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A variational principle for Kaluza–Klein type theories

Frédéric HÉLEIN*

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Abstract

For any positive integer n and any Lie group \mathfrak{G} , given a definite symmetric bilinear form on \mathbb{R}^n and an Ad-invariant scalar product on the Lie algebra of \mathfrak{G} , we construct a variational problem on fields defined on an arbitrary $(n + \dim\mathfrak{G})$ -dimensional manifold \mathcal{Y} . We show that, if \mathfrak{G} is compact and simply connected, any global solution of the Euler–Lagrange equations leads to identify \mathcal{Y} with the total space of a principal bundle over an n -dimensional manifold \mathcal{X} . Moreover \mathcal{X} is automatically endowed with a (pseudo-)Riemannian metric and a connection which are solutions of the Einstein–Yang–Mills system equation with a cosmological constant.

In 1919 T. Kaluza [11] (after an earlier attempt by G. Nordström [15] in 1914) discovered that solutions of the Einstein equations of gravity in vacuum on a 5-dimensional manifold could modelize Einstein equations coupled with Maxwell equations on a 4-dimensional space-time manifold, provided one assumes that the 5-dimensional manifold is a circle fiber bundle over space-time and that the metric is constant along these fibers. This was rediscovered more or less independently by O. Klein [13] in 1926 (and also by H. Mandel [14]), who proposed to assume that the size of the extra fifth dimension is sufficiently tiny in order to explain why this dimension is not directly observed. Since there this fascinating observation has been an important source of inspiration and questioning (see e.g. [7]). It has been extended to include non Abelian gauge theories [5, 12, 3, 4], in order to unify the Einstein equations with the Yang–Mills equations on a curved space-time and, in particular, it becomes an important ingredient of the 11-dimensional supergravity and the superstrings theories. It remains today a subject of questioning (see e.g. [1, 19]).

However some difficulties plags this beautiful idea:

First in the initial proposal by Kaluza and Klein the coefficient g_{55} of the metric on the 5-dimensional manifold along the fifth coordinates was assumed to be constant, which leads to inconsistency. This point was raised by P. Jordan [10] and Y. Thiry [17], who allowed this coefficient to be an extra scalar field. However this scalar field is a source of difficulties as to its physical interpretation.

A second problem is to explain why we don't 'see' the extra dimensions. The current answer since Klein relies on Heisenberg uncertainty principle: by

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expanding the fields in harmonic modes on each fiber and by assuming that the extra dimension is tiny we deduce that all modes excepted the zero one should be extremely massive and this would explain why we cannot observe their quantum excitations.

But lastly the fundamental question is to understand whether there is a dynamical mechanism which forces these extra dimensions to be compact (and tiny if we want to support the above hypothesis), instead of relying on an *ansatz* based on an *ad hoc* assumption.

In the following we present a model which aims to answer to the previous difficulties, even for classical solutions, provided that the structure Lie group is compact and simply connected (hence this works for instance for any group $SU(k)$ or for $SU(2) \times SU(3)$, but this excludes $U(1)$, for which our mechanism does not explain the compactness of the fibers without extra *ad hoc* hypotheses).

Of course this model leaves a number of well-known difficulties unanswered: it does not give a satisfactory answer for the electromagnetism, we cannot avoid a non vanishing cosmological constant in the the Einstein equations and it does not includes the fermions. However we hope that it could be an ingredient in further developments.

Notations — Let n be some positive integer and let \mathbf{b} be some symmetric non degenerate bilinear form on \mathbb{R}^n (e.g. a scalar product or the Minkowski pseudo metric on \mathbb{R}^n).

Let \mathfrak{G} be Lie group of finite dimension, equal to r , and let \mathfrak{g} be its Lie algebra. Let \mathbf{k} be a scalar product on \mathfrak{g} which is invariant by the adjoint action of \mathfrak{G} on \mathfrak{g} . We consider the vector space $\hat{\mathfrak{g}} := \mathbb{R}^n \oplus \mathfrak{g}$ and endow it with the bilinear form $\mathbf{h} := \mathbf{b} \oplus \mathbf{k}$, i.e. such that $\mathbf{h}((v, \xi), (w, \eta)) = \mathbf{b}(v, w) + \mathbf{k}(\xi, \eta)$, $\forall (v, \xi), (w, \eta) \in \mathbb{R}^n \oplus \mathfrak{g}$. We endow $\hat{\mathfrak{g}}$ with the Lie algebra structure $[(v, \xi), (w, \eta)] = (0, [\xi, \eta])$, $\forall (v, \xi), (w, \eta) \in \hat{\mathfrak{g}}$. We set $N := n + r$.

Let $(\mathbf{t}_A)_{1 \leq A \leq N}$ be a basis of $\hat{\mathfrak{g}}$ such that $(\mathbf{t}_a)_{1 \leq a \leq n}$ is a basis of \mathbb{R}^n and $(\mathbf{t}_\alpha)_{n+1 \leq \alpha \leq N}$ is a basis of \mathfrak{g} . We will systematically use the following conventions for the indices: $1 \leq A, B, C, \dots \leq N$ and $1 \leq a, b, c, \dots \leq n < n+1 \leq \alpha, \beta, \gamma, \dots \leq N$. We set $\mathbf{h}_{AB} := \mathbf{h}(\mathbf{t}_A, \mathbf{t}_B)$, $\mathbf{b}_{ab} := \mathbf{b}(\mathbf{t}_a, \mathbf{t}_b)$ and $\mathbf{k}_{\alpha\beta} := \mathbf{k}(\mathbf{t}_\alpha, \mathbf{t}_\beta)$. We denote by c_{BC}^A the structure constants of $\hat{\mathfrak{g}}$ in the basis $(\mathbf{t}_A)_{1 \leq A \leq n+r}$, defined by $[\mathbf{t}_B, \mathbf{t}_C] = \mathbf{t}_A c_{BC}^A$. We observe that

$$(\mathbf{h}_{AB}) = \begin{pmatrix} \mathbf{h}_{ab} & \mathbf{h}_{a\beta} \\ \mathbf{h}_{\alpha b} & \mathbf{h}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{ab} & 0 \\ 0 & \mathbf{k}_{\alpha\beta} \end{pmatrix}$$

and

$$(c_{BC}^A) = \begin{pmatrix} c_{bc}^a & c_{b\gamma}^a & c_{\beta\gamma}^a \\ c_{bc}^\alpha & c_{b\gamma}^\alpha & c_{\beta\gamma}^\alpha \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{\beta\gamma}^\alpha \end{pmatrix}.$$

We denote by $(\mathbf{t}^A)_{1 \leq A \leq N}$ the basis of $\hat{\mathfrak{g}}^*$ which is dual to $(\mathbf{t}_A)_{1 \leq A \leq N}$. We endow the Lie algebra $so(\hat{\mathfrak{g}}, \mathbf{h})$ of the rotation group $SO(\hat{\mathfrak{g}}, \mathbf{h})$ with the basis $(\mathbf{t}_{AB})_{1 \leq A < B \leq N}$ such that, if $u = \mathbf{t}_A u^A \in \hat{\mathfrak{g}}$, $\mathbf{t}_{AB}(\mathbf{t}_C) = \mathbf{t}_A \mathbf{h}_{BC} - \mathbf{t}_B \mathbf{h}_{AC}$ and we pose $\mathbf{t}_{AB} := -\mathbf{t}_{BA}$ if $A \geq B$.

The model — Let \mathcal{Y} be a smooth N -dimensional manifold. For any integer p denote by $\Omega^p(\mathcal{Y})$ the space of smooth p -forms on \mathcal{Y} . We define the space of

fields \mathcal{H} to be set of triplets (θ, φ, π) , where

$$\begin{cases} \theta \in \hat{\mathfrak{g}} \otimes \Omega^1(\mathcal{Y}), \text{ i.e. } \theta \text{ is a 1-form with coefficients in } \hat{\mathfrak{g}} \\ \varphi \in \mathfrak{so}(\hat{\mathfrak{g}}, \mathfrak{h}) \otimes \Omega^1(\mathcal{Y}), \text{ i.e. } \varphi \text{ is a 1-form with coefficients in } \mathfrak{so}(\hat{\mathfrak{g}}, \mathfrak{h}) \\ \pi \in \hat{\mathfrak{g}}^* \otimes \Omega^{n+r-2}(\mathcal{Y}), \text{ i.e. } \varphi \text{ is a } (n+r-2)\text{-form with coefficients in } \hat{\mathfrak{g}}^* \end{cases}$$

and which satisfy the two constraints that $\text{rank}\theta = N$ everywhere and, using the decomposition $\theta = \mathbf{t}_A \theta^A$,

$$\theta^a \wedge \theta^b \wedge \pi = 0, \forall a, b = 1, \dots, n. \quad (1)$$

We use the decompositions $\pi = \pi_A \mathbf{t}^A$ and $\varphi = \frac{1}{2} \mathbf{t}_{AB} \varphi^{AB}$, with $\varphi^{AB} + \varphi^{BA} = 0$, and we set $\varphi^A{}_B := \mathfrak{h}_{BB'} \varphi^{AB'}$. We note $d\varphi^{AB} + \frac{1}{2}[\varphi \wedge \varphi]^{AB} := d\varphi^{AB} + \varphi^A{}_C \wedge \varphi^{CB}$ and $d\theta^A + \frac{1}{2}[\theta \wedge \theta]^A := d\theta^A + \frac{1}{2}c_{BC}^A \theta^A \wedge \theta^B$. Lastly we set $\theta_{AB}^{(-2)} := \frac{\partial}{\partial \theta^A} \wedge \frac{\partial}{\partial \theta^B} \lrcorner \theta^1 \wedge \dots \wedge \theta^N$, where \lrcorner is the interior product.

Then to any (θ, φ, π) we associate the action

$$\mathcal{A}[(\theta, \varphi, \pi)] := \int_{\mathcal{Y}} \frac{1}{2} \theta_{AB}^{(-2)} \wedge \left(d\varphi^{AB} + \frac{1}{2}[\varphi \wedge \varphi]^{AB} \right) + \pi_A \wedge \left(d\theta^A + \frac{1}{2}[\theta \wedge \theta]^A \right). \quad (2)$$

Theorem 0.1 *Assume that \mathfrak{G} is compact and simply connected. Let \mathcal{Y} be a smooth connected N -dimensional manifold and let \mathcal{H} be as above. Then for any triplet $(\theta, \varphi, \pi) \in \mathcal{H}$ which is a **global** solution of the Euler–Lagrange equation of \mathcal{A} , there exist a Lie group \mathfrak{G}_0 whose universal covering is \mathfrak{G} , an n -dimensional manifold \mathcal{X} and a submersion $P : \mathcal{Y} \rightarrow \mathcal{X}$ which has the structure of a principal bundle with structure group \mathfrak{G}_0 over \mathcal{X} . Moreover \mathcal{X} is endowed with a pseudo-Riemannian metric \mathfrak{g} and a connection 1-form A which are solutions of the Einstein–Yang–Mills system of equation*

$$\begin{cases} \text{Ein}(\mathfrak{g})^a{}_d + \frac{1}{2} (F_{\gamma}{}^{ac} F_{\gamma dc} - \frac{1}{4} F_{\gamma}{}^{bc} F_{\gamma bc}) \delta_d^a & = \frac{1}{8} c_{\beta\gamma}^{\alpha} c_{\alpha\epsilon}^{\beta} h^{\gamma\epsilon} \delta_d^a \\ \nabla_c F_{\gamma}{}^{ca} - c_{\alpha\gamma}^{\beta} A_c^{\alpha} F_{\beta}{}^{ca} & = 0 \end{cases} \quad (3)$$

where $\text{Ein}(\mathfrak{g})^a{}_d := \text{Ric}(\mathfrak{g})^a{}_d - \frac{1}{2} R(\mathfrak{g}) \delta_d^a$ is the Einstein tensor of \mathfrak{g} and $F := dA + \frac{1}{2}[A \wedge A]$.

The hypothesis that (θ, φ, π) is a *global solution* in Theorem 0.1 means the following: by building a Riemannian metric associated to θ on \mathcal{Y} (e.g. $\eta_{ab} \theta^a \theta^b + \kappa_{\alpha\beta} \theta^{\alpha} \theta^{\beta}$, where (η_{ab}) is any scalar product on \mathbb{R}^n), we can endow \mathcal{Y} with the induced topology. Then the solution is said to be *global* if \mathcal{Y} is complete without boundary.

Comments on Theorem 0.1 — First note that, to any $\theta \in \hat{\mathfrak{g}} \otimes \Omega^1(\mathcal{Y})$ of maximal rank, one can associate the pseudo-Riemannian metric $\mathfrak{h} = \mathfrak{b}_{ab} \theta^a \theta^b + \kappa_{\alpha\beta} \theta^{\alpha} \theta^{\beta}$ on \mathcal{Y} .

The action \mathcal{A} is the sum of $\int_{\mathcal{Y}} \frac{1}{2} \theta_{AB}^{(-2)} \wedge (d\varphi^{AB} + \frac{1}{2}[\varphi \wedge \varphi]^{AB})$, the N -dimensional generalization of the so-called ‘Palatini’ functional on \mathcal{Y} , and of $\int_{\mathcal{Y}} \pi_A \wedge (d\theta^A + \frac{1}{2}[\theta \wedge \theta]^A)$, which is at the origin of the spontaneous symmetry breaking and fibration.

If one is familiar with the ‘Palatini’ action it is not difficult to see that variations with respect to φ implies that the connection on $T\mathcal{Y}$ associated to φ is torsion free and hence is the Levi-Civita connection for \mathfrak{h} . The variation

with respect to π , which plays here the role of a Lagrange multiplier, leads to the relation $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0 \text{ mod}[\theta^a]$. Using exterior differential calculus à la Cartan and Frobenius' theorem (as e.g. in [9, 18]), one deduce first that \mathcal{Y} is foliated by submanifolds \mathfrak{f} which are solutions of $\theta^a|_{\mathfrak{f}} = 0, \forall a = 1, \dots, n$. Further applications of Frobenius' theorem show that one can locally define a map $g : \mathcal{Y} \rightarrow \mathfrak{G}$ the restriction on each leaf \mathfrak{f} of which is a local diffeomorphism. The global solution hypothesis and the hypotheses on \mathfrak{G} then imply that the leaves actually close up and form the fibers of a fibration over a quotient manifold \mathcal{X} . Still using the relation $d\theta + \frac{1}{2}[\theta \wedge \theta] = 0 \text{ mod}[\theta^a]$ one deduces that $\theta^a = e^a$ and $\theta^\alpha = (gAg^{-1} + g^{-1}dg)^\alpha$, where e^a and A are pull-back forms of 1-forms on \mathcal{X} . We then set $F := dA + \frac{1}{2}[A \wedge A]$, the curvature 2-form of A .

Then the variation with respect to θ leads to the equation

$$\frac{1}{2}\theta_{ABC}^{(-3)} \wedge (d\varphi^{AB} + \varphi^A{}_D \wedge \varphi^{DB}) = -d\pi_C - c_{CA}^B \theta^A \wedge \pi_B \text{ mod}[\theta_\gamma^{(N-1)}].$$

One can recognize on the left hand side the Einstein tensor of \mathbf{h} on \mathcal{Y} . Hence, in order to prove (3), it would be nice to get that the left hand side vanish. But this is not obvious in the previous relation and actually this is not completely true. However a miracle occurs, analogous to what happens in [8, 9]. After a gauge transformation $e^\alpha = S_\beta^\alpha \theta^\beta$ and $\omega^\alpha{}_\beta = S_{\alpha'}^\alpha \varphi^{\alpha'}{}_{\beta'} (S^{-1})_\beta^{\beta'} - dS_{\beta'}^\alpha (S^{-1})_\beta^{\beta'}$, where $S = \text{Ad}_g$ the previous equation translates as

$$\frac{1}{2}e_{ABC}^{(-3)} \wedge (d\omega^{AB} + \omega^A{}_D \wedge \omega^{DB}) = -dp_C \text{ mod}[e_\gamma^{(-1)}]$$

The key observations are that the left hand side is constant on any fiber, whereas the restriction of the right hand side to any fiber is an exact form. Both observations lead to the conclusion that $\frac{1}{2}e_{ABC}^{(-3)} \wedge (d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB}) = 0 \text{ mod}[e_\gamma^{(-1)}]$, i.e. the two blocks $\text{Ein}(\mathbf{h})^{\alpha_c}$ and $\text{Ein}(\mathbf{h})^{\alpha_\gamma}$ of the Einstein tensor of ω vanish. This is enough to imply (3). The resulted picture is analogous to the one described in [2]: the manifold $(\mathcal{Y}, \mathbf{h})$ is not Einstein in general but leads to solutions of the system (3) anyway.

We remark also that in the case where \mathfrak{G} is not simply connected (e.g. for $U(1)$ or $U(1) \times SU(2) \times SU(3)$) similar conclusions can be reached if one assumes that the leaves of the foliation close up to compact fibers.

1 The Euler–Lagrange equations

In the following we assume that $(\theta, \varphi, \pi) \in \mathcal{H}$ is a critical point of \mathcal{A} . We denote by $\mathbf{h} = b_{ab}\theta^a\theta^b + k_{\alpha\beta}\theta^\alpha\theta^\beta$ on \mathcal{Y} the induced metric on \mathcal{Y} .

1.1 Variations with respect to coefficients of π

Since $\text{rank}\theta = N$, the family $(\theta^1, \dots, \theta^N)$ is a coframe on \mathcal{Y} , there exists unique coefficients H_{BC}^A such that $d\theta^A + \frac{1}{2}c_{BC}^A \theta^B \wedge \theta^C = \frac{1}{2}H_{BC}^A \theta^B \wedge \theta^C$ and $H_{BC}^A + H_{CB}^A = 0$. We note $\theta^{(N)} := \theta^1 \wedge \dots \wedge \theta^N$, $\theta_A^{(N-1)} := \frac{\partial}{\partial \theta^A} \lrcorner \theta^{(N)}$, $\theta_{AB}^{(N-2)} := \frac{\partial}{\partial \theta^A} \wedge \frac{\partial}{\partial \theta^B} \lrcorner \theta^{(N)}$, $\theta_{ABC}^{(N-3)} := \frac{\partial}{\partial \theta^A} \wedge \frac{\partial}{\partial \theta^B} \wedge \frac{\partial}{\partial \theta^C} \lrcorner \theta^{(N)}$. Hence we can decompose any $(N-2)$ -form π_A as $\pi_A = \frac{1}{2}\pi_A^{BC} \theta_{BC}^{(N-2)}$, where $\pi_A^{BC} + \pi_A^{CB} = 0$. The

hypothesis (1) then reads $\pi_A^{ab} = 0$ or

$$\pi_A = \pi_A^{b\gamma} \theta_{b\gamma}^{(N-2)} + \frac{1}{2} \pi_A^{\beta\gamma} \theta_{\beta\gamma}^{(N-2)} \quad (4)$$

A first order variation of (θ, φ, π) keeping θ and φ constant and respecting the constraint (4) thus induces a variation of π of the form $\delta\pi_A = \chi_A^{b\gamma} \theta_{b\gamma}^{(N-2)} + \frac{1}{2} \chi_A^{\beta\gamma} \theta_{\beta\gamma}^{(N-2)}$. The fact that the action \mathcal{A} is stationary with respect to such variations of π thus reads

$$\int_{\mathcal{Y}} \delta\pi_A \wedge \left(d\theta^A + \frac{1}{2} [\theta \wedge \theta]^A \right) = \int_{\mathcal{Y}} \left(\chi_A^{b\gamma} H_{b\gamma}^A + \frac{1}{2} \chi_A^{\beta\gamma} H_{\beta\gamma}^A \right) \theta^{(N)} = 0, \quad \forall \chi_A^{b\gamma}, \chi_A^{\beta\gamma}$$

and lead to the Euler–Lagrange equations $H_{b\gamma}^A = H_{\beta\gamma}^A = 0, \forall A, b, \beta, \gamma$. We thus deduce that

$$d\theta^A + \frac{1}{2} c_{BC}^A \theta^B \wedge \theta^C = \frac{1}{2} H_{bc}^A \theta^b \wedge \theta^c \quad (5)$$

or equivalently

$$\begin{cases} d\theta^a & = \frac{1}{2} H_{bc}^a \theta^b \wedge \theta^c \\ d\theta^\alpha + \frac{1}{2} c_{\beta\gamma}^\alpha \theta^\beta \wedge \theta^\gamma & = \frac{1}{2} H_{bc}^\alpha \theta^b \wedge \theta^c \end{cases} \quad (6)$$

1.2 Variations with respect to φ

Keeping θ and π fixed we look at first order variations $\delta\varphi = \psi$ of φ . This induces the condition that, for all ψ ,

$$\frac{1}{2} \int_{\mathcal{Y}} d \left(\psi^{AB} \wedge \theta_{AB}^{(N-2)} \right) + \psi^{AB} \wedge \left(d\theta_{AB}^{(N-2)} - \varphi^{B'}{}_B \wedge \theta_{AB'}^{(N-2)} - \varphi^{A'}{}_A \wedge \theta_{A'B}^{(N-2)} \right) = 0$$

Assuming that ψ has compact support we deduce the relation

$$\left(d\theta^A + \varphi^{A'}{}_{A'} \wedge \theta^{A'} \right) \wedge \theta_{ABC}^{(N-3)} = d\theta_{AB}^{(N-2)} - \varphi^{B'}{}_B \wedge \theta_{AB'}^{(N-2)} - \varphi^{A'}{}_A \wedge \theta_{A'B}^{(N-2)} = 0$$

which implies that the torsion 2-form $d\theta^A + \varphi^{A'}{}_{A'} \wedge \theta^{A'}$ vanishes. Hence the connection on $T\mathcal{Y}$ associated to φ is the Levi-Civita connection of $(\mathcal{Y}, \mathbf{h})$, where $\mathbf{h} = h_{AB} \theta^A \theta^B$.

1.3 Variations with respect to θ

We first observe that, through a variation $\delta\theta = \tau$ of θ keeping φ and the coefficients $\pi_A^{b\gamma}$ and $\pi_A^{\beta\gamma}$ fixed, we have

$$\delta \left(\theta_{AB}^{(N-2)} \right) = \tau^C \wedge \theta_{ABC}^{(N-3)},$$

$\delta \left(d\theta^A + \frac{1}{2} c_{BC}^A \theta^B \wedge \theta^C \right) = d\tau^A + c_{BC}^A \tau^B \wedge \theta^C$ which implies

$$\pi_A \wedge \delta \left(d\theta^A + \frac{1}{2} c_{BC}^A \theta^B \wedge \theta^C \right) = d(\tau^A \wedge \pi_A) + \tau^A \wedge (d\pi_A + c_{AB}^C \theta^B \wedge \pi_C)$$

and $\delta(\pi_A) = \pi_A^{b\gamma} \left(\tau^d \wedge \theta_{b\gamma d}^{(N-3)} + \tau^\delta \wedge \theta_{b\gamma\delta}^{(N-3)} \right) + \frac{1}{2} \pi_A^{\beta\gamma} \left(\tau^d \wedge \theta_{\beta\gamma d}^{(N-3)} + \tau^\delta \wedge \theta_{\beta\gamma\delta}^{(N-3)} \right)$
which implies using (5)

$$\delta(\pi_A) \wedge \left(d\theta^A + \frac{1}{2} c_{BC}^A \theta^B \wedge \theta^C \right) = -\pi_A^{b\gamma} H_{bd}^A \tau^d \wedge \theta_\gamma^{(N-1)}.$$

Hence assuming that τ has compact support we obtain

$$\int_{\mathcal{Y}} \tau^C \wedge \left(\frac{1}{2} \theta_{ABC}^{(N-3)} \wedge \Phi^{AB} - \pi_A^{b\gamma} H_{bc}^A \theta_\gamma^{(N-1)} + d\pi_C - c_{AC}^B \theta^A \wedge \pi_B \right) = 0$$

and we deduce the Euler–Lagrange equation

$$\frac{1}{2} \theta_{ABC}^{(N-3)} \wedge \Phi^{AB} + d\pi_C - c_{AC}^B \theta^A \wedge \pi_B = 0 \text{ mod } [\theta_\alpha^{(N-1)}] \quad (7)$$

where, for any 3-form ψ , $\psi = 0 \text{ mod } [\theta_\alpha^{(N-1)}]$ means that there exists coefficients ψ^α such that $\psi = \psi^\alpha \theta_\alpha^{(N-1)}$.

2 The fibration

From the first equation in (6) we deduce that $d\theta^a = 0 \text{ mod } [\theta^b]$, $\forall a = 1, \dots, n$. Since the rank of $(\theta^1, \dots, \theta^n)$ is equal to n everywhere, Frobenius' theorem implies that, for any point $y \in \mathcal{Y}$, there exists a neighbourhood of y in which there exists a unique submanifold \mathfrak{f} of dimension r crossing y such that $\theta^a|_{\mathfrak{f}} = 0$, $\forall a = 1, \dots, n$. Hence \mathcal{Y} is foliated by integral leaves of dimension r .

Consider any integral leaf \mathfrak{f} . It follows from the second equation in (6) that $d\theta^{\mathfrak{g}} + \frac{1}{2} [\theta^{\mathfrak{g}} \wedge \theta^{\mathfrak{g}}]|_{\mathfrak{f}} = 0$, where $\theta^{\mathfrak{g}} := \mathfrak{t}_\alpha \theta^\alpha$. Consider on the product manifold $\mathfrak{f} \times \mathfrak{G}$ with coordinates (y, h) the \mathfrak{g} -valued 1-form $\tau := dh - h\theta^{\mathfrak{g}}$. From the identity $d\tau + h(d\theta^{\mathfrak{g}} + \frac{1}{2} [\theta^{\mathfrak{g}} \wedge \theta^{\mathfrak{g}}]) + \tau \wedge \theta^{\mathfrak{g}} = 0$, we deduce that $d(\tau|_{\mathfrak{f} \times \mathfrak{G}}) = 0 \text{ mod } [\tau]$ and hence, again by Frobenius' theorem, the exterior differential system $\tau|_{\mathfrak{f} \times \mathfrak{G}} = 0$ is completely integrable. This implies, for any point $y_0 \in \mathfrak{f}$, the existence of an unique map $g : \mathfrak{f} \rightarrow \mathfrak{G}$ such that $g(y_0) = 1_{\mathfrak{G}}$ and $dg - g\theta^{\mathfrak{g}}|_{\mathfrak{f}} = 0$. Moreover g is locally invertible.

Consider any smooth path $\gamma : [0, 1] \rightarrow \mathfrak{G}$ such that $\gamma(0) = 1_{\mathfrak{G}}$. We can associate to it an unique path $u : [0, 1] \rightarrow \mathfrak{f}$ such that $u(0) = y_0$ and $(u, \gamma)^* \tau = 0$. In particular the hypothesis that $(\mathcal{Y}, \mathbf{h})$ is complete ensures that the differential equations defining u has a solution on the whole intervalle $[0, 1]$. Since the exterior differential system defined by τ is closed, to any path homotopic to γ in \mathfrak{G} with fixed extremities it corresponds a path homotopic to u in \mathfrak{f} with fixed extremities. Since \mathfrak{G} is simply connected we can thus define an unique map $T : \mathfrak{G} \rightarrow \mathfrak{f}$ such that $T(1_{\mathfrak{G}}) = y_0$ and $(T \times Id)^* \tau = 0$. Hence \mathfrak{G} is an universal cover of \mathfrak{f} and, in particular, since \mathfrak{G} is compact \mathfrak{f} is *compact*. By a theorem of C. Ehresmann [6], this implies that the set of integral leaves has the structure of an n -dimensional manifold \mathcal{X} and that the quotient map $P : \mathcal{Y} \rightarrow \mathcal{X}$ is a bundle fibration.

From $\frac{\partial}{\partial \theta^{\mathfrak{b}}} \lrcorner e^a = \frac{\partial}{\partial \theta^{\mathfrak{b}}} \lrcorner de^a = 0$ we deduce that there exists a coframe $(\underline{e}^a)_{1 \leq a \leq n}$ on \mathcal{X} such that $e^a = P^* \underline{e}^a$, $\forall a$. Thus we can equip \mathcal{X} with the pseudo Riemannian metric $\underline{\mathbf{g}} := \mathfrak{b}_{ab} \underline{e}^a \underline{e}^b$.

In the following we choose an n -dimensional submanifold $\Sigma \subset \mathcal{Y}$ transverse to the fibration and (restricting ourself to an open subset of \mathcal{Y} if necessary) we

define the map $g : \mathcal{Y} \rightarrow \mathfrak{G}$ which is constant equal to $1_{\mathfrak{G}}$ on Σ and such that $dg - g\theta^{\mathfrak{g}}|_f = 0$ for any integral leaf f . We then define $A := g\theta^{\mathfrak{g}}g^{-1} - dg \cdot g^{-1}$. Relation $dg - g\theta^{\mathfrak{g}}|_f = 0$ then translates as $A|_f = 0$ and hence we have the decomposition $A = A_a\theta^a$. Moreover since

$$\theta^{\mathfrak{g}} = g^{-1}Ag + g^{-1}dg, \quad (8)$$

we have $d\theta^{\mathfrak{g}} + \frac{1}{2}[\theta \wedge \theta]^{\mathfrak{g}} = g^{-1}(dA + \frac{1}{2}[A \wedge A])g = g^{-1}Fg$, where $F := dA + \frac{1}{2}[A \wedge A]$. In particular we deduce from (6) that $\frac{\partial}{\partial \theta^\alpha} \lrcorner dA = 0, \forall \alpha = n+1, \dots, N$, i.e. the coefficients A_a are constants on the fibers f . Moreover we have

$$F^\alpha = \frac{1}{2}F_{bc}^\alpha \theta^b \wedge \theta^c, \quad (9)$$

where the coefficients $F_{bc}^\alpha = gH_{bc}^\alpha g^{-1}$ are constant on the fibers. .

3 Gauge transformation

In the following we define the map $S : \mathcal{Y} \rightarrow \text{End}(\hat{\mathfrak{g}})$ such that $\forall (v, \xi) \in \hat{\mathfrak{g}} = \mathbb{R}^n \times \mathfrak{g}$, $S(v, \xi) := (v, g\xi g^{-1})$, where $g : \mathcal{Y} \rightarrow \mathfrak{g}$ is the map defined previously. Extending to $\hat{\mathfrak{g}}$ the adjoint action of \mathfrak{G} in a trivial way, we may write also $S(v, \xi) = g(v, \xi)g^{-1}$. Let $(S_B^A)_{1 \leq A, B \leq N}$ be the matrix of S in the basis $(\mathbf{t}_A)_{1 \leq A \leq N}$, i.e. such that $S(\mathbf{t}_A) = \mathbf{t}_B S_B^A$. We remark that S takes values in $SO(\hat{\mathfrak{g}}, \mathfrak{h})$ since in particular the scalar product \mathfrak{k} on \mathfrak{g} is invariant by the adjoint action of \mathfrak{G} . We define a new coframe $(e^A)_{1 \leq A \leq N}$ by $e^A = S_B^A \theta^B$. Equivalently

$$\begin{cases} e^a & := & \theta^a & \forall a = 1, \dots, n \\ e^\alpha & := & S_\beta^\alpha \theta^\beta & \forall \alpha = n+1, \dots, N \end{cases}$$

Then $e^\alpha = (g\theta^{\mathfrak{g}}g^{-1})^\alpha$ and (8) imply

$$e^\alpha = A^\alpha + (dg g^{-1})^\alpha, \quad \forall \alpha = n+1, \dots, N. \quad (10)$$

Let us translate the left hand side of (7) in the new coframe. First we define $e^{(N)} := e^1 \wedge \dots \wedge e^N$ and note that $e^{(N)} = \theta^{(N)}$. Moreover defining $e_A^{(N-1)} := \frac{\partial}{\partial e^A} \lrcorner e^{(N)}$, $e_{AB}^{(N-2)} := \frac{\partial}{\partial e^A} \wedge \frac{\partial}{\partial e^B} \lrcorner e^{(N)}$, we observe that, since $\frac{\partial}{\partial \theta^A} = \frac{\partial}{\partial e^B} S_B^A$, we have $\theta_A^{(N-1)} = e_{A'}^{(N-1)} S_{A'}^A$, $\theta_{AB}^{(N-2)} = e_{A'B'}^{(N-2)} S_{A'}^A S_{B'}^B$ and $\theta_{ABC}^{(N-3)} = e_{A'B'C'}^{(N-3)} S_{A'}^A S_{B'}^B S_{C'}^C$.

Second let ω be the $so(\hat{\mathfrak{g}}, \mathfrak{h})$ -valued connection 1-form in the coframe $(e^A)_{1 \leq A \leq N}$ defined by $\omega := S\varphi S^{-1} - dS S^{-1}$. Let $\Omega := d\omega + \frac{1}{2}[\omega \wedge \omega] = S\Phi S^{-1}$, where $\Phi := d\varphi + \frac{1}{2}[\varphi \wedge \varphi]$. Then $\Phi^{AB} = (S^{-1})_{A'}^A (S^{-1})_{B'}^B \Omega^{A'B'}$.

We deduce that $\theta_{ABC}^{(N-3)} \Omega^{AB} = e_{ABC'}^{(N-3)} \Omega^{AB} S_{C'}^C$. Hence (7) is equivalent to

$$\frac{1}{2} e_{ABC}^{(N-3)} \Omega^{AB} + (d\pi_{C'} - c_{AC'}^B \theta^A \wedge \pi_B) (S^{-1})_{C'}^{C'} = 0 \text{ mod}[e_\alpha^{(N-1)}]. \quad (11)$$

(Note that, for any $(N-1)$ -form ψ , $\psi = 0 \text{ mod}[e_\alpha^{(N-1)}]$ iff $\psi = 0 \text{ mod}[\theta_\alpha^{(N-1)}]$.)

Lemma 3.1 *We have*

$$(d\pi_{C'} - c_{AC'}^B \theta^A \wedge \pi_B) (S^{-1})_{C'}^{C'} = d \left(\pi_{C'} (S^{-1})_{C'}^{C'} \right) \text{ mod}[e_\gamma^{(N-1)}]. \quad (12)$$

Proof — From the definition of S we deduce $d(S^{-1})_C^{C'} = -(S^{-1})_A^{C'} c_{BC}^A (e^B - A^B)$. Hence

$$\begin{aligned} d\left((S^{-1})_C^{C'} \pi_{C'}\right) &= -(S^{-1})_A^{C'} c_{BC}^A (e^B - A^B) \wedge \pi_{C'} + (S^{-1})_C^{C'} (d\pi_{C'}) \\ &= -(S^{-1})_A^{C'} c_{BC}^A S_B^{B'} \theta^{B'} \wedge \pi_{C'} + (S^{-1})_A^{C'} c_{BC}^A A^B \wedge \pi_{C'} + (S^{-1})_C^{C'} (d\pi_{C'}) \end{aligned}$$

But because of $[\text{Ad}_g(\xi), \text{Ad}_g(\eta)] = \text{Ad}_g([\xi, \eta])$, $\forall \xi, \eta \in \mathfrak{g}$, which is equivalent to $c_{B'C'}^A S_B^{B'} S_C^{C'} = S_A^A c_{BC}^A$, we have $(S^{-1})_A^{C'} c_{BC}^A S_B^{B'} = c_{B'C''}^{C'} (S^{-1})_C^{C''}$. Thus for the first term on the r.h.s.,

$$(S^{-1})_A^{C'} c_{BC}^A S_B^{B'} \theta^{B'} \wedge \pi_{C'} = c_{B'C''}^{C'} (S^{-1})_C^{C''} \theta^{B'} \wedge \pi_{C'} = c_{AC}^B \theta^A \wedge \pi_B (S^{-1})_C^{C'}$$

and hence

$$d\left((S^{-1})_C^{C'} \pi_{C'}\right) = (d\pi_{C'} - c_{AC}^B \theta^A \wedge \pi_B) (S^{-1})_C^{C'} + (S^{-1})_A^{C'} c_{BC}^A A^B \wedge \pi_{C'}$$

However we deduce from (4) that $\pi_{C'} = \pi_{C'}^{b\gamma} e_{b\gamma}^{(N-2)} + \frac{1}{2} \pi_{C'}^{\beta\gamma} e_{\beta\gamma}^{(N-2)}$ and, since $A^B = A_c^B e^c$, we get $(S^{-1})_A^{C'} c_{BC}^A A^B \wedge \pi_{C'} = -(S^{-1})_A^{C'} c_{BC}^A A_b^B \pi_{C'}^{b\gamma} e_\gamma^{(N-1)} = 0 \text{ mod}[e_\gamma^{(N-1)}]$. Hence (12) follows. \square

Now let us define $p_C := \pi_{C'} (S^{-1})_C^{C'}$. We deduce from (12) that (11) is equivalent to

$$\frac{1}{2} e_{ABC}^{(N-3)} \wedge \Omega^{AB} + dp_C = 0 \text{ mod}[e_\alpha^{(N-1)}]. \quad (13)$$

We also observe the *a priori* decomposition $p_C = p_C^{b\gamma} e_{b\gamma}^{(N-2)} + \frac{1}{2} p_C^{\beta\gamma} e_{\beta\gamma}^{(N-2)}$, which, by writing $dp_C^{b\gamma} = p_{C,c}^{b\gamma} e^c + p_{C,\gamma}^{b\gamma} e^\gamma$ and $dp_C^{\beta\gamma} = p_{C,c}^{\beta\gamma} e^c + p_{C,\delta}^{\beta\gamma} e^\delta$, implies

$$dp_C = p_{C,\gamma}^{b\gamma} e_b^{(N-1)} - p_{C,b}^{b\gamma} e_\gamma^{(N-1)} + p_{C,\gamma}^{\beta\gamma} e_\beta^{(N-1)} = p_{C,\gamma}^{b\gamma} e_b^{(N-1)} \text{ mod}[e_\alpha^{(N-1)}]$$

On the other hand by decomposing $\Omega^{AB} = \frac{1}{2} \Omega^{AB}{}_{CD} e^C \wedge e^D$, we find that $\frac{1}{2} e_{ABC}^{(N-3)} \wedge \Omega^{AB} = -\text{Ein}(\Omega)^A{}_C e^{(N-1)}$, where $\text{Ein}(\Omega)^A{}_C := \text{Ric}(\Omega)^A{}_C - \frac{1}{2} \text{R}(\Omega) \delta^A{}_C$, $\text{Ric}(\Omega)^A{}_C := \Omega^{AB}{}_{CB}$ and $\text{R}(\Omega) := \text{Ric}(\Omega)^A{}_A$. Obviously $\text{Ric}(\Omega)^A{}_C$ is the Ricci curvature, $\text{R}(\Omega)$ the scalar curvature and $\text{Ein}(\Omega)^A{}_C$ the Einstein tensor, in the coframe $(e^A)_{1 \leq A \leq N}$. Hence we find that (13) is equivalent to $(\text{Ein}(\Omega)^a{}_C + p_{C,\gamma}^{a\gamma}) e_a^{(N-1)} = 0 \text{ mod}[e_\alpha^{(N-1)}]$, or

$$\text{Ein}(\Omega)^a{}_C = -p_{C,\gamma}^{a\gamma}, \quad \forall a = 1, \dots, n, \forall C = 1, \dots, N. \quad (14)$$

We will come back to this equation later on.

4 The connection and the curvature form in the coframe $(e^A)_{1 \leq A \leq +r}$

We need to compute the connection 1-form ω and the curvature 2-form. As a preliminary we first set $\underline{\gamma}^a{}_c$ to be the connection 1-form on $(\mathcal{X}, \underline{\mathfrak{g}})$ in the coframe \underline{e}^a , i.e. which satisfies $\underline{\gamma}^{ac} + \underline{\gamma}^{ca} = 0$ and $d\underline{e}^a + \underline{\gamma}^a{}_c \wedge \underline{e}^c = 0$. Then we

set $\gamma^a_c := P^* \overline{\gamma^a_c}$ which satisfies similar relations. Second we deduce from (10) and from the definition of e^α that

$$\begin{aligned} de^\alpha - \frac{1}{2}[e \wedge e]^\alpha + [A \wedge e]^\alpha &= de^\alpha + \frac{1}{2}[e \wedge e]^\alpha - [(dg g^{-1}) \wedge e]^\alpha \\ &= (gd\theta g^{-1} + [dgg^{-1} \wedge e])^\alpha + \frac{1}{2}[e \wedge e]^\alpha - [dgg^{-1} \wedge e]^\alpha \\ &= (g(d\theta + \frac{1}{2}[\theta \wedge \theta])g^{-1})^\alpha = F^\alpha = \frac{1}{2}F_{bc}^\alpha e^b \wedge e^c. \end{aligned}$$

All that can be summarized in the relations

$$\begin{cases} de^a + \gamma^a_c \wedge e^c &= 0 \\ de^\alpha - \frac{1}{2}F_{bc}^\alpha e^b \wedge e^c - \frac{1}{2}c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta) \wedge e^\gamma &= 0 \end{cases} \quad (15)$$

Now the connexion 1-form ω is uniquely characterized by the condition $\omega^{AB} + \omega^{BA} = 0$ (preservation of the metric) and $de^A + \omega^A_C \wedge e^C = 0$ (the torsion vanishes), which can be written

$$\begin{cases} de^a + \omega^a_c \wedge e^c + \omega^a_\gamma \wedge e^\gamma &= 0 \\ de^\alpha + \omega^\alpha_c \wedge e^c + \omega^\alpha_\gamma \wedge e^\gamma &= 0 \end{cases} \quad (16)$$

Comparing with (15) we are tempted to assume that $\omega^a_\gamma = -\frac{1}{2}c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta)$, which fulfills the condition $\omega^{\alpha\beta} + \omega^{\beta\alpha} = 0$, since $c_{\beta\gamma}^\alpha k^{\gamma\gamma} + c_{\beta\alpha}^\gamma k^{\alpha\alpha} = 0$ because the metric k is preserved by the adjoint action of \mathfrak{g} . We also assume that $\omega^a_c = -\frac{1}{2}F_{bc}^\alpha e^b$, which forces automatically $\omega^a_\gamma = \frac{1}{2}k_{\gamma\gamma'} F_{bc'}^{\gamma'} g^{c'c} e^b$, in order to satisfy $\omega^{\alpha b} + \omega^{b\alpha} = 0$. Then in order to fulfill the first relation of (16), one needs to assume that $\omega^a_c = \gamma^a_c - \frac{1}{2}k_{\gamma\gamma'} F_{a'c}^{\gamma'} b^{a'a} e^\gamma$. We then have $\omega^{ac} = \gamma^{ac} - \frac{1}{2}k_{\gamma\gamma'} F_{a'c}^{\gamma'} b^{a'a} b^{c'c} e^\gamma$, which is skew symmetric in (a, c) . Thus we see that the forms ω^{AC} defined by:

$$\begin{pmatrix} \omega^a_c & \omega^a_\gamma \\ \omega^\alpha_c & \omega^\alpha_\gamma \end{pmatrix} = \begin{pmatrix} \gamma^a_c - \frac{1}{2}k_{\gamma\gamma'} F_{a'c}^{\gamma'} b^{a'a} e^\gamma & \frac{1}{2}k_{\gamma\gamma'} F_{ba'}^{\gamma'} b^{a'a} e^b \\ -\frac{1}{2}F_{bc}^\alpha e^b & -\frac{1}{2}c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta) \end{pmatrix}$$

satisfy (16) and $\omega^{AC} + \omega^{CA} = 0$. Hence this is the Levi-Civita connection 1-form of $(\mathcal{Y}, \mathbf{h})$. In the following it will be convenient to set $F^{\gamma bc} := F_{bc}^\gamma$, $F^{\gamma a}_c := k_{\gamma\gamma'} F^{\gamma'}_{a'c} b^{a'a}$ and $F_{\gamma b}^c := k_{\gamma\gamma'} F^{\gamma'}_{bc'} b^{c'c}$. Then

$$\begin{pmatrix} \omega^a_c & \omega^a_\gamma \\ \omega^\alpha_c & \omega^\alpha_\gamma \end{pmatrix} = \begin{pmatrix} \gamma^a_c - \frac{1}{2}F^{\gamma a}_c e^\gamma & \frac{1}{2}F_{\gamma b}^a e^b \\ -\frac{1}{2}F_{bc}^\alpha e^b & -\frac{1}{2}c_{\beta\gamma}^\alpha (e^\beta - 2A^\beta) \end{pmatrix}$$

We can thus compute the curvature 2-form $\Omega^A_C = d\omega^A_C + \omega^A_B \wedge \omega^B_C$.

$$\Omega^a_c = d\left(\gamma^a_c - \frac{1}{2}F^{\gamma a}_c e^\gamma\right) + \left(\gamma^a_b - \frac{1}{2}F_{\beta' a}^b e^{\beta'}\right) \wedge \left(\gamma^b_c - \frac{1}{2}F_{\gamma' b}^c e^{\gamma'}\right) - \frac{1}{4}F_{\beta b'}^a F^{\beta c'}_{c'} e^{b'} \wedge e^{c'}$$

$$\Omega^a_\gamma = d(F_{\gamma b}^a e^b) + \frac{1}{2}\left(\gamma^a_b - \frac{1}{2}F_{\beta' a}^b e^{\beta'}\right) \wedge (F_{\gamma b'}^a e^{b'}) + \frac{1}{4}F_{\beta b'}^a c_{\beta'\gamma}^\beta e^{b'} \wedge (2A^{\beta'} - e^{\beta'})$$

$$\Omega^\alpha_c = -d(F_{bc}^\alpha e^b) - \frac{1}{2}(F^{\alpha b'}_{b'} e^{b'}) \wedge \left(\gamma^b_c - \frac{1}{2}F_{\gamma' b}^c e^{\gamma'}\right) - \frac{1}{4}F_{bc}^\beta c_{\beta'\gamma}^\alpha (2A^{\beta'} - e^{\beta'}) \wedge e^b$$

$$\Omega^\alpha_\gamma = \frac{1}{2}d(c_{\beta\gamma}^\alpha (2A^\beta - e^\beta)) - \frac{1}{4}(F^{\alpha b'}_{b'} e^{b'}) \wedge (F_{\gamma c'}^b e^{c'}) + \frac{1}{4}c_{\beta'\gamma}^\alpha c_{\beta'\gamma}^\beta (2A^{\beta'} - e^{\beta'}) \wedge (2A^{\gamma'} - e^{\gamma'}).$$

Lastly we obtain the components of the Ricci tensor $\text{Ric}(\omega)$ through a lengthy computation.

$$\text{Ric}(\omega)^a{}_d = \text{Ric}(\gamma)^a{}_d - \frac{1}{2}F_\beta{}^{ac}F^\beta{}_{dc} \quad (17)$$

where $\text{Ric}(\gamma)^a{}_d := (d\gamma^a{}_c + \gamma^a{}_b \wedge \gamma^b{}_c)_{de} \mathbf{b}^{ce}$ is the Ricci curvature of γ , and using the decomposition $dF_\delta{}^{ac} = F_\delta{}^{ac}{}_{,c}e^c$,

$$\text{Ric}(\omega)^a{}_\delta = \frac{1}{2} \left(F_\delta{}^{ac}{}_{,c} + (\omega^a{}_b)_c F_\delta{}^{bc} + (\omega^c{}_b)_c F_\delta{}^{ab} - c_{\alpha\delta}^\gamma A_c^\alpha F_\gamma{}^{ac} \right) \quad (18)$$

$$\text{Ric}(\alpha)^\alpha{}_\delta = \frac{1}{4}F_\delta{}^{bc}F^\alpha{}_{bc} - \frac{1}{4}c_{\beta\gamma}^\alpha c_{\delta\epsilon}^\beta \mathbf{k}^{\gamma\epsilon} \quad (19)$$

We deduce the scalar curvature $R(\omega)$ of ω in function of the scalar curvature $R(\gamma) := \text{Ric}(\gamma)^a{}_a$:

$$R(\alpha) = R(\gamma) - \frac{1}{4}F_\alpha{}^{ab}F^\alpha{}_{ab} - \frac{1}{4}c_{\beta\gamma}^\alpha c_{\alpha\delta}^\beta \mathbf{k}^{\gamma\delta} \quad (20)$$

Hence the Einstein tensor of ω is

$$\text{Ein}(\omega)^a{}_d = \text{Ein}(\gamma)^a{}_d - \frac{1}{2} \left(F_\beta{}^{ac}F^\beta{}_{dc} - \frac{1}{4}F_\alpha{}^{bc}F^\alpha{}_{bc}\delta^a{}_d \right) + \frac{1}{8}c_{\beta\gamma}^\alpha c_{\alpha\delta}^\beta \mathbf{k}^{\gamma\delta}\delta^a{}_d \quad (21)$$

and $\text{Ein}(\omega)^a{}_\delta = \text{Ric}(\omega)^a{}_\delta$ is given by (18).

An important observation is that *the components of $\text{Ein}(\omega)^a{}_d$ and $\text{Ein}(\omega)^a{}_\delta$ are constant on the fibers f .*

5 The Einstein–Yang–Mills equations

We conclude by exploiting the fact that the fibers f are compact without boundary. Let $\mu^{(r)} := e^{n+1} \wedge \dots \wedge e^N$ and set $\mu_\alpha^{(r-1)} := \frac{\partial}{\partial e^\alpha} \lrcorner \mu^{(r)}$, $\forall \alpha$. By integrating both sides of (14) on a fiber f we obtain

$$\int_f \text{Ein}(\Omega)^a{}_C \mu^{(r)} = - \int_f p_{C,\gamma}^{a\gamma} \mu^{(r)} = - \int_f d \left(p_C^{a\gamma} \mu_\gamma^{(r-1)} \right) = 0.$$

But on the one hand, the components of $\text{Ein}(\Omega)^a{}_C$ are constant on the fiber f , as seen in the previous section. Hence

$$\text{Ein}(\Omega)^a{}_C = \frac{\int_f \text{Ein}(\Omega)^a{}_C \mu^{(r)}}{\int_f \mu^{(r)}} = 0. \quad (22)$$

Thus, using (18) and (21) we deduce that ω and A are solutions of the Einstein–Yang–Mills system

$$\begin{cases} \text{Ein}(\gamma)^a{}_d - \frac{1}{2} \left(F_\beta{}^{ac}F^\beta{}_{dc} - \frac{1}{4}F_\alpha{}^{bc}F^\alpha{}_{bc}\delta^a{}_d \right) + \frac{1}{8}c_{\beta\gamma}^\alpha c_{\alpha\delta}^\beta \mathbf{k}^{\gamma\delta}\delta^a{}_d & = 0 \\ F_\delta{}^{ac}{}_{,c} + (\omega^a{}_b)_c F_\delta{}^{bc} + (\omega^c{}_b)_c F_\delta{}^{ab} - c_{\alpha\delta}^\gamma A_c^\alpha F_\gamma{}^{ac} & = 0 \end{cases}$$

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