)$  in an unbounded domain is a convolution of  $f$  with a Gaussian kernel of standard deviation  $\sigma = \sqrt{2t}$ , for a review see e.g.

[27]. Moreover, if  $f$  is an eigenfunction, namely, admits the linear eigenvalue problem

$$\Delta f = \lambda f, \quad \lambda \in \mathbb{R},$$

the variables  $x$  and  $t$  of the solution are separated, where

$$u(x, t) = e^{\lambda t} \cdot f(x). \quad (1)$$

We note that the heat equation is a gradient descent process with respect to the Dirichlet energy,

$$J_D(u) = \frac{1}{2} \int |\nabla u|^2 dx.$$

The Dirichlet energy and its associated flow smooth sharp transitions. Therefore, non-quadratic energies and non-linear flows are more useful for image processing tasks. A widely used edge preserving functional in image processing is the *Total Variation* (TV) energy,

$$J_{TV}(u) = \int |\nabla u| dx.$$

It can be perceived as the  $L^1$  analog of the Dirichlet energy. The associated gradient descent, known as TV-flow [1], is a nonlinear diffusion process of the form,

$$u_t = \operatorname{div} \left( \frac{\nabla u}{|\nabla u|} \right), \quad u(t=0) = f. \quad (2)$$

It can be shown ([23, 6]) that in the discrete 1D case the solution at time  $t$  coincides with the minimizer of the ROF model [22],

$$\arg \min_u \frac{1}{2} \|u - f\|^2 + t J_{TV}(u).$$

In an analog manner to the linear case, we can define a nonlinear eigenfunction with respect to the TV subdifferential by,

$$q = \lambda f, \quad -q \in \partial J_{TV}(f), \quad (3)$$

where  $\partial J(u)$  denotes the subdifferential of the functional  $J$  at  $u$ . Similarly to linear diffusion, if the initial condition admits (3) we obtain a simple analytic solution of the flow, where  $x$  and  $t$  are separated [1],

$$u(x, t) = (\lambda t + 1)^+ \cdot f(x). \quad (4)$$

The linear decay over time of (4) is the key to data representation by the TV-transform [12], where a nonlinear eigenfunction (3) becomes a delta in the TV transform domain.

One can generalize the aforementioned energies to a the  $p$ -Dirichlet functional:

$$J_p(u) = \frac{1}{p} \int |\nabla u|^p dx, \quad (5)$$

where  $p$  is classically in the range  $(1, \infty)$ . The gradient descent with respect to (5) is the nonlinear diffusion flow (hereinafter  $p$ -flow):

$$u_t = \Delta_p u, \quad u(0) = f, \quad (6)$$

where  $\Delta_p u$  is the  $p$ -Laplace operator defined by,

$$\Delta_p u = \operatorname{div} \left( |\nabla u|^{p-2} \nabla u \right). \quad (7)$$

We would like to investigate whether there are solutions to (6) in which  $x$  and  $t$  are separated, as in Eqs. (1) and (4). To address this issue we first consider a more general nonlinear flow

$$u_t = P(u(t)), \quad u(t=0) = f, \quad (\mathbf{P})$$

where  $P$  is a general nonlinear bounded operator. We seek conditions where the solution is in the form:

$$u(x, t) = a(t) \cdot f(x). \quad (\mathbf{ShP})$$

We denote these type of solutions as *shape preserving flows*, since there is no spatial change in  $u$  and the flow only modifies the contrast. The change of contrast is dictated by  $a(t)$ . The function  $a(t)$  is induced by the operator  $P$  and can be arbitrary. In our study we are interested in flows that vanish with time. We therefore term the function  $a(t)$  as *the decay profile*.

We begin by formulating necessary and sufficient conditions for solutions of  $(\mathbf{P})$  to be in the form of  $(\mathbf{ShP})$ . It turns out that the initial condition  $u(t=0) = f$  has to be an eigenfunction of  $P$ , i.e. admits

$$P(f) = \lambda f, \quad \lambda \in \mathbb{R}.$$

Moreover, we found that at  $t = 0$  the directional derivative of  $P$  (in the  $f$  direction) is proportional to  $f$ . When  $P$  is an homogeneous operator we can formulate analytic solutions for the flows. In the case of the  $p$ -Laplace operator, the decay profile is:

$$a(t) = [(2-p)\lambda t + 1]^{\frac{1}{2-p}}.$$

This decay profile holds for any value of  $p$  in the range  $[1, \infty)$ , for any eigenfunction  $f$  and any eigenvalue  $\lambda$ . As an immediate consequence, the  $p$ -flow has a finite extinction time  $T$  for  $1 \leq p < 2$ ,

$$T = \frac{1}{(p-2)\lambda}.$$

Following these results, a new family of nonlinear transforms is introduced. These transforms connect between eigenfunctions of the homogeneous operator and delta functions in the transform domain. Definitions for spectrum, filtering and data representation are formulated. Preliminary experiments illustrate this framework.

The plan of the paper is as follows: In Section 2 necessary and sufficient conditions are given for  $(\mathbf{P})$  to be shape preserving. Then we focus our discussion on homogeneous operators and show they can induce shape preserving flows. For these flows we formulate their decay profile and extinction time. In Section 3 we generalized the spectral one homogeneous representation [6] to  $p$ -homogeneous spectral representation for  $1 < p < 2$ . In Section 4 some numerical results are presented.

## 2 The theory of shape preserving flows

In this section we develop the theoretical basis for the  $p$ -spectra. We begin by formulating necessary and sufficient conditions for the flow  $(\mathbf{P})$  to be shape preserving. Then we show that operators of any homogeneity address these conditions.

### 2.1 Necessary and sufficient conditions for shape preserving flows

In the following theorem we define precise conditions for shape preservation under the assumption that in some initial time range the flow is infinitely differentiable, with respect to time.

**Theorem 1** (Shape preserving flow - necessary and sufficient condition). *Given a (non)linear scale-space flow, Eq.  $(\mathbf{P})$ , which exists for all  $t \in \mathbb{R}^+$ . We assume  $u(t)$  is infinitely differentiable with respect to  $t$  in the range  $D = [0, T] \subseteq \mathbb{R}^+$ . The solution  $u(t)$  is shape preserving (Eq.  $(\mathbf{ShP})$ ) in the range  $D$  iff for any  $n = 0, 1, ..$  we obtain*

$$\left. \frac{d^n}{dt^n} P(u(t)) \right|_{t=0} = c_n \cdot f, \quad (8)$$

where  $c_n \in \mathbb{R}$  is some constant.

*Proof.*

$\Rightarrow$  [We assume the solution is shape preserving (admits Eq.  $(\mathbf{ShP})$ ) and prove Eq. (8)].

Let us note that  $P(u(t))$  is infinitely differentiable in the range  $D$  using Eq.  $(\mathbf{P})$  and the fact that  $u(t)$  is infinitely differentiable. By substituting  $(\mathbf{P})$  and  $(\mathbf{ShP})$  in Eq. (8) we can calculate the  $n$ th derivative of  $P$ . For any  $n = 0, 1, ..$  we get:

$$\left. \frac{d^n}{dt^n} P(u(t)) \right|_{t=0} = \left. \frac{d^{n+1}}{dt^{n+1}} u(t) \right|_{t=0} = \left. a^{(n+1)}(t) \right|_{t=0} \cdot f = c_n \cdot f(x).$$

$\Leftarrow$ [We assume Eq. (8) and prove the solution is in the form of  $(\mathbf{ShP})$ ].  
The solution  $u(t)$  can be represented as a Taylor series at any time  $t$  in  $D$

because it is infinitely differentiable. Without loss of generality, we express  $u(t)$  at  $t = 0$ :

$$\begin{aligned}
u(t) &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dt^n} u(t) \Big|_{t=0} t^n \\
&= f + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{d^n}{dt^n} P(u(t)) \Big|_{t=0} t^{n+1} \\
&= f + \sum_{n=0}^{\infty} \frac{c_n f}{(n+1)!} t^{n+1} \\
&= \left( 1 + \sum_{n=1}^{\infty} \alpha_n t^n \right) f = a(t) f.
\end{aligned} \tag{9}$$

□

Let us discuss some implications of the above theorem. If the flow is shape preserving then (8) holds for any natural  $n$ , in particular for  $n = 0$ ,

$$P(f) = \lambda \cdot f, \tag{10}$$

where  $\lambda = c_0$ . Therefore,  $f$  being an eigenfunction of  $P$  is a necessary condition for a shape preserving flow. Let us further examine (8) for  $n = 1$ ,

$$\frac{d}{dt} P(u(t)) \Big|_{t=0} = \lim_{dt \rightarrow 0} \frac{P(u(t+dt)) - P(u(t))}{dt} \Big|_{t=0}.$$

When  $dt \rightarrow 0$  we can substitute  $P(u(t+dt))$  by  $P(u(t) + dt \cdot u_t)$ ,

$$\frac{d}{dt} P(u(t)) \Big|_{t=0} = \lim_{dt \rightarrow 0} \frac{P(u(t) + dt \cdot u_t) - P(u(t))}{dt} \Big|_{t=0}.$$

Substituting  $u_t$  by using **(P)** yields:

$$\frac{d}{dt} P(u(t)) \Big|_{t=0} = \lim_{dt \rightarrow 0} \frac{P(u(t) + dt \cdot P(u(t))) - P(u(t))}{dt} \Big|_{t=0}.$$

Evaluating the above expression at  $t = 0$  we get,

$$\frac{d}{dt} P(u(t)) \Big|_{t=0} = \lim_{dt \rightarrow 0} \frac{P(f + dt \cdot P(f)) - P(f)}{dt}.$$

But according to Eq. (10)  $f$  is an eigenfunction of  $P$ , therefore

$$\frac{d}{dt} P(u(t)) \Big|_{t=0} = \lim_{dt \rightarrow 0} \frac{P(f + \lambda dt \cdot f) - P(f)}{dt}. \tag{11}$$

One can observe that Eq. (11) is the directional derivative of the operator  $P$  at point  $f$  in the direction  $f$  (multiplied by  $\lambda$ ). From Theorem 1 it is linearly

dependent of  $f$ . That conclusion is valid for derivative of any order. It turns out that eigenfunctions play a crucial role in shape preserving flows.

As to the Taylor series at  $t = 0$  (Eq. (9)). This series uses the time derivatives of  $P(u)$  of any order at  $t = 0$ . These derivatives must be right-hand-sided since  $t \in [0, T)$ . The Taylor series is valid as  $u(t)$  is infinitely differentiable in  $D$ . The rest of this section is dedicated to a family of operators for which Theorem 1 holds.

## 2.2 Homogeneous operators

Here we focus on homogeneous operator of any positive degree. We show that if the operator  $P$  is homogeneous the solution of  $(\mathbf{P})$  is a shape preserving flow if and only if the initial condition  $u(t = 0) = f$  is an eigenfunction in the form of (10).

We begin by examining the general scale space flow  $(\mathbf{P})$ , showing that if the operator  $P$  is homogeneous then its time derivatives of any order are homogeneous as well (Lemma 1). We conclude that when the initial condition is an eigenfunction of  $P$  the conditions of Theorem 1 hold. Moreover, we obtain an analytic expression for the decay profile of the flow  $(\mathbf{P})$  (Theorem 2). Consequently, a simple expression for the extinction time for any eigenfunction is formulated for homogeneous operators of degree in the range  $(0, 1)$  (Prop. 2).

The homogeneity of the operator can be written in a general manner by,

$$P(au) = g(a) \cdot P(u), \quad \forall a \in \mathbb{R}, \quad (12)$$

given some function  $g(a)$ . However, since we should require restrictions such as  $g(ab) = g(a)g(b)$  ( $\forall a, b$ ) and that the sign of  $a$  is preserved, we assume  $g(a)$  takes the simple polynomial form,

$$g(a) = a|a|^{p-2}, \quad (13)$$

where  $p - 1$  is the degree of homogeneity. We use this definition as it coincides with the homogeneity of the  $p$ -Laplacian operator, which will be investigated more thoroughly later.

**Lemma 1.** *Given the scale-space  $(\mathbf{P})$  and  $u(t)$  exists and infinitely differentiable. If  $P$  is a  $g(a)$  homogeneous operator then  $(d^n/dt^n)P$  is a homogeneous operator with  $g(a) \cdot \left(\frac{g(a)}{a}\right)^n$  -homogeneity.*

*Proof.* Let us first show the homogeneity of the operator  $\partial P(u)/\partial u$ . On one hand we know that:

$$\frac{\partial P(au)}{\partial u} = g(a) \frac{\partial P(u)}{\partial u}.$$

On the other hand:

$$\frac{\partial P(au)}{\partial u} = a \frac{\partial P(au)}{\partial au}.$$

Thus we get:

$$\frac{\partial P(au)}{\partial(au)} = \frac{g(a)}{a} \frac{\partial P(u)}{\partial u}. \quad (14)$$

Another identity, which holds for  $u(t)$  admitting  $(\mathbf{P})$ , based on the chain rule, is,

$$\begin{aligned} \frac{d}{dt} P(u) &= \frac{\partial P(u)}{\partial u} \cdot u_t \\ &= \frac{\partial P(u)}{\partial u} \cdot P(u). \end{aligned} \quad (15)$$

We can now proceed and prove our claim by induction. According to Eqs. (14) and (15) we have,

$$\begin{aligned} \frac{d}{dt} P(au) &= \frac{\partial P(au)}{\partial(au)} \cdot P(au) \\ &= g(a) \frac{g(a)}{a} \frac{\partial P(u)}{\partial u} \cdot P(u) \\ &= g(a) \frac{g(a)}{a} \cdot \frac{d}{dt} P(u). \end{aligned} \quad (16)$$

Therefore, the homogeneity of the first order time derivative of  $P$  is  $g(a) \cdot \frac{g(a)}{a}$ . We note that  $a$  is in general a function of time,  $a(t)$ . According to the assumption of the induction, the homogeneity of  $d^{n-1}/dt^{n-1}P$  is  $g(a) \cdot (g(a)/a)^{n-1}$ . Let us denote the  $(n-1)$ th time derivative of  $P(u)$  as  $Q(u)$ . Then, in a similar manner to Eq. (15), we have

$$\frac{d}{dt} Q(u(t)) = \frac{\partial Q(u)}{\partial u} \cdot P(u).$$

Also, based on the assumption of the induction and similarly to Eq. (14) we get that the homogeneity of  $\partial Q/\partial u$  is  $(g(a)/a)^n$ . We therefore have,

$$\begin{aligned} \frac{d^n}{dt^n} P(au) &= \frac{d}{dt} Q(au) = \frac{\partial Q(au)}{\partial(au)} P(au) \\ &= \left( \frac{g(a)}{a} \right)^n \cdot \frac{\partial Q(u)}{\partial(u)} P(au) \\ &= g(a) \left( \frac{g(a)}{a} \right)^n \cdot \frac{\partial Q(u)}{\partial(u)} P(u) \\ &= g(a) \left( \frac{g(a)}{a} \right)^n \cdot \frac{d}{dt} Q(u), \end{aligned}$$

which proves the assertion of the lemma.  $\square$

As a conclusion of Theorem 1 and Lemma 1 we are able to state the following theorem.



**Theorem 2.** *If  $P$  is a  $p-1$  homogeneous operator, where  $g(a)$  as in (13), and  $u(t)$  is infinitely differentiable at  $t=0$ , then:*

1. *The solution of the scale space Eq. (P) exists for all  $t \in [0, T)$  and is shape preserving iff the initial condition  $u(t=0) = f$  is an eigenfunction in the sense (10).*
2. *The flow has the following analytic solution (for  $t \in [0, T)$ ):*

$$u(t) = a(t) \cdot f, \quad a(0) = 1, \quad (17a)$$

$$a(t) = [(2-p)\lambda t + 1]^{\frac{1}{2-p}} \quad (17b)$$

as long as  $a(t) \neq 0$ . Or equivalently, it is the solution of the following ODE,

$$\frac{d}{dt}a(t) = g(a(t)) \cdot \lambda, \quad a(0) = 1.$$

*Proof.*

1. According to Theorem 1 a shape preserving flow and Eq. (8) are equivalent.  
 $\Rightarrow$  [We assume (8) and prove  $u(t=0) = f$  is an eigenfunction].  
 Trivially, by assigning  $n=0$  in (8).  
 $\Leftarrow$  [We assume the initial condition is an eigenfunction and prove (8)].  
 We prove (8) by induction. The assumption of the induction is trivially valid for  $n=0$ , as  $f$  is an eigenfunction.

For any  $n$ , we assume  $(d^n/dt^n)P(u(t))|_{t=0} = c_n \cdot f$  and compute the  $(n+1)$ th time derivative of  $P(u(t))$ . Let us denote  $P^{(n)}(u(t)) := (d^n/dt^n)P(u(t))$ .  $P^{(n)}(u(t))$  is differentiable since  $u(t)$  is infinitely differentiable and using Eq. (P), thus we can directly compute  $P^{(n+1)}$  at  $t=0$  by

$$\begin{aligned} P^{(n+1)}(u(t)) \Big|_{t=0} &= \lim_{dt \rightarrow 0} \frac{P^{(n)}(u(t+dt)) - P^{(n)}(u(t))}{dt} \Big|_{t=0} \\ &= \lim_{dt \rightarrow 0} \frac{P^{(n)}(u(t) + P(u(t))dt) - P^{(n)}(u(t))}{dt} \Big|_{t=0} \\ &= \lim_{dt \rightarrow 0} \frac{P^{(n)}((1 + \lambda dt)f) - P^{(n)}(f)}{dt}. \end{aligned}$$

Using Lemma 1 we have

$$\begin{aligned} P^{(n+1)}(u(t)) \Big|_{t=0} &= \lim_{dt \rightarrow 0} \frac{g(1 + \lambda dt) \cdot \left(\frac{g(1 + \lambda dt)}{1 + \lambda dt}\right)^n \cdot P^{(n)}(f) - P^{(n)}(f)}{dt} \\ &= \lim_{dt \rightarrow 0} \frac{g(1 + \lambda dt) \cdot \left(\frac{g(1 + \lambda dt)}{1 + \lambda dt}\right)^n - 1}{dt} c_n f. \end{aligned}$$

Recall that  $g(1) = 1$  and therefore

$$g(1) \left(\frac{g(1)}{1}\right)^n = 1$$

and we have

$$= \lim_{dt \rightarrow 0} \frac{g(1 + \lambda dt) \cdot \left(\frac{g(1 + \lambda dt)}{1 + \lambda dt}\right)^n - g(1) \left(\frac{g(1)}{1}\right)^n}{dt} c_n f.$$

The above can be expressed by the following derivative

$$\begin{aligned} &= \frac{d}{d\tau} \left[ g(\tau) \left(\frac{g(\tau)}{\tau}\right)^n \right] \Big|_{\tau=1} \cdot \lambda c_n f \\ &= [(n+1)g'(1) - n] \lambda c_n f, \end{aligned}$$

and using  $g'(1) = p - 1$  yields

$$c_{n+1} = [(n+1)(p-1) - n] \lambda c_n.$$

2. Using the expression for  $c_{n+1}$  we can directly compute the decay profile and reach (17b) (see Appendix A). However, a simpler approach is to view the solution as a first order *ODE*. Since the solution is in the form of **(ShP)**, we can express the first time derivative as,

$$u_t = \frac{d}{dt} a(t) \cdot f.$$

In addition,  $u(t)$  admits the scale space equation,

$$\begin{aligned} u_t &= P(a(t)) \cdot f \\ &= g(a(t)) \lambda f. \end{aligned}$$

Thus, we get

$$\frac{d}{dt} a(t) = g(a) \lambda.$$

This is separable equation and using (13) yields,

$$\frac{d}{dt} a(t) = a|a|^{p-2} \lambda.$$

Under the assumption  $a > 0$  we have

$$a(t)^{2-p} = (2-p)\lambda t + C.$$

Based on the initial condition  $a(0) = 1$  gives

$$a(t) = [(2-p)\lambda t + 1]^{\frac{1}{2-p}}.$$

□

Let us examine the conditions for a shape preserving flow to blow up in finite time. From the analytic solution (17b) we can state the following corollary:

**Corollary 1** (Flows with finite blowup time). *If the conditions of Theorem 2 hold,  $p > 2$  and  $\lambda > 0$ , then the solution blows up in finite time  $T$ , where*

$$T = \frac{1}{(p-2)\lambda}.$$

This is an immediate consequence of (17b) for a negative power as the base tends to zero. As an example, the  $p$ -Laplacian (7) is a negative semidefinite operator. Therefore, its eigenvalues are negative, where the existence of eigenfunctions was proved in [11]. Consequently, the negative  $p$ -Laplacian induces positive eigenvalues and the inverse  $p$ -flow,  $u_t = -\Delta_p(u)$ , for  $p > 2$ , blows up in finite time.

Moreover, we can check when the flow vanishes. We refer to the first time  $T$  for which  $u(T) = 0$  as the *extinction time*. In a similar manner to Cor. 1 we can state a corollary for the extinction time.

**Corollary 2** (Flows with finite extinction time). *If the conditions of Theorem 2 hold,  $p < 2$  and  $\lambda < 0$ , then the solution has a finite extinction time*

$$T = \frac{1}{(p-2)\lambda}. \quad (18)$$

In the conditions of Cor. 2, the power in Eq. (17b) is positive and the base is zero for  $t = T$ . As for any positively homogeneous operator  $P(0) = 0$  we get that  $u$  is identically zero for  $t \geq T$ . Following the above discussion, the  $p$ -flow vanishes in finite time for  $1 < p < 2$ . In addition, if  $p \rightarrow 2$  Eq. (17a) becomes

$$a(t) = e^{\lambda t},$$

and the extinction time is infinite as expected for the heat equation. If  $p > 2$  the solution does not reach a steady state. This coincides with results in [28].

Another interesting case for the  $p$ -flow is when  $p = 1$ , where we obtain the TV-flow. For the case  $p = 1$  we get a linear decay profile (as proved in [3]). This is a subgradient flow and the conditions of Theorems 1 and 2 do not hold. However, we see that (17) still holds. Based on the decay profile of the TV-flow a new concept of nonlinear transform was introduced in [12]. This transform was defined as a second time derivative of  $u(t)$ ,

$$\phi_{TV}(t) = t \cdot u_{tt}, \quad (19)$$

for which an eigenfunction becomes a delta in the transform domain. This concept was extended to any absolutely one-homogeneous functionals (which have zero-homogeneous subgradients,  $p = 1$ ) in [6] or to the weighted problem introduced in [4]. This was used in several image-processing applications, e.g. for denoising [19], segmentation [29] and image fusion [14].

### 3 The $p$ -Laplace Spectra

In the last decade there is a rising interest in  $p$ -flows for  $1 < p < 2$  (see e.g. [16, 26, 7]). Therefore, it would be beneficial to develop a representation method based on these flows.

### 3.1 Extinction time for the $p$ -flow

In the previous section an expression for the time of extinction of the  $p$ -flow was given for eigenfunctions, ( $1 < p < 2$ ). We address now the issue of finite extinction time for arbitrary initial conditions  $f$ . Upper bounds on the extinction time were proposed in several studies. For example, in [24] the authors establish an upper bound (for  $1 < p < 2$ ) based on Barenblatt solution [2]. In [9] (see pages 188,189) the authors provide a bound for the continuous setting in a bounded domain in  $\mathbb{R}^N$ . In Proposition 3.12 in [6] an upper bound is derived for  $p = 1$  ( $TV$ -flow). In addition, it is worth mentioning that in [25] the behavior of the  $p$ -flow near the time of extinction is presented as an open problem. In Appendix B we show a different approach to proving finite extinction time of the  $p$ -flow using arguments similar to the ones of [6] in a semi-discrete setting.

### 3.2 The $p$ -Transform

In a similar manner to the  $TV$ -transform [12] and to the one-homogeneous transform [6], we would like to express an eigenfunction as a delta in time by applying a linear operator to the flow. We propose to take the time derivative of order  $\alpha$  for the  $p$ -flow, where

$$\alpha = \frac{1}{2-p} + 1. \quad (20)$$

As  $\alpha$  is not necessarily a natural number, fractional derivatives are required. There are many different definitions of the fractional calculus operators, see e.g. [17, 20, 18, 15]. All definitions coincide with derivatives in the classical sense (of integer orders) but can differ for non-integer orders. We chose to use a well known definition of Sonin and Laurent (see [20]), which is a generalization of Cauchy integral.

**Definition 1** (Fractional derivative). *The fractional derivative of order  $\alpha$  of the function  $u$  at point  $t$  is given by,*

$$\mathcal{D}^{(\alpha)}\{u(t)\} := \frac{\Gamma(\alpha+1)}{2\pi j} \oint_c \frac{u(\tau)}{(\tau-t)^{1+\alpha}} d\tau \quad (21)$$

where  $c$  is a closed curve around  $t$ ,  $\Gamma$  is the extension of the factorial function where  $\Gamma(n) = (n-1)!$  when  $n$  is natural.

We can now define the  $p$ -transform and the inverse  $p$ -transform as follows:

**Definition 2** (The  $p$ -transform). *The  $p$ -transform is defined by,*

$$\phi(t) = (-1)^{\lceil \alpha - 1 \rceil} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \mathcal{D}^\alpha \{u(t)\}, \quad (22)$$

where  $u(t)$  is the solution of (6),  $p \in (1,2)$ ,  $\alpha$  is as in (20) and  $\lceil \cdot \rceil$  is a ceil operator.

**Definition 3** (Inverse  $p$ -transform).

$$\hat{f} = \int_0^\infty \phi(t) dt. \quad (23)$$

Let us show the reconstruction property for  $\alpha \in \mathcal{N}$ .

**Proposition 1** (Reconstruction). *The function  $f$  can be reconstructed by the inverse transform (Eq. (23)), for  $\alpha \in \mathcal{N}$ .*

*Proof.* In this proof we limit ourselves to natural  $\alpha = n$ , i.e.  $p = \frac{2n-3}{n-1}$  where  $n \in \mathcal{N}$ . In this case the  $p$ -transform is:

$$\phi(t) = (-1)^{n-1} \frac{t^{n-1}}{(n-1)!} u^{(n)}(t), \quad n = 2, 3, \dots$$

Then using integration by parts we have

$$\begin{aligned} \int_0^\infty \phi(t) dt &= (-1)^{n-1} \int_0^\infty \frac{t^{n-1}}{(n-1)!} u^{(n)}(t) dt \\ &= (-1)^{n-1} \left[ \underbrace{\frac{t^{n-1}}{(n-1)!} u^{(n-1)}(t)}_{=0} \Big|_0^\infty - \int_0^\infty \frac{t^{n-2}}{(n-2)!} u^{(n-1)}(t) dt \right] \\ &= (-1)^n \left[ \underbrace{\frac{t^{n-2}}{(n-2)!} u^{(n-2)}(t)}_{=0} \Big|_0^\infty - \int_0^\infty \frac{t^{n-3}}{(n-3)!} u^{(n-2)}(t) dt \right] \\ &= \dots \\ &= (-1)^{2n-3} \int_0^\infty u'(t) dt \\ &= (-1)^{2n-2} f \\ &= f \end{aligned}$$

□

### 3.3 The $p$ -Spectra

**Definition 4** (The  $p$ -Spectrum).

$$S(t) = \langle f, \phi(t) \rangle.$$

The Parseval-type theorem holds,

$$\begin{aligned} \int S(t) dt &= \int \langle f, \phi(t) \rangle dt \\ &= \langle f, \int \phi(t) dt \rangle \\ &= \langle f, f \rangle = \|f\|^2. \end{aligned}$$

**Theorem 3** (Spectral response of an eigenfunction). *The  $p$ -transform, (22), of an eigenfunction  $f$  with eigenvalue  $\lambda$ , when  $\alpha \in \mathcal{N}$ , is:*

$$\phi(t) = \delta\left(t + \frac{\alpha - 1}{\lambda}\right) \cdot f. \quad (24)$$

*Proof.* The solution of (17) can be written as,

$$u(t) = \begin{cases} \left[\frac{\lambda}{\alpha-1}t + 1\right]^{\alpha-1} f & t \leq T \\ 0 & t > T. \end{cases}$$

When  $\alpha \in \mathcal{N}$  we can formulate the  $\alpha$ -th time derivative of  $u(t)$  as

$$\mathcal{D}^{(\alpha)}\{u(t)\} = (\alpha - 1)! \left(\frac{\lambda}{\alpha - 1}\right)^{\alpha-1} \delta\left(t + \frac{\alpha - 1}{\lambda}\right) \cdot f.$$

Assigning this in (22) gives,

$$\begin{aligned} \phi(t) &= (-1)^{\alpha-1} \frac{t^{\alpha-1}}{(\alpha - 1)!} (\alpha - 1)! \left(\frac{\lambda}{\alpha - 1}\right)^{\alpha-1} \delta\left(t + \frac{\alpha - 1}{\lambda}\right) \cdot f \\ &= (-1)^{\alpha-1} t^{\alpha-1} \left(\frac{\lambda}{\alpha - 1}\right)^{\alpha-1} \delta\left(t + \frac{\alpha - 1}{\lambda}\right) \cdot f. \end{aligned}$$

As the delta function is non zero for  $t = -(\alpha - 1)/\lambda$ , this is equivalent to

$$\begin{aligned} \phi(t) &= (-1)^{\alpha-1} \left(-\frac{\alpha - 1}{\lambda}\right)^{\alpha-1} \left(\frac{\lambda}{\alpha - 1}\right)^{\alpha-1} \delta\left(t + \frac{\alpha - 1}{\lambda}\right) \cdot f \\ &= \delta\left(t + \frac{\alpha - 1}{\lambda}\right) \cdot f. \end{aligned}$$

□

### 3.4 Filtering

The filtering is identical to the previous work on TV-transform (see e.g. [5]). The filtering is done in the time domain, as follows,

$$\hat{\phi}(t) = \phi(t) \cdot h(t), \quad (25)$$

where LP, HP, BP and BS are defined as in [12]. The filtered signal is respectively defined:

$$f_h = \int_0^\infty \hat{\phi}(t) dt. \quad (26)$$

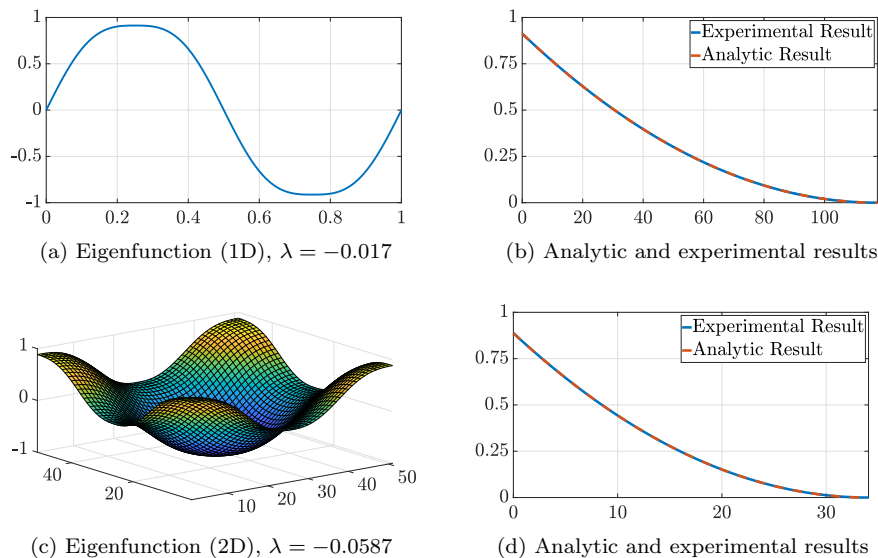


Figure 1: **Eigenfunction decay.** On the left, eigenfunctions of the  $p$ -Laplacian are shown for  $p = 1.5$  in one and two dimensions. On the right,  $u(t, x_0)$  vs. time and the theoretical plot  $a(t)f(x_0)$ .

## 4 Experiments

In this section we demonstrate numerically some of the theoretical findings presented earlier. The  $p$ -flow is implemented by simple explicit method using a standard  $3 \times 3$  spatial stencil. The step size is different for different values of  $p$ . Unless stated otherwise, our experiments are for  $p = 1.5$ . In this case the order of derivative is  $\alpha = 3$  and the step size is  $dt = 10^{-3}$ . For  $p = 1.3$  the step size is  $dt = 5 \cdot 10^{-4}$ .

**Decay profile.** In Fig. 1 we demonstrate the validity of the analytic solution of the decay profile, Eq. (17b) in one- and two-dimensions. On the left, two eigenfunctions of the  $p$ -Laplacian are shown, (with eigenvalues  $\lambda = -0.017$  (top) and  $\lambda = -0.0587$  (bottom)). On the right, the corresponding theoretical and experimental decays are plotted for a randomly selected point  $x_0$ . One can observe that the experimental results coincide well with the theoretical predictions.

**Transform and Spectrum.** We illustrate the application of the  $p$ -transform on eigenfunctions. Several eigenfunctions were generated (with different values of  $p$ ), using the algorithm of [8], see Fig. 2. We expect that the spectrum of a numerical eigenfunction will be composed of essentially one dominant peak, approximating a delta response at  $t = \frac{1}{(p-2)\lambda}$ .

In Fig. 3 we show the decay profile for  $p$ -flow for the eigenfunction whose eigenvalue  $\lambda = -0.216$ . One can see in the different subfigures the decay profile

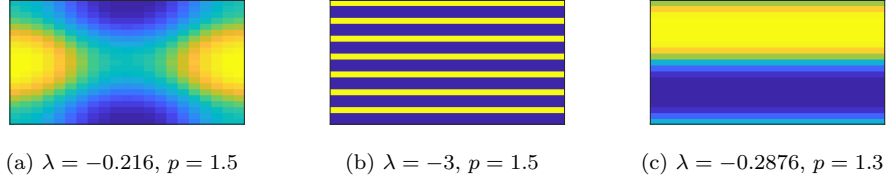


Figure 2: Eigenfunctions with different values of  $\lambda$  and  $p$ .

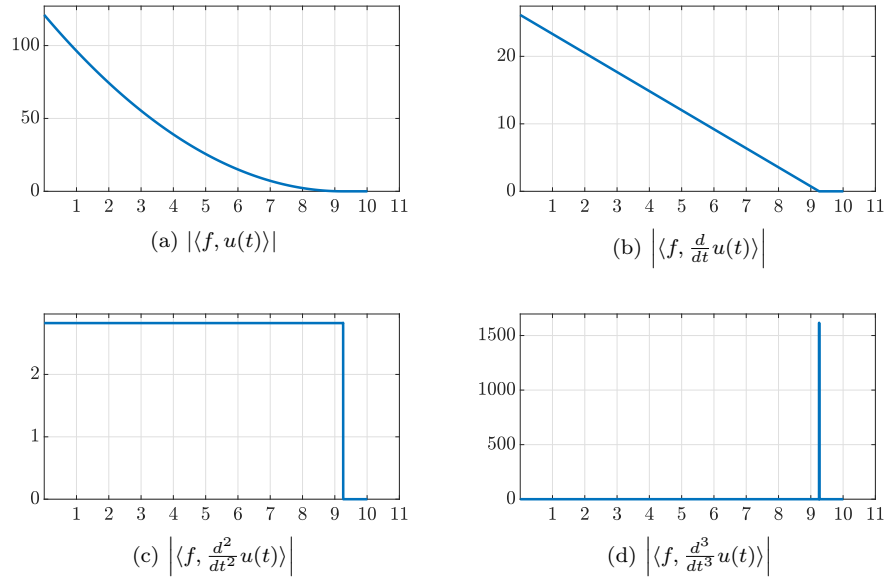


Figure 3:

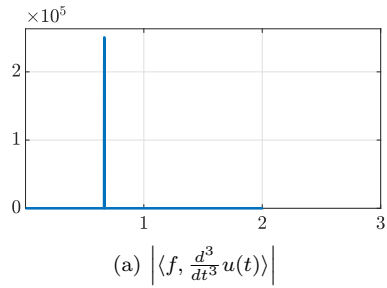


Figure 4:



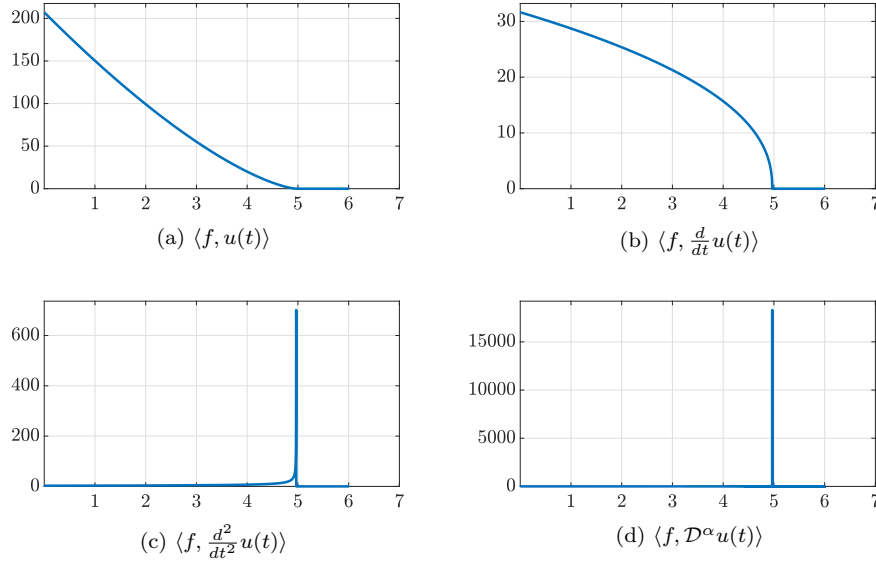


Figure 5: A demonstration of the general case where  $1/(2-p)$  is not natural and accordingly the delta function appears at  $t = 4.9667$ .

and its time derivatives. As expected, the decay profile is a polynomial of second order, its first order derivative is linear, the second derivative is a step function and the third is an isolated peak. In Fig. 4 we show the third order derivative of the  $p$ -flow and again, as  $\alpha = 3$ , we get an isolated peak.

For  $p = 1.5$ , the extinction time is  $T = -2/\lambda$ , following (18). It can be seen that the experimental results coincide with the analytic solution i.e. the extinction times are 9.24 and 0.667, as expected. We would like to note that for the eigenfunction with eigenvalue  $\lambda = -3$  we obtained numerically an almost exact eigenfunction and therefore the extinction time is predicted with very high accuracy. In general, these results fit the analytic solutions formulated in (17).

In Fig. 5 we demonstrate the transform for  $p = 1.3$ ,  $\alpha = 2\frac{3}{7}$ . Here the transform is based on fractional derivatives. There are several ways to implement fractional derivatives [21]. The derivative operator is a semigroup, therefore, we choose to separate the fractional derivative to the natural and the fractional parts. For the natural part (in our case 2) we apply standard derivation and for the fractional remainder (in our case  $\frac{3}{7}$ ) we approximate it via *Fast Fourier Transform* (FFT). The eigenvalue is  $\lambda = -0.2876$ , yielding a theoretical extinction time of  $T_{theory} = 4.967$ . The experimental result is very close where we get  $T = 4.9665$ .

**Filtering.** We implement the filtering process as suggested in Eqs. (25) and (26). Again, we use  $p$ -flow where  $p = 1.5$ , i.e.  $1/(2-p)$  is natural. In Fig. 6 we add two eigenfunctions with different eigenvalues. When the signal is

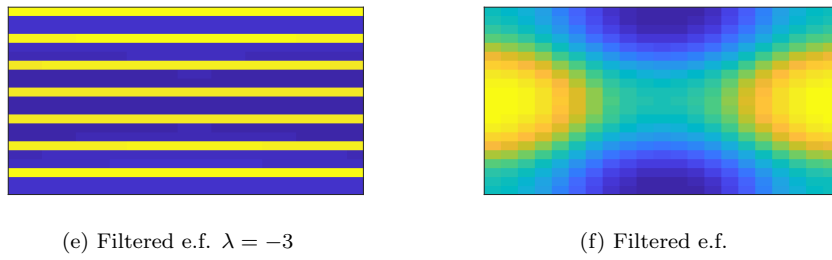
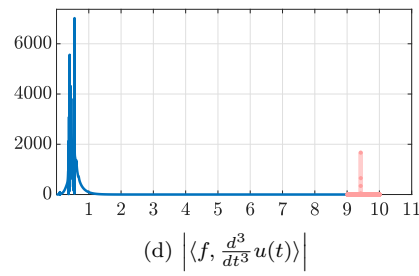
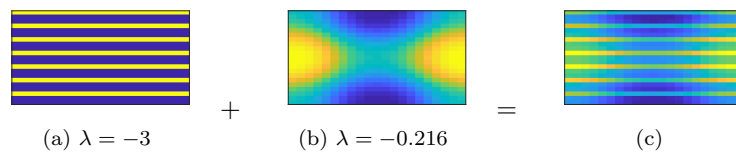
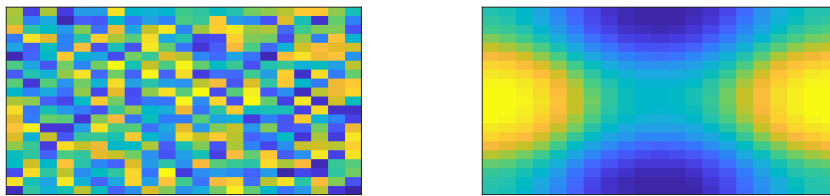
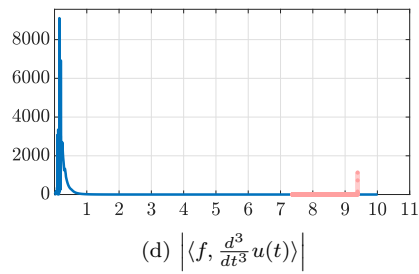
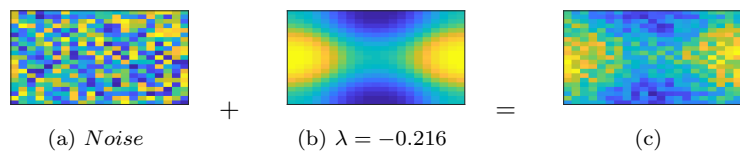


Figure 6: Distinguishing between two eigenfunctions using the time axis.



(e) The filtered out noise

(f) Recovered e.f.

Figure 7: Separation between noise and an eigenfunction.

composed of two independent eigenfunctions (see [13]) we can expect to obtain two distinct groups of peaks. And indeed we get that result in Fig. 6d. Filtering the red marked spectrum out of the signal results in recovering the eigenfunction with the lower eigenvalue. Naturally, subtracting this result from the initial image approximates well the second eigenfunction.

Now, we repeat the last experiment but with noise added to an eigenfunction (see Fig. 7). Again the signal is separated into two groups in the time (corresponding to eigenvalue or scale) axis Fig. 7d. We recovered the eigenfunction by filtering out the higher “frequencies”. The chosen part to be recovered is the red part in the Fig. 7d.

## 5 Conclusions

In this work we formulated necessary and sufficient conditions for general shape preserving flows. We have shown that homogeneous operators admit this condition when the flow is initialized with a nonlinear eigenfunction, with respect to the operator of the flow. Moreover, a closed form solution was formulated for any homogeneous operator and eigenfunction. Consequently, blowup and extinction times were easily derived.

Following these insights, we could extend the TV and one-homogeneous transforms to the  $p$ -Laplacian flow, when the flow becomes extinct in finite time,  $p \in (1, 2)$ . It was shown that the flow’s decay profile is a polynomial of order  $1/(2 - p)$ . Fractional derivatives were suggested to define the nonlinear  $p$ -transform. A full theory is still under-way, where currently we have developed results in the classical case of integer derivatives. Numerically, it was shown that the concept extends well to fractional derivatives. We have demonstrated our approach in filtering and reconstruction.

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## A An alternative proof of Theorem 2, part 2

We compute the decay profile using the first part of the proof of Theorem 2. The time derivative of  $P(u(t))$ , at  $t = 0$ , can be expressed as,

$$\begin{aligned}
 P^n(u(t)) \Big|_{t=0} &= \prod_{i=0}^n [i \cdot g'(1) - (i-1)] \lambda^{n+1} \cdot f \\
 &= [-(g'(1) - 1) \lambda]^{n+1} \prod_{i=0}^n \left[ -i - \frac{1}{g'(1) - 1} \right] \cdot f \quad (27) \\
 &= [(n+1)!] \binom{-\frac{1}{g'(1)-1}}{n+1} [-(g'(1) - 1) \lambda]^{n+1} \cdot f.
 \end{aligned}$$

Let us recall that

$$\frac{d^{n+1}}{dt^{n+1}}u(t) = \frac{d^n}{dt^n}P(u).$$

Since  $u(t)$  is infinitely differentiable, we can use a Taylor series to express  $u(t)$ ,

$$\begin{aligned} u(t) &= u(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{d^n}{dt^n} u(t) \Big|_{t=0} t^n \\ &= u(0) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{d^{n+1}}{dt^{n+1}} u(t) \Big|_{t=0} t^{n+1} \\ &= u(0) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \frac{d^n}{dt^n} P(u(t)) \Big|_{t=0} t^{n+1}. \end{aligned}$$

Using Eq. (27) and the initial condition  $u(0) = f$  we get

$$\begin{aligned} u(t) &= f + f \cdot \sum_{n=0}^{\infty} \binom{-\frac{1}{g'(1)-1}}{n+1} [-(g'(1)-1)\lambda]^{n+1} t^{n+1} \\ &= f + f \cdot \sum_{n=1}^{\infty} \binom{-\frac{1}{g'(1)-1}}{n} [-(g'(1)-1)\lambda]^n t^n \\ &= f \cdot \sum_{n=0}^{\infty} \binom{-\frac{1}{g'(1)-1}}{n} [-(g'(1)-1)\lambda]^n t^n \\ &= [-(g'(1)-1)\lambda t + 1]^{-\frac{1}{g'(1)-1}} \cdot f \end{aligned}$$

Assigning  $g'(1) = p - 1$  yields (17).

## B Finite extinction time of the $p$ -flow

The finite extinction time was proved in Theorem 2 for an eigenfunction as initial condition. In the following proposition we prove that the  $p$ -flow vanishes in finite time for arbitrary initial conditions. We show here a different approach based on [6] (Proposition 3.12).

**Proposition 2** (Finite extinction time in the semi-discrete setting). *Let  $u(t)$  be the solution to a semi-discrete scale-space flow (6) (continuous in time and discrete in space) with an initial condition  $u(0) = f$ . We assume  $f$  does not contain any null-space element of  $P$ . Then there exists  $T > 0$  such that  $u(t) = 0 \forall t \geq T$ .*

*Proof.* Let us denote  $\|\cdot\|_2$  as  $\|\cdot\|$ , then

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \|u(t)\|^2 &= \langle u(t), u_t \rangle \\ &= \langle u(t), \Delta_p(u(t)) \rangle \\ &= -\|\nabla u(t)\|_p^p. \end{aligned}$$

According to the norm equivalence we can write

$$\frac{d}{dt} \frac{1}{2} \|u(t)\|^2 \leq -c \|\nabla u(t)\|^p.$$

Under the assumption of Lipschitz continuity we have

$$\frac{d}{dt} \frac{1}{2} \|u(t)\|^2 \leq -C \|u(t)\|^p$$

Then

$$\frac{d}{dt} \|u(t)\| \leq -C \|u(t)\|^{p-1}$$

$$p - 1 \neq 1 > 0$$

$$\begin{aligned} \|u(t)\|^{1-p} \frac{d}{dt} \|u(t)\| &\leq -C \\ \frac{1}{2-p} \|u(t)\|^{2-p} &\leq \frac{1}{2-p} \|u(0)\|^{2-p} - Ct \\ \|u(t)\|^{2-p} &\leq \|f\|^{2-p} - (2-p)Ct. \end{aligned}$$

Finally, we have

$$u(t) = 0, \quad \forall t > T = \frac{\|f\|^{2-p}}{(2-p)C}$$

□