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► **To cite this version:**

Fabien Gensbittel, Marcin Peski, Jérôme Renault. NON-COMPACTNESS OF INFORMATION STRUCTURES. 2018. hal-01869139v1

HAL Id: hal-01869139

<https://hal.science/hal-01869139v1>

Preprint submitted on 1 Oct 2018 (v1), last revised 16 Sep 2021 (v3)

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NON-COMPACTNESS OF INFORMATION STRUCTURES

FABIEN GENSBITTEL, MARCIN PESKI, AND JÉRÔME RENAULT

ABSTRACT. We say that two type spaces over a fixed space of uncertainty are δ -away if there exists a zero-sum payoff function (uniformly bounded by 1) such that the values of the zero-sum game on the two type spaces are δ -away from each other. We show that the induced topology is not pre-compact: there exists $\delta > 0$ and a set of infinitely many type spaces such that any two of them are δ -away from each other. Thus, it is impossible to approximate the entire universe of type spaces with finite sets. Moreover, this construction shows that there exists type spaces having the same joint distribution of beliefs of arbitrarily high-order that are δ -away from each other

1. INTRODUCTION

In some situations, agents have an opportunity to choose information structure before they acquire information and make decisions in the game, or even before they know what game they are playing. Examples include a stock market trader who develops a software to process and analyze large amounts of financial information before he knows what stock he is going to trade, or, a spy master who allocates resources to different tasks or regions before she understands the nature of future conflicts.

A fundamental problem for such agents is to understand the scope of their choices. Many information structures are similar precisely in that they induce a similar strategic situation in any future game, and they lead to similar outcomes. An agent faced with a costly decision to replace the existing information structure with an alternative will decide against it if the latter leads to very similar outcomes. This leads to a question of classification: is it possible to divide all information structures into a finite number of classes such that all type spaces in the same class lead to similar outcomes? A related question is about the relation between the value of an information structure and the details with which information is described. The question is whether these details matter significantly or there is a limit to gains from improving information.

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Date: October 20, 2017.

In this paper, we show that the answers to the above questions are negative. There are infinitely many significantly distinct information structures. There is no limit to changes in payoffs induced by better information.

In order to give a precise statement of our result, we need to introduce some terminology. We model information structures as Harsanyi common prior type spaces over a fixed space of uncertainty $\Omega = \{0, 1\}$. (Because our results are negative, the restriction to binary fundamental uncertainty is without loss of generality). We focus on situations of strictly conflicting interests, which we model as a zero-sum games. We say that two type spaces over a fixed space of uncertainty are δ -distant if there exists a zero-sum game with a payoff function uniformly bounded by 1 such that the values of the induced games on the two type spaces are δ -away from each other. The main result of the paper is that such distance is not pre-compact: there exists $\delta > 0$ and a set of infinitely many type spaces such that each one of them is δ -away from any other.

In the proof, we show that there exist sequences of games and type spaces, in which the players essentially report their types and obtain payoffs that depend on whether their report is consistent with the report of their opponent. The payoffs are chosen so that the players have incentives to report their information truthfully. The main difficulty is to make sure that the payoffs in these games are uniformly bounded as the type spaces grow. (To compare, our idea is similar to [5]. However, notice that the latter paper requires unbounded payoffs that typically grow with the size of the type space).

All the type spaces in the sequence are all δ -away from each other for some $\delta > 0$. Nevertheless, we choose the type spaces in the sequence so that they are related with each other. For example, they can be nested within each other in a sense that each player receives a strictly better information as we move along the sequence. Moreover, for each $n < m, m'$, the joint distribution of beliefs of order n is the same in the m th type space as in the m' th type space.

The proof is nonconstructive. First, we identify two properties, that, if satisfied by a sequence of type spaces, ensure that the results that we describe above hold. The properties can be interpreted as a way of ensuring that the different levels of information that players receive are not correlated. Next, we describe a stochastic process of choosing the type spaces. We show that, with a large probability, the realization of the process will satisfy the required properties. In particular, a sequence of type spaces with required properties must exist. (The basic idea of the proof relies on the probabilistic method that was popularized by P. Erdős; for a general overview of the method, see [1]. We are not aware of any other application of the probabilistic method in the economic theory.).

Our paper answers in the negative the last open problem posed by J.F. Mertens in [11]¹. Specifically, Problem 2 asks about the equicontinuity of the family of value functions $\text{val}(T, G)$ over type spaces across all (uniformly bounded) zero-sum game. The positive answer would have imply the equicontinuity of the discounted and the average value in repeated games, and it would have consequences for the convergence of the limits theorems. However, our results imply that the answer to the problem is negative.

This paper adds to the literature on the topologies of type spaces. This literature was spurred by an observation in [14] that solution concepts are highly sensitive to higher-order beliefs. [6] introduces *uniform strategic topology* in which two types are close if, for any (not necessarily zero-sum) game, the sets of (almost) rationalizable outcomes are (almost) equal. They show that finite types are not dense in such a topology.² [3] and [4] provide a characterization of the uniform-strategic topology in terms of the uniform weak topology on belief hierarchies. The key difference between our notion of distance on type spaces and the uniform strategic topology is that we work with *ex ante* information structures and solution concepts, whereas the uniform strategic topology is designed to work on the *interim* level. The *ex ante* notion is more appropriate to the information design context, since the information structure is evaluated *before* players receive any information, perhaps because the same information structure is used multiple times in different situations. (In the examples listed at the beginning of the introduction, financial software is used to process information about multiple potential investments, and a spy network is used to collect information about various potential threats.)

Another potential difference is that the uniform strategic topology applies to all (including non-zero sum) games. We believe that it is important to restrict constructions to zero-sum games in which the conflict is a dominant aspect of the strategic situation. At the same time, we note that the non-compactness result easily extends to non-zero-sum games. Because one can draw from a larger space of test games, the non-zero-sum conclusion is weaker than our main result and it has an independent, direct, and simpler proof.

Finally, this paper contributes to a recent but rapidly growing field of information design ([8], [7], [2], to name a few). In that literature, a principal designs an information structure which the agents use to play a game with incomplete information. The objective is to maximize the principal's payoff from the equilibrium outcome of the game. We likewise

¹Problem 1 asked about the convergence of the value, and it was proved false in [15]. Problem 3 asked about the equivalence between the existence of the uniform value and the uniform convergence of the value functions, it was proved to be false by [12] and [9].

²(See also [13]). [6] focus mostly on the weaker notion of *strategic topology* that differs from the uniform strategic in the same way that the pointwise convergence differs from uniform convergence. The finite types are dense under the strategic topology.

analyze an agent who chooses or compares different information structures from the *ex ante* perspective. However, unlike the information design literature wherein the game is typically fixed, we compare type spaces uniformly across all games.

Section 2 describes the model, the main result, and some corollaries. Section 3 describes the properties of the type spaces and games. Section 4 describes the games and computes values of the type spaces that satisfy the properties from the previous section. Section 5 uses the probabilistic method to establish the existence of a sequence of type spaces that satisfies the properties from Section 3.

2. RESULTS

Let $\Omega = \{0, 1\}$ be a space of payoff-relevant uncertainty. A (finite) *type space* T is a tuple (U, V, μ) of finite sets of types U for the maximizer, V for the minimizer, and a probability measure $\mu \in \Delta(U \times V \times \Omega)$. Let \mathcal{T} be the collection of all type spaces.

A (zero-sum) game G is a tuple (X, Y, g) , where X and Y are finite action sets of the maximizer and the minimizer, respectively, and $g : X \times Y \times \Omega \rightarrow [-1, 1]$ is the payoff function of the maximizer. Let \mathcal{G} be the collection of games. For each game G and each type space T , let $\text{val}(T, G)$ denote the value of game G on type space T . We distinguish between two type spaces so far as they lead to different behavior in some game. For this reason, we define an equivalence relation \sim on \mathcal{T} , where $T \sim S$ if and only if $\text{val}(G, T) = \text{val}(G, S)$ for each $G \in \mathcal{G}$.

It is well-known that if T and S induce the same distribution over the space of Mertens-Zamir hierarchies of beliefs, then $T \sim S$ (see Proposition III.4.4 in [10]).

2.1. Non-compactness of the space of type spaces. Define the distance between two type spaces T and S as the supremum over the value differences across all zero-sum games:

$$d(T, S) = \sup_{G \in \mathcal{G}} |v(G, T) - v(G, S)|. \quad (2.1)$$

It is easy to show that $d(\cdot, \cdot)$ is a pseudo-metric on \mathcal{T} and a proper metric on \mathcal{T}/\sim . The induced topology is of *uniform convergence* of type spaces, uniformly across all games $G \in \mathcal{G}$.

Our two main results are corollaries of the next proposition.

Proposition 1. *There exists $\varepsilon > 0$ and a sequence of type spaces $(T^l)_{l \geq 1}$ and games $(G^p)_{p \geq 1}$ such that for all $1 \leq p \leq l$,*

$$\text{val}(T^l, G^p) \geq \varepsilon \text{ and } \text{val}(T^l, G^{l+1}) \leq -\varepsilon. \quad (2.2)$$

Moreover, for all integer n , the sequence of joint distributions of n -order beliefs induced by the sequence of type spaces T^l is eventually constant.

The proof of Proposition 1 is spread throughout all the remaining sections of the paper. The idea behind the construction of the type spaces is described in Section 3, with a proof of a key step postponed until Section 5. Section 4 describes the games and establishes the above bounds.

The first main result of the paper says that this topology is not pre-compact.

Theorem 1. *There exists $\varepsilon > 0$ such that for any finite subset $\mathcal{S} \subseteq \mathcal{T}$, there exists a type space T such that, for each $S \in \mathcal{S}$,*

$$d(T, S) \geq \varepsilon.$$

The space of type spaces \mathcal{T} cannot be covered by finitely many sets such that all type spaces within a set are ε -away from each other.

Proposition 1 follows from Lemma 1, Propositions 2, 3 and 4 below. Here, we use the Proposition to prove the Theorem.

Proof. (Proof of Theorem 1) We show that the Proposition leads to our main result. Suppose by contradiction that there exists a finite subset $\mathcal{S} \subseteq \mathcal{T}$ of type spaces such that for each m there exists $S^m \in \mathcal{S}$ such that

$$d(T^m, S^m) < \varepsilon.$$

Because \mathcal{S} is finite, there must exist $m \neq m'$ such that $S^m = S^{m'} = S$. By the triangle inequality,

$$d(T^m, T^{m'}) \leq d(T^m, S) + d(T^{m'}, S) < 2\varepsilon.$$

W.l.o.g. assume that $m < m'$. Then, (2.2) implies that

$$\text{val}(T^m, G^{m+1}) \leq -\varepsilon \text{ and } \text{val}(T^{m'}, G^{m+1}) \geq \varepsilon,$$

which implies that

$$d(T^m, T^{m'}) \geq 2\varepsilon.$$

Contradiction. □

The second main results is related to the hierarchies of beliefs.

Theorem 2. *There exists $\varepsilon > 0$ such that for any positive integer n , there exist two type spaces T, T' having the same joint distribution of n -order beliefs and such that,*

$$d(T, T') \geq 2\varepsilon.$$

This theorem says that knowing only the first n -beliefs (for arbitrarily large n) is not sufficient to play almost-optimally in a zero-sum game with incomplete information.

Proof. (Proof of Theorem 2) We show that the Proposition leads to our main result. n being fixed, there exists $m < m'$ such that T^m and $T^{m'}$ induce the same distribution of n -order beliefs. As in the proof of Theorem 1, (2.2) implies that

$$d(T^m, T^{m'}) \geq 2\varepsilon.$$

□

2.2. Non-compactness of the space of games. Our main result has the following dual form: for any two games $G, H \in \mathcal{G}$, define a distance

$$d(G, H) = \sup_{T \in \mathcal{T}} |\text{val}(G, T) - \text{val}(H, T)|.$$

Corollary 1. *There exists $\varepsilon > 0$ such that for each finite sequence of games G_1, \dots, G_k , there exists a game G such that*

$$d(G, G_l) \geq \varepsilon \text{ for each } l.$$

The interpretation of the result is analogous to the interpretation of Theorem 1. Suppose that a mechanism designer chooses a game (i.e., a set of actions, and a payoff function that depends on profiles and the (unknown) state of the world) before she knows the information structure used by the players. Many of the games that she can choose are similar in the sense that they lead to similar equilibrium outcomes across *all* type spaces. We may ask: is it possible to classify all games into a finite number of classes such that each class contains only similar games? Can the mechanism designer restrict herself to games with a bounded numbers of actions? Corollary 1 implies that the answer is negative.

2.3. Non-zero sum games and ex ante solution concepts. A non-zero-sum game with two players G is a tuple (X, Y, g_X, g_Y) , where X and Y are finite action sets and $g_X, g_Y : X \times Y \times \Omega \rightarrow [-1, 1]$ are payoff functions. Let \mathcal{G}^* be the collection of games. An *ex ante solution concept* $E : \mathcal{G}^* \times \mathcal{T} \rightrightarrows [-1, 1]^2$ is a correspondence that maps a type space and a game into a set of payoff profiles. For instance, if E is a Bayesian Nash equilibrium, then $(\pi_1, \pi_2) \in E(G, T)$ if and only if there exists a strategy profile $\sigma_X : U \rightarrow \Delta X$ and $\sigma_Y : V \rightarrow \Delta Y$ such that each type best responds and π_i is the ex ante payoff of player i .

Each solution concept E induces a pseudo-metric on the space of type spaces:

$$d_E(T, S) = \sup_{G \in \mathcal{G}^*} d_H(E(G, T), E(G, S)).$$

Here, $d_H(\cdot, \cdot)$ is the Hausdorff distance between the sets.

Corollary 2. *Suppose that E is either a Bayesian Nash equilibrium or Bayes Correlated equilibrium ([2]). Then, there exists $\varepsilon > 0$ such that for any finite subset $\mathcal{S} \subseteq \mathcal{T}$, there exists a type space T such that for each $S \in \mathcal{S}$,*

$$d_E(T, S) \geq \varepsilon.$$

Proof. Notice that if $G \in \mathcal{G} \subseteq \mathcal{G}^*$ is a zero-sum game, and E is either Bayesian Nash or Bayes Correlated equilibrium, then for any type space T , the solution set consists of one point,

$$E(G, T) = \{(v(G, T), -v(G, T))\}.$$

The observation follows from Theorem 1. □

Because this result allows for a larger number of test-games, the conclusion of Corollary 2 is strictly weaker than the thesis of Theorem 1. For the sake of completeness and clarity, we report a more direct and simpler proof in the last section of the paper.

2.4. Uniform vs. pointwise topologies. One can consider a topology on \mathcal{T} (or, more precisely, on the quotient space \mathcal{T}/\sim) generated by sets

$$U(T, G, \varepsilon) = \{S : |\text{val}(G, T) - \text{val}(G, S)| < \varepsilon\}.$$

This is a topology of the *pointwise convergence*: a sequence of (equivalence classes) of type spaces T_n converges to type space T if and only if for each game G , $\text{val}(G, T_n) \rightarrow \text{val}(G, T)$. Such topology is clearly weaker than the topology of the uniform convergence. One can easily show that there exists a countable sequence $G_1, G_2, \dots \in \mathcal{G}$ such that the topology of the pointwise convergence is metrizable with metric

$$d^*(T, S) = \sum_{i=1}^{\infty} \frac{1}{2^i} |\text{val}(G_i, T) - \text{val}(G_i, S)|.$$

Let \mathcal{P} denote the set of coherent probabilities over Mertens-Zamir hierarchies³. When endowed with the weak* topology denoted τ^* , (\mathcal{P}, τ^*) is a compact metrizable space, and the set \mathcal{P}_f of coherent probabilities with finite support is dense in (\mathcal{P}, τ^*) (see Theorem III.2.2 and Theorem III.3.1 in [10]). Note that for $T, S \in \mathcal{T}$, $d^*(T, S)$ depends only on the distributions induced by T and S over the space of Mertens-Zamir hierarchies of beliefs. We can therefore consider d^* as a pseudo-distance on the space \mathcal{P} by letting for $p, q \in \mathcal{P}$, $d^*(p, q) = d^*(T_p, T_q)$, where T_p, T_q are type spaces inducing respectively the distributions p and q . The map d^* is continuous on (\mathcal{P}, τ^*) (see Proposition III.4.3 in [10]). Therefore, for all $\varepsilon > 0$, the family of balls $(B_{(\mathcal{P}, d^*)}(p, \varepsilon))_{p \in \mathcal{P}}$ where $B_{(\mathcal{P}, d^*)}(p, \varepsilon) = \{q \in \mathcal{P} \mid d^*(p, q) < \varepsilon\}$ is an open cover of

³Coherent probabilities (resp. coherent probabilities with finite support) over the Mertens-Zamir hierarchies are exactly the distributions over hierarchies of beliefs induced by a type space (resp. finite type space) with common prior.

(\mathcal{P}, τ^*) , from which we can extract a finite subcover $(B_{(\mathcal{P}, d^*)}(p_i, \varepsilon))_{i=1, \dots, n}$. In each of the balls $B_{(\mathcal{P}, d^*)}(p_i, \varepsilon)$, we can choose a probability $q_i \in \mathcal{P}_f$ so that $(B_{(\mathcal{P}, d^*)}(q_i, 2\varepsilon))_{i=1, \dots, n}$ is a finite cover of \mathcal{P} . We deduce that the family $(B_{(\mathcal{T}, d^*)}(T_{q_i}, 2\varepsilon))_{i=1, \dots, n}$ where $B_{(\mathcal{T}, d^*)}(T_{q_i}, 2\varepsilon) = \{S \in \mathcal{T} \mid d^*(S, T_{q_i}) < 2\varepsilon\}$ is a finite cover of \mathcal{T} . Therefore (\mathcal{T}, d^*) is a totally bounded pseudo-metric space and $(\mathcal{T}/\sim, d^*)$ is a totally bounded metric space.

2.5. A negative answer to a problem posed by Mertens. As announced in the introduction, our first main result (Theorem 1) has for consequence that the answer to the last open problem posed by Mertens in [11] is negative. Precisely, this problem asks about the equicontinuity for the weak* topology of the family of value functions $p \in \mathcal{P} \rightarrow \text{val}(T_p, G)$ across all zero-sum games $G \in \mathcal{G}$. Let us show that a positive answer would contradict Theorem 1. Recall that for $T, S \in \mathcal{T}$, $d(T, S) = \sup_{G \in \mathcal{G}} |\text{val}(T, G) - \text{val}(S, G)|$ depends only on the induced distributions over the space of Mertens-Zamir hierarchies of beliefs. As we did for d^* , we can consider d as a pseudo-distance on the space \mathcal{P} by letting for $p, q \in \mathcal{P}$, $d^*(p, q) = d^*(T_p, T_q)$. If the family of value functions $p \in \mathcal{P} \rightarrow \text{val}(T_p, G)$ is equicontinuous for the weak* topology across all zero-sum games $G \in \mathcal{G}$, then it implies that d is continuous on (\mathcal{P}, τ^*) . Using the same argument as in the previous subsection, this implies that $(\mathcal{T}/\sim, d)$ is a totally bounded metric space, which contradicts Theorem 1.

3. CONSTRUCTIONS

In this section, we describe the construction of type spaces and games from Proposition 1.

3.1. Markov chain. Let N be a large even positive integer to be fixed later. We write $A = U = V = \{1, \dots, N\}$, with the idea of using U while speaking of actions or signals of the maximizer, and using V while speaking of actions and signals of the minimizer. We write A^l, U^l, V^l to denote Cartesian products of the l copies of the respective sets. (In all other uses, the superscripts denote simply the order of elements in a sequence.) We interpret U^l and V^l as the space of types or actions of each of the two players, and A^{2l} and A^{2l+1} as the spaces of action or type profiles.

We are going to construct a homogeneous Markov chain $(a_n)_{n \geq 1}$ taking values in A . The initial distribution of a_1 is uniform on A , and for each n , the conditional law of a_{n+1} given a_n is $P_{a_n, \cdot}$, where P is the transition matrix.

We are going to choose the transition matrix so that the induced Markov chain satisfies specific properties called conditions UI1 and UI2, detailed in subsection 3.5 below. The proof that a matrix P satisfying these conditions exist is nonconstructive and it relies on the probabilistic method (see [1]). We choose the matrix P in a stochastic way. Precisely, for all even integer N , let \mathcal{S}_A denote the collection of all subsets $S \subseteq A$ with cardinality $|S| = \frac{1}{2}N$.

Let $(S_a)_{a \in A}$ be a collection of N independent draws from the uniform distribution over \mathcal{S}_A . Let $X_{a,b} = \mathbf{1}_{\{b \in S_a\}}$ and define

$$P_{a,b} = \frac{2}{N} X_{a,b}.$$

Then, $P_{a,b} \in \{0, \frac{2}{N}\}$, and $\sum_b P_{a,b} = 1$ so that P is a stochastic matrix.

We use \mathbb{P}_N as the law that governs the choice of matrix P .

Proposition 2. *There exists $N^* < \infty$ such that for each $N > N^*$,*

$$\mathbb{P}_N (P \text{ induces the Markov chain that satisfies conditions UI1 and UI2}) > 0.$$

It follows that there exists a transition matrix P that induces the Markov chain that satisfies conditions UI1 and UI2.

The proof of the proposition can be found in Section 5.

From now on, assume that P is chosen from \mathbb{P}_N and fixed. Given P is chosen, we use ν^∞ to denote the law of the Markov chain $(a_n)_{n \geq 1}$. The choice of matrix P ensures that the conditional law of a_{n+1} given a_n is uniform a subset of A of cardinal $\frac{N}{2}$.

Let $\nu^n \in \Delta(A^n)$ denote the joint distribution of the first n elements (a_1, \dots, a_n) and $C^n \subset A^n$ denote its support. By construction, we have $C^1 = A$ and

$$(a_1, \dots, a_n) \in C^n \Leftrightarrow \forall i = 1, \dots, n-1, P_{a_i, a_{i+1}} > 0. \quad (3.1)$$

3.2. Consistency. Below, we are going to use the following notion of consistency. For all $l \geq 1$, $m \geq 0$, $u = (u_1, \dots, u_l)$ in U^l and $v = (v_1, \dots, v_m)$ in V^m , we write

$$\begin{aligned} a^{2q}(u, v) &= (u_1, v_1, \dots, u_q, v_q) \in A^{2q} \text{ for each } q \leq \min\{l, m\}, \\ a^{2q+1}(u, v) &= (u_1, v_1, \dots, u_q, v_q, u_{q+1}) \in A^{2q+1} \text{ for each } q \leq \min\{l-1, m\}. \end{aligned}$$

For $r \leq \min\{2l, 2m+1\}$, we say that u and v are *consistent at level r* , we write $u \smile_r v$, if $a^r(u, v) \in C^r$. Let

$$\begin{aligned} f(u, v) &= \min\{r \leq \min\{2l, 2m+1\}, \text{ not } u \smile_r v\}, \text{ and} \\ f(u, v) &= +\infty, \text{ if the set is empty, (i.e., if } u \smile_{\min\{2l, 2m+1\}} v). \end{aligned}$$

Function f identifies the first level at which u and v are not consistent.

Remark 1. *For any u and v , $u \smile_1 v$ since $C^1 = A$. By (3.1), $u \smile_r v$ implies $u \smile_{r-1} v$. And for u and v in A^l :*

$$u \smile_{2l} v \iff a^{2l}(u, v) \in C^{2l} \iff f(u, v) = +\infty.$$

3.3. Type Spaces. We construct the type spaces T^l in the following way. First, the Nature chooses the first $2l$ elements of the Markov chain $(u_1, v_1, u_2, v_2, \dots, v_l)$. Next, selects, conditionally on u_1 but independently from the rest, the state $\omega = 1$ with probability $\frac{u_1}{N+1}$ and the state $\omega = 0$ with the remaining probability $1 - \frac{u_1}{N+1}$. Player 1 observes the odd elements of the sequence $u = (u_1, \dots, u_l)$, and player 2 observes the even elements $v = (v_1, \dots, v_l)$ of the sequence. Formally,

Definition 1. We define the type space $T^l = (U^l, V^l, \mu^l)$ for $l \geq 1$, where the set of signals for player 1 is U^l , the set of signals of player 2 is V^l , and μ^l is the probability on $\Omega \times U^l \times V^l$ such that for all $u = (u_1, \dots, u_l) \in U^l$, $v = (v_1, \dots, v_l) \in V^l$, $\omega \in \Omega$, we have

$$\mu^l(u, v, \omega) = \nu^{2l}(a^{2l}(u, v)) \cdot \frac{\mathbf{1}_{\omega=1}u_1 + \mathbf{1}_{\omega=0}(1 - u_1)}{N + 1}.$$

It follows that the marginal of μ^{l+1} over $\Omega \times U^l \times V^l$ is equal to μ^l for each l . Additionally, by assumption, the distribution of types u^1 is uniform.

By construction, along the sequence of the type spaces, each player's type assigns probability 1 to the opponent's types being consistent at level $2l$ with her own type.

There is an equivalent description of the sequence of type spaces T^l . Notice that $T^l \simeq ((U^\infty, \mathcal{U}^l), (V^\infty, \mathcal{V}^l), \mu^\infty)$, where U^∞ and V^∞ are the spaces of types for, respectively, the maximizer and the minimizer, μ^∞ is a common prior, and \mathcal{U}^l and \mathcal{V}^l are filtrations on the spaces of types such that for each l , the σ -algebra \mathcal{U}^l is generated by sets $U^l(u) = \{u' : u'_r = u_r \text{ for } r \leq l\}$ and \mathcal{V}^l is generated by analogously defined sets $V^l(v)$. Each σ -algebra is finite, and it is traditionally referred to as an information partition. Thus, the type spaces T^l can be nested within each other in a sense that each player receives a strictly better information as we move along the sequence.

Additionally, the Markov property implies that the higher order beliefs become constant as we move along the sequence. Recall that n -order beliefs are defined inductively as conditional laws. Precisely, the first order beliefs θ_1^i of player i is the conditional law of ω given her signal. The n -order belief θ_n^i of player i is the conditional law of $(\omega, \theta_{n-1}^{-i})$ given her signal. In this construction, conditional laws are seen as random variables taking values in space of probability measures.

Lemma 1. For all $l \geq m$, the joint distribution of $(\omega, \theta_{2m-2}^1, \theta_{2m-2}^2)$ induced by the type space T^l is independent of l , where θ_n^i denote the n -order beliefs of player $i = 1, 2$.

Proof. Let $l \geq m$ and consider the type space T^l . We use the notation $\mathcal{L}(X|Y)$ for the conditional law of X given Y , and the identification $(a_1, \dots, a_{2l}) = (u_1, v_1, \dots, u_l, v_l)$.

At first, note that by construction ω and (a_2, \dots, a_{2l}) are conditionally independent given a_1 , so that the sequence $(\omega, a_1, a_2, \dots, a_{2l})$ is a Markov process. It follows that

$$\theta_1^1 = \mathcal{L}(\omega|u_1, \dots, u_l) = \mathcal{L}(\omega|u_1).$$

The Markov property implies that

$$\theta_1^2 = \mathcal{L}(\omega|v_1, \dots, v_l) = \mathcal{L}(\omega|v_1), \quad \theta_2^2 = \mathcal{L}(\omega, \theta_1^1(u_1)|v_1, \dots, v_l) = \mathcal{L}(\omega, \theta_1^1(u_1)|v_1),$$

and therefore we have

$$\theta_2^1 = \mathcal{L}(\omega, \theta_1^2(v_1)|u_1, \dots, u_l) = \mathcal{L}(\omega, \theta_1^2(v_1)|u_1, u_2).$$

By induction, and applying the same argument (future and past of a Markov process are conditionally independent given the current position), we deduce that for all $n \geq 1$,

$$\theta_{2n}^1 = \mathcal{L}(\omega, \theta_{2n-1}^2|u_1, \dots, u_{\min(l, n+1)})$$

$$\theta_{2n+1}^1 = \mathcal{L}(\omega, \theta_{2n}^2|u_1, \dots, u_{\min(l, n+1)})$$

$$\theta_{2n-1}^2 = \mathcal{L}(\omega, \theta_{2n-2}^1|v_1, \dots, v_{\min(l, n)})$$

$$\theta_{2n}^2 = \mathcal{L}(\omega, \theta_{2n-1}^1|v_1, \dots, v_{\min(l, n)})$$

As a consequence, for all $n \leq m - 1$, these conditional laws do not depend on which type space T^l we are using as soon as $l \geq m$ and the result follows. \square

3.4. Games. Let

$$g_0(\omega, a) = \frac{1}{2} - \left(\omega - \frac{a}{N+1}\right)^2.$$

For each $q \in [0, 1]$, define

$$g_0(q, a) = qg_0(1, a) + (1 - q)g_0(0, a)$$

be the expected payoff from action a if q is the probability of state 1. Given fixed N , there exists $\varepsilon \in (0, \frac{1}{10})$ such that for all $u_1 \in U$, for all $a \neq u_1$ in U ,

$$g_0\left(\frac{u_1}{N+1}, a\right) + 10\varepsilon < g_0\left(\frac{u_1}{N+1}, u_1\right). \quad (3.2)$$

Indeed, any $\varepsilon \in (0, \frac{1}{10(N+1)^2})$ can do. We fix such ε in the sequel. Let

$$h^p(u, v) = \begin{cases} \varepsilon & \text{if } f(u, v) = +\infty \\ 5\varepsilon & \text{if } f(u, v) \text{ is an even integer} \\ -5\varepsilon & \text{if } f(u, v) \text{ is an odd integer.} \end{cases}$$

Definition 2. For $p \geq 1$, define the game $G^p = (U^p, V^{p-1}, g^p)$ with payoffs

$$g^p(u, v, \omega) = g_0(\omega, u_1) - w + h^p(u, v)$$

where

$$w = \frac{1}{N} \sum_{u_1=1}^N g_0 \left(\frac{u_1}{N+1}, u_1 \right).$$

Notice that $|g^p| \leq 1$ for each p .

The first component of payoffs, $g_0(\omega, u_1) - w$, is designed to incentivize the maximizer to choose action that matches her belief about the state of the world. Indeed, suppose that player 1 assigns probability $\frac{a}{N+1}$ to state $\omega = 1$. In order to maximize the expected payoff from term g_0 , the unique optimal choice of player 1 is to choose $u_1 = a$. The choice of ε implies that the first-level information will be reported truthfully no matter what are the payoff consequences of the report from the second term, h^p . The normalization constant w is chosen so that the average payoffs from the first component are equal to 0. We use this observation in the sequel.

The second term, h^p , is designed to punish a player for misreporting higher-level information. If $f(u, v) < +\infty$, then we say that a mistake occurs at level $f(u, v)$, i.e., the first level at which the reports are inconsistent. If $f(u, v)$ is odd, then the maximizer is considered to blame and she is punished with payoff -5ε . If $f(u, v)$ is even, then we say that the minimizer is to blame for the mistake. In such a case, the maximizer gets payoff 5ε . Finally, if none of the players make a mistake, i.e., $f(u, v) = +\infty$, the maximizer gets ε . Of course, the optimal behavior of the players shall depend on their beliefs about the actions of their opponent.

3.5. Conditions UI. We need additional assumptions on the Markov chain to ensure that players have incentives to report their true types. The idea is that the players can be punished for misreports only if the misreports lead to a mistake (i.e., an inconsistent profile of types) with a sufficiently large probability. Consider the type space T^l and the game G^p for some $p \leq l$. Suppose that the minimizer type $\hat{v} \in V^l$ reports type $v' \in V^{p-1}$ with $v'_m \neq \hat{v}_m$ for some $m \leq p-1$. For $r = 2m-1, 2m$, we want that for some parameter $\alpha > 0$ to be fixed later:

$$|\nu^{2l}(u \smile_{r+1} v' | v = \hat{v}, u \smile_r v') - \frac{1}{2}| \leq \alpha. \quad (3.3)$$

In the above formulation, we consider the probability space $U^l \times V^l$ equipped with the probability ν^{2l} , we identify (u, v) with $a^{2l}(u, v) = (u_1, v_1, \dots, u_l, v_l)$, and (u, v) is a random variable while \hat{v}, v' are deterministic. The expression $\nu^{2l}(u \smile_{r+1} v' | v = \hat{v}, u \smile_r v')$ is therefore the conditional probability that the type u of player 1 is $(r+1)$ -consistent with v' given that the type v of player 2 is equal to \hat{v} and that u is r -consistent with v' , when the types (u, v) of the players are chosen according to the Markov chain of law ν^{2l} .

Note that by construction, if $v'_m = \hat{v}_m$ the above conditional probability is equal to 1. Indeed, in case $r = 2m - 1$ using (3.1),

$$\{u \smile_{2m} v'\} = \{u \smile_{2m-1} v'\} \cap \{P_{u_m, v'_m} > 0\} = \{u \smile_{2m-1} v'\} \cap \{P_{u_m, \hat{v}_m} > 0\},$$

so that

$$\nu^{2l}(u \smile_{2m} v' | v = \hat{v}, u \smile_{2m-1} v') = \nu^{2l}(\{P_{u_m, v'_m} > 0\} | v = \hat{v}, u \smile_{2m-1} v') = 1,$$

since the event $\{P_{u_m, v'_m} > 0\}$ has probability 1 (recall that by definition the true types are consistent with probability 1). A similar argument holds for $r = 2m$, and this fact will be used below in Propositions 3 and 4.

The condition (3.3) implies that, when the minimizer reports v'_m instead of \hat{v}_m , if $u \smile_{2m-1} v'$ (no mistake was detected for the reports $u_1, v'_1, \dots, v'_{m-1}, u_m$), the probability that v'_m leads to a mistake is close to 1/2. Similarly, if $u \smile_{2m} v'$, the probability that u_{m+1} leads to a mistake is close to 1/2.

We also require analogous conditions for the maximizer that are explicitly defined below using similar notation.

Definition 3. *We say that the conditions UI1 are satisfied if: for all $l \geq 1$, for all $\hat{u} = (\hat{u}_1, \dots, \hat{u}_l)$ in U^l and $u' = (u'_1, \dots, u'_p)$ in U^p with $p = l + 1$ such that $\hat{u}_1 = u'_1$, for all $m \in \{1, \dots, l\}$ such that $\hat{u}_m \neq u'_m$, for $r = 2m - 2, 2m - 1$*

$$\left| \nu^{2l}(u' \smile_{r+1} v | u = \hat{u}, u' \smile_r v) - \frac{1}{2} \right| \leq \alpha.$$

We also require that

$$\left| \nu^{2l}(u' \smile_{2l+1} v | u = \hat{u}, u' \smile_{2l} v) - \frac{1}{2} \right| \leq \alpha.$$

We say that the conditions UI2 are satisfied if: for all $1 \leq p \leq l$, for all $\hat{v} \in V^l$, for all $v' \in V^{p-1}$, for all $m \in \{1, \dots, p-1\}$ such that $\hat{v}_m \neq v'_m$, for $r = 2m - 1, 2m$

$$\left| \nu^{2l}(u \smile_{r+1} v' | v = \hat{v}, u \smile_r v') - \frac{1}{2} \right| \leq \alpha.$$

4. VALUES

The type space T^l and the game G^p together form the incomplete information game $\Gamma(T^l, G^p)$ where:

- First, (u, v, ω) in $U^l \times V^l \times \Omega$ is selected according to μ^l . u is told to player 1 and v is told to player 2.
- Then, simultaneously player 1 chooses a report u' in U^p , player 2 chooses a report v' in V^{p-1} , and the payoff of player 1 is $g^p(u', v', \omega)$.

For $p \leq l$, we say that a strategy of a player is *truthful* if each type of the player correctly reports the first p coordinates of her type. If $p = l + 1$, the strategy is truthful if the first l coordinates of the maximizer's report match her type. We are going to show, that under suitable conditions on the priors μ^l the values (2.2) can be computed assuming that players play truthful strategies.

The above discussion of the payoffs implies that the maximizer always reports the first level information truthfully. The particular form of μ^l (i.e., the uniform distribution of the first order information) and the choice of the normalizing constant imply that the expected payoff from the first component is equal to 0. As a special case, the value of $\Gamma(T^l, G^1)$ the expected payoff component, which given truthful reporting is equal to ε .

More generally, suppose that both players choose truthful strategy in the incomplete information game $\Gamma(T^l, G^p)$ for $p \leq l$. In such a case, none of the players ever makes a mistake and the expected payoff from the second component is equal to ε . We are going to show that, under conditions UI2, if the maximizer always chooses truthful strategy, the minimizer cannot do better by misreporting.

4.1. Value $\text{val}(T^l, G^p)$ for $p \leq l$.

Proposition 3. *Suppose the conditions UI2 are satisfied with $\alpha > 0$ such that $\alpha + \alpha^2 \leq \frac{1}{22}$. Then :*

$$\forall 1 \leq p \leq l, \text{val}(T^l, G^p) \geq \varepsilon.$$

Proof. We assume that maximizer chooses the truthful strategy. Consider type $\hat{v} = (\hat{v}_1, \dots, \hat{v}_l)$ of the minimizer who chooses to report (v'_1, \dots, v'_{p-1}) .

We work on the probability space $\Omega \times U^l \times V^l$ equipped with the probability μ^l , and (ω, u, v) is a random variable of law μ^l , while \hat{v} and v' are deterministic.

Define the non-increasing sequence of events $A_n = \{u \prec_n v'\}$. We claim that for all $n = 1, \dots, p$

$$\mathbb{E}_{\mu^l}[h^p(u, v') | v = \hat{v}, A_{2n-1}] \geq \varepsilon.$$

If $n = p$, $h^p(u, v') = \varepsilon$ on the event A_{2p-1} , implying the result.

Assume now that for some n such that $1 < n < p$, we have

$$\mathbb{E}_{\mu^l}[h^p(u, v') | v = \hat{v}, A_{2n+1}] \geq \varepsilon.$$

Since we have a non-increasing sequence of events,

$$\mathbb{1}_{A_{2n-1}} = \mathbb{1}_{A_{2n+1}} + \mathbb{1}_{A_{2n-1}} \mathbb{1}_{A_{2n}^c} + \mathbb{1}_{A_{2n}} \mathbb{1}_{A_{2n+1}^c},$$

and by definition:

$$h^p(u, v') \mathbb{1}_{A_{2n-1}} = h^p(u, v') \mathbb{1}_{A_{2n+1}} + 5\varepsilon \mathbb{1}_{A_{2n-1}} \mathbb{1}_{A_{2n}^c} - 5\varepsilon \mathbb{1}_{A_{2n}} \mathbb{1}_{A_{2n+1}^c}.$$

If $v'_n = \hat{v}_n$, then $\mu^l(A_{2n+1}|A_{2n-1}, v = \hat{v}) = 1$ (see the discussion in subsection 3.5), implying that

$$\mu^l(A_{2n}^c|A_{2n-1}, v = \hat{v}) = \mu^l(A_{2n} \cap A_{2n+1}^c|A_{2n-1}, v = \hat{v}) = 0$$

and

$$\begin{aligned} \mathbb{E}_{\mu^l}[h^p(u, v')|v = \hat{v}, A_{2n-1}] &= \mathbb{E}_{\mu^l}[\mathbb{E}_{\mu^l}[h^p(u, v')|v = \hat{v}, A_{2n+1}]\mathbb{1}_{A_{2n+1}}|v, A_{2n-1}] \\ &\quad + 5\varepsilon\mu^l(A_{2n}^c|A_{2n-1}, v = \hat{v}) - 5\varepsilon\mu^l(A_{2n} \cap A_{2n+1}^c|A_{2n-1}, v = \hat{v}) \\ &\geq \varepsilon. \end{aligned}$$

If $v'_n \neq \hat{v}_n$, assumption UI2 implies that

$$\begin{aligned} |\mu^l(A_{2n+1}|A_{2n-1}, v = \hat{v}) - \frac{1}{4}| &\leq \alpha^2 + \alpha. \\ |\mu^l(A_{2n}^c|A_{2n-1}, v = \hat{v}) - \frac{1}{2}| &\leq \alpha. \\ |\mu^l(A_{2n} \cap A_{2n+1}^c|A_{2n-1}, v = \hat{v}) - \frac{1}{4}| &\leq \alpha^2 + \alpha. \end{aligned}$$

It follows that

$$\begin{aligned} \mathbb{E}_{\mu^l}[h^p(u, v')|v = \hat{v}, A_{2n-1}] &= \mathbb{E}_{\mu^l}[\mathbb{E}_{\mu^l}[h^p(u, v')|v = \hat{v}, A_{2n+1}]\mathbb{1}_{A_{2n+1}}|v = \hat{v}, A_{2n-1}] \\ &\quad + 5\varepsilon\mu^l(A_{2n}^c|A_{2n-1}, v = \hat{v}) - 5\varepsilon\mu^l(A_{2n} \cap A_{2n+1}^c|A_{2n-1}, v = \hat{v}) \\ &\geq \varepsilon\left(\frac{1}{4} - \alpha^2 - \alpha\right) + 5\varepsilon\left(\frac{1}{2} - \alpha\right) - 5\varepsilon\left(\frac{1}{4} + \alpha^2 + \alpha\right) \\ &= \varepsilon\left(\frac{3}{2} - 11(\alpha^2 + \alpha)\right) \geq \varepsilon, \end{aligned}$$

where the last inequality uses the fact that $\alpha^2 + \alpha \leq \frac{1}{22}$. The claim follows by backward induction.

Since A_1 is an event which holds almost surely, we deduce that

$$\mathbb{E}_{\mu^l}[h^p(u, v')|v = \hat{v}] = \mathbb{E}_{\mu^l}[h^p(u, v')|v = \hat{v}, A_1] \geq \varepsilon,$$

which concludes the proof since it shows that the truthful strategy of the maximizer guarantees the payoff ε . \square

4.2. Value $\text{val}(T^l, G^{l+1})$. Next, we show that, under conditions UI1, the value of game G^{l+1} on type space T^l is not higher than $-\varepsilon$.

Proposition 4. *Suppose the conditions UI1 are satisfied with $\alpha > 0$ such that $\alpha + \alpha^2 \leq \frac{1}{22}$. Then,*

$$\forall l \geq 1, \text{val}(T^l, G^{l+1}) \leq -\varepsilon.$$

Proof. We assume that the minimizer chooses the truthful strategy and that the maximizer truthfully reports the first level belief. Consider a maximizer type $\hat{u} = (\hat{u}_1, \dots, \hat{u}_l)$ who chooses to report $u' = (u'_1, \dots, u'_l, u'_{l+1})$ such that $u_1 = u'_1$. We show that the expected value of the second payoff component is not larger $-\varepsilon$. We work on the probability space $\Omega \times U^l \times V^l$ equipped with the probability μ^l , and (ω, u, v) is a random variable of law μ^l , while \hat{u} and u' are deterministic. Consider the non-increasing sequence of events $B_n = \{u' \smile_n v\}$. We claim that for all $n = 1, \dots, l$

$$\mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_{2n}] \leq -\varepsilon.$$

If $n = l$, we have $\mathbb{1}_{B_{2n}} = \mathbb{1}_{B_{2l+1}} + \mathbb{1}_{B_{2n}} \mathbb{1}_{B_{2l+1}^c}$, and

$$h^{l+1}(u', v) \mathbb{1}_{B_{2n}} = \varepsilon \mathbb{1}_{B_{2l+1}} - 5\varepsilon \mathbb{1}_{B_{2n}} \mathbb{1}_{B_{2l+1}^c}.$$

UI1 implies that

$$|\mu^l(B_{2l+1}|u = \hat{u}, B_{2l}) - \frac{1}{2}| \leq \alpha.$$

$$|\mu^l(B_{2l+1}^c|u = \hat{u}, B_{2l}) - \frac{1}{2}| \leq \alpha.$$

It follows that

$$\begin{aligned} \mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_{2l}] &= \varepsilon \mu^l(B_{2l+1}|u = \hat{u}, B_{2l}) - 5\varepsilon \mu^l(B_{2l+1}^c|u = \hat{u}, B_{2l}) \\ &\leq \varepsilon\left(\frac{1}{2} + \alpha\right) - 5\varepsilon\left(\frac{1}{2} - \alpha\right) \\ &= \varepsilon\left(-\frac{4}{2} + 6\alpha\right) \leq -\varepsilon, \end{aligned}$$

where we used that $\alpha + \alpha^2 \leq \frac{1}{22}$ implies that $\alpha \leq \frac{1}{6}$.

Assume now that for n with $0 < n \leq l - 1$, we have

$$\mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_{2n+2}] \leq -\varepsilon.$$

Using that we have a non-increasing sequence of events,

$$\mathbb{1}_{B_{2n}} = \mathbb{1}_{B_{2n+2}} + \mathbb{1}_{B_{2n}} \mathbb{1}_{B_{2n+1}^c} + \mathbb{1}_{B_{2n+1}} \mathbb{1}_{B_{2n+2}^c}.$$

and by definition of h^{l+1} we have

$$h^{l+1}(u', v) \mathbb{1}_{B_{2n}} = h^{l+1}(u', v) \mathbb{1}_{B_{2n+2}} - 5\varepsilon \mathbb{1}_{B_{2n}} \mathbb{1}_{B_{2n+1}^c} + 5\varepsilon \mathbb{1}_{B_{2n+1}} \mathbb{1}_{B_{2n+2}^c}.$$

If $u'_{n+1} = \hat{u}_{n+1}$, then $\mu^l(B_{2n+2}|B_{2n}, u = \hat{u}) = 1$, implying that

$$\mu^l(B_{2n+1}^c|B_{2n}, u = \hat{u}) = \mu^l(B_{2n+1} \cap B_{2n+2}^c|B_{2n}, u = \hat{u}) = 0$$

and

$$\begin{aligned}\mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_{2n}] &= \mathbb{E}_{\mu^l}[\mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_{2n+2}]\mathbb{1}_{B_{2n+2}}|u = \hat{u}, B_{2n}] \\ &\quad - 5\varepsilon\mu^l(B_{2n+1}^c|B_{2n}, u = \hat{u}) + 5\varepsilon\mu^l(B_{2n+1} \cap B_{2n+2}^c|B_{2n}, u = \hat{u}) \\ &\leq -\varepsilon.\end{aligned}$$

If $u'_{n+1} \neq \hat{u}_{n+1}$, assumption UI1 implies that

$$\begin{aligned}|\mu^l(B_{2n+2}|B_{2n}, u = \hat{u}) - \frac{1}{4}| &\leq \alpha^2 + \alpha. \\ |\mu^l(B_{2n+1}^c|B_{2n}, u = \hat{u}) - \frac{1}{2}| &\leq \alpha. \\ |\mu^l(B_{2n+1} \cap B_{2n+2}^c|B_{2n}, u = \hat{u}) - \frac{1}{4}| &\leq \alpha^2 + \alpha.\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_{2n}] &= \mathbb{E}_{\mu^l}[\mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_{2n+2}]\mathbb{1}_{B_{2n+2}}|u = \hat{u}, B_{2n}] \\ &\quad - 5\varepsilon\mu^l(B_{2n+1}^c|B_{2n}, u = \hat{u}) + 5\varepsilon\mu^l(B_{2n+1} \cap B_{2n+2}^c|B_{2n}, u = \hat{u}) \\ &\leq -\varepsilon\left(\frac{1}{4} - \alpha^2 - \alpha\right) - 5\varepsilon\left(\frac{1}{2} - \alpha\right) + 5\varepsilon\left(\frac{1}{4} + \alpha^2 + \alpha\right) \\ &= -\varepsilon\left(\frac{3}{2} - 11(\alpha^2 + \alpha)\right) \leq -\varepsilon,\end{aligned}$$

where the last inequality uses the fact that $\alpha^2 + \alpha \leq \frac{1}{22}$.

Since B_2 is an event which holds almost surely, we deduce that

$$\mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}] = \mathbb{E}_{\mu^l}[h^{l+1}(u', v)|u = \hat{u}, B_2] \leq -\varepsilon,$$

which concludes the proof since it shows that the truthful strategy of the minimizer guarantees that the payoff of the maximizer is less or equal to $-\varepsilon$. \square

5. EXISTENCE

The goal of this section is to present the proof of Proposition 2.

5.1. Transition matrix. Recall that we generate the matrix P in the following way. \mathcal{S}_A is the collection of all subsets $S \subseteq A$ with cardinality $|S| = \frac{1}{2}N$ and $(S_a)_{a \in A}$ is a collection of N independent draws from the uniform distribution over \mathcal{S}_A defined on a probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$. Let $X_{a,b} = \mathbb{1}_{\{b \in S_a\}}$ and

$$P_{a,b} = \frac{2}{N}X_{a,b}.$$

We use the following probability bounds.

Lemma 2. For any $a \neq b$, each $\gamma > 0$

$$\mathbb{P}_N \left(\left| |S_a \cap S_b| - \frac{1}{4}N \right| \geq \gamma N \right) \leq \frac{1}{2} e^4 N e^{-2\gamma^2 N}.$$

Proof. Consider a family of i.i.d. Bernoulli variables $(\tilde{X}_{i,j})_{i=a,b,j \in A}$ of parameter $\frac{1}{2}$ defined on a space $(\Omega, \mathcal{F}, \mathbb{P})$. For $i = a, b$, define the events $\tilde{L}_i = \{\sum_{j \in A} \tilde{X}_{i,j} = \frac{N}{2}\}$ and the set-valued variables $\tilde{S}_i = \{j \in A \mid \tilde{X}_{i,j} = 1\}$. It is straightforward to check that the conditional law of $(\tilde{S}_a, \tilde{S}_b)$ given $\tilde{L}_a \cap \tilde{L}_b$ under \mathbb{P} is the same as the law of (S_a, S_b) under \mathbb{P}_N . It follows that

$$\begin{aligned} \mathbb{P}_N \left(\left| |S_a \cap S_b| - \frac{1}{4}N \right| \geq \gamma N \right) &= \mathbb{P} \left(\left| |\tilde{S}_a \cap \tilde{S}_b| - \frac{1}{4}N \right| \geq \gamma N \mid \tilde{L}_a \cap \tilde{L}_b \right) \\ &\leq \frac{\mathbb{P} \left(\left| |\tilde{S}_a \cap \tilde{S}_b| - \frac{1}{4}N \right| \geq \gamma N \right)}{\mathbb{P}(\tilde{L}_a \cap \tilde{L}_b)}. \end{aligned}$$

Using Hoeffding inequality, we have

$$\begin{aligned} \mathbb{P} \left(\left| |\tilde{S}_a \cap \tilde{S}_b| - \frac{1}{4}N \right| \geq \gamma N \right) &= \mathbb{P} \left(\left| \sum_{j \in A} \tilde{X}_{a,j} \tilde{X}_{b,j} - \frac{1}{4}N \right| \geq \gamma N \right) \\ &\leq 2e^{-2\gamma^2 N} \end{aligned}$$

On the other hand, using Stirling approximation⁴, we have

$$\mathbb{P}(\tilde{L}_a \cap \tilde{L}_b) = \left(\frac{1}{2^N} \frac{N!}{(\frac{N}{2}!)^2} \right)^2 \geq \left(\frac{2^{N+1} N^{-\frac{1}{2}}}{2^N e^2} \right)^2 = \frac{4}{Ne^4}.$$

We deduce that

$$\mathbb{P}_N \left(\left| |S_a \cap S_b| - \frac{1}{4}N \right| \geq \gamma N \right) \leq \frac{1}{2} e^4 N e^{-2\gamma^2 N}.$$

□

Lemma 3. For each $a \neq b$, for any subset $S \subseteq A$ such that $|S| \geq \frac{1}{8}N$, any $\gamma \geq \frac{1}{2N-2}$,

$$\begin{aligned} \mathbb{P}_N \left(\left| \sum_{i \in S} X_{i,a} - \frac{1}{2}|S| \right| \geq \gamma N \right) &\leq e^{-\frac{1}{16}N\gamma^2}, \\ \mathbb{P}_N \left(\left| \sum_{i \in S} X_{i,a} X_{i,b} - \frac{1}{4}|S| \right| \geq \gamma N \right) &\leq e^{-\frac{1}{16}N\gamma^2}. \end{aligned}$$

⁴ $n^{n+\frac{1}{2}}e^{-n} \leq n! \leq en^{n+\frac{1}{2}}e^{-n}$

Proof. For the first inequality, notice that $X_{i,a}$ are i.i.d. variables equal to 1 with probability $\frac{1}{2}$ and equal to 0 with the remaining probability. The Hoeffding inequality implies that

$$\mathbb{P}_N \left(\left| \sum_{i \in S} X_{i,a} - \frac{1}{2} |S| \right| \geq \gamma N \right) \leq e^{-2|S|\gamma^2} \leq e^{-\frac{1}{4}N\gamma^2}.$$

For the second inequality, let $Z_i = X_{i,a}X_{i,b}$. Notice that all variables Z_i are i.i.d. equal to 1 with probability $p = \frac{1}{2} \left(\frac{\frac{N}{2}-1}{N-1} \right) = \frac{1}{4} - \frac{1}{4N-4}$ and 0 with the remaining probability. The Hoeffding inequality implies that

$$\begin{aligned} \mathbb{P}_N \left(\left| \sum_{i \in S} Z_i - \frac{1}{4} |S| \right| \geq \gamma N \right) &\leq \mathbb{P}_N \left(\left| \sum_{i \in S} Z_i - p |S| \right| \geq \frac{1}{2} \gamma N \right) \\ &\leq e^{-2|S|(\frac{1}{2}\gamma)^2} \leq e^{-\frac{1}{16}N\gamma^2}, \end{aligned}$$

where we used that $|S||p - \frac{1}{4}| \leq \frac{N}{4N-4} \leq \frac{\gamma N}{2}$ for the first inequality, and that $|S| \geq \frac{1}{8}N$ for the last one. \square

For each $a \neq b$ and $c \neq d$, each $\gamma > 0$, define

$$\begin{aligned} Y_a &= 2 \sum_{i \in A} X_{i,a}, \quad Y^c = 2 \sum_{i \in A} X_{c,i} \\ Y_{a,b} &= 4 \sum_{i \in A} X_{i,a}X_{i,b}, \quad Y_a^c = 4 \sum_{i \in A} X_{i,a}X_{c,i}, \quad Y^{c,d} = 4 \sum_{i \in A} X_{a,i}X_{b,i}, \\ Y_{a,b}^c &= 8 \sum_{i \in A} X_{i,a}X_{i,b}X_{c,i}, \quad Y_a^{c,d} = 8 \sum_{i \in A} X_{i,a}X_{c,i}X_{d,i}, \\ Y_{a,b}^{c,d} &= 16 \sum_{i \in A} X_{i,a}X_{i,b}X_{c,i}X_{d,i}. \end{aligned}$$

Lemma 4. For each $a \neq b$ and $c \neq d$, each $\gamma \geq 64/N$, each of the variables $Z = Y_a, Y^c, Y_{a,b}, Y^{c,d}, Y_a^c, Y_{a,b}^c, Y_a^{c,d}, Y_{a,b}^{c,d}$,

$$\mathbb{P}_N (|Z - N| \geq \gamma N) \leq e^4 N e^{-\frac{1}{16}N(\frac{1}{64}\gamma)^2}.$$

Proof. In case $Z = Y_a, Y_{a,b}$, the bound follow from Lemma 3 (for $S = A$).

In case $Z = Y^c$, the bound is trivially satisfied.

If $Z = Y^{c,d}$, the bound follows from Lemma 2.

In case $Z = Y_{a,b}^{c,d}$, notice that

$$Y_{a,b}^{c,d} = 16 \sum_{i \in A} X_{i,a}X_{i,b}X_{c,i}X_{d,i} = 16 \sum_{i \in S_c \cap S_d} Z_i,$$

where $Z_i = X_{i,a}X_{i,b}$. All variables Z_i are i.i.d. equal to 1 with probability $p = \frac{1}{2} \left(\frac{\frac{N}{2}-1}{N-1} \right) = \frac{1}{4} - \frac{1}{4N-4}$ and 0 with the remaining probability. Moreover, $\{Z_i\}_{i \neq c,d}$ are independent of

$S_c \cap S_d$. Up to enlarge the probability space, we can construct a new collection of random Bernoulli i.i.d. variables Z'_i such that $Z'_i = Z_i$ for all $i \neq c, d$ and such that $\{Z'_i, S_c \cap S_d\}$ are all independent. Then,

$$\left| Y_{a,b}^{c,d} - 16 \sum_{i \in S_c \cap S_d} Z'_i \right| \leq 32,$$

and, because $\frac{1}{2}\gamma N \geq 32$, we have

$$\mathbb{P}_N \left(\left| Y_{a,b}^{c,d} - N \right| \geq \gamma N \right) \leq \mathbb{P}_N \left(\left| \sum_{i \in S_c \cap S_d} Z'_i - \frac{1}{16}N \right| \geq \frac{1}{32}\gamma N \right).$$

Define the events

$$\begin{aligned} A &= \left\{ \left| |S_c \cap S_d| - \frac{N}{4} \right| \geq \frac{1}{64}\gamma N \right\}, \\ B &= \left\{ \left| \sum_{i \in S_c \cap S_d} Z'_i - \frac{1}{4}|S_c \cap S_d| \right| \geq \frac{1}{64}\gamma N \right\}. \end{aligned}$$

Then, the probability can be further bounded by

$$\leq \mathbb{P}_N(A) + \mathbb{P}_N(B|A^c) \leq \frac{1}{2}e^4 N e^{-2N(\frac{1}{16}\gamma)^2} + e^{-\frac{1}{16}N(\frac{1}{64}\gamma)^2},$$

where the first bound comes from Lemma 2, and the second from the second bound in Lemma 3.

The remaining bounds have proofs similar to case $Z = Y_{a,b}^{c,d}$. We describe obvious changes:

- In case $Z = Y_a^{c,d}$, we define $Z_i = X_{i,a}$ as the i.i.d. random variable with probability $p = \frac{1}{2}$. We proceed in an analogous way to the more complicated case, but we rely on the first part of Lemma 3 to bound the probability $\mathbb{P}_N(B|A^c)$.
- In case $Z = Y_{a,b}^c$, we replace $|S_c \cap S_d|$ by the (deterministic) cardinality $|S_c|$. Event A (defined with $\frac{N}{4}$ replaced by $\frac{N}{2}$) fails and there is no need to condition on A^c .
- Finally, case $Z = Y_a^c$ combines the two simplifications.

□

Finally, we describe an event E that collects these bounds. Let $\alpha > 0$ satisfying $\alpha^2 + \alpha \leq \frac{1}{22}$. For each $a \neq b$ and $c \neq d$, define

$$\begin{aligned} E_{a,b,c,d} &= \left\{ \left| \frac{Y_{a,b}}{Y_a} - 1 \right| \leq \alpha \right\} \cap \left\{ \left| \frac{Y_{a,b}^c}{Y_a^c} - 1 \right| \leq \alpha \right\} \cap \left\{ \left| \frac{Y_a^{c,d}}{Y_a^c} - 1 \right| \leq \alpha \right\} \cap \left\{ \left| \frac{Y_{a,b}^{c,d}}{Y_a^{c,d}} - 1 \right| \leq \alpha \right\} \\ &\quad \left\{ \left| \frac{Y_a^{c,d}}{Y^c} - 1 \right| \leq \alpha \right\} \cap \left\{ \left| \frac{Y_a^c}{Y^c} - 1 \right| \leq \alpha \right\} \cap \left\{ \left| \frac{Y_a^{c,d}}{Y^{c,d}} - 1 \right| \leq \alpha \right\}. \end{aligned}$$

Finally, let

$$E = \bigcap_{a,b,c,d:a \neq b \text{ and } c \neq d} E_{a,b,c,d}.$$

Lemma 5. *We have*

$$\mathbb{P}_N(E) \leq 1 - N^4 \times 7 \times e^4 N e^{-\frac{1}{16}N(\frac{1}{64}\frac{\alpha}{3})^2}$$

In particular, there exists N_1 such that for all $N \geq N_1$, the above probability is strictly positive.

Proof. Take $\gamma = \frac{\alpha}{3}$ and let

$$F_{a,b,c,d} = \bigcap_{Z=Y_a, Y_{a,b}, Y^{c,d}, Y^{c,d}, Y_a^c, Y_{a,b}^c, Y_a^{c,d}, Y_{a,b}^{c,d}} \{|Z - N| \leq \gamma N\}.$$

It is easy to see that $F_{a,b,c,d} \subseteq E_{a,b,c,d}$. The probability that $F_{a,b,c,d}$ holds can be bounded from Lemma 4 as

$$\mathbb{P}_N(F_{a,b,c,d}) \geq 1 - 7 \times e^4 N e^{-\frac{1}{16}N(\frac{1}{64}\frac{\alpha}{3})^2}.$$

The result follows from the fact that there are at most N^4 ways of choosing (a, b, c, d) such that $a \neq b$ and $c \neq d$. \square

5.2. Conditions UI1, UI2. Finally, we show that event E implies conditions UI1 + UI2.

Proposition 5. *If event E holds, then the conditions UI1, UI2 are satisfied.*

Proof. Let us begin with condition UI2 which we recall here: for all $1 \leq p \leq l$, for all $\hat{v} \in V^l$, for all $v' \in V^{p-1}$, for all $m \in \{1, \dots, p-1\}$ such that $\hat{v}_m \neq v'_m$, then for $r = 2m-1, 2m$

$$\left| \nu^{2l}(u \smile_{r+1} v' | v = \hat{v}, u \smile_r v') - \frac{1}{2} \right| \leq \alpha. \quad (5.1)$$

In all the computations below, $\omega_N \in E$ is fixed, $U^l \times V^l$ is the probability space equipped with the probability ν^{2l} . (u, v) is a random variable, \hat{v}, v' are deterministic, and events refer to subsets of $U^l \times V^l$. The quantity $\nu^{2l}(u \smile_{r+1} v' | v = \hat{v}, u \smile_r v')$ is the conditional probability of the event $\{u \smile_{r+1} v'\}$ given that $u \smile_r v'$ and that the type v of player 2 is equal \hat{v} . We divide the problem into different cases.

Case $m > 1$ and $r = 2m-1$.

Note that the events $\{u \smile_{2m} v'\}$ and $\{u \smile_{2m-1} v'\}$ can be decomposed as follows

$$\begin{aligned} \{u \smile_{2m} v'\} &= \{u \smile_{2m-2} v'\} \cap \{X_{v'_{m-1}, u_m} = 1\} \cap \{X_{u_m, v'_m} = 1\}, \\ \{u \smile_{2m-1} v'\} &= \{u \smile_{2m-2} v'\} \cap \{X_{v'_{m-1}, u_m} = 1\}. \end{aligned}$$

In the both cases, the Markov property implies the first event on the left-hand side is conditionally independent from the subsequent event(s). It follows that

$$\begin{aligned} \nu^{2l}(u \smile_{2m} v' | v = \hat{v}, u \smile_{2m-1} v') &= \nu^{2l}\left(X_{u_m, v'_m} = 1 | X_{v'_{m-1}, u_m} = 1, (v_{m-1}, v_m) = (\hat{v}_{m-1}, \hat{v}_m)\right), \\ &= \frac{\sum_{i \in U} P_{\hat{v}_{m-1}, i} P_{i, \hat{v}_m} X_{v'_{m-1}, i} X_{i, v'_m}}{\sum_{i \in U} P_{\hat{v}_{m-1}, i} P_{i, \hat{v}_m} X_{v'_{m-1}, i}}. \end{aligned}$$

The latter is equal to

$$= \frac{\sum_{i \in U} X_{\hat{v}_{m-1}, i} X_{i, \hat{v}_m} X_{v'_{m-1}, i} X_{i, v'_m}}{\sum_{i \in U} X_{\hat{v}_{m-1}, i} X_{i, \hat{v}_m} X_{v'_{m-1}, i}} = \frac{1 Y_{\hat{v}_m, v'_m}^{\hat{v}_{m-1}, v'_{m-1}}}{2 Y_{\hat{v}_m}^{\hat{v}_{m-1}, v'_{m-1}}}.$$

If $v'_{m-1} \neq \hat{v}_{m-1}$, we conclude that event E implies (5.1).

If $v'_{m-1} = \hat{v}_{m-1}$, $X_{\hat{v}_{m-1}, u} = X_{v'_{m-1}, u} \in \{0, 1\}$, and the above can be simplified to

$$= \frac{\sum_{u \in U} X_{\hat{v}_{m-1}, u} X_{u, \hat{v}_m} X_{u, v'_m}}{\sum_{u \in U} X_{u, \hat{v}_{m-1}} X_{u, \hat{v}_m}} = \frac{1 Y_{\hat{v}_m, v'_m}^{\hat{v}_{m-1}}}{2 Y_{\hat{v}_m}^{\hat{v}_{m-1}}},$$

and (5.1) follows.

Case $r = 2m$

The event $\{u \smile_{2m+1} v'\}$ can be decomposed as follows

$$\{u \smile_{2m+1} v'\} = \{u \smile_{2m} v'\} \cap \{X_{v'_m, u_{m+1}} = 1\}.$$

We deduce that

$$\begin{aligned} \nu^{2l}(u \smile_{2m+1} v' | v = \hat{v}, u \smile_{2m} v') &= \nu^{2l}\left(X_{v'_m, u_{m+1}} = 1 | (v_m, v_{m+1}) = (\hat{v}_m, \hat{v}_{m+1})\right), \\ &= \frac{\sum_{i \in U} P_{\hat{v}_m, i} P_{i, \hat{v}_{m+1}} X_{v'_m, i}}{\sum_{i \in U} P_{\hat{v}_m, i} P_{i, \hat{v}_{m+1}}} \\ &= \frac{\sum_{i \in U} X_{\hat{v}_m, i} X_{i, \hat{v}_{m+1}} X_{v'_m, i}}{\sum_{i \in U} X_{\hat{v}_m, i} X_{i, \hat{v}_{m+1}}} = \frac{1 Y_{\hat{v}_{m+1}}^{\hat{v}_m, v'_m}}{2 Y_{\hat{v}_{m+1}}^{\hat{v}_m}}, \end{aligned}$$

which implies (5.1).

Case $m = 1, r = 1$

The event $\{u \smile_2 v'\}$ is equal to

$$\{u \smile_2 v'\} = \{X_{u_1, v'_1} = 1\},$$

and the event $\{u \smile_1 v'\}$ is trivial. It follows that

$$\begin{aligned} \nu^{2l}(u \smile_2 v' | v = \hat{v}, u \smile_1 v') &= \nu^{2l}(X_{u_1, v'_1} = 1 | v_1 = \hat{v}_1), \\ &= \frac{\sum_{i \in U} P_{i, \hat{v}_1} X_{i, v'_1}}{\sum_{i \in U} P_{i, \hat{v}_1}} \\ &= \frac{\sum_{i \in U} X_{i, \hat{v}_1} X_{i, v'_1}}{\sum_{i \in U} X_{i, \hat{v}_1}} = \frac{1}{2} \frac{Y^{\hat{v}_1, v'_1}}{Y^{\hat{v}_1}}, \end{aligned}$$

which implies (5.1).

Let us now consider the conditions UI1: At first we require that for all $l \geq 1$, for all $\hat{u} = (\hat{u}_1, \dots, \hat{u}_l)$ in U^l and $u' = (u'_1, \dots, u'_p)$ in U^p with $p = l + 1$ such that $\hat{u}_1 = u'_1$, for all $m \in \{2, \dots, l\}$ such that $\hat{u}_m \neq u'_m$, for $r = 2m - 2, 2m - 1$

$$\left| \nu^{2l}(u' \smile_{r+1} v | u = \hat{u}, u' \smile_r v) - \frac{1}{2} \right| \leq \alpha. \quad (5.2)$$

In all the computations below, $\omega_N \in S \cap E$ is fixed, $U^l \times V^l$ is the probability space equipped with the probability ν^{2l} . (u, v) is a random variable, \hat{u}, u' are deterministic, and events refer to subsets of $U^l \times V^l$. The quantity $\nu^{2l}(u' \smile_{r+1} v | u = \hat{u}, u' \smile_r v)$ is the conditional probability of the event $\{u' \smile_{r+1} v\}$ given that $u' \smile_r v$ and that the type u of player 1 is equal \hat{u} . We divide the problem into different cases.

Case $r = 2m - 1$ and $m = l$. The event $\{u' \smile_{2m} v\}$ can be decomposed as follows

$$\{u' \smile_{2m} v\} = \{u' \smile_{2m-1} v\} \cap \{X_{u'_m, v_m} = 1\},$$

We deduce that

$$\begin{aligned} \nu^{2l}(u' \smile_{2m} v | u = \hat{u}, u' \smile_{2m-1} v) &= \nu^{2l}(X_{u'_m, v_m} = 1 | u_m = \hat{u}_m), \\ &= \frac{\sum_{i \in V} P_{\hat{u}_m, i} X_{u'_m, i}}{\sum_{i \in V} P_{\hat{u}_m, i}}. \end{aligned}$$

The latter is equal to

$$= \frac{\sum_{i \in V} X_{\hat{u}_m, i} X_{u'_m, i}}{\sum_{i \in V} X_{\hat{u}_m, i}} = \frac{1}{2} \frac{Y^{\hat{u}_m, u'_m}}{Y^{\hat{u}_m}},$$

which implies (5.2).

Case $r = 2m - 1$ and $m < l$. The event $\{u' \smile_{2m} v\}$ can be decomposed as follows

$$\{u' \smile_{2m} v\} = \{u' \smile_{2m-1} v\} \cap \{X_{u'_m, v_m} = 1\},$$

It follows that

$$\begin{aligned} \nu^{2l}(u' \smile_{2m} v | u = \hat{u}, u' \smile_{2m-1} v) &= \nu^{2l}(X_{u'_m, v_m} = 1 | (u_m, u_{m+1}) = (\hat{u}_m, \hat{u}_{m+1})), \\ &= \frac{\sum_{i \in V} P_{\hat{u}_m, i} P_{i, \hat{u}_{m+1}} X_{u'_m, i}}{\sum_{i \in V} P_{\hat{u}_m, i} P_{i, \hat{u}_{m+1}}}. \end{aligned}$$

The latter is equal to

$$= \frac{\sum_{i \in V} X_{\hat{u}_m, i} X_{i, \hat{u}_{m+1}} X_{u'_m, i}}{\sum_{i \in V} X_{\hat{u}_m, i} X_{i, \hat{u}_{m+1}}} = \frac{1}{2} \frac{Y_{\hat{u}_{m+1}}^{\hat{u}_m, u'_m}}{Y_{\hat{u}_{m+1}}^{\hat{u}_m}},$$

which implies (5.2).

Case $r = 2m - 2$

The events $\{u' \smile_{2m-1} v\}$ and $\{u' \smile_{2m-2} v\}$ can be decomposed as follows

$$\begin{aligned} \{u' \smile_{2m-1} v\} &= \{u' \smile_{2m-2} v\} \cap \{X_{v_{m-1}, u'_m} = 1\}, \\ \{u' \smile_{2m-2} v\} &= \{u' \smile_{2m-3} v\} \cap \{X_{u'_{m-1}, v_{m-1}} = 1\}. \end{aligned}$$

We deduce that

$$\begin{aligned} \nu^{2l}(u' \smile_{2m-1} v | u = \hat{u}, u' \smile_{2m-2} v) &= \nu^{2l}\left(X_{v_{m-1}, u'_m} = 1 | X_{u'_{m-1}, v_{m-1}} = 1, (u_{m-1}, u_m) = (\hat{u}_{m-1}, \hat{u}_m)\right), \\ &= \frac{\sum_{i \in V} P_{\hat{u}_{m-1}, i} P_{i, \hat{u}_m} X_{u'_{m-1}, i} X_{i, u'_m}}{\sum_{i \in V} P_{\hat{u}_{m-1}, i} P_{i, \hat{u}_m} X_{u'_{m-1}, i}} \\ &= \frac{\sum_{i \in V} X_{\hat{u}_{m-1}, i} X_{i, \hat{u}_m} X_{u'_{m-1}, i} X_{i, u'_m}}{\sum_{i \in V} X_{\hat{u}_{m-1}, i} X_{i, \hat{u}_m} X_{u'_{m-1}, i}} = \frac{1}{2} \frac{Y_{\hat{u}_m, u'_m}^{\hat{u}_{m-1}, u'_{m-1}}}{Y_{\hat{u}_m}^{\hat{u}_{m-1}, u'_{m-1}}}. \end{aligned}$$

If $u'_{m-1} \neq \hat{u}_{m-1}$, we conclude that (5.2) holds.

If $u'_{m-1} = \hat{u}_{m-1}$, $X_{\hat{u}_{m-1}, i} = X_{u'_{m-1}, i} \in \{0, 1\}$, and the above can be simplified to

$$= \frac{1}{2} \frac{Y_{\hat{u}_m, u'_m}^{\hat{u}_{m-1}}}{Y_{\hat{u}_m}^{\hat{u}_{m-1}}},$$

so that (5.2) holds.

We also require that

$$\left| \nu^{2l}(u' \smile_{2l+1} v | u = \hat{u}, u' \smile_{2l} v) - \frac{1}{2} \right| \leq \alpha. \quad (5.3)$$

The events $\{u' \smile_{2l+1} v\}$ and $\{u' \smile_{2l} v\}$ can be decomposed as follows

$$\begin{aligned} \{u' \smile_{2l+1} v\} &= \{u' \smile_{2l} v\} \cap \{X_{v_l, u'_{l+1}} = 1\}, \\ \{u' \smile_{2l} v\} &= \{u' \smile_{2l-1} v\} \cap \{X_{u'_l, v_l} = 1\}. \end{aligned}$$

We deduce that

$$\begin{aligned}
\nu^{2l} (u' \smile_{2l+1} v | u = \hat{u}, u' \smile_{2l} v) &= \nu^{2l} \left(X_{v_l, u'_{l+1}} = 1 | X_{u'_l, v_l} = 1, u_l = \hat{u}_l \right), \\
&= \frac{\sum_{i \in V} P_{\hat{u}_l, i} X_{u'_l, i} X_{i, u'_{l+1}}}{\sum_{i \in V} P_{\hat{u}_l, i} X_{u'_l, i}} \\
&= \frac{\sum_{i \in V} X_{\hat{u}_l, i} X_{u'_l, i} X_{i, u'_{l+1}}}{\sum_{i \in V} X_{\hat{u}_l, i} X_{u'_l, i}} = \frac{1}{2} \frac{Y_{\hat{u}_l, u'_l}}{Y_{\hat{u}_l, u'_l}}.
\end{aligned}$$

If $u'_l \neq \hat{u}_l$, we conclude that (5.3) holds.

If $u'_l = \hat{u}_l$, $X_{\hat{u}_l, v} = X_{u'_l, v} \in \{0, 1\}$, and the above can be simplified to

$$= \frac{1}{2} \frac{Y_{\hat{u}_l}}{Y_{\hat{u}_l}},$$

and we conclude that (5.3) holds. \square

6. NON-ZERO SUM GAMES

In this section, we present a simple and direct proof of Corollary 2. The construction is based on the "email" type spaces from [14].

Type spaces: For each $\alpha \in (0, \frac{1}{2})$, let $T_\alpha = (\mathbb{N}, \mathbb{N}, \mu_\alpha)$ be a Rubinstein's type space over the binary space of uncertainty $\Omega = \{0, 1\}$ with common prior μ_α such that

$$\mu_\alpha(\omega = 0, t_1 = t_2 = 0) = \alpha,$$

$$\forall \hat{t}_1 \geq 1, \forall \hat{t}_2 \in \{\hat{t}_1 - 1, \hat{t}_1\}, \mu_\alpha(\{\omega = 1, t_1 = \hat{t}_1, t_2 = \hat{t}_2\}) = \alpha(1 - \alpha)^{\hat{t}_1 + \hat{t}_2}.$$

The interpretation is that the state is $\omega = 0$ with probability α . If the state is 1, player 1 observes it and sends a message to the other player. Whenever a player receives the message, she immediately sends another message back. Each turn, the message gets lost and does not reach the other player with probability α . The type of a player is equal to the number of messages she sent.

Notice that each of the types $t_1 \geq 1$ of player 1 assigns probability $\frac{1}{2-\alpha} \in (\frac{1}{2}, \frac{2}{3})$ to the opponent's type $t_2 = t_1 - 1$ and the remaining probability to the opponent's type $t_2 = t_1$. Similarly, each of the types $t_2 \geq 0$ of player 2 assigns probability $\frac{1}{2-\alpha} \in (\frac{1}{2}, \frac{2}{3})$ to the opponent's type $t_1 = t_2$ and the remaining probability to the opponent's type $t_1 = t_2 + 1$.

Games: For each increasing function $f : \mathbb{N} \rightarrow [0, \frac{1}{4}]$, we construct a game $G^f = (A_1, A_2, u_1^f, u_2^f)$, where $A_i = \mathbb{N}$ is the action set for player i , and $u_i^f : A_i \times A_{-i} \times \Omega \rightarrow \mathbb{R}$ is the payoff function

so that

$$\begin{aligned}
 u_1^f(a_1, a_2, 0) &= \begin{cases} 0, & a_1 = 0, \\ -\frac{1}{4}, & a_1 > 0 \end{cases}, \\
 u_1^f(a_1, a_2, 1) &= \begin{cases} -\frac{1}{4}, & a_1 < a_2, \\ f(a_1), & a_1 \in \{a_2, a_2 + 1\}, \\ -1 & a_1 > a_2 + 1, \end{cases} \\
 u_2^f(a_1, a_2, \cdot) &= \begin{cases} -\frac{1}{4}, & a_2 < a_1 - 1, \\ f(a_2), & a_2 \in \{a_1 - 1, a_1\}, \\ -1 & a_2 > a_1, \end{cases}
 \end{aligned}$$

Observe that

- action $a_1 = 0$ is strictly dominant for type $t_1 = 0$ of player 1,
- If type $t_1 \geq 1$ player 1 believes that the type $t_2 = t_1 - 1$ plays action a_2 , then none of actions $a_1 > a_2 + 1$ are t_1 's best responses. Indeed, the payoff from such an action is not larger than

$$\frac{1}{2}(-1) + \frac{1}{2}\left(\frac{1}{4}\right) < -\frac{1}{4},$$

whereas playing 0 gives a payoff not smaller than $-\frac{1}{4}$.

- similarly, if player 2 believes with probability higher than $\frac{1}{2}$ that the opponent plays action a_1 , then none of actions $a_2 > a_1$ are best responses.

Claim 1. *The best payoff for player 1 in any Bayesian Nash equilibrium of game G^f on type space T_α is attained at*

$$v(f, \alpha) := \sum_{m \geq 1} f(m) \alpha (1 - \alpha)^{2m-1} (2 - \alpha). \quad (6.1)$$

Proof. First, we show that such a payoff is attained. Consider a profile of strategies (σ_1, σ_2) such that $\sigma_i(t_i) = t_i$. This is an equilibrium since each of the players types is receiving payoff $f(t_i) \in [0, \frac{1}{40}]$ (or 0 in case of type $t_1 = 0$). Given the above remark, deviations of player 1 of type t_1 to actions $a_1 > t_1$ give a negative payoff, and deviations to actions $a_1 < t_1$ give a payoff smaller than $f(t_1)$. A similar argument holds for player 2. The ex ante expected payoff of player 1 in such an equilibrium is equal to (6.1).

We will show that there is no better equilibrium payoff. Indeed, take any equilibrium (σ_1, σ_2) . For each action $a \geq 0$, let

$$t_i(a) = \inf \{t : \sigma_i(\{a' \in \mathbb{N} | a' \geq a\} | t) > 0\}$$

be the first type of player i that assigns positive probability to actions higher than a . Using the above three observations about best responses, induction on types shows that that $t_i(a) \geq a$. But then, the equilibrium payoff cannot be higher than (6.1). \square

Let \mathcal{F} be the space of increasing functions $f : \mathbb{N} \rightarrow [0, \frac{1}{4}]$. Let $f_n(m) = \frac{1}{4} \cdot \mathbf{1}\{m \geq n\}$.

Claim 2. *There exists n^* such that for all $m > n \geq n^*$,*

$$v\left(\frac{1}{2^m}, f_{2^n}\right) - v\left(\frac{1}{2^n}, f_{2^n}\right) > \frac{1}{16e^2},$$

where e is the base of the natural logarithm.

Proof. Observe that for each α and $n \geq 1$,

$$\begin{aligned} 4v(\alpha, f_n) &= \sum_{m \geq n} \alpha(1-\alpha)^{2^m-1} (2-\alpha) \\ &= \alpha(2-\alpha)(1-\alpha)^{2^n-1} \sum_{t \geq 0} (1-\alpha)^{2^t} \\ &= (1-\alpha)^{2^n-1} \frac{\alpha(2-\alpha)}{1-(1-\alpha)^2} \\ &= (1-\alpha)^{2^n-1}. \end{aligned}$$

Moreover, for any $0 < \alpha < \beta < 1$,

$$\begin{aligned} 4(v(\alpha, f_n) - v(\beta, f_n)) &= (1-\alpha)^{2^n-1} - (1-\beta)^{2^n-1} \\ &= (\beta-\alpha) \left(\sum_{l=1}^{k=2^n-2} (1-\alpha)^{2^n-1-l} (1-\beta)^l \right) \\ &\geq (\beta-\alpha)(2^n-2)(1-\beta)^{2^n-1}. \end{aligned}$$

Thus,

$$\begin{aligned} 4 \left(v\left(\frac{1}{2^m}, f_{2^n}\right) - v\left(\frac{1}{2^n}, f_{2^n}\right) \right) &\geq \left(\frac{1}{2^n} - \frac{1}{2^m}\right) (2^{n+1}-2) \left(1 - \frac{1}{2^n}\right)^{2^{n+1}-1} \\ &\geq \frac{2^n}{2^{n+1}} \left(1 - \frac{1}{2^n}\right)^{2^{n+1}-1} = \frac{1}{2} \left(1 - \frac{1}{2^n}\right)^{2^{n+1}-1}. \end{aligned}$$

For sufficiently high n , $\left(1 - \frac{1}{2^n}\right)^{2^{n+1}-1} > \frac{1}{2}e^{-2}$. \square

Finally, we can present a direct proof of Corollary 2.

Proof. It is enough to show that there exists $\varepsilon > 0$ and a sequence of type spaces (T^m) such that for each k, l

$$d_E(T^k, T^l) > 2\varepsilon.$$

Let $\varepsilon = \frac{1}{2} \frac{1}{16e^2}$, and let $T^n = T_{\frac{1}{2^{n+n^*}}}$. The result is implied by the previous claim. \square

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