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# A VARIATIONAL PRINCIPLE FOR KALUZA–KLEIN TYPE THEORIES

FRÉDÉRIC HÉLEIN

ABSTRACT. For any positive integer  $n$  and any Lie group  $\mathfrak{G}$ , given a definite symmetric bilinear form on  $\mathbb{R}^n$  and an Ad-invariant scalar product on the Lie algebra of  $\mathfrak{G}$ , we construct a variational problem on fields defined on an arbitrary oriented  $(n + \dim\mathfrak{G})$ -dimensional manifold  $\mathcal{Y}$ . We show that, if  $\mathfrak{G}$  is compact and simply connected, any global solution of the Euler–Lagrange equations leads, through a spontaneous symmetry breaking, to identify  $\mathcal{Y}$  with the total space of a principal bundle over an  $n$ -dimensional manifold  $\mathcal{X}$ . Moreover  $\mathcal{X}$  is then endowed with a (pseudo-)Riemannian metric and a connection which are solutions of the Einstein–Yang–Mills system equation with a cosmological constant.

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## 1. INTRODUCTION

In 1919 T. Kaluza [10] (after an earlier attempt by G. Nordström [14] in 1914) discovered that solutions of the Einstein equations of gravity in vacuum on a 5-dimensional manifold could modelize Einstein equations coupled with Maxwell equations on a 4-dimensional space-time manifold, provided one assumes that the 5-dimensional manifold is a circle fiber bundle over space-time and that the metric is constant along these fibers. This was rediscovered more or less independently by O. Klein [12] in 1926 (and also by H. Mandel [13]), who proposed to assume that the size of the extra fifth dimension is sufficiently tiny in order to explain why this dimension is not directly observed. Since then this fascinating observation has

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been an important source of inspiration and questioning (see e.g. [6]). It has been extended to include non Abelian gauge theories [5, 11, 3, 4], in order to unify the Einstein equations with the Yang–Mills equations on a curved space-time and, in particular, it becomes an important ingredient of the 11-dimensional supergravity and the superstrings theories. It remains today a subject of questioning (see e.g. [1, 18]).

However some difficulties plag this beautiful idea:

First in the initial proposal by Kaluza and Klein the coefficient  $g_{55}$  of the metric on the 5-dimensional manifold along the fifth coordinates was assumed to be constant, which leads to inconsistency. This point was raised by P. Jordan [9] and Y. Thiry [16], who allowed this coefficient to be an extra scalar field. However this scalar field is a source of difficulties as to its physical interpretation.

A second problem is to explain why we don't 'see' the extra dimensions. The current answer since Klein relies on Heisenberg uncertainty principle: by expanding the fields in harmonic modes on each fiber and by assuming that the extra dimension is tiny we deduce that all modes excepted the zero one should be extremely massive and this would explain why we cannot observe their quantum excitations.

But lastly the fundamental question is to understand **why** these extra dimensions are fibered and compact (and tiny if we want to support the above hypothesis): could a dynamical mechanism explain these assumptions, instead of relying on an *ansatz* based on an *ad hoc* hypothesis?

In the following we present a model which answers to these questions, without relying on a smallness of the fibers assumption, provided that the structure Lie group is compact and simply connected (hence this excludes  $U(1)$ , for which our mechanism does not explain the compactness of the fibers without extra *ad hoc* hypotheses).

**Notations** — Let  $n$  be some positive integer and let  $\mathbf{b}$  be some symmetric non degenerate bilinear form on  $\mathbb{R}^n$  (e.g. a scalar product or the Minkowski pseudo metric on  $\mathbb{R}^n$ ).

Let  $\mathfrak{G}$  be Lie group of finite dimension, equal to  $r$ , and let  $\mathfrak{g}$  be its Lie algebra. Let  $\mathbf{k}$  be a scalar product on  $\mathfrak{g}$  which is invariant by the adjoint action of  $\mathfrak{G}$  on  $\mathfrak{g}$ . We consider the vector space  $\hat{\mathfrak{g}} := \mathbb{R}^n \oplus \mathfrak{g}$  and endow it with the bilinear form  $\mathbf{h} := \mathbf{b} \oplus \mathbf{k}$ , i.e. such that  $\mathbf{h}((v, \xi), (w, \eta)) = \mathbf{b}(v, w) + \mathbf{k}(\xi, \eta)$ ,  $\forall (v, \xi), (w, \eta) \in \mathbb{R}^n \oplus \mathfrak{g}$ . We endow  $\hat{\mathfrak{g}}$  with the Lie algebra structure  $[(v, \xi), (w, \eta)] = (0, [\xi, \eta])$ ,  $\forall (v, \xi), (w, \eta) \in \hat{\mathfrak{g}}$ . We set  $N := n + r$ .

Let  $(\mathbf{t}_A)_{1 \leq A \leq N}$  be a basis of  $\hat{\mathfrak{g}}$  such that  $(\mathbf{t}_a)_{1 \leq a \leq n}$  is a basis of  $\mathbb{R}^n$  and  $(\mathbf{t}_\alpha)_{n+1 \leq \alpha \leq N}$  is a basis of  $\mathfrak{g}$ . We will systematically use the following conventions for the indices:  $1 \leq A, B, C, \dots \leq N$  and  $1 \leq a, b, c, \dots \leq n < \alpha, \beta, \gamma, \dots \leq N$ . We set  $\mathbf{h}_{AB} := \mathbf{h}(\mathbf{t}_A, \mathbf{t}_B)$ ,  $\mathbf{b}_{ab} := \mathbf{b}(\mathbf{t}_a, \mathbf{t}_b)$  and  $\mathbf{k}_{\alpha\beta} := \mathbf{k}(\mathbf{t}_\alpha, \mathbf{t}_\beta)$ . We denote by  $c_{BC}^A$  the structure constants of  $\hat{\mathfrak{g}}$  in the basis  $(\mathbf{t}_A)_{1 \leq A \leq N}$ , defined by  $[\mathbf{t}_B, \mathbf{t}_C] = \mathbf{t}_A c_{BC}^A$ . We observe that

$$(\mathbf{h}_{AB}) = \begin{pmatrix} \mathbf{h}_{ab} & \mathbf{h}_{a\beta} \\ \mathbf{h}_{\alpha b} & \mathbf{h}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{ab} & 0 \\ 0 & \mathbf{k}_{\alpha\beta} \end{pmatrix}$$

and

$$(c_{BC}^A) = \begin{pmatrix} c_{bc}^a & c_{b\gamma}^a & c_{\beta\gamma}^a \\ c_{bc}^\alpha & c_{b\gamma}^\alpha & c_{\beta\gamma}^\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{\beta\gamma}^\alpha \end{pmatrix}.$$

We denote by  $(\mathbf{t}^A)_{1 \leq A \leq N}$  the basis of  $\hat{\mathfrak{g}}^*$  which is dual to  $(\mathbf{t}_A)_{1 \leq A \leq N}$ . We endow the Lie algebra  $so(\hat{\mathfrak{g}}, \mathfrak{h})$  of the rotation group  $SO(\hat{\mathfrak{g}}, \mathfrak{h})$  with the basis  $(\mathbf{t}_{AB})_{1 \leq A < B \leq N}$  such that, if  $u = \mathbf{t}_A u^A \in \hat{\mathfrak{g}}$ ,  $\mathbf{t}_{AB}(\mathbf{t}_C) = \mathbf{t}_A \mathfrak{h}_{BC} - \mathbf{t}_B \mathfrak{h}_{AC}$  and we pose  $\mathbf{t}_{AB} := -\mathbf{t}_{BA}$  if  $A \geq B$ .

For any smooth manifold  $\mathcal{M}$  and any nonnegative integer  $p$  denote by  $\Omega^p(\mathcal{M})$  the space of smooth  $p$ -forms on  $\mathcal{M}$ . When considering  $\hat{\mathfrak{g}}$ -valued  $p$ -forms  $\phi \in \hat{\mathfrak{g}} \otimes \Omega^p(\mathcal{M})$  we will use the decomposition  $\phi = \mathbf{t}_A \phi^A$ , where  $\phi^A \in \Omega^p(\mathcal{M})$ ,  $\forall A$ . Similarly if  $\phi \in so(\hat{\mathfrak{g}}, \mathfrak{h}) \otimes \Omega^p(\mathcal{M})$ , we set  $\phi = \frac{1}{2} \mathbf{t}_{AB} \phi^{AB}$ , with  $\phi^{AB} + \phi^{BA} = 0$ , and we pose  $\phi^A_B = \mathfrak{h}_{BB'} \phi^{AB'}$ . Lastly if  $\phi \in \hat{\mathfrak{g}}^* \otimes \Omega^p(\mathcal{M})$ , we decompose it as  $\phi = \phi_A \mathbf{t}^A$ .

**The model** — Let  $\mathcal{Y}$  be a smooth oriented  $N$ -dimensional manifold. We define the space of fields  $\mathcal{H}$  to be set of triplets  $(\theta, \varphi, \pi)$ , where

$$\begin{cases} \theta \in \hat{\mathfrak{g}} \otimes \Omega^1(\mathcal{Y}), \text{ i.e. } \theta \text{ is a 1-form with coefficients in } \hat{\mathfrak{g}} \\ \varphi \in so(\hat{\mathfrak{g}}, \mathfrak{h}) \otimes \Omega^1(\mathcal{Y}), \text{ i.e. } \varphi \text{ is a 1-form with coefficients in } so(\hat{\mathfrak{g}}, \mathfrak{h}) \\ \pi \in \hat{\mathfrak{g}}^* \otimes \Omega^{N-2}(\mathcal{Y}), \text{ i.e. } \pi \text{ is a } (N-2)\text{-form with coefficients in } \hat{\mathfrak{g}}^* \end{cases}$$

and which satisfy the two constraints

$$(1) \quad \text{rank}(\theta^1, \dots, \theta^N) = N \quad \text{everywhere,}$$

so that  $(\theta^1, \dots, \theta^N)$  is a coframe on  $\mathcal{Y}$ , and

$$(2) \quad \theta^a \wedge \theta^b \wedge \pi = 0, \quad \forall a, b = 1, \dots, n.$$

We note  $\frac{1}{2}[\varphi \wedge \varphi]^{AB} := \varphi^A_C \wedge \varphi^{CB}$  and  $\frac{1}{2}[\theta \wedge \theta]^A := \frac{1}{2} c_{BC}^A \theta^B \wedge \theta^C$  and lastly we set  $\theta_{AB}^{(-2)} := \frac{\partial}{\partial \theta^A} \wedge \frac{\partial}{\partial \theta^B} \lrcorner \theta^1 \wedge \dots \wedge \theta^N$ , where  $\lrcorner$  is the interior product and  $(\frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^N})$  is the frame on  $\mathcal{Y}$  which is dual to  $(\theta^1, \dots, \theta^N)$ .

Then to any  $(\theta, \varphi, \pi) \in \mathcal{H}$  we associate the action

$$(3) \quad \mathcal{A}[(\theta, \varphi, \pi)] := \int_{\mathcal{Y}} \frac{1}{2} \theta_{AB}^{(-2)} \wedge \left( d\varphi^{AB} + \frac{1}{2} [\varphi \wedge \varphi]^{AB} \right) + \pi_A \wedge \left( d\theta^A + \frac{1}{2} [\theta \wedge \theta]^A \right).$$

**Theorem 1.1.** *Assume that  $\mathfrak{G}$  is compact and simply connected. Let  $\mathcal{Y}$  be a smooth oriented  $N$ -dimensional manifold and let  $\mathcal{H}$  be as above. Then for any triplet  $(\theta, \varphi, \pi) \in \mathcal{H}$  which is a **global** solution of the Euler–Lagrange equation of  $\mathcal{A}$ , there exists a Lie group  $\mathfrak{G}_0$  whose universal covering is  $\mathfrak{G}$ , an  $n$ -dimensional manifold  $\mathcal{X}$  and a submersion  $P : \mathcal{Y} \rightarrow \mathcal{X}$  which has the structure of a principal bundle with structure group  $\mathfrak{G}_0$  over  $\mathcal{X}$ . Moreover  $\mathcal{X}$  is endowed with a pseudo-Riemannian metric  $\mathfrak{g}$  and a connection 1-form  $A$  which are solutions of the Einstein–Yang–Mills system of equation*

$$(4) \quad \begin{cases} \text{Ein}(\mathfrak{g})^a_d + \frac{1}{2} (F_\gamma^{ac} F_{\gamma dc} - \frac{1}{4} F_\gamma^{bc} F_{\gamma bc}) \delta_d^a & = \kappa \delta_d^a \\ \nabla_c F_\gamma^{ca} - c_{\alpha\gamma}^\beta A_c^\alpha F_\beta^{ca} & = 0 \end{cases}$$

where  $\text{Ein}(\mathfrak{g})^a_d := \text{Ric}(\mathfrak{g})^a_d - \frac{1}{2} R(\mathfrak{g}) \delta_d^a$  is the Einstein tensor of  $\mathfrak{g}$ ,  $\kappa := \frac{1}{8} c_{\beta\gamma}^\alpha c_{\alpha\epsilon}^\beta h^{\gamma\epsilon}$  and  $F := dA + \frac{1}{2} [A \wedge A]$ .

The hypothesis that  $(\theta, \varphi, \pi)$  is a *global solution* in Theorem 1.1 means the following: by building a Riemannian metric associated to  $\theta$  on  $\mathcal{Y}$  (e.g.  $\sum_{A=1}^N (\theta^A)^2$ ), we can endow  $\mathcal{Y}$  with the induced topology. Then the solution is said to be *global* if  $\mathcal{Y}$  is complete without boundary.

**Comments on Theorem 1.1** — First note that, to any  $\theta \in \hat{\mathfrak{g}} \otimes \Omega^1(\mathcal{Y})$  satisfying (1), one can associate the pseudo-Riemannian metric  $\mathfrak{h} = \mathfrak{b}_{ab} \theta^a \theta^b + \mathfrak{k}_{\alpha\beta} \theta^\alpha \theta^\beta$  on  $\mathcal{Y}$ .

The action  $\mathcal{A}$  is the sum of  $\int_{\mathcal{Y}} \frac{1}{2} \theta_{AB}^{(-2)} \wedge (d\varphi^{AB} + \frac{1}{2}[\varphi \wedge \varphi]^{AB})$ , the  $N$ -dimensional generalization of the so-called ‘Palatini’ functional on  $\mathcal{Y}$ , and of  $\int_{\mathcal{Y}} \pi_A \wedge (d\theta^A + \frac{1}{2}[\theta \wedge \theta]^A)$ , which is at the origin of the spontaneous symmetry breaking and fibration.

If one is familiar with the ‘Palatini’ action it is not difficult to see that, if the action is stationary with respect to variations of  $\varphi$ , then the connection on  $T\mathcal{Y}$  associated to  $(\theta, \varphi)$  is torsion free and hence is the Levi-Civita connection for  $\mathbf{h}$ . The variation with respect to  $\pi$ , which plays here the role of a Lagrange multiplier, leads to the relation  $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0 \text{ mod}[\theta^a]$ . Using exterior differential calculus à la Cartan and Frobenius’ theorem (as e.g. in [8, 17]), one deduces first that  $\mathcal{Y}$  is foliated by submanifolds  $\mathfrak{f}$  which are solutions of  $\theta^a|_{\mathfrak{f}} = 0, \forall a = 1, \dots, n$ . Further applications of Frobenius’ theorem show that one can locally define a map  $g : \mathcal{Y} \rightarrow \mathfrak{G}$  the restriction on each leaf  $\mathfrak{f}$  of which is a local diffeomorphism. The global solution hypothesis and the hypotheses on  $\mathfrak{G}$  then imply that the leaves actually close up and form the fibers of a fibration over a quotient manifold  $\mathcal{X}$ . Still using the relation  $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0 \text{ mod}[\theta^a]$  one deduces that  $\theta^a = e^a$  and  $\theta^\alpha = (gAg^{-1} + g^{-1}dg)^\alpha$ , where  $e^a$  and  $A$  are pull-back forms of 1-forms on  $\mathcal{X}$ . We then set  $F := dA + \frac{1}{2}[A \wedge A]$ , the curvature 2-form of  $A$ .

Then the variation with respect to  $\theta$  leads to the equation

$$\frac{1}{2} \theta_{ABC}^{(-3)} \wedge (d\varphi^{AB} + \varphi^A{}_D \wedge \varphi^{DB}) = -d\pi_C - c_{CA}^B \theta^A \wedge \pi_B \text{ mod}[\theta_\gamma^{(N-1)}].$$

One can recognize on the left hand side the Einstein tensor of  $\mathbf{h}$  on  $\mathcal{Y}$ . Hence, in order to prove the first equation in (4), things would be easy if the right hand side of this equation would vanish. But this is obviously not true. However a miracle occurs, analogous to what happens in [7, 8]. After a gauge transformation  $e^\alpha = S^\alpha_\beta \theta^\beta$  and  $\omega^\alpha{}_\beta = S^\alpha_{\alpha'} \varphi^{\alpha'}{}_{\beta'} (S^{-1})^{\beta'}_\beta - dS^\alpha_{\beta'} (S^{-1})^{\beta'}_\beta$ , where  $S = \text{Ad}_g$ , the previous equation translates as

$$\frac{1}{2} e_{ABC}^{(-3)} \wedge (d\omega^{AB} + \omega^A{}_D \wedge \omega^{DB}) = -dp_C \text{ mod}[e_\gamma^{(-1)}]$$

The key observations are that the left hand side is constant on any fiber, whereas the restriction of the right hand side to any fiber is an exact form. Both observations lead to the conclusion that  $\frac{1}{2} e_{ABC}^{(-3)} \wedge (d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB}) = 0 \text{ mod}[e_\gamma^{(-1)}]$ , i.e. the two blocks  $\text{Ein}(\mathbf{h})^{a_c}$  and  $\text{Ein}(\mathbf{h})^{a_\gamma}$  of the Einstein tensor of  $\omega$  vanish. This is enough to imply (4). The resulted picture is analogous to the one described in [2]: the manifold  $(\mathcal{Y}, \mathbf{h})$  is not Einstein in general but leads to solutions of the system (4) anyway.

## 2. THE EULER-LAGRANGE EQUATIONS

In the following we assume that  $(\theta, \varphi, \pi) \in \mathcal{H}$  is a critical point of  $\mathcal{A}$ . We denote by  $\mathbf{h} = \mathfrak{b}_{ab} \theta^a \theta^b + \mathfrak{k}_{\alpha\beta} \theta^\alpha \theta^\beta$  the induced metric on  $\mathcal{Y}$ .

**2.1. Variations with respect to coefficients of  $\pi$ .** Since  $\text{rank} \theta = N$ , the family  $(\theta^1, \dots, \theta^N)$  is a coframe on  $\mathcal{Y}$ , there exists unique coefficients  $H_{BC}^A$  such that  $d\theta^A + \frac{1}{2} c_{BC}^A \theta^B \wedge \theta^C = \frac{1}{2} H_{BC}^A \theta^B \wedge \theta^C$  and  $H_{BC}^A + H_{CB}^A = 0$ . We note  $\theta^{(N)} := \theta^1 \wedge \dots \wedge \theta^N$ ,  $\theta_A^{(N-1)} := \frac{\partial}{\partial \theta^A} \lrcorner \theta^{(N)}$ ,  $\theta_{AB}^{(N-2)} := \frac{\partial}{\partial \theta^A} \wedge \frac{\partial}{\partial \theta^B} \lrcorner \theta^{(N)}$ ,  $\theta_{ABC}^{(N-3)} := \frac{\partial}{\partial \theta^A} \wedge \frac{\partial}{\partial \theta^B} \wedge \frac{\partial}{\partial \theta^C} \lrcorner \theta^{(N)}$ . Hence we can decompose any  $(N-2)$ -form  $\pi_A$  as  $\pi_A = \frac{1}{2} \pi_{BC}^A \theta_{BC}^{(N-2)}$ ,

where  $\pi_A^{BC} + \pi_A^{CB} = 0$ . The hypothesis (2) then reads  $\pi_A^{ab} = 0$  or

$$(5) \quad \pi_A = \pi_A^{b\gamma} \theta_{b\gamma}^{(N-2)} + \frac{1}{2} \pi_A^{\beta\gamma} \theta_{\beta\gamma}^{(N-2)}$$

A first order variation of  $(\theta, \varphi, \pi)$  keeping  $\theta$  and  $\varphi$  constant and respecting the constraint (5) thus induces a variation of  $\pi$  of the form  $\delta\pi_A = \chi_A^{b\gamma} \theta_{b\gamma}^{(N-2)} + \frac{1}{2} \chi_A^{\beta\gamma} \theta_{\beta\gamma}^{(N-2)}$ . The fact that the action  $\mathcal{A}$  is stationary with respect to such variations of  $\pi$  thus reads

$$\int_{\mathcal{Y}} \delta\pi_A \wedge \left( d\theta^A + \frac{1}{2} [\theta \wedge \theta]^A \right) = \int_{\mathcal{Y}} \left( \chi_A^{b\gamma} H_{b\gamma}^A + \frac{1}{2} \chi_A^{\beta\gamma} H_{\beta\gamma}^A \right) \theta^{(N)} = 0, \quad \forall \chi_A^{b\gamma}, \chi_A^{\beta\gamma}$$

and lead to the Euler–Lagrange equations  $H_{b\gamma}^A = H_{\beta\gamma}^A = 0$ ,  $\forall A, b, \beta, \gamma$ . We thus deduce that

$$(6) \quad \Theta^A := d\theta^A + \frac{1}{2} c_{BC}^A \theta^B \wedge \theta^C = \frac{1}{2} H_{bc}^A \theta^b \wedge \theta^c$$

or equivalently

$$(7) \quad \begin{cases} d\theta^a & = \frac{1}{2} H_{bc}^a \theta^b \wedge \theta^c \\ d\theta^\alpha + \frac{1}{2} c_{\beta\gamma}^\alpha \theta^\beta \wedge \theta^\gamma & = \frac{1}{2} H_{bc}^\alpha \theta^b \wedge \theta^c \end{cases}$$

**2.2. Variations with respect to  $\varphi$ .** Keeping  $\theta$  and  $\pi$  fixed we look at first order variations  $\delta\varphi = \psi$  of  $\varphi$ . This induces the condition that, for all  $\psi$ ,

$$\frac{1}{2} \int_{\mathcal{Y}} d \left( \psi^{AB} \wedge \theta_{AB}^{(N-2)} \right) + \psi^{AB} \wedge \left( d\theta_{AB}^{(N-2)} - \varphi^{B'}{}_B \wedge \theta_{AB'}^{(N-2)} - \varphi^{A'}{}_A \wedge \theta_{A'B}^{(N-2)} \right) = 0$$

Assuming that  $\psi$  has compact support we deduce the relation

$$\left( d\theta^A + \varphi^{A'}{}_A \wedge \theta^{A'} \right) \wedge \theta_{ABC}^{(N-3)} = d\theta_{AB}^{(N-2)} - \varphi^{B'}{}_B \wedge \theta_{AB'}^{(N-2)} - \varphi^{A'}{}_A \wedge \theta_{A'B}^{(N-2)} = 0$$

which implies that the torsion 2-form  $d\theta^A + \varphi^{A'}{}_A \wedge \theta^{A'}$  vanishes. Hence the connection on  $T\mathcal{Y}$  associated to  $\varphi$  is the Levi-Civita connection of  $(\mathcal{Y}, \mathbf{h})$ , where  $\mathbf{h} = h_{AB} \theta^A \theta^B$ .

**2.3. Variations with respect to  $\theta$ .** We first observe that, through a variation  $\delta\theta = \tau$  of  $\theta$  keeping  $\varphi$  and the coefficients  $\pi_A^{b\gamma}$  and  $\pi_A^{\beta\gamma}$  fixed, we have

$$\delta \left( \theta_{AB}^{(N-2)} \right) = \tau^C \wedge \theta_{ABC}^{(N-3)},$$

plus the relation  $\delta\Theta^A = d\tau^A + c_{BC}^A \tau^B \wedge \theta^C$  which implies

$$\pi_A \wedge \delta\Theta^A = d(\tau^A \wedge \pi_A) + \tau^A \wedge (d\pi_A + c_{AB}^C \theta^B \wedge \pi_C)$$

and lastly  $\delta\pi_A = \pi_A^{b\gamma} \left( \tau^d \wedge \theta_{b\gamma d}^{(N-3)} + \tau^\delta \wedge \theta_{b\gamma\delta}^{(N-3)} \right) + \frac{1}{2} \pi_A^{\beta\gamma} \left( \tau^d \wedge \theta_{\beta\gamma d}^{(N-3)} + \tau^\delta \wedge \theta_{\beta\gamma\delta}^{(N-3)} \right)$

which, thanks to  $\Theta^A \wedge \theta_{b\gamma\delta}^{(N-3)} = \Theta^A \wedge \theta_{\beta\gamma d}^{(N-3)} = \Theta^A \wedge \theta_{\beta\gamma\delta}^{(N-3)} = 0$  by (6), leads to

$$(\delta\pi_A) \wedge \Theta^A = -\pi_A^{b\gamma} H_{bd}^A \tau^d \wedge \theta_\gamma^{(N-1)}.$$

In conclusion, by assuming that  $\tau$  has compact support, we obtain

$$\int_{\mathcal{Y}} \tau^C \wedge \left( \frac{1}{2} \theta_{ABC}^{(N-3)} \wedge \Phi^{AB} - \pi_A^{b\gamma} H_{bc}^A \theta_\gamma^{(N-1)} + d\pi_C - c_{AC}^B \theta^A \wedge \pi_B \right) = 0$$

where we set  $\Phi := d\varphi + \frac{1}{2} [\varphi \wedge \varphi]$ . Hence we deduce the Euler–Lagrange equation

$$(8) \quad \frac{1}{2} \theta_{ABC}^{(N-3)} \wedge \Phi^{AB} + d\pi_C - c_{AC}^B \theta^A \wedge \pi_B = 0 \text{ mod } [\theta_{\mathfrak{g}}^{(N-1)}]$$

where, for any 3-form  $\psi$ ,  $\psi = 0 \bmod[\theta_{\mathfrak{g}}^{(N-1)}]$  means that there exists coefficients  $\psi^\alpha$  such that  $\psi = \psi^\alpha \theta_\alpha^{(N-1)}$ .

### 3. THE FIBRATION

For simplicity we assume in the following that  $\mathcal{Y}$  is connected. If not it suffices to apply the following in each connected component of  $\mathcal{Y}$ .

From the first equation in (7) we deduce that  $d\theta^a = 0 \bmod[\theta^b]$ ,  $\forall a = 1, \dots, n$ . Since the rank of  $(\theta^1, \dots, \theta^n)$  is equal to  $n$  everywhere, Frobenius' theorem implies that, for any point  $y \in \mathcal{Y}$ , there exists a neighbourhood of  $y$  in which there exists a unique submanifold  $\mathfrak{f}$  of dimension  $r$  crossing  $y$  such that  $\theta^a|_{\mathfrak{f}} = 0$ ,  $\forall a = 1, \dots, n$ . Hence  $\mathcal{Y}$  is foliated by integral leaves of dimension  $r$ .

Consider any integral leaf  $\mathfrak{f}$ . It follows from the second equation in (7) that  $d\theta^{\mathfrak{g}} + \frac{1}{2}[\theta^{\mathfrak{g}} \wedge \theta^{\mathfrak{g}}]|_{\mathfrak{f}} = 0$ , where  $\theta^{\mathfrak{g}} := \mathfrak{t}_\alpha \theta^\alpha$ . Consider on the product manifold  $\mathfrak{f} \times \mathfrak{G}$  the  $\mathfrak{g}$ -valued 1-form  $\tau := dh - h\theta^{\mathfrak{g}}$ , where  $(y, h)$  denotes a point in  $\mathfrak{f} \times \mathfrak{G}$ . From the identity  $d\tau + h(d\theta^{\mathfrak{g}} + \frac{1}{2}[\theta \wedge \theta]^{\mathfrak{g}}) + \tau \wedge \theta^{\mathfrak{g}} = 0$ , we deduce that  $d(\tau|_{\mathfrak{f} \times \mathfrak{G}}) = 0 \bmod[\tau]$  and hence, again by Frobenius' theorem, the exterior differential system  $\tau|_{\mathfrak{f} \times \mathfrak{G}} = 0$  is completely integrable. This implies, for any point  $y_0 \in \mathfrak{f}$ , the existence of a unique map  $g : \mathfrak{f} \rightarrow \mathfrak{G}$  such that  $g(y_0) = 1_{\mathfrak{G}}$  and  $dg - g\theta^{\mathfrak{g}}|_{\mathfrak{f}} = 0$ . Moreover  $g$  is locally invertible.

Consider any smooth path  $\gamma : [0, 1] \rightarrow \mathfrak{G}$  such that  $\gamma(0) = 1_{\mathfrak{G}}$ . We can associate to it a unique path  $u : [0, 1] \rightarrow \mathfrak{f}$  such that  $u(0) = y_0$  and  $(u, \gamma)^* \tau = 0$ . In particular the hypothesis that  $(\mathcal{Y}, \mathfrak{h})$  is complete ensures that the differential equations defining  $u$  has a solution on the whole interval  $[0, 1]$ . Since the exterior differential system defined by  $\tau$  is closed, to any path homotopic to  $\gamma$  in  $\mathfrak{G}$  with fixed extremities it corresponds a path homotopic to  $u$  in  $\mathfrak{f}$  with fixed extremities. Since  $\mathfrak{G}$  is simply connected we can thus define a unique map  $T : \mathfrak{G} \rightarrow \mathfrak{f}$  such that  $T(1_{\mathfrak{G}}) = y_0$  and  $(T \times Id)^* \tau = 0$ . Hence  $\mathfrak{G}$  is a universal cover of  $\mathfrak{f}$  and, in particular, since  $\mathfrak{G}$  is compact  $\mathfrak{f}$  is *compact*.

For any  $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ , let  $X$  be the vector field on  $\mathcal{Y}$  defined by  $X = x^a \frac{\partial}{\partial \theta^a}$ . Let  $\mathfrak{f}_0$  be some integral leaf. Since  $\mathfrak{f}_0$  is compact, there exists a neighbourhood  $\mathcal{T}$  of  $\mathfrak{f}_0$  in  $\mathcal{Y}$  and some  $\varepsilon > 0$  such that, for any  $x$  in the unit ball  $B^n$  of  $\mathbb{R}^n$ , the flow map  $(t, y) \mapsto e^{tX}(y)$  is defined on  $[-\varepsilon, \varepsilon] \times \mathcal{T}$ . We observe that, due to (7),  $L_X \theta^a = H_{bc}^a x^b \theta^c$ ,  $\forall a$ . Hence there exists functions  $M_{bc}^a$  on  $\mathcal{Y}$  (depending on  $x$ ) such that  $(e^{tX})^* \theta^a = M_c^a \theta^c$ ,  $\forall a$ . For any leaf  $\mathfrak{f} \subset \mathcal{T}$ , let  $\iota : \mathfrak{f} \rightarrow \mathcal{Y}$  its embedding map and  $\iota_t := e^{tX} \circ \iota$ . Note that the image of  $\iota_t$  is  $e^{tX}(\mathfrak{f})$ . We have then

$$\iota_t^* \theta^a = (e^{tX} \circ \iota)^* \theta^a = \iota^* (e^{tX})^* \theta^a = \iota^* (M_c^a \theta^c), \quad \forall a.$$

Thus the 1-form  $\mathfrak{t}_a \theta^a$  vanishes on  $e^{tX}(\mathfrak{f})$  iff it vanishes on  $\mathfrak{f}$ , i.e.  $\mathfrak{f}$  is an integral leaf iff  $e^{tX}(\mathfrak{f})$  is also an integral leaf. In particular the map  $B^n \times \mathfrak{f}_0 \ni (x, y) \mapsto e^{\varepsilon x^a \frac{\partial}{\partial \theta^a}}(y)$  is a local diffeomorphism onto a neighbourhood of  $\mathfrak{f}_0$ , which provides us with a local trivialization of the set of leaves. Hence the set  $\mathcal{X}$  of integral leaves has the structure of an  $n$ -dimensional manifold and the quotient map  $P : \mathcal{Y} \rightarrow \mathcal{X}$  is a bundle fibration.

Set  $e^a := \theta^a$ , for  $1 \leq a \leq n$ . From  $\frac{\partial}{\partial \theta^b} \lrcorner e^a = \frac{\partial}{\partial \theta^b} \lrcorner de^a = 0$  we deduce that there exists a coframe  $(\underline{e}^a)_{1 \leq a \leq n}$  on  $\mathcal{X}$  such that  $e^a = P^* \underline{e}^a$ ,  $\forall a$ . Thus we can equip  $\mathcal{X}$  with the pseudo Riemannian metric  $\underline{\mathfrak{g}} := \mathfrak{b}_{ab} \underline{e}^a \underline{e}^b$ .

In the following we choose an  $n$ -dimensional submanifold  $\Sigma \subset \mathcal{Y}$  transverse to the fibration and (restricting ourself to an open subset of  $\mathcal{Y}$  if necessary) we

define the map  $g : \mathcal{Y} \rightarrow \mathfrak{G}$  which is constant equal to  $1_{\mathfrak{G}}$  on  $\Sigma$  and such that  $dg - g\theta^{\mathfrak{g}}|_{\mathfrak{f}} = 0$  for any integral leaf  $\mathfrak{f}$ . We then define  $A := g\theta^{\mathfrak{g}}g^{-1} - dg \cdot g^{-1}$ . Relation  $dg - g\theta^{\mathfrak{g}}|_{\mathfrak{f}} = 0$  then translates as  $A|_{\mathfrak{f}} = 0$  and hence we have the decomposition  $A = A_a\theta^a$ . Moreover since

$$(9) \quad \theta^{\mathfrak{g}} = g^{-1}Ag + g^{-1}dg,$$

we have  $d\theta^{\mathfrak{g}} + \frac{1}{2}[\theta \wedge \theta]^{\mathfrak{g}} = g^{-1}(dA + \frac{1}{2}[A \wedge A])g = g^{-1}Fg$ , where  $F := dA + \frac{1}{2}[A \wedge A]$ . In particular we deduce from (7) that  $\frac{\partial}{\partial \theta^\alpha} \lrcorner dA = 0$ ,  $\forall \alpha = n+1, \dots, N$ , i.e. the coefficients  $A_a$  are constants on the fibers  $\mathfrak{f}$ . Moreover we have

$$(10) \quad F^\alpha = \frac{1}{2}F_{bc}^\alpha e^b \wedge e^c,$$

where the coefficients  $F_{bc} = gH_{bc}^{\mathfrak{g}}g^{-1}$  are constant on the fibers.

#### 4. GAUGE TRANSFORMATION

In the following we define the map  $S : \mathcal{Y} \rightarrow \text{End}(\hat{\mathfrak{g}})$  such that  $\forall (v, \xi) \in \hat{\mathfrak{g}} = \mathbb{R}^n \times \mathfrak{g}$ ,  $S(v, \xi) := (v, g\xi g^{-1})$ , where  $g : \mathcal{Y} \rightarrow \mathfrak{G}$  is the map defined previously. Extending to  $\hat{\mathfrak{g}}$  the adjoint action of  $\mathfrak{G}$  in a trivial way, we may write also  $S(v, \xi) = g(v, \xi)g^{-1}$ . Let  $(S_B^A)_{1 \leq A, B \leq N}$  be the matrix of  $S$  in the basis  $(\mathbf{t}_A)_{1 \leq A \leq N}$ , i.e. such that  $S(\mathbf{t}_A) = \mathbf{t}_B S_B^A$ . We remark that  $S$  takes values in  $SO(\hat{\mathfrak{g}}, \mathfrak{h})$  since the scalar product  $\mathfrak{k}$  on  $\mathfrak{g}$  is invariant by the adjoint action of  $\mathfrak{G}$ . We define a new coframe  $(e^A)_{1 \leq A \leq N}$  by  $e^A = S_B^A \theta^B$ . Equivalently

$$\begin{cases} e^a & := & \theta^a & \forall a = 1, \dots, n \\ e^\alpha & := & S_\beta^\alpha \theta^\beta & \forall \alpha = n+1, \dots, N \end{cases}$$

Then  $e^\alpha = (g\theta^{\mathfrak{g}}g^{-1})^\alpha$  and (9) imply

$$(11) \quad e^\alpha = A^\alpha + (dg g^{-1})^\alpha, \quad \forall \alpha = n+1, \dots, N.$$

We deduce that

$$\begin{aligned} de^\alpha - \frac{1}{2}[e \wedge e]^\alpha + [A \wedge e]^\alpha &= de^\alpha - \frac{1}{2}[e \wedge e]^\alpha + [(e - dg g^{-1}) \wedge e]^\alpha \\ &= de^\alpha + \frac{1}{2}[e \wedge e]^\alpha - [dg g^{-1} \wedge e]^\alpha \\ &= (gd\theta g^{-1} + [dgg^{-1} \wedge e])^\alpha + \frac{1}{2}[e \wedge e]^\alpha - [dgg^{-1} \wedge e]^\alpha \\ &= (g(d\theta + \frac{1}{2}[\theta \wedge \theta])g^{-1})^\alpha \end{aligned}$$

from which we deduce the useful identity

$$(12) \quad de^\alpha - \frac{1}{2}[e \wedge e]^\alpha + [A \wedge e]^\alpha = F^\alpha := \frac{1}{2}F_{bc}^\alpha e^b \wedge e^c.$$

Let us translate the left hand side of (8) in the new coframe. First we define  $e^{(N)} := e^1 \wedge \dots \wedge e^N$  and note that  $e^{(N)} = \theta^{(N)}$ . Moreover defining  $e_A^{(N-1)} := \frac{\partial}{\partial e^A} \lrcorner e^{(N)}$ ,  $e_{AB}^{(N-2)} := \frac{\partial}{\partial e^A} \wedge \frac{\partial}{\partial e^B} \lrcorner e^{(N)}$ , we observe that, since  $\frac{\partial}{\partial \theta^A} = \frac{\partial}{\partial e^B} S_B^A$ , we have  $\theta_A^{(N-1)} = e_{A'}^{(N-1)} S_{A'}^A$ ,  $\theta_{AB}^{(N-2)} = e_{A'B'}^{(N-2)} S_{A'}^A S_{B'}^B$  and  $\theta_{ABC}^{(N-3)} = e_{A'B'C'}^{(N-3)} S_{A'}^A S_{B'}^B S_{C'}^C$ .

Second let  $\omega$  be the  $so(\hat{\mathfrak{g}}, \mathfrak{h})$ -valued connection 1-form in the coframe  $(e^A)_{1 \leq A \leq N}$ , which is equal to  $\omega := S\varphi S^{-1} - dS S^{-1}$ . Let  $\Omega := d\omega + \frac{1}{2}[\omega \wedge \omega] = S\Phi S^{-1}$ , where  $\Phi = d\varphi + \frac{1}{2}[\varphi \wedge \varphi]$ . Then  $\Phi^{AB} = (S^{-1})_{A'}^A (S^{-1})_{B'}^B \Omega^{A'B'}$ .

We deduce that  $\theta_{ABC}^{(N-3)} \Omega^{AB} = e_{ABC'}^{(N-3)} \Omega^{AB} S_{C'}^C$ . Hence (8) is equivalent to

$$(13) \quad \frac{1}{2}e_{ABC}^{(N-3)} \Omega^{AB} + (d\pi_{C'} - c_{AC'}^B \theta^A \wedge \pi_B) (S^{-1})_{C'}^C = 0 \text{ mod } [e_{\mathfrak{g}}^{(N-1)}],$$



where, for any  $(N-1)$ -form  $\psi$ ,  $\psi = 0 \bmod[e_{\mathfrak{g}}^{(N-1)}]$  iff there exists forms  $\psi^\alpha$  such that  $\psi = \psi^\alpha e_\alpha^{(N-1)}$ .

**Lemma 4.1.** *We have*

$$(14) \quad (d\pi_{C'} - c_{AC'}^B \theta^A \wedge \pi_B) (S^{-1})_{C'}^{C'} = d\left(\pi_{C'} (S^{-1})_{C'}^{C'}\right) \bmod[e_{\mathfrak{g}}^{(N-1)}].$$

*Proof* — From the definition of  $S$  we deduce  $d(S^{-1})_{C'}^{C'} = -(S^{-1})_{A'}^{C'} c_{BC}^A (e^B - A^B)$ . Hence

$$\begin{aligned} d\left((S^{-1})_{C'}^{C'} \pi_{C'}\right) &= -(S^{-1})_{A'}^{C'} c_{BC}^A (e^B - A^B) \wedge \pi_{C'} + (S^{-1})_{C'}^{C'} (d\pi_{C'}) \\ &= -(S^{-1})_{A'}^{C'} c_{BC}^A S_B^{B'} \theta^{B'} \wedge \pi_{C'} + (S^{-1})_{A'}^{C'} c_{BC}^A A^B \wedge \pi_{C'} + (S^{-1})_{C'}^{C'} (d\pi_{C'}) \end{aligned}$$

But because of  $[\text{Ad}_g(\xi), \text{Ad}_g(\eta)] = \text{Ad}_g([\xi, \eta])$ ,  $\forall \xi, \eta \in \mathfrak{g}$ , which is equivalent to  $c_{B'C'}^A S_B^{B'} S_C^{C'} = S_A^A c_{BC}^A$ , we have  $(S^{-1})_{A'}^{C'} c_{BC}^A S_B^{B'} = c_{B'C''}^{C'} (S^{-1})_{C''}^{C'}$ . Thus for the first term on the r.h.s.,

$$(S^{-1})_{A'}^{C'} c_{BC}^A S_B^{B'} \theta^{B'} \wedge \pi_{C'} = c_{B'C''}^{C'} (S^{-1})_{C''}^{C'} \theta^{B'} \wedge \pi_{C'} = c_{AC'}^B \theta^A \wedge \pi_B (S^{-1})_{C'}^{C'}$$

and hence

$$d\left((S^{-1})_{C'}^{C'} \pi_{C'}\right) = (d\pi_{C'} - c_{AC'}^B \theta^A \wedge \pi_B) (S^{-1})_{C'}^{C'} + (S^{-1})_{A'}^{C'} c_{BC}^A A^B \wedge \pi_{C'}$$

However we deduce from (5) that  $\pi_{C'} = \pi_{C'}^{b\gamma} e_{b\gamma}^{(N-2)} + \frac{1}{2} \pi_{C'}^{\beta\gamma} e_{\beta\gamma}^{(N-2)}$  and, since  $A^B = A_c^B e^c$ , we get  $(S^{-1})_{A'}^{C'} c_{BC}^A A^B \wedge \pi_{C'} = -(S^{-1})_{A'}^{C'} c_{BC}^A A_b^B \pi_{C'}^{b\gamma} e_\gamma^{(N-1)} = 0 \bmod[e_{\mathfrak{g}}^{(N-1)}]$ . Hence (14) follows.  $\square$

Thus if we define  $p_C := \pi_{C'} (S^{-1})_{C'}^{C'}$  we deduce from (14) that (13) is equivalent to

$$(15) \quad \frac{1}{2} e_{ABC}^{(N-3)} \wedge \Omega^{AB} + dp_C = 0 \bmod[e_{\mathfrak{g}}^{(N-1)}].$$

We need to compute  $dp_C$ . For that purpose we use the *a priori* decomposition  $p_C = p_C^{b\gamma} e_{b\gamma}^{(N-2)} + \frac{1}{2} p_C^{\beta\gamma} e_{\beta\gamma}^{(N-2)}$ . We first compute using (7) and (12)

$$\begin{aligned} de_{b\gamma}^{(N-2)} &= de^a \wedge e_{b\gamma a}^{(N-3)} + de^\alpha \wedge e_{b\gamma \alpha}^{(N-3)} \\ &= H_{ab}^a e_\gamma^{(N-1)} + c_{\gamma\alpha}^\alpha e_b^{(N-1)} - c_{\beta\gamma}^\alpha (A^\beta)_b e_\gamma^{(N-1)} \\ &= \left(H_{ab}^a - c_{\beta\gamma}^\alpha (A^\beta)_b\right) e_\gamma^{(N-1)} = 0 \bmod[e_{\mathfrak{g}}^{(N-1)}], \end{aligned}$$

where we have used the fact that, since  $\mathfrak{G}$  is compact, its Lie algebra  $\mathfrak{g}$  is *unimodular*, which reads  $c_{\gamma\alpha}^\alpha = 0$ . Similarly

$$\begin{aligned} de_{\beta\gamma}^{(N-2)} &= de^a \wedge e_{\beta\gamma a}^{(N-3)} + de^\alpha \wedge e_{\beta\gamma \alpha}^{(N-3)} \\ &= 0 + c_{\gamma\alpha}^\alpha e_\beta^{(N-1)} + c_{\alpha\beta}^\alpha e_\gamma^{(N-1)} + c_{\beta\gamma}^\alpha e_\alpha^{(N-1)} \\ &= 0 \bmod[e_{\mathfrak{g}}^{(N-1)}], \end{aligned}$$

Then by writing  $dp_C^{b\gamma} = p_{C,c}^{b\gamma} e^c + p_{C,\gamma}^{b\gamma} e^\gamma$  and  $dp_C^{\beta\gamma} = p_{C,c}^{\beta\gamma} e^c + p_{C,\delta}^{\beta\gamma} e^\delta$ , we get

$$dp_C = p_{C,\gamma}^{b\gamma} e_b^{(N-1)} - p_{C,b}^{b\gamma} e_\gamma^{(N-1)} + p_{C,\gamma}^{\beta\gamma} e_\beta^{(N-1)} = p_{C,\gamma}^{b\gamma} e_b^{(N-1)} \bmod[e_{\mathfrak{g}}^{(N-1)}]$$

Lastly by decomposing  $\Omega^{AB} = \frac{1}{2} \Omega^{AB}{}_{CD} e^C \wedge e^D$ , we find that  $\frac{1}{2} e_{ABC}^{(N-3)} \wedge \Omega^{AB} = -\text{Ein}(\omega)^A{}_C e_A^{(N-1)}$ , where  $\text{Ein}(\omega)^A{}_C := \text{Ric}(\omega)^A{}_C - \frac{1}{2} \text{R}(\omega) \delta^A{}_C$ ,  $\text{Ric}(\omega)^A{}_C := \Omega^{AB}{}_{CB}$  and  $\text{R}(\omega) := \text{Ric}(\omega)^A{}_A$ . Obviously  $\text{Ric}(\omega)^A{}_C$  is the Ricci curvature,  $\text{R}(\omega)$  the scalar

curvature and  $\text{Ein}(\omega)^A_C$  the Einstein tensor, in the coframe  $(e^A)_{1 \leq A \leq N}$ . Hence we find that (15) is equivalent to  $(\text{Ein}(\omega)^a_C - p_{C,\gamma}^{a\gamma}) e_a^{(N-1)} = 0 \text{ mod}[e_{\mathfrak{g}}^{(N-1)}]$ , or

$$(16) \quad \text{Ein}(\omega)^a_C = p_{C,\gamma}^{a\gamma}, \quad \forall a = 1, \dots, n, \forall C = 1, \dots, N.$$

We will come back to this equation later on.

### 5. THE CONNECTION AND THE CURVATURE FORM IN THE COFRAME $(e^A)_{1 \leq A \leq +r}$

We need to compute the connection 1-form  $\omega$  and the curvature 2-form. As a preliminary we first set  $\underline{\gamma}^a_c$  to be the connection 1-form on  $(\mathcal{X}, \underline{\mathfrak{g}})$  in the coframe  $\underline{e}^a$ , i.e. which satisfies  $\underline{\gamma}^{ac} + \underline{\gamma}^{ca} = 0$  and  $d\underline{e}^a + \underline{\gamma}^a_c \wedge \underline{e}^b = 0$ . Then we set  $\gamma^a_c := P^* \underline{\gamma}^a_c$  which satisfies similar relations, which, together with (12), leads to

$$(17) \quad \begin{cases} de^a + \gamma^a_c \wedge e^c & = 0 \\ de^a - \frac{1}{2} F_{bc}^\alpha e^b \wedge e^c - \frac{1}{2} c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta) \wedge e^\gamma & = 0 \end{cases}$$

Now the connexion 1-form  $\omega$  is uniquely characterized by the condition  $\omega^{AB} + \omega^{BA} = 0$  (preservation of the metric) and  $de^A + \omega^A_C \wedge e^C = 0$  (the torsion vanishes), which can be written

$$(18) \quad \begin{cases} de^a + \omega^a_c \wedge e^c + \omega^a_\gamma \wedge e^\gamma & = 0 \\ de^a + \omega^a_c \wedge e^c + \omega^a_\gamma \wedge e^\gamma & = 0 \end{cases}$$

Comparing with (17) we are tempted to assume that  $\omega^a_\gamma = -\frac{1}{2} c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta)$ , which fulfills the condition  $\omega^{\alpha\beta} + \omega^{\beta\alpha} = 0$ , since  $c_{\beta\gamma}^\alpha k^{\gamma'\gamma} + c_{\beta\alpha'}^\gamma k^{\alpha'\alpha} = 0$  because the metric  $k$  is preserved by the adjoint action of  $\mathfrak{g}$ . We also guess that  $\omega^a_c = -\frac{1}{2} F_{bc}^\alpha e^b$ , which forces automatically  $\omega^a_\gamma = \frac{1}{2} k_{\gamma\gamma'} F_{bc}^{\gamma'} g^{c'e^b}$ , in order to satisfy  $\omega^{ab} + \omega^{ba} = 0$ . Then in order to fulfill the first relation of (18), one needs to assume that  $\omega^a_c = \gamma^a_c - \frac{1}{2} k_{\gamma\gamma'} F_{a'c}^{\gamma'} b^{a'a} e^\gamma$ . We then check that  $\omega^{ac} = \gamma^{ac} - \frac{1}{2} k_{\gamma\gamma'} F_{a'c}^{\gamma'} b^{a'a} b^{c'e^\gamma}$  is skew symmetric in  $(a, c)$ . Thus we see that the forms  $\omega^{AC}$  defined by:

$$\begin{pmatrix} \omega^a_c & \omega^a_\gamma \\ \omega^\alpha_c & \omega^\alpha_\gamma \end{pmatrix} = \begin{pmatrix} \gamma^a_c - \frac{1}{2} k_{\gamma\gamma'} F_{a'c}^{\gamma'} b^{a'a} e^\gamma & \frac{1}{2} k_{\gamma\gamma'} F_{ba'}^{\gamma'} b^{a'a} e^b \\ -\frac{1}{2} F_{bc}^\alpha e^b & -\frac{1}{2} c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta) \end{pmatrix}$$

satisfy (18) and  $\omega^{AC} + \omega^{CA} = 0$ . Hence this is the Levi-Civita connection 1-form of  $(\mathcal{Y}, \mathbf{h})$ . In the following it will convenient to set  $F^\gamma_{bc} := F_{bc}^\gamma$ ,  $F_\gamma^a_c := k_{\gamma\gamma'} F_{a'c}^{\gamma'} b^{a'a}$  and  $F_{\gamma b^c} := k_{\gamma\gamma'} F_{bc}^{\gamma'} b^{c'e}$ . Then

$$\begin{pmatrix} \omega^a_c & \omega^a_\gamma \\ \omega^\alpha_c & \omega^\alpha_\gamma \end{pmatrix} = \begin{pmatrix} \gamma^a_c - \frac{1}{2} F_\gamma^a_c e^\gamma & \frac{1}{2} F_{\gamma b^a} e^b \\ -\frac{1}{2} F_{bc}^\alpha e^b & -\frac{1}{2} c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta) \end{pmatrix}$$

We can thus compute the curvature 2-form  $\Omega^A_C = d\omega^A_C + \omega^A_B \wedge \omega^B_C$ .

$$\Omega^a_c = d\left(\gamma^a_c - \frac{1}{2} F_\gamma^a_c e^\gamma\right) + \left(\gamma^a_b - \frac{1}{2} F_{\beta'}^a_b e^{\beta'}\right) \wedge \left(\gamma^b_c - \frac{1}{2} F_{\gamma'}^b_c e^{\gamma'}\right) - \frac{1}{4} F_{\beta\beta'}^a F_{c'e}^{\beta'} \wedge e^{c'}$$

$$\Omega^a_\gamma = d(F_{\gamma b^a} e^b) + \frac{1}{2} \left(\gamma^a_b - \frac{1}{2} F_{\beta'}^a_b e^{\beta'}\right) \wedge (F_{\gamma b'^a} e^{b'}) + \frac{1}{4} F_{\beta\beta'}^a c_{\beta'\gamma}^\beta e^{b'} \wedge (2A^{\beta'} - e^{\beta'})$$

$$\Omega^\alpha_c = -d(F_{bc}^\alpha e^b) - \frac{1}{2} (F_{b'\beta'}^\alpha e^{b'}) \wedge \left(\gamma^b_c - \frac{1}{2} F_{\gamma'}^b_c e^{\gamma'}\right) - \frac{1}{4} F_{bc}^\beta c_{\beta'\beta}^\alpha (2A^{\beta'} - e^{\beta'}) \wedge e^b$$

$$\Omega^\alpha_\gamma = \frac{1}{2} d(c_{\beta\gamma}^\alpha (2A^\beta - e^\beta)) - \frac{1}{4} (F_{b'\beta'}^\alpha e^{b'}) \wedge (F_{\gamma c'^b} e^{c'}) + \frac{1}{4} c_{\beta'\beta}^\alpha c_{\beta'\gamma}^\beta (2A^{\beta'} - e^{\beta'}) \wedge (2A^{\gamma'} - e^{\gamma'}).$$

Lastly we obtain the components of the Ricci tensor  $\text{Ric}(\omega)$  through a lengthy computation.

$$(19) \quad \text{Ric}(\omega)^a{}_d = \text{Ric}(\gamma)^a{}_d - \frac{1}{2}F_\beta{}^{ac}F^\beta{}_{dc}$$

where  $\text{Ric}(\gamma)^a{}_d := (d\gamma^a{}_c + \gamma^a{}_b \wedge \gamma^b{}_c)_{de} \mathbf{b}^{ce}$  is the Ricci curvature of  $\gamma$ , and using the decompositions  $dF_\delta{}^{ac} = F_\delta{}^{ac}{}_{,c}e^c$  and  $\gamma^a{}_b = (\gamma^a{}_b)_c e^c$ ,

$$(20) \quad \text{Ric}(\omega)^a{}_\delta = \frac{1}{2} (F_\delta{}^{ac}{}_{,c} + (\gamma^a{}_b)_c F_\delta{}^{bc} + (\gamma^c{}_b)_c F_\delta{}^{ab} - c_{\alpha\delta}^\gamma A_c^\alpha F_\gamma{}^{ac})$$

$$(21) \quad \text{Ric}(\omega)^\alpha{}_\delta = \frac{1}{4}F_\delta{}^{bc}F^\alpha{}_{bc} - \frac{1}{4}c_{\beta\gamma}^\alpha c_{\delta\epsilon}^\beta \mathbf{k}^{\gamma\epsilon}$$

We deduce the scalar curvature  $R(\omega)$  of  $\omega$  in function of the scalar curvature  $R(\gamma) := \text{Ric}(\gamma)^a{}_a$ :

$$(22) \quad R(\omega) = R(\gamma) - \frac{1}{4}F_\alpha{}^{ab}F^\alpha{}_{ab} - \frac{1}{4}c_{\beta\gamma}^\alpha c_{\alpha\delta}^\beta \mathbf{k}^{\gamma\delta}$$

Hence the Einstein tensor of  $\omega$  is

$$(23) \quad \text{Ein}(\omega)^a{}_d = \text{Ein}(\gamma)^a{}_d - \frac{1}{2} \left( F_\beta{}^{ac}F^\beta{}_{dc} - \frac{1}{4}F_\alpha{}^{bc}F^\alpha{}_{bc}\delta^a{}_d \right) + \frac{1}{8}c_{\beta\gamma}^\alpha c_{\alpha\delta}^\beta \mathbf{k}^{\gamma\delta}\delta^a{}_d$$

and  $\text{Ein}(\omega)^a{}_\delta = \text{Ric}(\omega)^a{}_\delta$  is given by (20).

An important observation is that *the components of  $\text{Ein}(\omega)^a{}_d$  and  $\text{Ein}(\omega)^a{}_\delta$  are constant on the fibers  $f$ .*

## 6. THE EINSTEIN–YANG–MILLS EQUATIONS

We conclude by exploiting the fact that the fibers  $f$  are compact without boundary. Let  $\mu^{(r)} := e^{n+1} \wedge \dots \wedge e^N$  and set  $\mu_\alpha^{(r-1)} := \frac{\partial}{\partial e^\alpha} \lrcorner \mu^{(r)}$ ,  $\forall \alpha$ . By integrating both sides of (16) on a fiber  $f$  we obtain

$$\int_f \text{Ein}(\omega)^a{}_C \mu^{(r)} = \int_f p_{C,\gamma}^{a\gamma} \mu^{(r)} = \int_f d \left( p_C^{a\gamma} \mu_\gamma^{(r-1)} \right) = 0.$$

But on the one hand, the components of  $\text{Ein}(\omega)^a{}_C$  are constant on the fiber  $f$ , as seen in the previous section. Hence

$$(24) \quad \text{Ein}(\omega)^a{}_C = \frac{\int_f \text{Ein}(\omega)^a{}_C \mu^{(r)}}{\int_f \mu^{(r)}} = 0.$$

Thus, using (20) and (23) we deduce that  $\gamma$  and  $A$  are solutions of the Einstein–Yang–Mills system

$$\begin{cases} \text{Ein}(\gamma)^a{}_d - \frac{1}{2} (F_\beta{}^{ac}F^\beta{}_{dc} - \frac{1}{4}F_\alpha{}^{bc}F^\alpha{}_{bc}\delta^a{}_d) + \frac{1}{8}c_{\beta\gamma}^\alpha c_{\alpha\delta}^\beta \mathbf{k}^{\gamma\delta}\delta^a{}_d & = 0 \\ F_\delta{}^{ac}{}_{,c} + (\gamma^a{}_b)_c F_\delta{}^{bc} + (\gamma^c{}_b)_c F_\delta{}^{ab} - c_{\alpha\delta}^\gamma A_c^\alpha F_\gamma{}^{ac} & = 0 \end{cases}$$

## 7. CONCLUSION

Given a compact, simply connected structure Lie group  $\mathfrak{G}$  and an  $(n + \dim\mathfrak{G})$ -dimensional manifold without any structure *a priori* we defined an action functional on a space of fields on this manifold, such that each critical point of this action which satisfies fairly broad assumptions produces spontaneously a fiber bundle structure on the manifold and solutions of an Einstein–Yang–Mills system on the quotient manifold.

Klein’s hypothesis that the metric is covariantly constant along spontaneously created fibers was abandoned a long time ago because *apart from its physical meaning, this assumption seems to be artificial*, to quote [6]. However, our model shows the existence of dynamic mechanisms that naturally imply that this hypothesis can be verified.

Our construction works in particular if  $\mathfrak{G}$  is equal to  $SU(2)$ ,  $SU(2)$  or  $SU(2) \times SU(3)$ , but not for  $U(1)$  or  $U(1) \times SU(2) \times SU(3)$  in full generality. In the latter case one needs to assume the further hypothesis that the leaves of the foliation close up to compact fibers to reach similar conclusions. Additional mechanisms should therefore be used to avoid the necessity of the latter hypothesis. Another challenging question is how to include fermions in the model.

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