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# SOLUTIONS FOR A NONLINEAR NEUMANN PROBLEM IN THREE-DIMENSIONAL EXTERIOR DOMAINS 

# SOLUTIONS POUR UN PROBLÈME DE NEUMANN NON-LINÉAIRE DANS DES DOMAINES EXTÉRIEURS EN DIMENSION 3 

ADÉLAÏDE OLIVIER AND OLIVIER REY


#### Abstract

We prove the existence of multipeak solutions for an nonlinear elliptic Neumann problem involving nearly critical Sobolev exponent, in threedimensional exterior domains.


RÉsumé. Nous démontrons, pour un problème elliptique de Neunmann avec non-linéarité presque critique, dans un domaine extérieur en dimension trois, l'existence de solutions qui se concentrent en plusieurs points de la frontière lorsque la non-linéarité devient critique.

To Prof. Norman Dancer

## 1. Introduction and Results

There have been innumerable articles devoted over the last three decades to the study of elliptic partial differential equations of second order with critical nonlinearity. One thing, however, is to be noticed: virtually all articles consider problems in bounded domains. A work like Yan's one [15] is an exception. In that paper, Yan considers the following Neumann problem:

$$
\left\{\begin{array}{rlrl}
-\Delta u & =u^{2^{*}-1-\varepsilon} & , u>0 &  \tag{1.1}\\
\frac{\partial u}{\partial \nu} & =0 & & \mathbb{R}^{N} \backslash \Omega \\
\text { on } \partial \Omega
\end{array}\right.
$$

where $\Omega$ is a smooth and bounded domain in $\mathbb{R}^{N}, N \geq 3$, such that $R^{N} \backslash \Omega$ is connected, $2^{*}=2 N /(N-2)$ is the limiting Sobolev exponent for the embedding of the Sobolev space $W^{1,2}(\Omega)$ into the $L^{p}(\Omega)$-spaces, and $\varepsilon$ is a strictly positive number assumed to be small. Although formally, the problem is subcritical, it is asymptotically critical, and the techniques to be used to study it as $\varepsilon$ goes to zero are the same as those to be used in the case of critical nonlinearities.

Considering exterior domains is all but arbitrary. Indeed, various models lead to study the equation $-\Delta u=u^{p}$ in domains with small holes. As the size of these

[^0]holes goes to zero, the limiting problem which is obtained through a rescaling centered on one of the holes looks precisely as (1.1).

In order to state the results proved by Yan, as well as those that we propose to establish, some notations have first to be introduced. For $\lambda \in \mathbb{R}_{+}^{*}$ and $x \in \mathbb{R}^{N}$, we denote by $U_{\lambda, x}$ the function defined in $\mathbb{R}^{N}$ by

$$
\begin{equation*}
U_{\lambda, x}(y)=[N(N-2)]^{\frac{N-2}{4}} \frac{\lambda^{\frac{N-2}{2}}}{\left(1+\lambda^{2}|y-x|^{2}\right)^{\frac{N-2}{2}}} \tag{1.2}
\end{equation*}
$$

The $U_{\lambda, x}$ 's are the only nontrivial solutions to the equation $-\Delta U=U^{2^{*}-1}, U \geq 0$ in $\mathbb{R}^{N}$ (see for example [1], [13] or [7]) and induce, as $\lambda$ goes to infinity, a lack of compactness of the embedding of $W^{1,2}$ into $L^{2^{*}}$. In the following, $D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)$ refers to the completion of the set of smooth functions with compact support in $\mathbb{R}^{N} \backslash \Omega$ for the norm

$$
\|u\|=<u, u>^{1 / 2}
$$

with

$$
<u, v>=\int_{R^{N} \backslash \Omega} \nabla u . \nabla v
$$

Lastly, we denote by $H(y)$ the mean curvature of $\partial \Omega$ at a point $y$ of this boundary. Yan proves:

Theorem 1.1. [15]. Assume that $N \geq 4$.
(1) [Case of a positive local maximum of $H$.] Suppose that $S$ is a connected subset of $\partial \Omega$ satisfying: $H(y)=H_{m}>0$ for any $y \in S$; and there exists $\delta>0$ such that $H(y)<H_{m}$ for any $y \in S_{\delta} \backslash S$, and $H$ has no critical point in $S_{\delta} \backslash S$, with $S_{\delta}=\{y \in \partial \Omega$ s.t. $d(x, y) \leq \delta\}$. Then, for any positive integer $k$, there exists $\varepsilon_{0}>0$ such that for any $\varepsilon_{0} \in\left(0, \varepsilon_{0}\right)$, (1.1) has a solution

$$
\begin{equation*}
u_{\varepsilon}=\sum_{i=1}^{k} \alpha_{\varepsilon, i} U_{\lambda_{\varepsilon, i}, x_{\varepsilon, i}}+v_{\varepsilon} \tag{1.3}
\end{equation*}
$$

where, as $\varepsilon$ goes to zero
$\alpha_{\varepsilon, i} \rightarrow 1$
$\varepsilon \lambda_{\varepsilon, i} \rightarrow c^{*} H_{m} \quad c^{*}$ a positive constant depending on $N$ only
$x_{\varepsilon, i} \in S_{\delta} \quad$ and $\quad x_{\varepsilon, i} \rightarrow x_{i} \in S$
for any $i, i \leq i \leq k$, and

$$
v_{\varepsilon} \rightarrow 0 \quad \text { in } \quad D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)
$$

(2) [Case of a positive local minimum of $H$.] Suppose that $S$ is a connected subset of $\partial \Omega$ satisfying: $H(y)=H_{m}>0$ for any $y \in S$; and there exists $\delta>0$ such that $H(y)>H_{m}$ for any $y \in S_{\delta} \backslash S$, and $H$ has no critical point in $S_{\delta} \backslash S$, with $S_{\delta}=\{y \in \partial \Omega \quad$ s.t. $d(x, y) \leq \delta\}$. Then, there exists $\varepsilon_{0}>0$ such that for any $\varepsilon_{0} \in\left(0, \varepsilon_{0}\right)$, (1.1) has a solution

$$
\begin{equation*}
u_{\varepsilon}=\alpha_{\varepsilon} U_{\lambda_{\varepsilon}, x_{\varepsilon}}+v_{\varepsilon} \tag{1.4}
\end{equation*}
$$

where $\alpha_{\varepsilon, i} \rightarrow 1, \varepsilon \lambda_{\varepsilon} \rightarrow c^{*} H_{m}, x_{\varepsilon} \rightarrow x_{0} \in S$ and $v_{\varepsilon} \rightarrow 0$ in $D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)$ as $\varepsilon$ goes to zero.

The arguments developed by Yan to prove Theorem 1.1 allow him to consider also the problem

$$
\left\{\begin{array}{rlrl}
-\Delta u+\mu u & =u^{2^{*}-1} & , u>0 &  \tag{1.5}\\
\text { in } \Omega \\
\frac{\partial u}{\partial \nu} & =0 & & \text { on } \partial \Omega
\end{array}\right.
$$

where $\mu$ is a positive number assumed to be large. Yan proves, for $N \geq 5$ and a positive local minimum of $H$, an equivalent of Theorem 1.1 (1) as $\mu$ goes to infinity ( $1 / \mu$ plays a role similar to that of $\varepsilon$ previously). Such a result has been extended to the cases $N=3,4$ by Wei and Yan in [14]. On the other hand, it has never been demonstrated until now that the statement of Theorem 1.1 itself is valid in the case $N=3$. As Brezis and Nirenberg [6] have shown, there are problems involving elliptic equations with critical nonlinearity where cases $N=3$ and $N \geq 4$ are qualitatively different. This is not the case here, and we are able to prove:

Theorem 1.2. Let $N=3$.
(1) [Case of a positive local maximum of $H$.] Suppose that $S \subset \partial \Omega$ is such that $0<\sup _{y \in \partial S} H(y)<\sup _{y \in S} H(y)=H_{m}$. Then, for any positive integer $k$, there exists $\varepsilon_{0}>0$ such that for any $\varepsilon_{0} \in\left(0, \varepsilon_{0}\right)$, (1.1) has a solution

$$
\begin{equation*}
u_{\varepsilon}=\sum_{i=1}^{k} \alpha_{\varepsilon, i} U_{\lambda_{\varepsilon, i}, x_{\varepsilon, i}}+v_{\varepsilon} \tag{1.6}
\end{equation*}
$$

where, as $\varepsilon$ goes to zero

$$
\begin{aligned}
& \alpha_{\varepsilon, i} \rightarrow 1 \\
& \frac{\ln \lambda_{\varepsilon, i}}{\varepsilon \lambda_{\varepsilon, i}} \rightarrow \frac{\pi}{16 H_{m}} \quad\left(\text { i.e. } \lambda_{\varepsilon, i} \sim \frac{16 H_{m}}{\pi \varepsilon} \ln \frac{1}{\varepsilon}\right) \\
& x_{\varepsilon, i} \rightarrow x_{i} \in S \quad \text { such that } \quad H\left(x_{i}\right)=\max _{y \in S} H(y)
\end{aligned}
$$

for any $i, 1 \leq i \leq k$, and

$$
v_{\varepsilon} \rightarrow 0 \quad \text { in } \quad D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)
$$

(2) [Case of a positive local minimum of $H$.] Suppose that $S \subset \partial \Omega$ is such that $0<\inf _{y \in S} H(y)<\inf _{y \in \partial S} H(y)=H_{m}$. Then, there exists $\varepsilon_{0}>0$ such that for any $\varepsilon_{0} \in\left(0, \varepsilon_{0}\right)$, (1.1) has a solution

$$
\begin{equation*}
u_{\varepsilon}=\alpha_{\varepsilon} U_{\lambda_{\varepsilon}, x_{\varepsilon}}+v_{\varepsilon} \tag{1.7}
\end{equation*}
$$

where $\alpha_{\varepsilon, i} \rightarrow 1, \frac{\ln \lambda_{\varepsilon, i}}{\varepsilon \lambda_{\varepsilon, i}} \rightarrow \frac{\pi}{16 H_{m}}, x_{\varepsilon} \rightarrow x_{0} \in S$ such that $H\left(x_{i}\right)=\min _{y \in S} H(y)$ and $v_{\varepsilon} \rightarrow 0$ in $D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)$ as $\varepsilon$ goes to zero.

Remark 1. As $\partial S$ is closed and bounded, $\sup _{y \in \partial S} H(y)$ and $\inf _{y \in \partial S} H(y)$ are achieved. The same holds for $\sup _{y \in S} H(y)$ in case (1) and $\inf _{y \in S} H(y)$ in case (2). Indeed, let us consider in case (1) a maximizing sequence $\left(y_{n}\right)$ in $S$ for $H$. A
subsequence of $\left(y_{n}\right)$ converges to some limit $\bar{y}$ which satisfies $H(\bar{y})=H_{m}$. As, by assumption, $\sup _{y \in \partial S} H(y)<H_{m}, \bar{y}$ has to be in the interior of $S$. Consequently, there exist one or several points $y \in S$ such that $H(y)=H_{m}$, and which are local maxima of H . We conclude in the same way in case (2).
Remark 2. As $\Omega$ is assumed to be bounded in $\mathbb{R}^{N}, H$ has a strictly positive maximum on $\partial \Omega$, and case (1) always occurs (in the particular case of a ball, we can take $S=\partial \Omega, \partial S=\varnothing$ ).

Remark 3. The method we use to prove Theorem 1.2 allows us to get rid of the unessential assumption in the statement of Theorem 1.1, that $H$ has no critical point in $S_{\delta} \backslash S$. Our argument to eliminate such an assumption applies as well to the case $N \geq 4$.

Problem (1.1) can be formulated in a variational way: formally, $u \in D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)$ solves (1.1) if and only if $u$ is a nontrivial critical point of the functional

$$
\begin{equation*}
I_{\varepsilon}=\frac{1}{2} \int_{R^{N} \backslash \Omega}|\nabla u|^{2}-\frac{1}{2^{*}-\varepsilon} \int_{R^{N} \backslash \Omega}\left(u^{+}\right)^{2^{*}-\varepsilon} \tag{1.8}
\end{equation*}
$$

with $u^{+}=\max (u, 0)$. Indeed, a critical point of $I_{\varepsilon}$ satisfies

$$
\begin{equation*}
-\Delta u=\left(u^{+}\right)^{2^{*}-\varepsilon} \text { in } R^{N} \backslash \Omega, \quad ; \quad \frac{\partial u}{\partial \nu}=0 \text { on } \partial \Omega \tag{1.9}
\end{equation*}
$$

Multiplying the equation by $u^{-}=\max (-u, 0)$ and integrating on $R^{N} \backslash \Omega$, we see that $u^{-} \equiv 0$, whence $u \geq 0$ in $R^{N} \backslash \Omega$. If $u \not \equiv 0$, the strong maximum principle implies that $u>0$ in $R^{N} \backslash \Omega$; as a consequence, $u$ is a solution of (1.1).

However, a difficulty arises: $I_{\varepsilon}$ is not defined in whole $D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)$ for, as $R^{N} \backslash \Omega$ is not bounded, this space does not embed into $L^{2^{*}-\varepsilon}\left(R^{N} \backslash \Omega\right)$. Our first task will therefore be, in the next section, to build a functional $\tilde{I}_{\varepsilon}$ well defined in $D^{1,2}\left(\mathbb{R}^{N} \backslash \Omega\right)$, and whose critical points which write as in Theorem 1.1 or 1.2 are solutions of (1.1).

In Section 3 we shall perform a parametrization of the problem in a neighbourhood of the solutions of type (1.3) (1.6) we look for, in order to obtain a functional depending on the $\alpha_{i}$ 's, $\lambda_{i}$ 's, $x_{i}$ 's and $v$.

In Section 4, an optimization of that functional with respect to the $\alpha_{i}$ 's and $v$ will provide us with a function now dependent only on the $\lambda_{i}$ 's and the $x_{i}$ 's. Then we will be able, in the last section (Section 5), to deduce from the assumptions of Theorem 1.2, that the reduced function has a critical point - whence, by construction, the existence of a solution of (1.1) with the properties specified in Theorem 1.2.

The proof of a number of technical results necessary for the exposition of the main argument is given in Appendix.

## 2. The variational formulation

Although this article is intended to establish Theorem 1.2, and is therefore concerned only with space dimension $N=3$, we consider in this section any dimension $N \geq 3$ - insofar the argument is identical in all dimensions. Moreover, we shall make explicit some points which are sketched in [15]. To obtain, from $I_{\varepsilon}$ defined
by (1.8), a functional $\widetilde{I}_{\varepsilon}$ well defined in $D^{1,2}\left(R^{N} \backslash \Omega\right)$, the nonlinearity has to be truncated at infinity. By adapting Yan's strategy in [15], we proceed as follows.

We choose $R>0$ such that $\Omega \subset B_{R / 2}(0)$, and $\tau>0$. Let $\varphi: \mathbb{R}_{+} \times \mathbb{R} \rightarrow \mathbb{R}$, $0 \leq \varphi \leq 1, \varphi$ smooth in $\mathbb{R}_{+} \times \mathbb{R}_{+}$, such that

$$
\begin{cases}\varphi(s, t)=0 & \text { if } t<0  \tag{2.1}\\ \varphi(s, t)=1 & \text { if } 0 \leq s \leq R \text { and } t \geq 0, \text { or } s \geq R \text { and } 0 \leq s^{N-2} t \leq \tau \\ \varphi(s, t)=0 & \text { if } s \geq R+1 \text { and } s^{N-2} t \geq \tau+1\end{cases}
$$

Then, we define $g_{\varepsilon}: \mathbb{R}^{N} \times \mathbb{R} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
g_{\varepsilon}(y, t)=|t|^{2^{*}-1-\varepsilon} \varphi(|y|, t) \tag{2.2}
\end{equation*}
$$

which, for $\varepsilon$ small enough, is $C^{1}$, and we consider the problem

$$
\left\{\begin{array}{rlrl}
-\Delta u & =g_{\varepsilon}(y, u) & & \text { in }  \tag{2.3}\\
R^{N} \backslash \Omega \\
\frac{\partial u}{\partial \nu} & =0 & & \text { on } \partial \Omega
\end{array}\right.
$$

By construction, $g_{\varepsilon}(y, u)=\left(u^{+}\right)^{2^{*}-1-\varepsilon}$ for $|y| \leq R$, or $|y| \geq R$ and $|y|^{N-2} u^{+} \leq$ $\tau ; g_{\varepsilon}(y, u)=0$ for $|y| \geq R+1$ and $|y|^{N-2} u^{+} \geq \tau+1$. We notice that, according to definition (1.2) of the $U_{\lambda, x}$ 's, when $x \in \partial \Omega$ (whence $\left.|x|<R\right),|y|^{N-2} U_{\lambda, x}(y)$ goes to zero as $\lambda$ goes to infinity, uniformly with respect to $y,|y| \geq R$. Consequently, when the $x_{i}$ 's belong to $\partial \Omega$, the $\alpha_{i}$ 's are close to 1 and the $\lambda_{i}$ 's are large enough, we have

$$
g_{\varepsilon}\left(y, \sum_{i=1}^{k} \alpha_{i} U_{\lambda_{i}, x_{i}}(y)\right)=\left(\sum_{i=1}^{k} \alpha_{i} U_{\lambda_{i}, x_{i}}(y)\right)^{2^{*}-1-\varepsilon}
$$

in all $R^{N} \backslash \Omega$ : the truncation does not affect the nonlinearity when $u=\sum_{i=1}^{k} \alpha_{i} U_{\lambda_{i}, x_{i}}$, in the neighborhood of which we look for a solution of (1.1). We set now

$$
\begin{equation*}
\widetilde{I}_{\varepsilon}(u)=\frac{1}{2} \int_{R^{N} \backslash \Omega}|\nabla u|^{2}-\int_{R^{N} \backslash \Omega} G_{\varepsilon}(y, u) \tag{2.4}
\end{equation*}
$$

with

$$
G_{\varepsilon}(y, u)=\int_{0}^{u} g_{\varepsilon}(y, t) d t
$$

According to the definition of $g_{\varepsilon}, G_{\varepsilon}(y, u)=0$ if $u \leq 0, G_{\varepsilon}(y, u)=\frac{1}{2^{*}-\varepsilon} u^{2^{*}-\varepsilon}$ if $u \geq 0$ and $|y| \leq R$, and $\left|G_{\varepsilon}(y, u)\right| \leq \frac{1}{2^{*}-\varepsilon}|u|^{2^{*}-\varepsilon}$ everywhere. Moreover, (2.1) and (2.2) imply that if $|y| \geq R+1$ and $|y|^{N-2}|u| \geq \tau+1, g_{\varepsilon}(y, u)=0$, and if $|y| \geq R+1$ and $|y|^{N-2}|u| \leq \tau+1$

$$
\begin{equation*}
\left|\frac{g_{\varepsilon}(y, u)}{u}\right| \leq|u|^{2^{*}-2-\varepsilon} \leq \frac{(\tau+1)^{\frac{4}{N-2}-\varepsilon}}{|y|^{4-(N-2) \varepsilon}} \tag{2.5}
\end{equation*}
$$

Then, using Young's inequality, we can write, for $|y| \geq R+1$,

$$
0 \leq G_{\varepsilon}(y, u) \leq \frac{u^{2}(\tau+1)^{\frac{4}{N-2}-\varepsilon}}{2|y|^{4-(N-2) \varepsilon}} \leq(\tau+1)^{\frac{4}{N-2}-\varepsilon}\left(\frac{|u|^{2^{*}}}{2^{*}}+\frac{1}{N|y|^{2 N-\frac{N(N-2)}{2} \varepsilon}}\right)
$$

As $D^{1,2}\left(R^{N} \backslash \Omega\right)$ embeds into $L^{2^{*}}\left(R^{N} \backslash \Omega\right)$, we see that $\widetilde{I}_{\varepsilon}$ is well defined in this functional space.

Arguing as we did before for the solutions of (1.8), we know that a nontrivial solution of (2.3) is strictly positive in $R^{N} \backslash \Omega$. Let us show that a solution $u_{\varepsilon}$ of (2.3), writing as (1.3) (1.6), is actually a solution of (1.1). To this end, we shall show that $|y|^{N-2} u_{\varepsilon}(y)$ goes uniformly to 0 in $\mathbb{R}^{N} \backslash B_{R}(0)$ as $\varepsilon$ goes to zero - this entails, by definition of $g_{\varepsilon}$, that $g_{\varepsilon}\left(y, u_{\varepsilon}\right)=u_{\varepsilon}^{2^{*}-1-\varepsilon}$ in $R^{N} \backslash \Omega$, whence the result. Setting

$$
w_{\varepsilon}(y)=\frac{1}{|y|^{N-2}} u_{\varepsilon}\left(\frac{y}{|y|^{2}}\right)
$$

this amounts to showing $w_{\varepsilon}$ goes uniformly to 0 in $B_{1 / R}(0)$. As $u_{\varepsilon}$ is assumed to solve (2.3), $w_{\varepsilon}$ satisfies, in $B_{2 / R}(0) \backslash\{0\}$

$$
-\Delta w_{\varepsilon}=\frac{1}{|y|^{N+2}} \Delta u_{\varepsilon}\left(\frac{y}{|y|^{2}}\right)=\frac{1}{|y|^{N+2}} g_{\varepsilon}\left(\frac{y}{|y|^{2}},|y|^{N-2} w_{\varepsilon}(y)\right)
$$

We may also write

$$
\begin{equation*}
-\Delta w_{\varepsilon}=a_{\varepsilon}(y) w_{\varepsilon} \quad \text { in } B_{2 / R}(0) \backslash\{0\} \tag{2.6}
\end{equation*}
$$

where, according to the definition (2.1) (2.2) of $g_{\varepsilon}, a_{\varepsilon}$ satisfies

$$
\begin{equation*}
0<a_{\varepsilon}(y) \leq \frac{1}{|y|^{(N-2) \varepsilon}} w_{\varepsilon}^{2^{*}-2-\varepsilon} \quad \text { in } B_{2 / R}(0) \tag{2.7}
\end{equation*}
$$

and also, because of (2.5)

$$
\begin{equation*}
0<a_{\varepsilon}(y) \leq \frac{C}{|y|^{(N-2) \varepsilon}} \quad \text { in } B_{1 /(R+1)}(0) \tag{2.8}
\end{equation*}
$$

where $C$ is a constant depending only on $\tau$ and $N$.
Actually, we can check that (2.6) holds in whole $B_{2 / R}(0)$. To this end, we consider $\psi \in C^{\infty}\left(\mathbb{R}^{N}, \mathbb{R}\right)$ such that $\psi(y)=0$ if $|y| \leq 1 / 2$ and $\psi(y)=1$ if $|y| \geq 1$, and we set, for $n \in \mathbb{N}^{*}, \psi_{n}(y)=\psi(n y)$. For any $\varphi \in C_{0}^{\infty}\left(B_{2 / R}(0)\right), \psi_{n} \varphi \in$ $C_{0}^{\infty}\left(B_{2 / R}(0) \backslash\{0\}\right)$, and (2.6) yields

$$
\left\langle-\Delta w_{\varepsilon}, \psi_{n} \varphi\right\rangle=\int_{B_{2 / R}(0)} a_{\varepsilon}(y) w_{\varepsilon} \psi_{n} \varphi
$$

Through dominated convergence, it is easily checked that the right hand side goes to $\int_{B_{2 / R}(0)} a_{\varepsilon}(y) w_{\varepsilon} \varphi$ as $n$ goes to infinity. On the left hand side we have

$$
\left\langle-\Delta w_{\varepsilon}, \psi_{n} \varphi\right\rangle=-\left\langle w_{\varepsilon}, \psi_{n} \Delta \varphi+2 \nabla \psi_{n} . \nabla \varphi+\varphi \Delta \psi_{n}\right\rangle
$$

and

$$
\begin{aligned}
-\left\langle w_{\varepsilon}, \varphi \Delta \psi_{n}\right\rangle & =\int_{1 / 2 n \leq|y| \leq 1 / n} w_{\varepsilon} \varphi \Delta \psi_{n} \\
& =\mathrm{O}\left(n^{2}\left(\int_{1 / 2 n \leq|y| \leq 1 / n} w_{\varepsilon}^{2^{*}}\right)^{\frac{1}{2^{*}}}\left(\int_{1 / 2 n \leq|y| \leq 1 / n} d y\right)^{\frac{N+2}{2 N}}\right)
\end{aligned}
$$

As

$$
\int_{1 / 2 n \leq|y| \leq 1 / n} w_{\varepsilon}^{2^{*}}=\int_{n \leq|y| \leq 2 n} \frac{1}{|y|^{2 N}} w_{\varepsilon}^{2^{*}}\left(\frac{y}{|y|^{2}}\right)=\int_{n \leq|y| \leq 2 n} u_{\varepsilon}^{2^{*}}
$$

we see that

$$
-\left\langle w_{\varepsilon}, \varphi \Delta \psi_{n}\right\rangle=\mathrm{o}\left(n^{-\frac{N-2}{2}}\right)
$$

The term involving $\nabla \psi_{n} . \nabla \varphi$ may be treated in the same way, so that

$$
\lim _{n \rightarrow \infty}\left\langle-\Delta w_{\varepsilon}, \psi_{n} \varphi\right\rangle=-\lim _{n \rightarrow \infty}\left\langle w_{\varepsilon}, \psi_{n} \Delta \varphi\right\rangle=-\left\langle w_{\varepsilon}, \Delta \varphi\right\rangle=\left\langle-\Delta w_{\varepsilon}, \varphi\right\rangle
$$

and (2.6) holds in whole $B_{2 / R}(0)$, as announced.
Now, in view (2.7) and(2.8), for any $y_{0} \in B_{3 /(2 R)}(0)$ and any $\delta, 0<\delta<\frac{1}{4 R}$, we can write

$$
\begin{aligned}
\int_{B_{\delta}\left(y_{0}\right)} a_{\varepsilon}^{N / 2} & =\int_{B_{\delta}\left(y_{0}\right) \cap B_{1 /(R+1)}(0)} a_{\varepsilon}^{N / 2}+\int_{B_{\delta}\left(y_{0}\right) \backslash B_{1 /(R+1)}(0)} a_{\varepsilon}^{N / 2} \\
& \leq C_{1} \int_{B_{\delta}\left(y_{0}\right)}|y|^{-\frac{N(N-2)}{2} \varepsilon} d y+C_{2}(R+1)^{\frac{N(N-2)}{2} \varepsilon} \int_{B_{7 /(4 R)}(0)} w_{\varepsilon}^{\frac{N}{2}\left(2^{*}-2-\varepsilon\right)} \\
& \leq C_{3} \delta^{N-\frac{N(N-2)}{2} \varepsilon}+C_{4}\left(\int_{B_{7 /(4 R)}(0)} w_{\varepsilon}^{2^{*}}\right)^{1-\frac{N-2}{4} \varepsilon}
\end{aligned}
$$

where $C_{1}, \ldots, C_{4}$ are constants depending only on $\tau, R$ and $N$. Furthermore, we have

$$
\begin{equation*}
\int_{B_{7 /(4 R)}(0)} w_{\varepsilon}^{2^{*}}=\int_{\mathbb{R}^{N} \backslash B_{4 R / 7}(0)} u_{\varepsilon}^{2^{*}} \tag{2.9}
\end{equation*}
$$

and, when $u_{\varepsilon}$ write as (1.3) (1.6), the last integral goes to zero as $\varepsilon$ goes to zero (remember that the $U_{\lambda_{\varepsilon, i}, x_{\varepsilon, i}}$ 's concentrate at points located on the boundary of $\Omega$, with $\Omega \subset B_{R / 2}(0)$, and $v_{\varepsilon}$ goes to zero in $D^{1,2}\left(R^{N} \backslash \Omega\right)$ whence also in $L^{2^{*}}\left(R^{N} \backslash \Omega\right)$ ). Consequently, we see that for every $\eta>0$ we can choose $\delta>0$ such that, for $\varepsilon$ small enough,

$$
\begin{equation*}
\int_{B_{\delta}\left(y_{0}\right)} a_{\varepsilon}^{N / 2}<\eta \tag{2.10}
\end{equation*}
$$

uniformly with respect to $y_{0} \in B_{3 /(2 R)}(0)$. This will imply that $w_{\varepsilon}$ goes to zero in $L^{\infty}\left(B_{1 / R}(0)\right)$.

Indeed, let us consider $\delta^{\prime}, 0<\delta^{\prime}<\delta$, and $\psi \in C^{\infty}\left(\mathbb{R}^{N}\right)$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $B_{\delta^{\prime}}(0), \psi \equiv 0$ in $\mathbb{R}^{N} \backslash B_{\delta}(0)$. For $y_{0} \in B_{3 /(2 R)}(0)$, let $\psi_{y_{0}} \in C^{\infty}\left(\mathbb{R}^{3}\right)$ be the function defined by $\psi_{y_{0}}(y)=\psi\left(y-y_{0}\right)$. Multiplying (2.6) by $\psi_{y_{0}}^{2} w_{\varepsilon}^{\gamma}, \gamma>1$, and integrating in $B_{\delta}\left(y_{0}\right)$, we obtain

$$
\begin{equation*}
-\int_{B_{\delta}\left(y_{0}\right)} \Delta w_{\varepsilon} \cdot \psi_{y_{0}}^{2} w_{\varepsilon}^{\gamma}=\int_{B_{\delta}\left(y_{0}\right)} a_{\varepsilon} \psi_{y_{0}}^{2} w_{\varepsilon}^{\gamma+1} \tag{2.11}
\end{equation*}
$$

provided that the integrals are well defined. On the one hand, integrating by parts, we have

$$
\begin{aligned}
&-\int_{B_{\delta}\left(y_{0}\right)} \Delta w_{\varepsilon} \cdot \psi_{y_{0}}^{2} w_{\varepsilon}^{\gamma}=\frac{4 \gamma}{(\gamma+1)^{2}} \int_{B_{\delta}\left(y_{0}\right)}\left|\nabla\left(\psi_{y_{0}} w_{\varepsilon}^{\frac{\gamma+1}{2}}\right)\right|^{2} \\
&- \frac{4(\gamma-1)}{(\gamma+1)^{2}} \int_{B_{\delta}\left(y_{0}\right)} w_{\varepsilon}^{\frac{\gamma+1}{2}} \nabla \psi_{y_{0}} \cdot \nabla\left(\psi_{y_{0}} w_{\varepsilon}^{\frac{\gamma+1}{2}}\right) \\
& \quad-\frac{4}{(\gamma+1)^{2}} \int_{B_{\delta}\left(y_{0}\right)}\left|\nabla \psi_{y_{0}}\right|^{2} w_{\varepsilon}^{\gamma+1}
\end{aligned}
$$

On the other hand, Hölder's inequality yields

$$
\int_{B_{\delta}\left(y_{0}\right)} a_{\varepsilon} \psi_{y_{0}}^{2} w_{\varepsilon}^{\gamma+1} \leq\left(\int_{B_{\delta}\left(y_{0}\right)} a_{\varepsilon}^{N / 2}\right)^{\frac{2}{N}}\left(\int_{B_{\delta}\left(y_{0}\right)} \psi_{y_{0}}^{2^{*}} w_{\varepsilon}^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{N}}
$$

Coming back to (2.11), we deduce from (2.10) and Cauchy-Schwarz inequality

$$
\begin{aligned}
& \frac{4 \gamma}{(\gamma+1)^{2}} \int_{B_{\delta}\left(y_{0}\right)}\left|\nabla\left(\psi_{y_{0}} w_{\varepsilon}^{\frac{\gamma+1}{2}}\right)\right|^{2} \\
& \quad-\frac{4(\gamma-1)}{(\gamma+1)^{2}}\left(\int_{B_{\delta}\left(y_{0}\right)}\left|\nabla \psi_{y_{0}}\right|^{2} w_{\varepsilon}^{\gamma+1}\right)^{\frac{1}{2}}\left(\int_{B_{\delta}\left(y_{0}\right)}\left|\nabla\left(\psi_{y_{0}} w_{\varepsilon}^{\frac{\gamma+1}{2}}\right)\right|^{2}\right)^{\frac{1}{2}} \\
& \quad \leq \frac{4}{(\gamma+1)^{2}} \int_{B_{\delta}\left(y_{0}\right)}\left|\nabla \psi_{y_{0}}\right|^{2} w_{\varepsilon}^{\gamma+1}+\eta^{\frac{2}{N}}\left(\int_{B_{\delta}\left(y_{0}\right)} \psi_{y_{0}}^{2^{*}} w_{\varepsilon}^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{2}} .
\end{aligned}
$$

Therefore, there exists a constant $C$ such that

$$
\int_{B_{\delta}\left(y_{0}\right)}\left|\nabla\left(\psi_{y_{0}} w_{\varepsilon}^{\frac{\gamma+1}{2}}\right)\right|^{2} \leq C\left[\int_{B_{\delta}\left(y_{0}\right)} w_{\varepsilon}^{\gamma+1}+\eta^{\frac{2}{N}}\left(\int_{B_{\delta}\left(y_{0}\right)} \psi_{y_{0}}^{2^{*}} w_{\varepsilon}^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{2}}\right]
$$

Still assuming that the integrals are well defined, $\psi_{y_{0}} w_{\varepsilon}^{\frac{\gamma+1}{2}} \in W_{0}^{1,2}\left(B_{\delta}\left(y_{0}\right)\right)$, and as $W_{0}^{1,2}\left(B_{\delta}\left(y_{0}\right)\right)$ embeds continuously into $L^{2^{*}}\left(B_{\delta}\left(y_{0}\right)\right)$, we have

$$
\left(\int_{B_{\delta}\left(y_{0}\right)} \psi_{y_{0}}^{2^{*}} w_{\varepsilon}^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{2}} \leq \frac{1}{S} \int_{B_{\delta}\left(y_{0}\right)}\left|\nabla\left(\psi_{y_{0}} w_{\varepsilon}^{\frac{\gamma+1}{2}}\right)\right|^{2}
$$

where $S$ is a strictly positive constant (which depends only on $N$ ). We see that, choosing $\eta$ small enough, there exists a constant $C^{\prime}$, depending only on $\delta$ and $N$, such that

$$
\begin{equation*}
\left(\int_{B_{\delta}\left(y_{0}\right)} \psi_{y_{0}}^{2^{*}} w_{\varepsilon}^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{2}} \leq C^{\prime} \int_{B_{\delta}\left(y_{0}\right)} w_{\varepsilon}^{\gamma+1} \tag{2.12}
\end{equation*}
$$

With $\gamma=2^{*}-1$, we obtain

$$
\begin{equation*}
\left(\int_{B_{\delta^{\prime}}\left(y_{0}\right)} w_{\varepsilon}^{\frac{N}{N-2} 2^{*}}\right)^{\frac{N-2}{2}} \leq C^{\prime} \int_{B_{\delta}\left(y_{0}\right)} w_{\varepsilon}^{2^{*}} \tag{2.13}
\end{equation*}
$$

The right hand side integral is well defined, which proves that the left hand side is also well defined. (Actually, to make the previous argument perfectly rigorous, we should have multiplied (2.11) by $\psi_{y_{0}}^{2} w_{\varepsilon, n}^{\gamma}$, where, for $n \in \mathbb{N} *, w_{\varepsilon, n}(y)=w_{\varepsilon}(y)$ if $w_{\varepsilon}(y) \leq n, w_{\varepsilon, n}(y)=n$ otherwise. Then, all the integrals in the previous computations are well defined and, letting $n$ go to infinity, we obtain (2.13).)

The right hand side of (2.13) goes to zero as $\varepsilon$ goes to zero, as (2.9) proves. As a consequence, $w_{\varepsilon}$ goes to zero in $L^{\frac{N}{N-2}} 2^{*}\left(B_{\delta^{\prime}}\left(y_{0}\right)\right)$, uniformly with respect to $y_{0}$, and thus in $L^{\frac{N}{N-2}} 2^{*}\left(B_{3 /(2 R)}(0)\right)$. Iterating the process, we find through (2.12) that $w_{\varepsilon}$ goes to zero in every $L^{p}\left(B_{3 /(2 R)}(0)\right), p<\infty$. This implies, taking into account (2.7), that $a_{\varepsilon}$ goes to zero in $L^{q}\left(B_{3 /(2 R)}(0)\right)$ for some $q>N / 2$ (actually, for any $q<\infty$ ). Then, standard theory for elliptic equations (see e.g. Theorem 8.24 in [9]) ensures that $w_{\varepsilon}$ goes to zero in $L^{\infty}\left(B_{1 / R}(0)\right)$. Therefore, $g_{\varepsilon}\left(y, u_{\varepsilon}\right)=u_{\varepsilon}^{2^{*}-1-\varepsilon}$ in $R^{N} \backslash \Omega$ and $u_{\varepsilon}$ is a solution of (1.1).

## 3. Parametrization of the variational problem

In this section, we shall still consider the general case $N \geq 3$. Let $k \in \mathbb{N}^{*}$. We set

$$
\begin{align*}
\mathcal{D}_{\varepsilon}=\left\{(\Lambda, X) \in\left(\mathbb{R}_{+}^{*}\right)^{k} \times(\partial \Omega)^{k}\right. & \text { s.t. } f_{1}(\varepsilon)<\lambda_{i}<f_{2}(\varepsilon)  \tag{3.1}\\
& \text { and } \left.\left|x_{i}-x_{j}\right|>h(\varepsilon), 1 \leq i, j \leq k, i \neq j\right\}
\end{align*}
$$

where $\Lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right), X=\left(x_{1}, \ldots, x_{n}\right)$ and $f_{1}, f_{2}, h$ are positive functions such that $f_{1}(\varepsilon)$ and $f_{2}(\varepsilon)$ go to infinity, $h(\varepsilon)$ goes to zero as $\varepsilon$ goes to zero, and whose precise expression will be determined later. We define also, for $(\Lambda, X) \in$ $\left(\mathbb{R}_{+}^{*}\right)^{k} \times(\partial \Omega)^{k}$

$$
\begin{align*}
E_{\Lambda, X}=\left\{v \in D^{1,2}\left(R^{N} \backslash \Omega\right) \text { s.t. }\left\langle v, U_{i}\right\rangle=\left\langle v, \frac{\partial U_{i}}{\partial \lambda_{i}}\right\rangle\right. & =\left\langle v, \frac{\partial U_{i}}{\partial \tau_{i, j}}\right\rangle=0  \tag{3.2}\\
1 & \leq i \leq k, 1 \leq j \leq N-1\}
\end{align*}
$$

where $U_{i}=U_{\lambda_{i}, x_{i}}$, and $\left(\tau_{i, 1}, \ldots, \tau_{i, j}\right)$ is an orthogonal system of coordinates of the tangent space to $\partial \Omega$ at $x_{i}$. Lastly, for $\delta>0$, we define

$$
\begin{align*}
\mathcal{M}_{\varepsilon, \delta}=\{ & (A, \Lambda, X, v) \in \mathbb{R}^{k} \times\left(\mathbb{R}_{+}^{*}\right)^{k} \times(\partial \Omega)^{k} \times D^{1,2}\left(R^{N} \backslash \Omega\right)  \tag{3.3}\\
& \text { s.t. } \left.\left|\alpha_{i}-1\right|<\delta, 1 \leq i \leq k ;(\Lambda, X) \in \mathcal{D}_{\varepsilon} ; v \in E_{\Lambda, X},\|v\|<\delta\right\}
\end{align*}
$$

where $A=\left(\alpha_{1}, \ldots, \alpha_{k}\right)$, and we consider on $\mathcal{M}_{\varepsilon, \delta}$ the functional

$$
\begin{equation*}
J_{\varepsilon}(A, \Lambda, X, v)=\widetilde{I}_{\varepsilon}\left(\sum_{i=1}^{k} \alpha_{i} U_{i}+v\right) \tag{3.4}
\end{equation*}
$$

According to [3, 4], we know that for $\varepsilon$ and $\delta$ small enough, $(A, \Lambda, X, v)$ is a critical point of $J_{\varepsilon}$ in $\mathcal{M}_{\varepsilon, \delta}$ if and only if $u=\sum_{i=1}^{k} \alpha_{i} U_{i}+v$ is a critical point of $\widetilde{I}_{\varepsilon}$ in $D^{1,2}\left(R^{N} \backslash \Omega\right)$. Let us remark that, in consideration of (3.2), $(A, \Lambda, X, v)$ is a critical point of $J_{\varepsilon}$ in $\mathcal{M}_{\varepsilon, \delta}$ if and only if there exists Lagrange multipliers $A_{i}, B_{i}$, $C_{i j}, 1 \leq i \leq k, 1 \leq j \leq N-1$ such that

$$
\begin{align*}
\frac{\partial J_{\varepsilon}}{\partial \alpha_{i}} & =0  \tag{3.5}\\
\frac{\partial J_{\varepsilon}}{\partial \lambda_{i}} & =B_{i}\left\langle\frac{\partial^{2} U_{i}}{\partial \lambda_{i}^{2}}, v\right\rangle+\sum_{\ell=1}^{N-1} C_{i \ell}\left\langle\frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \tau_{i j}}, v\right\rangle  \tag{3.6}\\
\frac{\partial J_{\varepsilon}}{\partial \tau_{i j}} & =B_{i}\left\langle\frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \tau_{i j}}, v\right\rangle+\sum_{\ell=1}^{N-1} C_{i \ell}\left\langle\frac{\partial^{2} U_{i}}{\partial \tau_{i j} \partial \tau_{i \ell}}, v\right\rangle  \tag{3.7}\\
\frac{\partial J_{\varepsilon}}{\partial v} & =\sum_{i=1}^{k} A_{i} U_{i}+\sum_{i=1}^{k} B_{i} \frac{\partial U_{i}}{\partial \lambda_{i}}+\sum_{i=1}^{k} \sum_{\ell=1}^{N-1} C_{i \ell} \frac{\partial U_{i}}{\partial \tau_{i \ell}} \tag{3.8}
\end{align*}
$$

In order to find critical points of $J_{\varepsilon}$ in $\mathcal{M}_{\varepsilon, \delta}$ we shall proceed in two steps. Firstly we shall eliminate the non-significant parameters : the $\alpha_{i}$ 's and $v$. More precisely, we shall prove the existence of a $C^{1}$-map which to every $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$ associates $A_{\varepsilon}(\Lambda, X) \in \mathbb{R}^{k}$, each $\alpha_{\varepsilon, i}$ close to 1 , and $v_{\varepsilon} \in E_{\Lambda, X},\left\|v_{\varepsilon}\right\|$ close to zero, such that (3.5) and (3.8) are satisfied. It will be left to us, in a second time, to show, using
a topological argument, that the function $(\Lambda, X) \mapsto J_{\varepsilon}\left(A_{\varepsilon}(\Lambda, X), \Lambda, X, v_{\varepsilon}(\Lambda, X)\right)$ has a critical point in $\mathcal{D}_{\varepsilon}$.

## 4. The reduced functional

We first perform a change of variables concerning the parameters $\alpha_{i}$ 's, setting

$$
\begin{equation*}
A^{\prime}=A-\mathbf{1} \tag{4.1}
\end{equation*}
$$

where $\mathbf{1}=(1, \ldots, 1)$, that is $A^{\prime}=\left(\alpha_{1}^{\prime}, \ldots, \alpha_{k}^{\prime}\right)$, with

$$
\begin{equation*}
\alpha_{i}^{\prime}=\alpha_{i}-1, \quad 1 \leq i \leq k \tag{4.2}
\end{equation*}
$$

We define also, on $\mathbb{R}^{k} \times D^{1,2}\left(R^{N} \backslash \Omega\right)$, the scalar product

$$
\begin{equation*}
\ll(A, f),(B, g) \gg=\sum_{i=1}^{k} a_{i} b_{i}+\int_{R^{N} \backslash \Omega} \nabla f . \nabla g \tag{4.3}
\end{equation*}
$$

and $\|\|\cdot\|\|$ will denote the associated norm. Let us expand $J_{\varepsilon}(A, \Lambda, X, v)$ with respect to $w=\left(A^{\prime}, v\right)$ in a neighborhood of $(\mathbf{1}, 0)$ in $\mathbb{R}^{k} \times E_{\Lambda, X}$. We can write, using Riesz representation theorem

$$
\begin{array}{r}
J_{\varepsilon}(A, \Lambda, X, v)=J_{\varepsilon}(\mathbf{1}, \Lambda, X, 0)+\ll f_{\varepsilon, \Lambda, X}, w \gg+\frac{1}{2} \ll Q_{\varepsilon, \Lambda, X} w, w \gg  \tag{4.4}\\
+R_{\varepsilon, \Lambda, X}(w)
\end{array}
$$

where $f_{\varepsilon, \Lambda, X} \in \mathbb{R}^{k} \times E_{\Lambda, X}$ is such that

$$
\begin{align*}
& \ll f_{\varepsilon, \Lambda, X}, w \gg=\sum_{i=1}^{k}\left[\left\langle\sum_{j=1}^{k} U_{j}, U_{i}\right\rangle-\int_{R^{N} \backslash \Omega} U_{i} g_{\varepsilon}\left(y, \sum_{j=1}^{k} U_{j}\right)\right] \alpha_{i}^{\prime}  \tag{4.5}\\
&-\int_{R^{N} \backslash \Omega} g_{\varepsilon}\left(y, \sum_{j=1}^{k} U_{j}\right) v
\end{align*}
$$

$Q_{\varepsilon, \Lambda, X}$ is an endomorphism of $\mathbb{R}^{k} \times E_{\Lambda, X}$ such that

$$
\begin{align*}
\ll Q_{\varepsilon, \Lambda, X} w, w \gg \sum_{i, j=1}^{k} & {\left[\left\langle U_{i}, U_{j}\right\rangle-\int_{R^{N} \backslash \Omega} U_{i} U_{j} \frac{\partial g_{\varepsilon}}{\partial t}\left(y, \sum_{\ell=1}^{k} U_{\ell}\right)\right] \alpha_{i}^{\prime} \alpha_{j}^{\prime} }  \tag{4.6}\\
& -\sum_{i=1}^{k}\left(\int_{R^{3} \backslash \Omega} \frac{\partial g_{\varepsilon}}{\partial t}\left(y, \sum_{\ell=1}^{k} U_{\ell}\right) U_{i} v\right) \alpha_{i}^{\prime} \\
& +\|v\|^{2}-\int_{R^{N} \backslash \Omega} \frac{\partial g_{\varepsilon}}{\partial t}\left(y, \sum_{\ell=1}^{k} U_{\ell}\right) v^{2}
\end{align*}
$$

and $R_{\varepsilon, \Lambda, X}$ satisfies

$$
\begin{equation*}
R_{\varepsilon, \Lambda, X}^{(m)}(w)=\mathrm{O}\left(\|w\|^{\min \left(2^{*}-\varepsilon, 3\right)-m}\right), \quad m=0,1,2 \tag{4.7}
\end{equation*}
$$

(In the special case $N=3,2^{*}=6$ and $R_{\varepsilon, \Lambda, X}^{(m)}(w)=\mathrm{O}\left(\|w\|^{3-m}\right), m=0,1,2$.) The fact that equations (3.5) and (3.8) are satisfied is equivalent to

$$
\begin{equation*}
f_{\varepsilon, \Lambda, X}+Q_{\varepsilon, \Lambda, X} w+R_{\varepsilon, \Lambda, X}^{\prime}=0 \tag{4.8}
\end{equation*}
$$

In the special case $N=3$ we choose in the definition (3.1) of $\mathcal{D}_{\varepsilon}$ (for reasons that will become clear later)

$$
\begin{equation*}
f_{1}(\varepsilon)=\frac{a_{1}}{\varepsilon} \ln \frac{1}{\varepsilon}, \quad f_{2}(\varepsilon)=\frac{a_{2}}{\varepsilon} \ln \frac{1}{\varepsilon}, \quad h(\varepsilon)=b\left(\ln \frac{1}{\varepsilon}\right)^{-3 / 4} \tag{4.9}
\end{equation*}
$$

where $a_{1}, a_{2}, b$, with $a_{1}<a_{2}$, are strictly positive constants which will be determined later. (For $N \geq 4$, following [15], we would choose $f_{1}(\varepsilon)=a_{1} / \varepsilon$, $f_{2}(\varepsilon)=a_{2} / \varepsilon, h(\varepsilon)=\varepsilon^{1-\frac{1+\theta}{N-2}}$ with $\theta$ a small positive number.) Then it is proved in Appendix B1 that

$$
\begin{equation*}
\left\|f_{\varepsilon, \Lambda, X}\right\|=\mathrm{O}\left(\varepsilon^{1 / 2}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right) \tag{4.10}
\end{equation*}
$$

Moreover it is proved in Appendix B2 that $Q_{\varepsilon, \Lambda, X}$ is invertible and there exists $\rho$, independent of $\varepsilon$ and $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$, such that

$$
\begin{equation*}
\left\|\left\|Q_{\varepsilon, \Lambda, X}^{-1}\right\|\right\| \leq \rho \tag{4.11}
\end{equation*}
$$

Let us consider now the map $F_{\varepsilon, \Lambda, X}$, from a neighborhood of $(0,0)$ in $\left(\mathbb{R}^{k} \times E_{\Lambda, X}\right)^{2}$ to $\mathbb{R}^{k} \times E_{\Lambda, X}$ defined by

$$
F_{\varepsilon, \Lambda, X}(f, w)=f+Q_{\varepsilon, \Lambda, X} w+R_{\varepsilon, \Lambda, X}^{\prime}(w)
$$

We have

$$
F_{\varepsilon, \Lambda, X}(0,0)=0 \quad \text { and } \quad \frac{\partial F_{\varepsilon, \Lambda, X}}{\partial w}=Q_{\varepsilon, \Lambda, X}+R_{\varepsilon, \Lambda, X}^{\prime \prime}(w)
$$

From (4.11) and (4.7), we deduce the existence of $\eta>0$ such that for $\varepsilon$ small enough and $\|\mid w\|<\eta, \frac{\partial F_{\varepsilon, \Lambda, X}}{\partial w}$ is invertible. Then, taking into account (4.10), the implicit function theorem allows us to state:

Proposition 4.1. There exists $\varepsilon_{0}>0$ such that for every $\varepsilon \in\left(0, \varepsilon_{0}\right)$, a $C^{1}$-map exists which to every $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$ associates $w_{\varepsilon, \Lambda, X}=\left(A_{\varepsilon, \Lambda, X}^{\prime}, v_{\varepsilon, \Lambda, X}\right) \in \mathbb{R}^{k} \times E_{\Lambda, X}$ such that $F_{\varepsilon, \Lambda, X}\left(f_{\varepsilon, \Lambda, X}, w_{\varepsilon, \Lambda, X}\right)=0$. This means that (3.5) and (3.8) are satisfied when the $\alpha^{i}$ 's and $v$ are such that $A=\mathbf{1}+A_{\varepsilon, \Lambda, X}^{\prime}$ and $v=v_{\varepsilon, \Lambda, X}$. Moreover, when $N=3$,

$$
\begin{align*}
\left|\alpha_{i ; \varepsilon, \Lambda, X}^{\prime}\right| & =\left|\alpha_{i ; \varepsilon, \Lambda, X}-1\right|=\mathrm{O}\left(\varepsilon^{1 / 2}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right), \quad 1 \leq i \leq k  \tag{4.12}\\
\left\|v_{\varepsilon, \Lambda, X}\right\| & =\mathrm{O}\left(\varepsilon^{1 / 2}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right) \tag{4.13}
\end{align*}
$$

as $\varepsilon$ goes to zero.
Remark. Actually, estimate (4.12) could be improved: we could prove that $\left|\alpha_{i}^{\prime}\right|=\mathrm{O}\left(\varepsilon \ln \frac{1}{\varepsilon}\right)$. However, (4.12) is sufficient for our purposes.

We consider now the reduced functional

$$
\begin{equation*}
\widetilde{J}_{\varepsilon}(\Lambda, X)=J_{\varepsilon}\left(\alpha_{\varepsilon}(\Lambda, X), \Lambda, X, v_{\varepsilon}(\Lambda, X)\right) \tag{4.14}
\end{equation*}
$$

in $\mathcal{D}_{\varepsilon}$. For $\widetilde{J}_{\varepsilon}$ and its derivatives with respect to the $\lambda_{i}$ 's, the following expansion holds:

Proposition 4.2. Let $N=3$. For $\varepsilon \in\left(0, \varepsilon_{0}\right)$ and $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$, we have:

$$
\begin{align*}
& \widetilde{J}_{\varepsilon}(\Lambda, X)=k K_{1, \varepsilon}+  \tag{4.15}\\
& +\sum_{i=1}^{k}\left(K_{2} H\left(x_{i}\right) \frac{\ln \lambda_{i}}{\lambda_{i}}+K_{3} \varepsilon \ln \lambda_{i}\right) \\
&  \tag{4.16}\\
& -K_{4} \sum_{\substack{i, j=1 \\
i \neq j}}^{k} \frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}\left|x_{i}-x_{j}\right|}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1}\right) \\
& \frac{\partial \widetilde{J}_{\varepsilon}}{\partial \lambda_{i}}(\Lambda, X)=-K_{2} H\left(x_{i}\right) \frac{\ln \lambda_{i}}{\lambda_{i}^{2}}+K_{3} \frac{\varepsilon}{\lambda_{i}}+\mathrm{O}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}\right)
\end{align*}
$$

where $K_{\varepsilon, 1}, K_{2}, K_{3}, K_{4}$ are strictly positive constants.
This proposition is proved in Appendix A. We are now able to prove Theorem 1.2.

## 5. Proof of Theorem 1.2

For sake of simplicity, we always assume now that $N=3$. We shall concentrate our attention on part (1) of Theorem 1.2 (the proof of part (2) will then be straightforward). We look for a critical point $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$ of $\widetilde{J}_{\varepsilon}$. The $x_{i}$ 's will be supposed to belong to the subset $S$ which, according to the assumptions of Theorem 1.2, is such that

$$
0<\max _{y \in \partial S} H(y)<\max _{y \in S} H(y)=H_{m} .
$$

More precisely, the $x_{i}$ 's will be supposed to converge, as $\varepsilon$ goes to zero, to points $\bar{x}_{i}$ which satisfy $H\left(\bar{x}_{i}\right)=H_{m}$. Then according to (4.16), the $\lambda_{i}$ 's should be close to $\lambda(\varepsilon)$ defined by

$$
\begin{equation*}
\frac{\ln \lambda(\varepsilon)}{\lambda(\varepsilon)}=\frac{K_{3} \varepsilon}{K_{2} H_{m}} \tag{5.1}
\end{equation*}
$$

A simple computation yields the expansions

$$
\begin{align*}
& \lambda(\varepsilon)=\frac{K_{2} H_{m}}{K_{3} \varepsilon} \ln \frac{1}{\varepsilon}+\frac{K_{2} H_{m}}{K_{3} \varepsilon} \ln \ln \frac{1}{\varepsilon}+\mathrm{O}\left(\frac{1}{\varepsilon}\right)  \tag{5.2}\\
& \ln \lambda(\varepsilon)=\ln \frac{1}{\varepsilon}+\ln \ln \frac{1}{\varepsilon}+\mathrm{O}(1) \tag{5.3}
\end{align*}
$$

Now, we are able to fix $a_{1}$ and $a_{2}$ in the definition (4.9) of functions $f_{1}$ and $f_{2}$ which occur in (3.1): we choose $a_{1}$ and $a_{2}$ such that

$$
\begin{equation*}
0<a_{1}<\frac{K_{2} H_{m}}{K_{3}}<a_{2} \tag{5.4}
\end{equation*}
$$

(For example, in view of (A.23), we may take $a_{1}=15 H_{m} / \pi$ and $a_{2}=17 H_{m} / \pi$.)
In order to prove the theorem, we are going to argue by contradiction. The general strategy is the following: we define two levels, $c_{\varepsilon, 1}$ and $c_{\varepsilon, 2}, c_{\varepsilon, 1}<c_{\varepsilon, 2}$, and assume that $\widetilde{J}_{\varepsilon}$ has no critical value between them. Then we use the gradient flow to deform the level sets of $\widetilde{J}_{\varepsilon}$ corresponding to $c_{\varepsilon, 2}$ and $c_{\varepsilon, 1}$ one into the other. The difference of topology between the two level sets provides us with a contradiction.

Let us make the argument precise. Firstly, we define the two levels $c_{\varepsilon, 1}$ and $c_{\varepsilon, 2}$ as follows. In view of (4.15), we set

$$
\begin{equation*}
c_{1, \varepsilon}=k K_{\varepsilon, 1}+\sum_{i=1}^{k}\left(K_{2} H_{m} \frac{\ln \lambda(\varepsilon)}{\lambda(\varepsilon)}+K_{3} \varepsilon \ln \lambda(\varepsilon)\right)-\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4} \tag{5.5}
\end{equation*}
$$

Taking into account (5.1), we can also write

$$
\begin{equation*}
c_{1, \varepsilon}=k K_{\varepsilon, 1}+k K_{3} \varepsilon(\ln \lambda(\varepsilon)+1)-\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4} \tag{5.6}
\end{equation*}
$$

Concerning $c_{2, \varepsilon}$, we just set $c_{2, \varepsilon}=k K_{\varepsilon, 1}+\tau$ where $\tau$ is some small positive constant. It is clear that for $\varepsilon$ small enough, $c_{2, \varepsilon}>c_{1, \varepsilon}$. Next, we have to define the subset of $\mathcal{D}_{\varepsilon}$ on which the deformation argument will be implemented. We set

$$
\begin{equation*}
S^{\prime}=\left\{x \in S \text { s.t. } H(x)>H_{m}-a_{0}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right\} \tag{5.7}
\end{equation*}
$$

where $a_{0}$ is a strictly positive constant to be determined later,

$$
\begin{align*}
& M=\left\{X \in(\partial \Omega)^{k} \text { s.t. } x_{i} \in S^{\prime} \text { and }\left|x_{i}-x_{j}\right|>b\left(\ln \frac{1}{\varepsilon}\right)^{-3 / 4}\right.  \tag{5.8}\\
& \qquad1 \leq i, j \leq k, i \neq j\}
\end{align*}
$$

( $b$, which already occurs in the definition (4.9), will be determined later) and $T$ is the interval

$$
\begin{equation*}
T=]\left(1-D\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right) \lambda(\varepsilon),\left(1+D\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right) \lambda(\varepsilon)[ \tag{5.9}
\end{equation*}
$$

where $D$ is a large constant which will be chosen later. We note that for $\varepsilon$ small enough, every $\lambda \in T$ satisfies $\frac{a_{1}}{\varepsilon} \ln \frac{1}{\varepsilon}<\lambda<\frac{a_{2}}{\varepsilon} \ln \frac{1}{\varepsilon}$, and $T^{k} \times M \subset \mathcal{D}_{\varepsilon}$. We note also that for $\varepsilon$ small enough, $T^{k} \times M \subset \widetilde{J}_{\varepsilon}{ }^{c_{\varepsilon, 2}}$, where $\widetilde{J}_{\varepsilon}{ }^{\omega}, \omega \in \mathbb{R}$, is the level set $\left\{(\Lambda, X) \in \mathcal{D}_{\varepsilon}\right.$ s.t. $\left.\widetilde{J}_{\varepsilon}(\Lambda, X)<\omega\right\}$.

We consider now the gradient flow

$$
\left\{\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t}(\Lambda(t), X(t)) & =-\nabla \widetilde{J}_{\varepsilon}(\Lambda(t), X(t))  \tag{5.10}\\
(\Lambda(0), X(0)) & \in\left(T^{k} \times M\right)
\end{align*}\right.
$$

Arguing by contradiction, we assume :
(H) $\widetilde{J}_{\varepsilon}$ has no critical value between $c_{\varepsilon, 1}$ and $c_{\varepsilon, 2}$ in $T^{k} \times M$.

The following holds:
Lemma 5.1. The flow line $(\Lambda(t), X(t))$ defined by (5.8) and starting from a point of $T^{k} \times M$ does not leave $T^{k} \times M$ before it reaches $\widetilde{J}_{\varepsilon}^{c_{\varepsilon, 1}}$.

Proof. We have to check that for any $(\Lambda, X) \in \partial\left(T^{k} \times M\right)$ either $-\nabla \widetilde{J}_{\varepsilon}$ points inwards, or $\widetilde{J}_{\varepsilon}$ is less than $c_{\varepsilon, 1}$.

Case 1. $X \in \partial M$.
Case 1 . There is some $j$ such that $H\left(x_{j}\right)=H_{m}-a_{0}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}$.
According to (4.15), we have

$$
\begin{align*}
\widetilde{J}_{\varepsilon}(\Lambda, X) \leq k K_{1, \varepsilon}+\sum_{i=1}^{k}\left(K_{2}\right. & \left.H\left(x_{i}\right) \frac{\ln \lambda_{i}}{\lambda_{i}}+K_{3} \varepsilon \ln \lambda_{i}\right)  \tag{5.11}\\
& +K_{2}\left(H\left(x_{j}\right)-H_{m}\right) \frac{\ln \lambda_{j}}{\lambda_{j}}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1}\right)
\end{align*}
$$

When $\lambda_{i}=\lambda(\varepsilon)\left(1+\zeta_{i}\right), \zeta_{i}$ small, a simple computations yields

$$
K_{2} H_{m} \frac{\ln \lambda_{i}}{\lambda_{i}}+K_{3} \varepsilon \ln \lambda_{i}=K_{2} H_{m} \frac{\ln \lambda(\varepsilon)}{\lambda(\varepsilon)}+K_{3} \varepsilon \ln \lambda(\varepsilon)+\mathrm{O}\left(\frac{\zeta_{i}}{\lambda(\varepsilon)}+\varepsilon \zeta_{i}^{2}\right)
$$

and, if $\zeta_{i}=\mathrm{O}\left(\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right)$, as it is the case when $\lambda_{i} \in T$

$$
\frac{\zeta_{i}}{\lambda(\varepsilon)}+\varepsilon \zeta_{i}^{2}=\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right)
$$

Moreover, we have, using (5.9) and (5.1) (5.2)

$$
\frac{\ln \lambda_{j}}{\lambda_{j}}=\frac{\ln \lambda(\varepsilon)}{\lambda(\varepsilon)}+\mathrm{O}\left(\zeta_{j} \frac{\ln \lambda(\varepsilon)}{\lambda(\varepsilon)}\right)=\frac{K_{3} \varepsilon}{K_{2} H_{m}}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right)
$$

Therefore, taking into account (5.5),

$$
\widetilde{J}_{\varepsilon}(\Lambda, X) \leq c_{\varepsilon, 1}+\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}-a_{0} \frac{K_{3}}{H_{m}} \varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right)
$$

so that $\widetilde{J}_{\varepsilon}(\Lambda, X)<c_{\varepsilon, 1}$ for $\varepsilon$ small enough, provided that $a_{0}>\frac{H_{m}}{K_{3}}$. For example we set in (5.7)

$$
\begin{equation*}
a_{0}=2 \frac{H_{m}}{K_{3}} \tag{5.12}
\end{equation*}
$$

(or, in view of (A.23), $a_{0}=\frac{64}{\sqrt{3} \pi^{2}} H_{m}$ ).
Case $1 b$. There is some $i, j, i \neq j$, such that $\left|x_{i}-x_{j}\right|=b\left(\ln \frac{1}{\varepsilon}\right)^{-3 / 4}$.
According to (4.15) and using the previous computations, we have in that case

$$
\widetilde{J}_{\varepsilon}(\Lambda, X) \leq c_{\varepsilon, 1}+\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}-\frac{K_{4}}{b \lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}} \varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{3 / 4}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right)
$$

As we have also, from (5.9)

$$
\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}=\frac{1}{\lambda(\varepsilon)}\left(1+\mathrm{O}\left(\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right)\right)
$$

and from (5.2)

$$
\begin{equation*}
\frac{1}{\lambda(\varepsilon)}=\frac{K_{3} \varepsilon}{K_{2} H_{m}}\left[\frac{1}{\ln \frac{1}{\varepsilon}}+\mathrm{O}\left(\frac{\ln \ln \frac{1}{\varepsilon}}{\left(\ln \frac{1}{\varepsilon}\right)^{2}}\right)\right] \tag{5.13}
\end{equation*}
$$

we obtain

$$
\widetilde{J}_{\varepsilon}(\Lambda, X) \leq c_{\varepsilon, 1}+\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}-\frac{K_{3} K_{4}}{b K_{2} H_{m}} \varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}+\mathrm{o}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right)
$$

Therefore, $\widetilde{J}_{\varepsilon}(\Lambda, X)<c_{\varepsilon, 1}$ for $\varepsilon$ small enough, provided that $b<\frac{K_{3} K_{4}}{K_{2} H_{m}}$. For example, we set in (4.9) and (5.8)

$$
\begin{equation*}
b=\frac{K_{3} K_{4}}{2 K_{2} H_{m}} \tag{5.14}
\end{equation*}
$$

(or, in view of (A.23), $b=\frac{\sqrt{3} \pi^{2}}{32 H_{m}}$ ).
Case 2. $\Lambda \in \partial T$.
Case 2a. There is some $j$ such that $\lambda_{j}=\left(1+D\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right) \lambda(\varepsilon)$.

In view of (4.16), we compute, using the fact that $H\left(x_{i}\right) \leq H_{m}$, (5.1)-(5.3) and (5.13)

$$
\begin{aligned}
-K_{2} H\left(x_{i}\right) \frac{\ln \lambda_{j}}{\lambda_{j}^{2}}+K_{3} \frac{\varepsilon}{\lambda_{j}} & \geq-K_{2} H_{m} \frac{\ln \lambda_{j}}{\lambda_{j}^{2}}+K_{3} \frac{\varepsilon}{\lambda_{j}} \\
& \geq \frac{K_{3}^{2}}{K_{2} H_{m}} D \varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}+\mathrm{o}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}\right)
\end{aligned}
$$

From (4.16), we know that there exists a constant $C$, independent of $D$, such that

$$
\frac{\partial \widetilde{J}_{\varepsilon}}{\partial \lambda_{i}}(\Lambda, X) \geq-K_{2} H\left(x_{i}\right) \frac{\ln \lambda_{i}}{\lambda_{i}^{2}}+K_{3} \frac{\varepsilon}{\lambda_{i}}-C\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}\right)
$$

Therefore, if $D$ is chosen large enough, so that $\frac{K_{3}^{2}}{K_{2} H_{m}} D>C$, the inequality implies $\frac{\partial \widetilde{J}_{\varepsilon}}{\partial \lambda_{j}}(\Lambda, X)>0$, so that $-\nabla \widetilde{J}_{\varepsilon}(\Lambda, X)$ is directed toward the interior of $T^{k} \times M$.
Case 2b. There is some $j$ such that $\lambda_{j}=\left(1-D\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right) \lambda(\varepsilon)$.
According to (4.16), (5.7) and (5.12), we have

$$
\frac{\partial \widetilde{J}_{\varepsilon}}{\partial \lambda_{j}}(\Lambda, X) \leq-K_{2} H_{m}\left(1-\frac{2}{K_{3}}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right) \frac{\ln \lambda_{j}}{\lambda_{j}^{2}}+K_{3} \frac{\varepsilon}{\lambda_{j}}+\mathrm{O}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}\right)
$$

and the same kind of computations as in the previous subcase yield

$$
\frac{\partial \widetilde{J}_{\varepsilon}}{\partial \lambda_{j}}(\Lambda, X) \leq-\frac{K_{3}^{2}}{K_{2} H_{m}}\left(D-\frac{2}{K_{3}}\right) \varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}+\mathrm{O}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}\right)
$$

where once again the last term denotes a quantity whose absolute value is less than $C \varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}$ for some constant $C$ independent of $D$. Consequently, if $D$ is chosen large enough, $\frac{\partial \widetilde{J}_{\varepsilon}}{\partial \lambda_{j}}(\Lambda, X)<0$, so that $-\nabla \widetilde{J}_{\varepsilon}(\Lambda, X)$ is directed toward the interior of $T^{k} \times M$.

We can now complete the proof of Theorem 1.2. We define $S^{\prime \prime} \subset S^{\prime}$ and $M^{\prime} \subset M$ as

$$
\begin{equation*}
S^{\prime \prime}=\left\{x \in S \text { s.t. } H(x)>H_{m}-\frac{H_{m}}{4 k K_{3}}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right\} \tag{5.15}
\end{equation*}
$$

and

$$
\begin{align*}
M^{\prime}= & \left\{X \in(\partial \Omega)^{k} \text { s.t. } x_{i} \in S^{\prime \prime}\right.  \tag{5.16}\\
& \text { and } \left.\left|x_{i}-x_{j}\right|>4 k(k-1) \frac{K_{3} K_{4}}{K_{2} H_{m}}\left(\ln \frac{1}{\varepsilon}\right)^{-3 / 4}, 1 \leq i, j \leq k, i \neq j\right\}
\end{align*}
$$

We claim that for $\varepsilon$ small enough

$$
\begin{equation*}
\forall(\Lambda, X) \in T^{k} \times M^{\prime} \quad \widetilde{J}_{\varepsilon}(\Lambda, X)>c_{\varepsilon, 1} \tag{5.17}
\end{equation*}
$$

Let us assume that (5.17) is true. For $X \in M^{\prime}$, we take $(\Lambda(\varepsilon), X)$ as initial value in (5.10) - with $\Lambda(\varepsilon)=(\lambda(\varepsilon), \ldots, \lambda(\varepsilon))$. Lemma 5.1 and (5.17) imply that the flow line has to meet $\partial M^{\prime}$ before $(\Lambda(t), X(t))$ reaches $\widetilde{J}_{\varepsilon}^{c_{\varepsilon, 1}}$. Consequently the flow, projected onto the $X$-variable, provides us with a deformation of $M^{\prime}$ onto $\partial M^{\prime}$. However, $M^{\prime}$ is topologically different from $\partial M^{\prime}$ - see Proposition B1 in [8] - hence a contradiction. Consequently, assumption $(\mathrm{H})$ is not true - that is, $\widetilde{J}_{\varepsilon}$ has
a critical point in $T^{k} \times M$, which provides us with a solution of (1.1) satisfying the statements of Theorem 1.2 (1).

It only remains to prove (5.17). Let $(\Lambda, X) \in T^{k} \times M^{\prime}$. According to (4.15), we have

$$
\begin{array}{r}
\widetilde{J}_{\varepsilon}(\Lambda, X)=k K_{1, \varepsilon}+\sum_{i=1}^{k}\left(K_{2} H_{m} \frac{\ln \lambda_{i}}{\lambda_{i}}+K_{3} \varepsilon \ln \lambda_{i}\right)-\sum_{i=1}^{k} K_{2}\left(H_{m}-H\left(x_{i}\right)\right) \frac{\ln \lambda_{i}}{\lambda_{i}} \\
-K_{4} \sum_{\substack{i, j=1 \\
i \neq j}}^{k} \frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}\left|x_{i}-x_{j}\right|}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1}\right) .
\end{array}
$$

We remark that for $\lambda_{i} \in T$

$$
\frac{\ln \lambda_{i}}{\lambda_{i}}=\frac{K_{3} \varepsilon}{K_{2} H_{m}}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right)
$$

so that, using (5.15),

$$
\sum_{i=1}^{k} K_{2}\left(H_{m}-H\left(x_{i}\right)\right) \frac{\ln \lambda_{i}}{\lambda_{i}} \leq \frac{\varepsilon}{4}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right)
$$

We have also, using (5.16) and (5.2),

$$
K_{4} \sum_{\substack{i, j=1 \\ i \neq j}}^{k} \frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}\left|x_{i}-x_{j}\right|} \leq \frac{\varepsilon}{4}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4} \ln \ln \frac{1}{\varepsilon}\right)
$$

Lastly, we remark that in view of (5.1) and (5.9)

$$
K_{2} H_{m} \frac{\ln \lambda_{i}}{\lambda_{i}}+K_{3} \varepsilon \ln \lambda_{i}=K_{2} H_{m} \frac{\ln \lambda(\varepsilon)}{\lambda(\varepsilon)}+K_{3} \varepsilon \ln \lambda(\varepsilon)+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right)
$$

Then, taking into account definition (5.5) of $c_{\varepsilon, 1}$, we obtain

$$
\widetilde{J}_{\varepsilon}(\Lambda, X) \geq c_{\varepsilon, 1}+\frac{\varepsilon}{2}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\right)
$$

and (5.17) follows. This ends the proof of the first part of Theorem 1.2.
The proof of the second part is straightforward. Indeed, in that case, the only thing we have to do is to minimize $\widetilde{J}_{\varepsilon}(\lambda, x)$ in $T \times S^{\prime}$. One can easily deduce from the previous computations that such a minimum cannot lie on the boundary of $T \times S^{\prime}$ - whence the existence of a critical point of $\widetilde{J}_{\varepsilon}$ which provides us with the desired solution of (1.1).

## Appendix A.

We begin this appendix by a number of integral estimates, which will be useful in establishing Proposition 4.2.
A.1. Integral estimates. We recall that $U_{i}$ denotes the function $U_{\lambda_{i}, x_{i}}$ defined by (1.2), where $x_{i}$ is assumed to belong to the boundary of $\Omega$. In the first place, we state some results concerning integrals involving only $U_{i}$ and its derivatives with respect to $\lambda_{i}$ and $x_{i}$ :

Lemma A.1. As $\lambda_{i} \rightarrow \infty$, we have

$$
\begin{align*}
& \left\|U_{i}\right\|^{2}=\frac{3 \sqrt{3} \pi^{2}}{8}+\sqrt{3} \pi H\left(x_{i}\right) \frac{\ln \lambda_{i}}{\lambda_{i}}+\mathrm{O}\left(\frac{1}{\lambda_{i}}\right)  \tag{A.1}\\
& \int_{R^{3} \backslash \Omega} U_{i}^{6}=  \tag{A.2}\\
& \int_{R^{3} \backslash \Omega} U_{i}^{6-\varepsilon}=\frac{3 \sqrt{3} \pi^{2}}{8}+\frac{3 \sqrt{3} \pi}{4} \frac{H\left(x_{i}\right)}{\lambda_{i}}+\mathrm{O}\left(\frac{1}{\lambda_{i}^{2}}\right)  \tag{A.3}\\
& \\
& \quad+\frac{3 \sqrt{3} \pi^{2}}{4} \frac{3 \sqrt{3} \pi^{2}}{16} \varepsilon \ln \lambda_{i}+\frac{3 \sqrt{3} \pi^{2}}{32}(4 \ln 2-1-\ln 3) \varepsilon  \tag{A.4}\\
& \left\langle U_{i}, \frac{\partial U_{i}}{\partial \lambda_{i}}\right\rangle=  \tag{A.5}\\
& -\frac{\sqrt{3} \pi}{2} H\left(\varepsilon^{2} \ln ^{2} \lambda_{i}+\frac{\ln \lambda_{i}}{\lambda_{i}^{2}}+\mathrm{O}\left(\frac{1}{\lambda_{i}^{2}}\right) \text { as } \varepsilon \ln \lambda_{i} \rightarrow 0\right.  \tag{A.6}\\
&
\end{aligned} \begin{aligned}
&\left\langle U_{i}, \frac{\partial U_{i}}{\partial \tau_{i \ell}}\right\rangle=-\frac{\sqrt{3} \pi}{2} \frac{\partial H}{\partial \tau_{i \ell}}\left(x_{i}\right) \frac{\ln \lambda_{i}}{\lambda_{i}}+\mathrm{O}\left(\frac{1}{\lambda_{i}}\right)  \tag{A.7}\\
&\left\|\frac{\partial U_{i}}{\partial \lambda_{i}}\right\|^{2}=\frac{15 \sqrt{3} \pi^{2}}{128 \lambda_{i}^{2}}+\mathrm{O}\left(\frac{\ln \lambda_{i}}{\lambda_{i}^{3}}\right)  \tag{A.8}\\
&\left\langle\frac{\partial U_{i}}{\partial \lambda_{i}}, \frac{\partial U_{i}}{\partial \tau_{i \ell}}\right\rangle=\mathrm{O}\left(\frac{\ln \lambda_{i}}{\lambda_{i}^{2}}\right) \\
&\left\langle\frac{\partial U_{i}}{\partial \tau_{1 \ell}}, \frac{\partial U_{i}}{\partial \tau_{i m}}\right\rangle=\frac{15 \sqrt{3} \pi^{2}}{128} \lambda_{i}^{2} \delta_{\ell m}+\mathrm{O}\left(\lambda_{i} \ln \lambda_{i}\right)
\end{align*}
$$

Proof. (A.1) and (A.2) are proved in Appendix C of [11]. Let us prove (A.3). Outside of $B_{\tau}\left(x_{i}\right)$, where $\tau>0$ is some fixed number, $U_{i}(y)$ behaves as $\lambda_{i}^{-1 / 2} \mid y-$ $\left.x_{i}\right|^{-1}$, whence

$$
\int_{\left(R^{3} \backslash \Omega\right) \cap B_{\tau}\left(x_{i}\right)^{c}} U_{i}^{6-\varepsilon}=\mathrm{O}\left(\frac{1}{\lambda_{i}^{3}}\right)
$$

In $B_{\tau}\left(x_{i}\right)$, we can write

$$
U_{i}^{-\varepsilon}(y)=\exp \left(-\varepsilon \ln U_{i}(y)\right)=1-\frac{\varepsilon}{2} \ln \lambda_{i}+\frac{\varepsilon}{2} \ln \left(1+\lambda_{i}^{2}\left|y-x_{i}\right|^{2}\right)-\frac{\varepsilon}{4} \ln 3+\mathrm{O}\left(\varepsilon^{2}\left(\ln \lambda_{i}\right)^{2}\right)
$$

On the one hand

$$
\int_{\left(R^{3} \backslash \Omega\right) \cap B_{\tau}\left(x_{i}\right)} U_{i}^{6}=\int_{R^{3} \backslash \Omega} U_{i}^{6}+\mathrm{O}\left(\frac{1}{\lambda_{i}^{3}}\right)
$$

and the integral of $U_{i}^{6}$ over $R^{3} \backslash \Omega$ is given by (A.2). On the other hand

$$
\int_{\left(R^{3} \backslash \Omega\right) \cap B_{\tau}\left(x_{i}\right)} U_{i}^{6} \ln \left(1+\lambda_{i}^{2}\left|y-x_{i}\right|^{2}\right)=\int_{R^{3} \backslash \Omega} U_{i}^{6} \ln \left(1+\lambda_{i}^{2}\left|y-x_{i}\right|^{2}\right)+\mathrm{O}\left(\frac{1}{\lambda_{i}^{3}}\right)
$$

and similarly to (A.2)

$$
\int_{R^{3} \backslash \Omega} U_{i}^{6} \ln \left(1+\lambda_{i}^{2}\left|y-x_{i}\right|^{2}\right)=\frac{1}{2} \int_{\mathbb{R}^{3}} U_{i}^{6} \ln \left(1+\lambda_{i}^{2}\left|y-x_{i}\right|^{2}\right)+\mathrm{O}\left(\frac{1}{\lambda_{i}}\right)
$$

Lastly, using the residue theorem, we compute

$$
\int_{\mathbb{R}^{3}} U_{i}^{6} \ln \left(1+\lambda_{i}^{2}\left|y-x_{i}\right|^{2}\right)=\frac{3 \sqrt{3} \pi^{2}}{8}(4 \ln 2-1)
$$

Gathering these results, we obtain (A.3).
Estimates (A.4)-(A.8) follow from Appendix D of [12] - although in that paper $N$ is assumed to be larger than 5, the computations carried out therein extend straightforwardly to the case $N=3$, with the minor needed modifications.

In the next lemma, we collect the estimates relative to integrals involving both $U_{i}$ and $U_{j}, i \neq j$. Following [3], we set

$$
\begin{equation*}
\varepsilon_{i j}=\left(\frac{\lambda_{i}}{\lambda_{j}}+\frac{\lambda_{j}}{\lambda_{i}}+\lambda_{i} \lambda_{j}\left|x_{i}-x_{j}\right|^{2}\right)^{-1 / 2} \tag{A.9}
\end{equation*}
$$

Lemma A.2. Let $i \neq j$. As $\lambda_{i}, \lambda_{j} \rightarrow \infty$ and $\varepsilon_{i j} \rightarrow 0$, we have

$$
\begin{align*}
& \int_{R^{3} \backslash \Omega} U_{i}^{5} U_{j}=2 \sqrt{3} \pi \varepsilon_{i j}+\mathrm{O}\left(\frac{\varepsilon_{i j}}{\lambda_{i}^{1 / 2}}+\varepsilon_{i j}^{3}\right) \text { as } \varepsilon_{i j} \rightarrow 0,  \tag{A.10}\\
& \int_{\partial \Omega} \frac{\partial U_{i}}{\partial \nu} U_{j}=\mathrm{O}\left(\frac{|\ln | x_{i}-x_{j}| |}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}\right) \text { as } \lambda_{i}\left|x_{i}-x_{j}\right|, \lambda_{j}\left|x_{i}-x_{j}\right| \rightarrow \infty  \tag{A.11}\\
& \left\langle U_{i}, U_{j}\right\rangle=2 \sqrt{3} \pi \varepsilon_{i j}+\mathrm{O}\left(\frac{|\ln | x_{i}-x_{j}| |}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}+\frac{\varepsilon_{i j}}{\lambda_{i}^{1 / 2}}+\varepsilon_{i j}^{3}\right),  \tag{A.12}\\
& \text { as } \lambda_{i}\left|x_{i}-x_{j}\right|, \lambda_{j}\left|x_{i}-x_{j}\right| \rightarrow \infty \\
& \left\langle U_{i}, \frac{\partial U_{j}}{\partial \lambda_{j}}\right\rangle=\mathrm{O}\left(\frac{\varepsilon_{i j}}{\lambda_{j}}+\frac{|\ln | x_{i}-x_{j}| |}{\lambda_{i}^{1 / 2} \lambda_{j}^{3 / 2}}\right),  \tag{A.13}\\
& \left\langle U_{i}, \frac{\partial U_{j}}{\partial \tau_{j \ell}}\right\rangle=\mathrm{O}\left(\lambda_{j} \varepsilon_{i j}+\frac{\lambda_{j}^{1 / 2}}{\lambda_{i}^{1 / 2}}|\ln | x_{i}-x_{j}| |\right),  \tag{A.14}\\
& \left\langle\frac{\partial U_{i}}{\partial \lambda_{i}}, \frac{\partial U_{j}}{\partial \lambda_{j}}\right\rangle=\mathrm{O}\left(\frac{\varepsilon_{i j}}{\lambda_{i} \lambda_{j}}+\frac{|\ln | x_{i}-x_{j}| |}{\lambda_{i}^{3 / 2} \lambda_{j}^{3 / 2}}\right),  \tag{A.15}\\
& \left\langle\frac{\partial U_{i}}{\partial \lambda_{i}}, \frac{\partial U_{j}}{\partial \tau_{j \ell}}\right\rangle=\mathrm{O}\left(\frac{\lambda_{j} \varepsilon_{i j}}{\lambda_{i}}+\frac{\lambda_{j}^{1 / 2}}{\lambda_{i}^{3 / 2}}|\ln | x_{i}-x_{j}| |\right),  \tag{A.16}\\
& \left\langle\frac{\partial U_{i}}{\partial \tau_{i \ell}}, \frac{\partial U_{j}}{\partial \tau_{j m}}\right\rangle=\mathrm{O}\left(\lambda_{i} \lambda_{j} \varepsilon_{i j}+\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}|\ln | x_{i}-x_{j}| |\right) . \tag{A.17}
\end{align*}
$$

Proof. We know, from formula (E.1) in [3], that for $N \geq 3$

$$
\int_{\mathbb{R}^{N}} U_{i}^{\frac{N+2}{N-2}} U_{j}=(N(N-2))^{N / 2} C_{N} \varepsilon_{i j}+\mathrm{O}\left(\varepsilon_{i j}^{\frac{N}{N-2}}\right)
$$

with

$$
C_{N}=\int_{\mathbb{R}^{N}} \frac{\mathrm{~d} x}{\left(1+|x|^{2}\right)^{\frac{N+2}{2}}}=\frac{\sigma_{N-1}}{N}
$$

Therefore, for $N=3$,

$$
\int_{\mathbb{R}^{3}} U_{i}^{5} U_{j}=4 \sqrt{3} \pi \varepsilon_{i j}+\mathrm{O}\left(\varepsilon_{i j}^{3}\right)
$$

It is easily checked, through a rescaling, that the integral of $U_{i}^{5} U_{j}$ over $R^{3} \backslash \Omega$ is equal to half of the integral over the whole space, to within a small amount which can be estimated proceeding as in Appendix C of [11]. (A.10) follows.

We turn to (A.11). Up to a translation and a rotation of the coordinates in $\mathbb{R}^{3}$, we can assume that $x_{i}=0$, and that for $\tau>0$ small enough

$$
\begin{equation*}
\Omega \cap B_{\tau}(0)=\left\{y=\left(y^{\prime}, y_{3}\right) \in \mathbb{R}^{2} \times \mathbb{R} \text { s.t. }|y|<R, y_{3}>f\left(y^{\prime}\right)\right\} \tag{A.18}
\end{equation*}
$$

where $f$ is a smooth function such that $f(0)=0, f^{\prime}(0)=0$. We have, for $y \in$ $\partial \Omega \cap B_{\tau}(0)$,

$$
\frac{\partial U_{i}}{\partial \nu}(y)=\sum_{\ell=1,2} \frac{\partial U_{i}}{\partial x_{i \ell}}(y) \frac{\partial f}{\partial y_{\ell}}\left(y^{\prime}\right)-\frac{\partial U_{i}}{\partial x_{i 3}}=\mathrm{O}\left(\frac{\lambda_{i}^{5 / 2}\left|y^{\prime}\right|^{2}}{\left(1+\lambda_{i}^{2}\left|y^{\prime}\right|^{2}\right)^{3 / 2}}\right)
$$

Let us assume first that $d=\left|x_{i}-x_{j}\right|>\frac{\tau}{2}$. In $B_{\tau}(0), U_{j}=\mathrm{O}\left(\lambda_{j}^{-1 / 2}\right)$, so that

$$
\int_{\partial \Omega \cap B_{\tau}(0)} \frac{\partial U_{i}}{\partial \nu} U_{j}=\mathrm{O}\left(\frac{1}{\lambda_{j}^{1 / 2}} \int_{0}^{\tau} \frac{\lambda_{i}^{5 / 2} r^{3} \mathrm{~d} r}{\left(1+\lambda_{i}^{2} r^{2}\right)^{3 / 2}}\right)=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}\right)
$$

Outside of $B_{\tau}(0), \frac{\partial U_{i}}{\partial \nu}=\mathrm{O}\left(\lambda_{i}^{-1 / 2}\right)$, so that

$$
\begin{aligned}
\int_{\partial \Omega \backslash B_{\tau}(0)} \frac{\partial U_{i}}{\partial \nu} U_{j} & =\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2}} \int_{\partial \Omega \backslash B_{\tau}(0)} \frac{\lambda_{j}^{1 / 2}}{\left(1+\lambda_{j}^{2}\left|y-x_{j}\right|^{2}\right)^{1 / 2}} \mathrm{~d} y\right) \\
& =\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}} \int_{\partial \Omega \backslash B_{\tau}(0)} \frac{\mathrm{d} y}{\left|y-x_{j}\right|}\right)=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}\right) .
\end{aligned}
$$

Let us assume now that $d=\left|x_{i}-x_{j}\right|$ goes to zero. In $B_{d / 2}(0), U_{j}=\mathrm{O}\left(\lambda_{j}^{-1 / 2} d^{-1}\right)$, so that

$$
\int_{\partial \Omega \cap B_{d / 2}(0)} \frac{\partial U_{i}}{\partial \nu} U_{j}=\mathrm{O}\left(\frac{1}{\lambda_{j}^{1 / 2} d} \int_{0}^{d / 2} \frac{\lambda_{i}^{5 / 2} r^{3} \mathrm{~d} r}{\left(1+\lambda_{i}^{2} r^{2}\right)^{3 / 2}}\right)=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}\right)
$$

In the same way, as $\frac{\partial U_{i}}{\partial \nu}=\mathrm{O}\left(\lambda_{i}^{-1 / 2} d^{-1}\right)$ in $\partial \Omega \cap B_{d / 2}\left(x_{j}\right)$, we have

$$
\int_{\partial \Omega \cap B_{d / 2}\left(x_{j}\right)} \frac{\partial U_{i}}{\partial \nu} U_{j}=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}} \int_{\partial \Omega \cap B_{d / 2}\left(x_{j}\right)} \frac{\mathrm{d} y}{\left|y-x_{j}\right|}\right)=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}\right)
$$

and the same estimate holds for the integral over $\partial \Omega \backslash B_{\tau}(0)$, since outside of $B_{\tau}(0)$, $\frac{\partial U_{i}}{\partial \nu}=\mathrm{O}\left(\lambda_{i}^{-1 / 2}\right)$. Finally we notice that in $\omega=\left(\partial \Omega \cap B_{\tau}(0)\right) \backslash\left(B_{d / 2}(0) \cup B_{d / 2}\left(x_{j}\right)\right)$, $\left|y-x_{j}\right| \leq \frac{1}{3}|y|$, and

$$
\frac{\partial U_{i}}{\partial \nu} U_{j}=\mathrm{O}\left(\frac{\lambda_{i}^{5 / 2}\left|y^{\prime}\right|^{2}}{\left(1+\lambda_{i}^{2}\left|y^{\prime}\right|^{2}\right)^{3 / 2}} \frac{\lambda_{i}^{1 / 2}}{\left(1+\lambda_{j}^{2}\left|y^{\prime}\right|^{2}\right)^{1 / 2}}\right)=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}\left|y^{\prime}\right|^{2}}\right)
$$

Consequently

$$
\int_{\omega} \frac{\partial U_{i}}{\partial \nu} U_{j}=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}} \int_{d / 2}^{\tau} \frac{\mathrm{d} r}{r}\right)=\mathrm{O}\left(\frac{|\ln d|}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}}\right)
$$

and (A.11) is proved.
(A.12) follows from (A.10) and (A.11), since

$$
\left\langle U_{i}, U_{j}\right\rangle=\int_{R^{3} \backslash \Omega} U_{i}^{5} U_{j}-\int_{\partial \Omega} \frac{\partial U_{i}}{\partial \nu} U_{j}
$$

(with $\nu$ the outward normal to $\Omega$ ). (A.13)-(A.17) follow from computations quite similar to those above (some of these estimates could be improved, but they are sufficient for our purposes).

We are now able to prove Proposition 4.2.
A.2. Proof of Proposition 4.2. From now on, in accordance with the hypotheses of Proposition 4.2, we assume that $\varepsilon \in\left(0, \varepsilon_{0}\right)$ and $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$. This means, in view of (3.1) and (4.9),

$$
\begin{equation*}
\frac{a_{1}}{\varepsilon} \ln \frac{1}{\varepsilon}<\lambda_{i}<\frac{a_{2}}{\varepsilon} \ln \frac{1}{\varepsilon}, 1 \leq i \leq k ; \quad\left|x_{i}-x_{j}\right|>b\left(\ln \frac{1}{\varepsilon}\right)^{-3 / 4} \tag{A.19}
\end{equation*}
$$

We notice that in that case, $\varepsilon_{i j}$ defined by (A.9) is such that

$$
\begin{equation*}
\varepsilon_{i, j}=\frac{1}{\lambda_{i}^{1 / 2} \lambda_{j}^{1 / 2}\left|x_{i}-x_{j}\right|}+\mathrm{O}\left(\varepsilon^{3}\left(\ln \frac{1}{\varepsilon}\right)^{-3 / 4}\right) ; \quad \varepsilon_{i, j}=\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right) \tag{A.20}
\end{equation*}
$$

Proof of (4.15). From (4.14) and Proposition 4.1, we have

$$
\widetilde{J}_{\varepsilon}(\Lambda, X)=J_{\varepsilon}(\mathbf{1}, \Lambda, X, 0)+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1}\right)
$$

Next we have

$$
J_{\varepsilon}(\mathbf{1}, \Lambda, X, 0)=\frac{1}{2} \int_{R^{3} \backslash \Omega}\left|\nabla\left(\sum_{i=1}^{k} U_{i}\right)\right|^{2}-\frac{1}{6-\varepsilon} \int_{R^{3} \backslash \Omega}\left(\sum_{i=1}^{k} U_{i}\right)^{6-\varepsilon}
$$

since, in accordance with the definition of $g_{\varepsilon}, g_{\varepsilon}\left(\sum_{i=1}^{k} U_{i}\right)=\left(\sum_{i=1}^{k} U_{i}\right)^{5-\varepsilon}$ provided that $\varepsilon$ is small enough (implying, through (A.19), that the $\lambda_{i}$ 's are large enough). On the one hand

$$
\int_{R^{3} \backslash \Omega}\left|\nabla\left(\sum_{i=1}^{k} U_{i}\right)\right|^{2}=\sum_{i=1}^{k} \int_{R^{3} \backslash \Omega}\left|\nabla U_{i}\right|^{2}+\sum_{\substack{1 \leq i, j \leq k \\ i \neq j}} \int_{R^{3} \backslash \Omega} \nabla U_{i} \cdot \nabla U_{j}
$$

- quantities which are estimated by (A.1) and (A.12). On the other hand

$$
\begin{align*}
& \int_{R^{3} \backslash \Omega}\left(\sum_{i=1}^{k} U_{i}\right)^{6-\varepsilon}=\sum_{i=1}^{k} \int_{R^{3} \backslash \Omega} U_{i}^{6-\varepsilon}+(6-\varepsilon) \sum_{\substack{1 \leq i, j \leq k \\
i \neq j}} \int_{R^{3} \backslash \Omega} U_{i}^{5-\varepsilon} U_{j}  \tag{A.21}\\
&+\mathrm{O}\left(\sum_{\substack{1 \leq i, j \leq k \\
i \neq j}} \int_{R^{3} \backslash \Omega} U_{i}^{4-\varepsilon} U_{j}^{2}\right)
\end{align*}
$$

The first integral on the right hand side is estimated by (A.3). Concerning the second one, we remark that $U_{i}^{-\varepsilon}=1+\mathrm{O}\left(\varepsilon \ln \lambda_{i}\right)$ as $\varepsilon \ln \lambda_{i}$ goes to zero. Therefore, (A.19) yields $U_{i}^{-\varepsilon}=1+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1}\right)$, and

$$
\int_{R^{3} \backslash \Omega} U_{i}^{5-\varepsilon} U_{j}=\left(1+\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1}\right)\right) \int_{R^{3} \backslash \Omega} U_{i}^{5} U_{j}
$$

the last integral being estimated by (A.10). Lastly, we know from formula (E.3) in [3] that for $N \geq 3, i \neq j$,

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} U_{i}^{a} U_{j}^{b}=\left(\varepsilon_{i j}\left(\ln \frac{1}{\varepsilon_{i j}}\right)^{\frac{N-2}{N}}\right)^{\min (a, b)}, \quad a, b>1 ; a+b=\frac{2 N}{N-2} \tag{A.22}
\end{equation*}
$$

as $\varepsilon_{i, j}$ goes to zero. In our case, with $N=3, a=4, b=2$, we obtain, using (A.19)

$$
\int_{R^{3} \backslash \Omega} U_{i}^{4-\varepsilon} U_{j}^{2}=\mathrm{O}\left(\int_{\mathbb{R}^{3}} U_{i}^{4} U_{j}^{2}\right)=\mathrm{O}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{1 / 2}\right)
$$

Gathering these results, we obtain (4.15), with

$$
\left\{\begin{align*}
K_{\varepsilon, 1} & =\frac{\sqrt{3} \pi^{2}}{8}\left(1-\left(\frac{1}{12}+\frac{1}{8}(4 \ln 2-1-\ln 3)\right) \varepsilon\right)  \tag{A.23}\\
K_{2} & =\frac{\sqrt{3} \pi}{2}, \quad K_{3}=\frac{\sqrt{3} \pi^{2}}{32}, \quad K_{4}=\sqrt{3} \pi
\end{align*}\right.
$$

Proof of (4.16). As $\widetilde{J}_{\varepsilon}(\Lambda, X)=J_{\varepsilon}\left(A_{\varepsilon}(\Lambda, X), \Lambda, X, v_{\varepsilon}(\Lambda, X)\right)$, and $\frac{\partial J_{\varepsilon}}{\partial \alpha_{i}}=0$ for all $j$,

$$
\begin{equation*}
\frac{\partial \widetilde{J}_{\varepsilon}}{\partial \lambda_{i}}(\Lambda, X)=\frac{\partial J_{\varepsilon}}{\partial \lambda_{i}}(\Lambda, X)+\left\langle\frac{\partial J_{\varepsilon}}{\partial v}, \frac{\partial v_{\varepsilon}}{\partial \lambda_{i}}\right\rangle \tag{A.24}
\end{equation*}
$$

Let us first compute $\frac{\partial J_{\varepsilon}}{\partial \lambda_{i}}(\Lambda, X)$. According to (2.4) and (3.4), we have

$$
\frac{1}{\alpha_{i}} \frac{\partial J_{\varepsilon}}{\partial \lambda_{i}}(\Lambda, X)=\int_{R^{3} \backslash \Omega} \nabla\left(\sum_{j=1}^{k} \alpha_{j} U_{j}\right) \cdot \nabla \frac{\partial U_{i}}{\partial \lambda_{i}}-\int_{R^{3} \backslash \Omega} g_{\varepsilon}\left(y, \sum_{j=1}^{k} \alpha_{j} U_{j}+v_{\varepsilon}\right) \frac{\partial U_{i}}{\partial \lambda_{i}}
$$

The first integral on the right hand side is estimated by (A.4) and (A.13). Concerning the second one, we remark that according to the definition of $g_{\varepsilon}$,

$$
\begin{aligned}
\int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)} & g_{\varepsilon}\left(y, \sum_{j=1}^{k} \alpha_{j} U_{j}+v_{\varepsilon}\right) \frac{\partial U_{i}}{\partial \lambda_{i}} \\
& =\int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)}\left(\left(\sum_{j=1}^{k} \alpha_{j} U_{j}+v_{\varepsilon}\right)^{+}\right)^{5-\varepsilon} \frac{\partial U_{i}}{\partial \lambda_{i}} \\
& =\int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)}\left[\left(\sum_{j=1}^{k} \alpha_{j} U_{j}\right)^{5-\varepsilon}+(5-\varepsilon)\left(\sum_{j=1}^{k} \alpha_{j} U_{j}\right)^{4-\varepsilon} v_{\varepsilon}\right. \\
& \left.+\mathrm{O}\left(\left(\sum_{j=1}^{k} \alpha_{j} U_{j}\right)^{3} v_{\varepsilon}^{2}+\left|v_{\varepsilon}\right|^{5-\varepsilon}\right)\right] \frac{\partial U_{i}}{\partial \lambda_{i}}
\end{aligned}
$$

We notice that $\frac{\partial U_{i}}{\partial \lambda_{i}}=\mathrm{O}\left(\frac{1}{\lambda_{i}} U_{i}\right)$. Then we have

$$
\begin{align*}
& \int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)}\left(\sum_{j=1}^{k} \alpha_{j} U_{j}\right)^{5-\varepsilon} \frac{\partial U_{i}}{\partial \lambda_{i}}  \tag{A.25}\\
& \quad=\alpha_{i}^{5-\varepsilon} \int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)} U_{i}^{5-\varepsilon} \frac{\partial U_{i}}{\partial \lambda_{i}}+\mathrm{O}\left(\frac{1}{\lambda_{i}} \sum_{\substack{1 \leq i, j \leq k \\
i \neq j}} \int_{R^{3} \backslash \Omega}\left(U_{i}^{5} U_{j}+U_{j}^{5} U_{i}\right)\right)
\end{align*}
$$

(we recall that $\varepsilon \ln \lambda_{j} \rightarrow 0$ and $U_{j}^{-\varepsilon}=\mathrm{O}(1)$ for all $j$ ). The first integral on the right hand side may be estimated exactly in the same way as (A.3), and we obtain

$$
\int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)} U_{i}^{5-\varepsilon} \frac{\partial U_{i}}{\partial \lambda_{i}}=-\frac{\sqrt{3} \pi^{2}}{32} \frac{\varepsilon}{\lambda_{i}}+\mathrm{O}\left(\frac{1}{\lambda_{i}^{2}}+\varepsilon^{2} \frac{\ln ^{2} \lambda_{i}}{\lambda_{i}}\right) .
$$

The remaining terms are estimated through (A.10). Next, we write

$$
\left(\sum_{j=1}^{k} \alpha_{j} U_{j}\right)^{4-\varepsilon}=\alpha_{i}^{4-\varepsilon} U_{i}^{4-\varepsilon}+\mathrm{O}\left(\varepsilon \lambda_{i} U_{i}^{4}+\left(\sum_{\substack{1 \leq j \leq k \\ j \neq i}} U_{i}^{3} U_{j}+U_{j}^{4}\right)\right)
$$

and, as $-\Delta \frac{\partial U_{i}}{\partial \lambda_{i}}=5 U_{i}^{4} \frac{\partial U_{i}}{\partial \lambda_{i}}$ in $\mathbb{R}^{3}$, and $\left\langle\frac{\partial U_{i}}{\partial \lambda_{i}}, v_{\varepsilon}\right\rangle=0$

$$
\begin{aligned}
\int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)} U_{i}^{4} \frac{\partial U_{i}}{\partial \lambda_{i}} v_{\varepsilon} & =\int_{R^{3} \backslash \Omega} U_{i}^{4} \frac{\partial U_{i}}{\partial \lambda_{i}} v_{\varepsilon}+\mathrm{O}\left(\frac{1}{\lambda_{i}} \int_{B_{R}^{c}(0)} U_{i}^{5}\left|v_{\varepsilon}\right|\right) \\
& =-\frac{1}{5} \int_{\partial \Omega} \frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \nu} v_{\varepsilon}+\mathrm{O}\left(\frac{\|v\|}{\lambda_{i}^{7 / 2}}\right) .
\end{aligned}
$$

Since $D^{1,2}\left(R^{3} \backslash \Omega\right)$ embeds into $L^{4}(\partial \Omega)$,

$$
\int_{\partial \Omega} \frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \nu} v_{\varepsilon}=\mathrm{O}\left(\left(\int_{\partial \Omega}\left|\frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \nu}\right|^{4 / 3}\right)^{3 / 4}\left\|v_{\varepsilon}\right\|\right)
$$

Proceeding as in the proof of Lemma A.2, we remark that outside of $B_{\tau}\left(x_{i}\right), \frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \nu}=$ $\mathrm{o}\left(\frac{1}{\lambda_{i}^{3 / 2}}\right)$, and in $B_{\tau}\left(x_{i}\right)$, with the notation (A.18),

$$
\frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \nu}(x)=\mathrm{O}\left(\frac{\lambda_{i}^{3 / 2}\left|x^{\prime}\right|^{2}}{\left(1+\lambda_{i}^{2}\left|x^{\prime}\right|^{2}\right)^{3 / 2}}\right)
$$

Then we compute

$$
\int_{\partial \Omega \cap B_{\tau}\left(x_{i}\right)}\left|\frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \nu}\right|^{4 / 3}=\mathrm{O}\left(\lambda_{i}^{2} \int_{0}^{\tau} \frac{r^{\frac{8}{3}+1}}{\left(1+\lambda_{i}^{2} r^{2}\right)^{2}} \mathrm{~d} r\right)=\mathrm{O}\left(\frac{1}{\lambda_{i}^{2}}\right)
$$

so that

$$
\int_{\partial \Omega} \frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \nu} v_{\varepsilon}=\mathrm{O}\left(\frac{\left\|v_{\varepsilon}\right\|}{\lambda_{i}^{3 / 2}}\right) .
$$

We have also

$$
\varepsilon \ln \lambda_{i} \int_{R^{3} \backslash \Omega} U_{i}^{4}\left|\frac{\partial U_{i}}{\partial \lambda_{i}}\right|\left|v_{\varepsilon}\right|=\mathrm{O}\left(\varepsilon \frac{\ln \lambda_{i}}{\lambda_{i}}\left\|v_{\varepsilon}\right\|\right)
$$

and, for $j \neq i$,

$$
\begin{aligned}
\int_{R^{3} \backslash \Omega}\left(U_{i}^{3} U_{j}+U_{j}^{4}\right)\left|\frac{\partial U_{i}}{\partial \lambda_{i}}\right|\left|v_{\varepsilon}\right| & =\mathrm{O}\left(\frac{1}{\lambda_{i}} \int_{R^{3} \backslash \Omega}\left(U_{i}^{4} U_{j}+U_{i} U_{j}^{4}\right)\left|v_{\varepsilon}\right|\right) \\
& =\mathrm{O}\left(\frac{\left\|v_{\varepsilon}\right\|}{\lambda_{i}} \int_{R^{3} \backslash \Omega}\left(U_{i}^{20 / 5} U_{j}^{6 / 5}+U_{i}^{6 / 5} U_{j}^{20 / 5}\right)\left|v_{\varepsilon}\right|\right)
\end{aligned}
$$

This last quantity may be estimated through (A.22). Lastly

$$
\int_{\left(R^{3} \backslash \Omega\right) \cap B_{R}(0)}\left(\left(\sum_{j=1}^{k} \alpha_{j} U_{j}\right)^{3} v_{\varepsilon}^{2}+\left|v_{\varepsilon}\right|^{5-\varepsilon}\right) \frac{\partial U_{i}}{\partial \lambda_{i}}=\mathrm{O}\left(\frac{\left\|v_{\varepsilon}\right\|^{2}}{\lambda_{i}}\right)
$$

Finally, in $B_{R+1}(0) \backslash B_{R}(0)$, we use the fact that $\left|g_{\varepsilon}(y, u)\right| \leq|u|^{5-\varepsilon}$ and $U_{j}=$ $\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2}}\right)$ for all $j$, and in $B_{R+1}^{c}(0)$, we use (2.5) and, taking into account (4.12) (4.13) and (A.19) (A.20), we obtain

$$
\begin{equation*}
\frac{\partial J_{\varepsilon}}{\partial \lambda_{i}}\left(A_{\varepsilon}, \Lambda, X, v_{\varepsilon}\right)=-\frac{\sqrt{3} \pi}{2} H\left(x_{i}\right) \frac{\ln \lambda_{i}}{\lambda_{i}^{2}}+\frac{\sqrt{3} \pi^{2}}{32} \frac{\varepsilon}{\lambda_{i}}+\mathrm{O}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}\right) \tag{A.26}
\end{equation*}
$$

Comparing this expansion with (4.16) it appears, in view of (A.24), that it only remains to prove that $\left\langle\frac{\partial J_{\varepsilon}}{\partial \nu}, \frac{\partial v_{\varepsilon}}{\partial \lambda_{i}}\right\rangle=\mathrm{O}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-5 / 4}\right)$. According to (3.8)

$$
\begin{aligned}
\left\langle\frac{\partial J_{\varepsilon}}{\partial v}, \frac{\partial v_{\varepsilon}}{\partial \lambda_{i}}\right\rangle & =\sum_{j=1}^{k}\left\langle A_{j} U_{j}+B_{j} \frac{\partial U_{j}}{\partial \lambda_{j}}+\sum_{\ell=1,2} C_{j \ell} \frac{\partial U_{j}}{\partial \tau_{j \ell}}, \frac{\partial v_{\varepsilon}}{\partial \lambda_{i}}\right\rangle \\
& =-B_{i}\left\langle\frac{\partial^{2} U_{i}}{\partial \lambda_{i}^{2}}, v_{\varepsilon}\right\rangle-\sum_{\ell=1,2} C_{i \ell}\left\langle\frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \tau_{i \ell}}, v_{\varepsilon}\right\rangle
\end{aligned}
$$

since $\left\langle U_{j}, v_{\varepsilon}\right\rangle=\left\langle\frac{\partial U_{j}}{\partial \lambda_{j}}, v_{\varepsilon}\right\rangle=\left\langle\frac{\partial U_{j}}{\partial \tau_{j \ell}}, v_{\varepsilon}\right\rangle=0$ for all $j$ and $\ell$. On the one hand,

$$
\begin{equation*}
\left\langle\frac{\partial^{2} U_{i}}{\partial \lambda_{i}^{2}}, v_{\varepsilon}\right\rangle=\mathrm{O}\left(\frac{\left\|v_{\varepsilon}\right\|}{\lambda_{i}^{2}}\right) ; \quad\left\langle\frac{\partial^{2} U_{i}}{\partial \lambda_{i} \partial \tau_{i \ell}}, v_{\varepsilon}\right\rangle=\mathrm{O}\left(\left\|v_{\varepsilon}\right\|\right) \tag{A.27}
\end{equation*}
$$

On the other hand, the multipliers $A_{i}, B_{i}, C_{i \ell}$ which occur in (3.5)-(3.8) can easily be estimated. Namely, let us take the scaler product of (3.8) with $U_{i}, \frac{\partial U_{i}}{\partial \lambda_{i}}, \frac{\partial U_{i}}{\partial \tau_{i \ell}}$, $1 \leq i \leq k, \ell=1,2$ respectively. On one side, we find

$$
\left\langle\frac{\partial J_{\varepsilon}}{\partial v}, U_{i}\right\rangle=\frac{\partial J_{\varepsilon}}{\partial \alpha_{i}}=0 ; \quad\left\langle\frac{\partial J_{\varepsilon}}{\partial v}, \frac{\partial U_{i}}{\partial \lambda_{i}}\right\rangle=\frac{1}{\alpha_{i}} \frac{\partial J_{\varepsilon}}{\partial \lambda_{i}} ; \quad\left\langle\frac{\partial J_{\varepsilon}}{\partial v}, \frac{\partial U_{i}}{\partial \tau_{i \ell}}\right\rangle=\frac{1}{\alpha_{i}} \frac{\partial J_{\varepsilon}}{\partial \tau_{i \ell}}
$$

$\frac{\partial J_{\varepsilon}}{\partial \lambda_{i}}$ is given by (A.26), and computations quite similar to those establishing (A.26) show that $\frac{\partial J_{\varepsilon}}{\partial \tau_{i \ell}}=\mathrm{O}(1)$. On the other side, we have linear equations involving the $A_{i}$ 's, $B_{i}$ 's, $C_{i \ell}$ 's, whose coefficients are given by estimates (A.1), (A.4)-(A.8) and (A.12)-(A.17). Such a linear system is quasi-diagonal and invertible, and provides us with the estimates

$$
B_{i}=\mathrm{O}\left(\ln \frac{1}{\varepsilon}\right) ; \quad C_{i \ell}=\mathrm{O}\left(\varepsilon^{2}\left(\ln \frac{1}{\varepsilon}\right)^{-2}\right)
$$

Then, we deduce from (A.27), (4.13) and (A.19) that

$$
\left\langle\frac{\partial J_{\varepsilon}}{\partial v}, \frac{\partial v_{\varepsilon}}{\partial \lambda_{i}}\right\rangle=\mathrm{O}\left(\varepsilon^{5 / 2}\left(\ln \frac{1}{\varepsilon}\right)^{-3 / 2}\right)
$$

and the proof of (4.16) is complete.

## Appendix B.

B.1. Proof of (4.10). Let $f=f_{\varepsilon, \Lambda, X}$ be defined by (4.5), that is, for $w=\left(A^{\prime}, v\right) \in$ $\mathbb{R}^{k} \times E_{\Lambda, X}, A^{\prime}=\left(\alpha_{1}^{\prime}, \ldots, \alpha_{k}^{\prime}\right)$,

$$
\ll f, w \gg=\sum_{i=1}^{k}\left[\left\langle\sum_{j=1}^{k} U_{j}, U_{i}\right\rangle-\int_{R^{3} \backslash \Omega} U_{i} g_{\varepsilon}\left(y, \sum_{j=1}^{k} U_{j}\right)\right] \alpha_{i}^{\prime}-\int_{R^{3} \backslash \Omega} g_{\varepsilon}\left(y, \sum_{j=1}^{k} U_{j}\right) v
$$

First, we remark that according to the definition of $g_{\varepsilon}, g_{\varepsilon}\left(y, \sum_{j} U_{j}\right)=\left(\sum_{j} U_{j}\right)^{5-\varepsilon}$ everywhere, provided that $\varepsilon$ is small enough, so that

$$
\begin{aligned}
& \left\langle\sum_{j=1}^{k} U_{j}, U_{i}\right\rangle-\int_{R^{3} \backslash \Omega} U_{i} g_{\varepsilon}\left(y, \sum_{j=1}^{k} U_{j}\right)=\sum_{j=1}^{k}\left\langle U_{j}, U_{i}\right\rangle-\int_{R^{3} \backslash \Omega}\left(\sum_{j=1}^{k} U_{j}\right)^{5-\varepsilon} U_{i} \\
& \quad=\sum_{j=1}^{k}\left[\left\langle U_{j}, U_{i}\right\rangle-\int_{R^{3} \backslash \Omega} U_{j}^{5} U_{i}\right]+\mathrm{O}\left(\varepsilon \ln \lambda_{i}+\sum_{\substack{1 \leq i, j \leq k \\
i \neq j}} \int_{R^{3} \backslash \Omega}\left(U_{j}^{5}+U_{j} U_{i}^{4}\right) U_{i}\right)
\end{aligned}
$$

and, using (A.1) (A.2), (A.10), (A.12) and (A.19) (A.20), we obtain

$$
\left\langle\sum_{j=1}^{k} U_{j}, U_{i}\right\rangle-\int_{R^{3} \backslash \Omega} U_{i} g_{\varepsilon}\left(y, \sum_{j=1}^{k} U_{j}\right)=\mathrm{O}\left(\varepsilon \ln \frac{1}{\varepsilon}\right) .
$$

We turn now to the integral involving $v$. We have

$$
\begin{aligned}
\int_{R^{3} \backslash \Omega} g_{\varepsilon}\left(y, \sum_{j=1}^{k} U_{j}\right) v & =\int_{R^{3} \backslash \Omega}\left(\sum_{j=1}^{k} U_{j}\right)^{5-\varepsilon} v \\
& =\int_{R^{3} \backslash \Omega}\left(\sum_{j=1}^{k} U_{j}\right)^{5} v+\mathrm{O}\left(\varepsilon \ln \left(\max _{j} \lambda_{j}\right)\|v\|\right)
\end{aligned}
$$

since $\left(\sum_{j} U_{j}\right)^{-\varepsilon}=1+\mathrm{O}\left(\varepsilon \ln \left(\max _{j} \lambda_{j}\right)\right)$ and

$$
\begin{aligned}
\int_{R^{3} \backslash \Omega}\left(\sum_{j=1}^{k} U_{j}\right)^{5}|v| & \leq k^{4} \sum_{j=1}^{k} \int_{R^{3} \backslash \Omega} U_{j}^{5}|v| \\
& \leq k^{4} \sum_{j=1}^{k}\left(\int_{R^{3} \backslash \Omega} U_{j}^{5}\right)^{\frac{5}{6}}\left(\int_{R^{3} \backslash \Omega} v^{6}\right)^{\frac{1}{6}} \leq C\|v\|
\end{aligned}
$$

where $C$ is a constant. Next, we have

$$
\int_{R^{3} \backslash \Omega}\left(\sum_{j=1}^{k} U_{j}\right)^{5} v=\sum_{j=1}^{k} \int_{R^{3} \backslash \Omega} U_{j}^{5} v+\mathrm{O}\left(\sum_{\substack{1 \leq i, j \leq k \\ i \neq j}} \int_{R^{3} \backslash \Omega} U_{j}^{4} U_{i}|v|\right)
$$

and

$$
\int_{R^{3} \backslash \Omega} U_{j}^{4} U_{i}|v| \leq\left(\int_{\mathbb{R}^{3}} U_{j}^{\frac{24}{5}} U_{i}^{\frac{6}{5}}\right)^{\frac{5}{6}}\|v\|
$$

with, according to (A.22) and (A.20)

$$
\left(\int_{\mathbb{R}^{3}} U_{j}^{\frac{24}{5}} U_{i}^{\frac{6}{5}}\right)^{\frac{5}{6}}=\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{1 / 2}\right)
$$

Lastly, we have

$$
\int_{R^{3} \backslash \Omega} U_{j}^{5} v=\int_{R^{3} \backslash \Omega}-\Delta U_{j} v=-\int_{\partial \Omega} \frac{\partial U_{i}}{\partial \nu} v
$$

and we proceed, to estimate the last integral, as we did to estimate the integral over $\partial \Omega$ of $U_{i}^{4} \frac{\partial U_{i}}{\partial \lambda_{i}} v_{\varepsilon}$. Namely, we observe that $D^{1,2}\left(R^{3} \backslash \Omega\right)$ embeds into $L^{4}(\partial \Omega)$, from which we deduce

$$
\int_{\partial \Omega} \frac{\partial U_{i}}{\partial \nu} v=\mathrm{O}\left(\left(\int_{\partial \Omega}\left|\frac{\partial U_{j}}{\partial \nu}\right|^{4 / 3}\right)^{3 / 4}\|v\|\right)
$$

Far from $x_{i}, \frac{\partial U_{i}}{\partial \nu}=\mathrm{O}\left(\frac{1}{\lambda_{i}^{1 / 2}}\right)$, whereas close to $x_{i}$ we can write, with the notation of (A.18)

$$
\frac{\partial U_{j}}{\partial \nu}(x)=\mathrm{O}\left(\frac{\lambda_{j}^{5 / 2}\left|x^{\prime}\right|^{2}}{\left(1+\lambda_{j}^{2}\left|x^{\prime}\right|^{2}\right)^{3 / 2}}\right)
$$

so that

$$
\int_{\partial \Omega \cap B_{\tau}\left(x_{i}\right)}\left|\frac{\partial U_{j}}{\partial \nu}\right|^{4 / 3}=\mathrm{O}\left(\lambda_{j}^{\frac{10}{3}} \int_{0}^{\tau} \frac{r^{\frac{8}{3}+1}}{\left(1+\lambda_{j}^{2} r^{2}\right)^{2}} \mathrm{~d} r\right)=\mathrm{O}\left(\frac{1}{\lambda_{j}^{2 / 3}}\right)
$$

and

$$
\int_{\partial \Omega} \frac{\partial U_{i}}{\partial \nu} v=\mathrm{O}\left(\frac{\|v\|}{\lambda_{j}^{1 / 2}}\right)
$$

Then, (4.10) follows from (A.19).
B.2. Proof of the invertibility of $Q_{\varepsilon, \Lambda, X}$. Let $Q=Q_{\varepsilon, \Lambda, X}$ by defined by (4.6), that is, for $w=\left(A^{\prime}, v\right) \in \mathbb{R}^{k} \times E_{\Lambda, X}$

$$
\begin{aligned}
& \ll Q w, w \gg \sum_{i, j=1}^{k}\left[\left\langle U_{i}, U_{j}\right\rangle-(5-\varepsilon) \int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right)^{4-\varepsilon} U_{i} U_{j}\right] \alpha_{i}^{\prime} \alpha_{j}^{\prime} \\
&-(5-\varepsilon) \sum_{i=1}^{k}( \left.\int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right)^{4-\varepsilon} U_{i} v\right) \alpha_{i}^{\prime} \\
& \quad+\|v\|^{2}-(5-\varepsilon) \int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right) v^{2}
\end{aligned}
$$

since, as we previously noticed, $g_{\varepsilon}\left(y, \sum_{j} U_{j}\right)=\left(\sum_{j} U_{j}\right)^{5-\varepsilon}$ and $\frac{\partial g_{\varepsilon}}{\partial t}\left(y, \sum_{j} U_{j}\right)=$ $(5-\varepsilon)\left(\sum_{j} U_{j}\right)^{4-\varepsilon}$ everywhere, provided that $\varepsilon$ is small enough. Using (A.1) (A.2), (A.10), (A.12) and (A.19)-(A.22), we obtain

$$
\begin{align*}
&\left\langle U_{i}, U_{j}\right\rangle-(5-\varepsilon) \int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right)^{4-\varepsilon} U_{i} U_{j}  \tag{B.1}\\
&= \begin{cases}-\frac{3 \sqrt{3} \pi^{2}}{2}+\mathrm{O}\left(\varepsilon \ln \frac{1}{\varepsilon}\right) & \text { if } i=j, \\
\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right) & \text { if } i \neq j\end{cases}
\end{align*}
$$

Next, we write

$$
\begin{aligned}
& \int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right)^{4-\varepsilon} U_{i} v \\
& \quad=\int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right)^{4} U_{i} v+\mathrm{O}\left(\varepsilon \ln \left(\max _{\ell} \lambda_{\ell}\right)\|v\|\right) \\
& \quad=\int_{R^{3} \backslash \Omega} U_{i}^{5} v+\mathrm{O}\left(\varepsilon \ln \frac{1}{\varepsilon}\|v\|+\sum_{\substack{1 \leq i, j \leq k \\
i \neq j}} \int_{R^{3} \backslash \Omega}\left(U_{j}^{4} U_{i}+U_{j} U_{i}^{4}\right)|v|\right)
\end{aligned}
$$

We just proved, in the previous subsection, that

$$
\int_{R^{3} \backslash \Omega} U_{i}^{5} v=\mathrm{O}\left(\frac{\|v\|}{\lambda_{i}^{1 / 2}}\right)
$$

Then, using (A.22) and (A.20), we find

$$
\begin{equation*}
\int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right)^{4-\varepsilon} U_{i} v=\mathrm{O}\left(\varepsilon^{1 / 2}\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 2}\|v\|\right) \tag{B.2}
\end{equation*}
$$

Lastly, we have to consider the term involving $v^{2}-$ that is, the quadratic form in $v$ which writes

$$
\begin{aligned}
\widetilde{Q}(v) & =\|v\|^{2}-(5-\varepsilon) \int_{R^{3} \backslash \Omega}\left(\sum_{\ell=1}^{k} U_{\ell}\right)^{4-\varepsilon} v^{2} \\
& =\|v\|^{2}-5 \sum_{\ell=1}^{k} \int_{R^{3} \backslash \Omega} U_{\ell}^{4} v^{2}+o\left(\|v\|^{2}\right)
\end{aligned}
$$

We want to prove that $\widetilde{Q}$ is coercive, with a modulus of coercivity independent of $\varepsilon$ and $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$. Together with (B.1) and (B.2), this will prove that $Q$ is invertible, and the existence of $\rho$ independent of $\varepsilon$ and $(\Lambda, X) \in \mathcal{D}_{\varepsilon}$ such that $\left\|\left\|Q^{-1}\right\| \leq \rho\right.$.

We begin by considering the eigenvalue and eigenvector problem in $H^{1}\left(\mathbb{R}^{N}\right)$, $N \geq 3$

$$
\begin{equation*}
-\Delta \omega=\mu U_{\lambda, x}^{\frac{4}{N-2}} \omega \tag{B.3}
\end{equation*}
$$

The spectrum and the eigenvectors of (B.3) are linked to the spectrum and the eigenvectors of $-\Delta$ of $S^{N}$, which are known [5]. Namely, the eigenvalues of $-\Delta_{S^{N}}$ are $\lambda_{n}=n(N+k-1), n \in \mathbb{N}$, with multiplicity $m_{n}=\frac{(N+n-2)!(N+2 n-1)}{n!(N-1)!}$, and the corresponding eigenvectors are harmonic polynomials of degree $n$ : $\lambda_{0}=0, m_{0}=1$, $u_{0}=1 ; \lambda_{1}=N, m_{1}=N+1, u_{1, i}=x_{i}, 1 \leq i \leq N+1 ; \lambda_{2}=2(N+1), \ldots$ For $u$ a function defined on $S^{3}$, we define a function $v$ in $\mathbb{R}^{3}$ by

$$
u(x)=\left(1+|y|^{2}\right)^{\frac{N-2}{N}} v(y)
$$

where $y=\Pi x$ is the stereographic projection of $S^{N}$, the unit sphere of $\mathbb{R}^{N+1}$, with respect to the north pole $x_{i}=0,1 \leq i \leq N, x_{N+1}=1$, onto $\mathbb{R}^{N}$ identified to the hyperplane of $\mathbb{R}^{N+1}$ defined by $x_{N+1}=0 . u$ is an eigenvector of $-\Delta_{S^{N}}$ with eigenvalue $\lambda_{n}$ if and only if $v$ solves (B.3) with $(\lambda, x)=(1,0)$ and $\mu=\mu_{n}=$ $\frac{4 \lambda_{n}}{N(N-2)}+1$. In particular, the eigenvectors of (B.3) are $U_{1,0}$ for $\mu_{0}, \frac{\partial U_{1,0}}{\partial x_{i}}, 1 \leq i \leq N$, and $\frac{\partial U_{1,0}}{\partial \lambda}$ for $\mu_{1}$. The orthogonality of $v$ to $U_{1,0}, \frac{\partial U_{1,0}}{\partial \lambda}$, and $\frac{\partial U_{1,0}}{\partial x_{i}}, 1 \leq i \leq N$, in $D^{1,2}\left(\mathbb{R}^{N}\right)$ means that $u$ is orthogonal to 1 and the $x_{i}$ 's, $1 \leq i \leq N+1$, so that

$$
\int_{S^{N}}\left|\nabla_{S_{N}} u\right|^{2} \geq \lambda_{2} \int_{S^{N}} u^{2}
$$

From such an inequality we deduce, through straightforward computations

$$
\begin{equation*}
\int_{\mathbb{R}^{N}}\left(|\nabla v|^{2}-5 U_{1,0}^{4} v^{2}\right) \geq\left(1-\frac{N(N+2)}{4 \lambda_{2}+N(N-2)}\right) \int_{\mathbb{R}^{N}}|\nabla v|^{2} \tag{B.4}
\end{equation*}
$$

Through rescaling, we see that the same inequality holds replacing $U_{1,0}$ by $U_{\lambda, x}$, when $u$ is assumed to be orthogonal to $U_{\lambda, x}, \frac{\partial U_{\lambda, x}}{\partial \lambda}$, and $\frac{\partial U_{\lambda, x}}{\partial x_{i}}, 1 \leq i \leq N$, in $D^{1,2}\left(\mathbb{R}^{N}\right)$.

Let us consider now the eigenvalue and eigenvector problem in $H^{1}\left(\mathbb{R}_{+}^{N}\right)$,

$$
\begin{equation*}
-\Delta \omega=\mu U_{\lambda, 0}^{\frac{4}{N-2}} \omega \text { in } \mathbb{R}_{+}^{N} ; \quad \frac{\partial \omega}{\partial \nu}=0 \text { on } \partial \mathbb{R}_{+}^{N} \tag{B.5}
\end{equation*}
$$

with $\mathbb{R}_{+}^{N}=\left\{x=\left(x_{1}, \ldots, x_{N}\right) \in \mathbb{R}^{N}\right.$ s.t. $\left.x_{N}>0\right\}$. A symmetry argument shows that a solution of (B.5) provides us with a solution of (B.3). Therefore, for this new problem, the eigenvalues are the same as for (B.3), and the eigenvectors are the eigenvectors $\omega$ of (B.3) such that $\frac{\partial \omega}{\partial \nu}=0$ on $\partial \mathbb{R}_{+}^{N}$. We notice that $U_{\lambda, 0}, \frac{\partial U_{\lambda, 0}}{\partial \lambda}$, and $\frac{\partial U_{\lambda, 0}}{\partial x_{i}}, 1 \leq i \leq N-1$ satisfy that condition. Consequently, for $v$ orthogonal to $U_{\lambda, 0}, \frac{\partial U_{\lambda, 0}}{\partial \lambda}$, and $\frac{\partial U_{\lambda, 0}}{\partial x_{i}}, 1 \leq i \leq N-1$, in $D^{1,2}\left(\mathbb{R}^{N}\right)$, we have the same inequality as (B.4) replacing $\mathbb{R}^{N}$ by $\mathbb{R}_{+}^{N}$ (and $U_{1,0}$ by $U_{\lambda, 0}$ ). From such an inequality we can deduce, proceeding as in [2], that for $\rho<\mu_{2}, x \in \partial \Omega$ and $\lambda$ large enough,

$$
\int_{\mathbb{R}^{N} \backslash \Omega}|\nabla v|^{2}-5 U_{\lambda, x}^{4} v^{2} \geq \rho \int_{\mathbb{R}^{N} \backslash \Omega}|\nabla v|^{2}
$$

for any $v$ orthogonal to $U_{\lambda, x}, \frac{\partial U_{\lambda, x}}{\partial \lambda}$, and $\frac{\partial U_{\lambda, x}}{\partial \tau_{\ell}}, 1 \leq \ell \leq N$, in $D^{1,2}\left(\mathbb{R}^{N}\right)$. To complete the proof of the coercivity of $\widetilde{Q}$, we proceed as in [12]. We set

$$
d=\min _{i \neq j}\left|x_{i}-x_{j}\right|, \quad \Omega_{i}=\left(R^{3} \backslash \Omega\right) \cap B_{d / 2}\left(x_{i}\right)
$$

and, defining $v_{i}=v \mathbf{1}_{\Omega_{i}}$, we write

$$
v_{i}=v_{i}^{+}+v_{i}^{-}
$$

with

$$
v_{i}^{-} \in \operatorname{Span}\left(U_{i} ; \frac{\partial U_{i}}{\partial \lambda_{i}} ; \frac{\partial U_{i}}{\partial \tau_{i \ell}}, \ell=1,2\right)
$$

and $v_{i}^{+}$is orthogonal to $v_{i}^{-}$for the scalar product $\langle$,$\rangle in B_{i}$. The previous arguments imply that

$$
\int_{\Omega_{i}}\left|\nabla v_{i}^{+}\right|^{2}-5 U_{i}^{4} v^{2} \geq \rho \int_{\Omega_{i}}\left|\nabla v_{i}^{+}\right|^{2}
$$

for $\lambda_{i}$ large enough. On the other hand, multiplying the gradient of

$$
v_{i}^{-}=a_{i} U_{i}+b_{i} \frac{\partial U_{i}}{\partial \lambda_{i}}+\sum_{\ell=1,2} c_{i \ell} \frac{\partial U_{i}}{\partial \tau_{i \ell}}
$$

by the gradient of $U_{i}, \frac{\partial U_{i}}{\partial \lambda_{i}}, \frac{\partial U_{i}}{\partial \tau_{i \ell}}$ respectively, and integrating over $\Omega_{i}$, we obtain a quasi-diagonal and invertible linear system which allows to estimate $a_{i}, b_{i}, c_{i l}$ with respect to

$$
\int_{\Omega_{i}} \nabla v_{i}^{-} \nabla U_{i}=\int_{\Omega_{i}} \nabla v \nabla U_{i}=-\int_{\Omega \backslash \Omega_{i}} \nabla v \nabla U_{i}=\mathrm{O}\left(\left(\int_{\Omega \backslash \Omega_{i}}\left|\nabla U_{i}\right|^{2}\right)^{1 / 2}\|v\|\right)
$$

and similar formulas for the integrals involving $\frac{\partial U_{i}}{\partial \lambda_{i}}$ and $\frac{\partial U_{i}}{\partial \tau_{i \ell}}$. We check that

$$
\int_{\Omega \backslash \Omega_{i}}\left|\nabla U_{i}\right|^{2}=\mathrm{O}\left(\frac{1}{\lambda_{i} d}\right)=\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\right)
$$

and omitting here the details, we obtain

$$
\int_{\Omega_{i}}\left|\nabla v_{i}^{-}\right|^{2}=\mathrm{O}\left(\varepsilon\left(\ln \frac{1}{\varepsilon}\right)^{-1 / 4}\|v\|^{2}\right)
$$

Consequently, we have

$$
\int_{\Omega_{i}}|\nabla v|^{2}-5 U_{i}^{4} v^{2} \geq \rho \int_{\Omega_{i}}|\nabla v|^{2}+\mathrm{o}\left(\|v\|^{2}\right)
$$

and

$$
\begin{aligned}
\widetilde{Q}(v) & =\|v\|^{2}-\sum_{i=1}^{k} \int_{\Omega_{i}}|\nabla v|^{2}+\sum_{i=1}^{k}\left(\int_{\Omega_{i}}|\nabla v|^{2}-5 \int_{\Omega_{i}} U_{i}^{4} v^{2}\right) \\
& +5 \sum_{i=1}^{k} \int_{\left(R^{3} \backslash \Omega\right) \backslash \Omega_{i}} U_{i}^{4} v^{2} \\
& \geq \int_{\left(R^{3} \backslash \Omega\right) \backslash \Omega_{i}}|\nabla v|^{2}+\rho \sum_{i=1}^{k} \int_{\Omega_{i}}|\nabla v|^{2}+\mathrm{o}\left(\|v\|^{2}\right) \\
\geq & \rho^{\prime}\|v\|^{2}
\end{aligned}
$$

for $\varepsilon$ small enough and $\rho^{\prime}>0$ a suitable constant independent of $\varepsilon$.

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