Supplementary file on "Semiparametric density testing in the contamination model"

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1. Proofs of Lemmas 1-4

Lemma 1. Since the parameters (μ, s) belong to a compact set we can fix: $s_0 < s < s_1$ and $|\mu| < \mu_1$. We consider for simplicity $k = 2\ell$, $\ell \geq 0$, in the k-th order Hermite polynomial expression (6.1) and notice that for all $(\mu, s) \in \Lambda$,

$$|\mathbb{E}(H_k(\sqrt{s}Z + \mu))| \leq k! \sum_{m=0}^{\ell} \frac{\mathbb{E}((\sqrt{s}Z + \mu)^{2(\ell-m)})}{m!(2(\ell-m))!2^m}, \tag{1.1}$$

where Z is $\mathcal{N}(0,1)$ distributed. Now since

$$\begin{aligned} \mathbb{E}((\sqrt{s}Z + \mu)^{2(\ell-m)}) &= \sum_{j=0}^{2(\ell-m)} C_j^{2(\ell-m)} \sqrt{s}^j \mathbb{E}(Z^j)|\mu|^{2(\ell-m)-j} \\ &\leq \mathbb{E}(Z^{2(\ell-m)})(\sqrt{s} + |\mu|)^{2(\ell-m)} \\ &= \frac{2(\ell-m)!}{2^{(\ell-m)}(\ell-m)!}(\sqrt{s} + |\mu|)^{2(\ell-m)}, \end{aligned} \tag{1.2}$$

including (1.2) in (1.1), we obtain:

$$\begin{aligned} |\mathbb{E}(H_k(\sqrt{s}Z + \mu))| &\leq k! \sum_{m=0}^{\ell} \frac{(\sqrt{s} + |\mu|)^{2(\ell-m)}}{m!(\ell-m)!2^\ell} \\ &= \frac{k!}{\ell!} \left(\frac{(\sqrt{s} + |\mu|)^2 + 1}{2} \right)^\ell. \end{aligned} \tag{1.3}$$

Since $\alpha_k(\mu,s)=\mathbb{E}(H_k(\sqrt{s}Z+\mu))/q_k^2$, with $q_k^2=k!$, we deduce from (1.3) that for all $k\geq 0$ and for all $(\mu,s)\in\Lambda$,

$$\begin{aligned}|\alpha_k(\mu,s)| &\leq \frac{1}{\ell!}\left(\frac{(\sqrt{s}+|\mu|)^2+1}{2}\right)^{\ell}\\ &\leq \exp\left(\frac{(\sqrt{s}_1+\mu_1)^2+1}{2}\right),\end{aligned}$$

which proves the first part of **(A2)**.
For the second part of condition **(A2)**, we detail for simplicity the majorization of $\left|\frac{\partial}{\partial s}\mathbb{E}(H_k(sZ+\mu))\right|$ for $k=2\ell$, $\ell\geq 1$:

$$\left|\frac{\partial}{\partial s}\mathbb{E}(H_k(\sqrt{s}Z+\mu))\right| \leq \frac{k!}{2\sqrt{s}}\sum_{m=0}^{\ell-1}\frac{2(\ell-m)\mathbb{E}\left(\left|Z(\sqrt{s}Z+\mu)^{2(\ell-m)-1}\right|\right)}{m!(2(\ell-m))!2^m}.$$

Now since

$$\begin{aligned}\mathbb{E}\left(\left|Z(\sqrt{s}Z+\mu)^{2(\ell-m)-1}\right|\right) &\leq \sum_{j=0}^{2(\ell-m)-1} C_j^{2(\ell-m)-1}\sqrt{s}^j\mathbb{E}(Z^{j+1})|\mu|^{2(\ell-m)-1-j},\\ &\leq \mathbb{E}(Z^{2(\ell-m)})(\sqrt{s}+|\mu|)^{2(\ell-m)-1},\end{aligned}$$

we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial s}\mathbb{E}(H_k(\sqrt{s}Z+\mu))\right| &\leq \frac{k!}{2\sqrt{s}}\sum_{m=0}^{\ell-1}\frac{(s+|\mu|)^{2(\ell-m)-1}}{m!(\ell-m-1))!2^{\ell-1}}\\ &= \frac{k!}{2\sqrt{s}(\ell-1)!}\left(\frac{(\sqrt{s}+|\mu|)^2+1}{2}\right)^{\ell-1}\\ &\leq \frac{k!}{2\sqrt{s_0}}\exp\left(\frac{(\sqrt{s}_1+\mu_1)^2+1}{2}\right),\end{aligned}$$

which concludes the proof of **(A2)**.
We now consider condition **(A3)**. We have

$$\begin{aligned}\mathbb{E}(H_k^2(X_1)) &= (1-p)\mathbb{E}(H_k^2(Z))+p\mathbb{E}(H_k^2(\sqrt{s}Z+\mu))\\ &= (1-p)k!+p\mathbb{E}(H_k^2(\sqrt{s}Z+\mu)).\end{aligned}\tag{1.4}$$

Let us consider the last term of the above right-hand side equality, for $k=2\ell$ and $\ell\geq 0$:

$$\mathbb{E}(H_k^2(\sqrt{s}Z+\mu))=(k!)^2\sum_{m,q=0}^{\ell}\frac{\mathbb{E}\left((\sqrt{s}Z+\mu)^{2(2\ell-(m+q))}\right)}{m!q!(2(\ell-m))!(2(\ell-q))!2^{m+q}}.$$

By the Cauchy-Schwartz inequality, and the fact that for all $n \geq 1$, we have $\sqrt{2\pi}n^{n+1/2}e^{-n} \leq n! \leq en^{n+1/2}e^{-n}$, we obtain:

$$\begin{aligned}
\mathbb{E}(H_k^2(\sqrt{s}Z+\mu)) &\leq (k!)^2 \left(\sum_{m=0}^{\ell} \frac{\sqrt{\mathbb{E}\left((\sqrt{s}Z+\mu)^{4(\ell-m)}\right)}}{m!(2(\ell-m))!2^m} \right)^2 \\
&= (k!)^2 \left(\frac{1}{\ell!2^\ell} + \sum_{m=0}^{\ell-1} \frac{\sqrt{\mathbb{E}\left((\sqrt{s}Z+\mu)^{4(\ell-m)}\right)}}{m!(2(\ell-m))!2^m} \right)^2 \\
&\leq (k!)^2 \left(\frac{1}{\ell!2^\ell} + \sum_{m=0}^{\ell-1} \frac{\sqrt{(4(\ell-m))!}(\sqrt{s}+|\mu|)^{2(\ell-m)}}{2^\ell(2(\ell-m)!)^{3/2}m!} \right)^2 \\
&\leq \frac{(k!)^2 e}{2^{2\ell+1/2}(2\pi)^3} \left(\frac{(2\pi)^3 2^{\ell+1/2}}{e\ \ell!} + \right. \\
&\qquad \left. \sum_{m=0}^{\ell-1} 2^{\ell-m}(\ell-m)^{-(\ell-m)-1}e^{\ell-m}(\sqrt{s}+|\mu|)^{2(\ell-m)} \right)^2 \\
&\leq \frac{(k!)^2 e}{2^{2\ell+1/2}(2\pi)^3} \left(\frac{(2\pi)^3 2^{\ell+1/2}}{e\ \ell!} + \sum_{u=1}^{\ell} \rho^u u^{-u-1} \right)^2, \qquad (1.5)
\end{aligned}$$

where $u = \ell - m$ and $\rho = 2e(\sqrt{s}+|\mu|)^2$. Clearly, $\rho \leq \rho_0 = 2e(\sqrt{s_0}+\mu_0)^2$, and the series on the right hand side converges. Combining (1.4) and (1.5) we obtain

$$\begin{aligned}
\mathrm{var}(Q_k(X_1)^2/q_k^2) &\leq \mathbb{E}(Q_k^2(X_1))/q_k^4 \\
&= (1-p)/(k!) + p\mathbb{E}(H_k^2(\sqrt{s}Z+\mu))/(k!)^2
\end{aligned}$$

and we get the wanted result. □

Lemma 3. The polynomials defined by (6.2) satisfy the following relations:

$$xh_k(x) = h_{k+1}(x)/2 + kh_{k-1}(x) \text{ and } h_k'(x) = 2kh_{k-1}(x), \text{ for all } x \in \mathbb{R}.$$

It is also well known (see for instance Szegö, 1939) that there exists a constant $C > 0$ such that, for all $x \in \mathbb{R}$:

$$|\mathcal{H}_k(x)| = \exp(-x^2/2)|h_k(x)| \leq C\sqrt{k!2k}. \qquad (1.6)$$

Since $\alpha_k(\mu, s) = \mathbb{E}(\mathcal{H}_k(sY+\mu))/q_k^2$, we deduce that for all $s > 0$, and $\mu \in \mathbb{R}$,

$$\alpha_k(\mu, s) \leq C/\sqrt{k!2k},$$

which gives the first bound in **(A2)**. Moreover, we have

$$\begin{aligned}
\mathcal{H}_k'(x) &= \exp(-x^2/2)\left(-xh_k(x) + h_k'(x)\right) \\
&= \exp(-x^2/2)\left(-(h_{k+1}(x)/2 - kh_{k-1}(x)) + 2kh_{k-1}(x)\right) \\
&= -\mathcal{H}_{k+1}(x)/2 + k\mathcal{H}_{k-1}(x),
\end{aligned}$$

which leads to

$$\frac{\mathcal{H}_k'(x)}{q_k^2} = -\frac{\mathcal{H}_{k+1}(x)}{2^{k+1}k!} + \frac{\mathcal{H}_{k-1}(x)}{2^k(k-1)!}.$$

Combining this equality with (1.6) we obtain

$$\left|\frac{\mathcal{H}_k'(x)}{q_k^2}\right| \leq C\left(\frac{\sqrt{k+1}}{\sqrt{2^{k+1}k!}} + \frac{1}{\sqrt{2^{k+1}(k-1)!}}\right). \tag{1.7}$$

Now since $\dot{\alpha_k}(\mu,s) = \mathbb{E}\left((s^{-1/2}\mathcal{H}_k(\sqrt{s}Y+\mu), \mathcal{H}_k(\sqrt{s}Y+\mu))\right)/2$ it follows that for all $s>0$ and $\mu\in\mathbb{R}$:

$$\|\dot{\alpha_k}(\mu,s)\| \leq (s^{-1/2}/2+1)C\left(\frac{\sqrt{k+1}}{\sqrt{(k)!}\sqrt{2^{k+1}}} + \frac{1}{2\sqrt{2^{k-1}(k-1)!}}\right),$$

which gives the second bound in **(A2)**.

Finally from (1.6) we obtain $\mathrm{var}(Q_k(X_1)/q_k^2) = \mathrm{var}(\mathcal{H}_k(X_1)/q_k^2) \leq C^2/(k!2k)$, which directly insures **(A3)**. □

Remark 1. *Lemma 3 is very general. Lemma 1 can be extended to any null distribution f with known moments such that the series given in (1.1) is bounded. This is obviously the case for distributions with bounded support.*

Lemma 2. By the Taylor formula we have

$$\begin{aligned}
|\mathbb{E}^{(n)}\left(Q_k(X_0+\delta_n\varepsilon_1) - Q_k(X_0)\right)| &= |\sum_{j=1}^{k}\mathbb{E}^{(n)}\left((\delta_n\varepsilon_1)^j Q_k^{(j)}(X_0)/j!\right)| \\
&\leq \sum_{j=1}^{k}\frac{\delta_n^j}{j!}\mathbb{E}^{(n)}\left(|\varepsilon_1|^j\right)\mathbb{E}^{(n)}\left(|Q_k^{(j)}(X_0)|\right),
\end{aligned}$$

where $Q_k^{(j)}$ denotes the j-th derivative of the Hermite polynomial Q_k. These polynomials (see for instance Szegö, 1939) satisfy $Q_k^{(1)} = kQ_{k-1}$, which implies that $Q_k^{(j)} = \frac{k!}{(k-j)!}Q_{k-j}$, for $j\leq k$. It follows that

$$\begin{aligned}
\mathbb{E}^{(n)}\left(|Q_k^{(j)}(X_0)|\right) &= \int |Q_k^{(j)}(y)|g(y)\nu(dy) \\
&= \frac{k!}{(k-j)!}\int |Q_{k-j}(y)|g(y)\nu(dy) \\
&\leq \frac{k!}{(k-j)!}\sqrt{\int (Q_{k-j}(y))^2\nu(dy)\int g^2(y)\nu(dy)} \\
&= \frac{k!}{(k-j)!}q_{k-j}G,
\end{aligned}$$

where $G = \sqrt{\int g^2(y)\nu(dy)} < \infty$ since g belongs to $L^2(\nu)$. Since $q_k^2 = k!$ and $\delta_n < 1$, we get

$$\begin{aligned} \left|\mathbb{E}^{(n)}\left(Q_k(X_0+\delta_n\varepsilon_1)-Q_k(X_0)\right)\right|/q_k^2 &\leq G\delta_n \sum_{j=1}^{k} \frac{1}{\sqrt{(k-j)!j!}}\mathbb{E}^{(n)}\left(|\varepsilon_1|^j\right) \\ &\leq G\delta_n\mathbb{E}^{(n)}\left(e^{|\varepsilon_1|}\right), \end{aligned}$$

which is the desired conclusion. □

Lemma 4. By the Mean Value Theorem there exists a random variable ξ such that

$$\mathbb{E}^{(n)}\left(Q_k(X_0+\delta_n\varepsilon_1)-Q_k(X_0)\right) = \mathbb{E}^{(n)}\left(\delta_n\varepsilon_1 Q_k'(\xi)\right).$$

From (1.7) we have $|Q_k'(x)|/q_k^2 < 2C$ for all $x \in \mathbb{R}$, and we get

$$\begin{aligned} \left|\mathbb{E}^{(n)}\left(Q_k(X_0+\delta_n\varepsilon_1)-Q_k(X_0)\right)\right|/q_k^2 &\leq \delta_n\mathbb{E}^{(n)}\left(|\varepsilon_1||Q_k'(\xi)|\right)/q_k^2 \\ &\leq 2C\delta_n\mathbb{E}^{(n)}\left(|\varepsilon_1|\right), \end{aligned}$$

and the lemma follows. □

2. Contiguous alternative modelling

We study in this section the asymptotic behavior of the semiparametric estimator $(\bar{p}_n, \bar{\mu}_n)$ introduced in Bordes and Vandekerkhove (2010) when their model is no longer fixed but depends on n through the following transformation:

$$g^{(n)}(x) = pf_0(x) + (1-p)f^{(n)}(x-\mu), \quad x \in \mathbb{R}, \tag{2.1}$$

where $(f^{(n)})_{n\geq 1}$ is a sequence of ν-pdfs converging towards the limiting pdf f. For simplicity, when $f^{(n)}$ is replaced by f in (2.1), the resulting model will be so-called the *asymptotic model*. In this framework, for each $n \geq 1$, we consider a sample $(X_1^n, \ldots, X_n^n)$ independent and identically drawn from the n-local probability density function g_n. In addition, we suppose that for any $(n,m) \in \mathbb{N}^* \times \mathbb{N}$ such that $n \neq m$, we have $(X_1^n, \ldots, X_n^n)$ independent from $(X_1^m, \ldots, X_m^m)$. The sequence $(X_1^n, \ldots, X_n^n)_{n\geq 1}$ is commonly called a *row independent triangular-array*. To handle easily the asymptotic normality of the Bordes and Vandekerkhove (2010) semiparametric estimator based on the "corrupted" sample $(X_1^n, \ldots, X_n^n)$, we consider the *coupling*:

$$\begin{cases} X_i^n &:= (1-U_i)Y_i + U_i Z_i^n, \quad i = 1, \ldots, n \\ X_i &:= (1-U_i)Y_i + U_i Z_i, \quad i \geq 1, \end{cases} \tag{2.2}$$

where $(U_i)_{i\geq 1}$ and $(Y_i, Z_i, \varepsilon_i)_{i\geq 1}$ are independent and identically distributed samples respectively drawn from a Bernoulli distribution with parameter p and a $f_0 \otimes f(\cdot - \mu) \otimes f_1$-distribution. The random variable $Z_i^n := Z_i + \delta_n\varepsilon_i$ is by construction distributed according to $f^{(n)}$. Note that we have the following stochastic bound:

$$|X_i^n - X_i| \leq \delta_n|\varepsilon_i|, \quad i = 1, \ldots, n. \tag{2.3}$$

3. Estimation method

The cumulative distribution function (cdf) $G^{(n)}$ associated with model (2.1) is defined by

$$G^{(n)}(x) = (1-p)F_0(x) + pF^{(n)}(x-\mu), \quad x \in \mathbb{R},$$

where $G^{(n)}$, F_0 and $F^{(n)}$ are cdfs corresponding to the dfs $g^{(n)}$, f_0 and $f^{(n)}$ respectively. Let us denote by ϑ the Euclidean part (p,μ) of the model parameters taking values in Θ. Assume that the asymptotic model is identifiable and denote by $\vartheta_0 = (p_0, \mu_0)$ the true value of its unknown parameter ϑ. A way to estimate consistently ϑ_0, based on the triangular array $(X_1^n, \ldots, X_n^n)$, is to follow step by step the Bordes and Vandekerkhove (2010) procedure. Let us define

$$F^{(n)}(x) = \frac{1}{p}\left(G^{(n)}(x+\mu) - (1-p)F_0(x+\mu)\right), \quad x \in \mathbb{R}. \tag{3.1}$$

Because $F^{(n)}$ approximates the symmetric cdf F, we have $F^{(n)}(x) \approx 1 - F^{(n)}(-x)$, for all $x \in \mathbb{R}$. Let us introduce, for all $x \in \mathbb{R}$, the functions

$$H_1^{(n)}(x;\vartheta,G^{(n)}) = \frac{1}{p}G^{(n)}(x+\mu) - \frac{1-p}{p}F_0(x+\mu),$$

and

$$H_2^{(n)}(x;\vartheta,G^{(n)}) = 1 - \frac{1}{p}G^{(n)}(-x+\mu) + \frac{1-p}{p}F_0(-x+\mu).$$

We have, using (3.1) and the *almost*-symmetry of $F^{(n)}$,

$$H^{(n)}(x;\vartheta_0,G^{(n)}) = H_1^{(n)}(x;\vartheta_0,G^{(n)}) - H_2^{(n)}(x;\vartheta_0,G^{(n)}) \approx 0, \tag{3.2}$$

whereas we can expect that for all $\vartheta \neq \vartheta_0$ an *ad hoc* norm of the function $H^{(n)}$ will have a significant departure from zero. In Bordes *et al.* (2006a) the authors considered the $L^2_G(\mathbb{R})$-norm that proved to be interesting from both theoretical and numerical point of view. Considering such a norm leads to the following function $d^{(n)}$ on Θ:

$$d^{(n)}(\vartheta) := \int_{\mathbb{R}} (H^{(n)}(x;\vartheta,G^{(n)}))^2 dG^{(n)}(x),$$

which will likely converge towards the contrast function

$$d(\vartheta) = \int_{\mathbb{R}} (H(x;\vartheta,G))^2 dG(x),$$

associated with the asymptotic model (1.1), see Bordes and Vandekerkhove (2010, p.24).

Because $G^{(n)}$ is unknown it is natural to replace it by its empirical version $\widehat{G}_n^{(n)}$ obtained from the n-sample $(X_1^n, \ldots, X_n^n)$. However, because we aim to estimate ϑ by the minimum argument of the empirical version of $d^{(n)}$ using a differentiable optimization routine, we need to replace $G^{(n)}$ in $H^{(n)}$ by a regular

version $\widetilde{G}_n^{(n)}$ of $\widehat{G}_n^{(n)}$. Therefore we obtain an emprical version $d_n^{(n)}$ of $d^{(n)}$ defined by

$$d_n^{(n)}(\vartheta) = \int_{\mathbb{R}} (H^{(n)}(x;\vartheta,\widetilde{G}_n^{(n)}))^2 d\widehat{G}_n^{(n)}(x) = \frac{1}{n}\sum_{i=1}^{n} (H^{(n)}(X_i^n;\vartheta,\widetilde{G}_n^{(n)}))^2$$

where

$$\widehat{G}_n^{(n)}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_{X_i^n \le x}, \quad x \in \mathbb{R},$$

and $\widetilde{G}_n^{(n)}(x) = \displaystyle\int_{-\infty}^{x} \widehat{g}_n^{(n)}(t)dt$ denotes the smoothed version of the empirical cdf $\widehat{G}_n^{(n)}$ since $\widehat{g}_n^{(n)}$ is a kernel density estimator of $g^{(n)}$ defined by (5.1). Note that additional conditions on the bandwidth h_n and the kernel function q will be specified afterward.

In the sequel, when the above quantities are considered without superscript (n) this will simply means that $G^{(n)}$ has been replaced by G and $\mathbf{X}^{(n)} := (X_1^n, \ldots, X_n^n)$ by $\mathbf{X}^n := (X_1, \ldots, X_n)$ accordingly in their respective analytical expressions. Note that these estimators are then exactly the ones considered in Bordes and Vandekerkhove (2010, Section 2). Finally we propose to estimate ϑ_0 by

$$\bar{\vartheta}_n^{(n)} = (\bar{p}_n^{(n)}, \bar{\mu}_n^{(n)}) = \arg\min_{\vartheta\in\Theta} d_n^{(n)}(\vartheta).$$

4. Identifiability, consistency and asymptotic normality

4.1. General conditions and identifiability

In this section we give a set of conditions for which we obtain identifiability of the asymptotic model parameters, consistency and asymptotic normality of our estimators. Let us denote by m_0 and m the second-order moments of f_0 and f respectively. We introduce the set

$$\Phi = \mathbb{R}^*\times]0,+\infty[\setminus \cup_{k\in\mathbb{N}^*} \Phi_k$$

where

$$\Phi_k = \left\{ (\mu, m)) \in \mathbb{R}^*\times]0,+\infty[; m = m_0 + \mu^2 \frac{k \pm 2}{3k} \right\}.$$

Let us define $\mathcal{F}_q = \{f \in \mathcal{F}; \int_{\mathbb{R}} |x|^q f(x)dx < +\infty\}$ for $q \ge 1$. Denoting by $\bar{f}_0$ the Fourier transform of the df f_0 we consider one assumption, for which the semi-parametric identifiability of the model (1.1) parameters is obtained, see Bordes *et al.*, (2006b, Proposition 2, p. 736).

Identifiability condition (I). For all $n \ge 1$, let $(f_0, f) \in \mathcal{F}_3^2$, $\bar{f}_0 > 0$ and $(\mu_0, m)) \in \Phi_c^{(n)}$ where Φ_c a compact subset of Φ. We have $\vartheta_0 = (p_0, \mu_0) \in \Theta$ where Θ is a compact subset of $(0,1)\times\Xi$ where $\Xi = \{\mu; (\mu, m) \in \Phi_c\}$.

Kernel conditions (K).

(i) The even kernel density function K is bounded, uniformly continuous, square integrable, of bounded variations and has second order moment.
(ii) The function K has first order derivative $K' \in L^1(\mathbb{R})$ and $K'(x) \to 0$ as $|x| \to +\infty$. In addition if γ is the square root of the continuity modulus of K, we have

$$\int_0^1 (\log(1/u))^{1/2}\, d\gamma(u) < \infty.$$

Approximation conditions (A). The even kernel density function K is bounded, twice differentiable with bounded first and second derivatives.
Bandwidth conditions (B).

(i) $h_n \searrow 0$, $nh_n \to +\infty$ and $\sqrt{n}h_n^2 = o(1)$,
(ii) $nh_n/|\log h_n| \to +\infty$, $|\log h_n|/\log\log n \to +\infty$ and there exists a real number c such that $h_n \leq ch_{2n}$ for all $n \geq 1$,
(iii) $|\log h_n|/(nh_n^3) \to 0$.

Comments. The two first conditions in **(B)** (i) are necessary to obtain the pointwize consistency of the $\widehat{g}_n$ sequence of kernel estimators towards g. The third condition allows to control the distance between the empirical cdf $\widehat{G}_n$ and its regularized version $\widetilde{G}_n$. By using Corollary 1 in Shorack and Wellner (1986, p. 766) we obtain

$$\|\widetilde{G}_n - \widehat{G}_n\|_\infty = O_{a.s.}(h_n^2),$$

which by (i) and the law of iterated logarithm, leads to

$$\|\widetilde{G}_n - G\|_\infty = O_{a.s.}\left(\left(\frac{\log\log n}{n}\right)^{-1/2}\right). \tag{4.1}$$

Lemma 1. *Suppose that the kernel function q satisfies Conditions* **(K)** *and* **(A)** *and that the bandwidth (h_n) satisfies Conditions* **(B)**, *then we have:*

$$\begin{aligned}
&(i) \qquad \|\widetilde{G}_n^{(n)} - \widetilde{G}_n\|_\infty = O_{a.s.}\left(\delta_n/h_n\right),\\
&(ii) \qquad \|\widehat{g}_n^{(n)} - \widehat{g}_n\|_\infty = O_{a.s.}\left(\delta_n/h_n^2\right),\\
&(iii) \qquad \|(\widehat{g}_n^{(n)})' - (\widehat{g}_n)'\|_\infty = O_{a.s.}\left(\delta_n/h_n^3\right).
\end{aligned}$$

Proof. Let us detail the proof of result (ii). For all $x \in \mathbb{R}$, the stochastic error between $\widehat{g}_n^{(n)}(x)$ and $\widehat{g}_n(x)$ is controlled as follows:

$$\begin{aligned}
\left|\widehat{g}_n^{(n)}(x) - \widehat{g}_n(x)\right| &= \left|\frac{1}{nh_n}\sum_{i=1}^n \left(K\left(\frac{x - X_i^n}{h_n}\right) - K\left(\frac{x - X_i}{h_n}\right)\right)\right|, \quad x \in \mathbb{R}\\
&\leq \frac{1}{nh_n}\sum_{i=1}^n \left|K\left(\frac{x - X_i^n}{h_n}\right) - K\left(\frac{x - X_i}{h_n}\right)\right|\\
&\leq \frac{1}{nh_n^2}\sum_{i=1}^n ||K'||_\infty\, |X_i^n - X_i|\\
&\leq \frac{||K'||_\infty \delta_n}{h_n^2} \times \left(\frac{\sum_{i=1}^n |\varepsilon_i|}{n}\right),
\end{aligned} \tag{4.2}$$

where the last inequality comes from (2.3). The above result shows that, according to the Strong Law of Large numbers, $\|\widehat{g}_n^{(n)} - \widehat{g}_n\|_\infty = O_{a.s.}(\delta_n/h_n^2)$. The proofs of (i) and (iii) are identic to the proof (ii). □

4.2. Consistency and preliminary convergence rate

We denote for simplicity by $\dot{h}(\vartheta)$ and $\ddot{h}(\vartheta)$ the gradient vector and hessian matrix of any real function h (when it makes sense) with respect to argument $\vartheta \in \mathbb{R}^2$.

Lemma 2. *Assume that Conditions* **(K)**, **(A)** *and* **(B)** *are satisfied and that* Θ *is a compact subset of* $(0,1) \times \Phi_c$.
(i) If K *is bounded over* $\mathbb{R}$ *then* $\sup_{\vartheta\in\Theta} \left|d_n^{(n)}(\vartheta) - d_n(\vartheta)\right| = O_{a.s.}(\delta_n/h_n)$.

(ii) If K' *is bounded over* $\mathbb{R}$ *then* $\left\|\dot{d}_n^{(n)}(\vartheta_0) - \dot{d}_n(\vartheta_0)\right\| = O_{a.s.}(\delta_n^2/h_n^3)+O_{a.s.}(\delta_n/h_n)$.

(iii) If K'' *is bounded over* $\mathbb{R}$ *then* $\sup_{\vartheta\in\Theta} \left\|\ddot{d}_n^{(n)}(\vartheta) - \ddot{d}_n(\vartheta)\right\| = O_{a.s.}(\delta_n/h_n^3)$.

Proof. For the proof of (i) let us write for all $\vartheta \in \Theta$:

$$\begin{aligned}
\left|d_n^{(n)}(\vartheta) - d_n(\vartheta)\right| &= \left|\frac{1}{n}\sum_{i=1}^n \left(H^2(X_i^n;\vartheta,\widetilde{G}_n^{(n)}) - H^2(X_i;\vartheta,\widetilde{G}_n)\right)\right| \\
&\le \frac{1}{n}\sum_{i=1}^n \left|H^2(X_i^n;\vartheta,\widetilde{G}_n^{(n)}) - H^2(X_i^n;\vartheta,\widetilde{G}_n)\right| \\
&\quad + \frac{1}{n}\sum_{i=1}^n \left|H^2(X_i^n;\vartheta,\widetilde{G}_n) - H^2(X_i;\vartheta,\widetilde{G}_n)\right| \\
&\le O_{a.s.}\left(\|\widetilde{G}_n^{(n)} - \widetilde{G}_n\|_\infty\right) \\
&\quad + O_{a.s.}\left(\frac{1}{n}\sum_{i=1}^n \left|\widetilde{G}_n(X_i^n+\mu) - \widetilde{G}_n(X_i+\mu)\right|\right). \qquad (4.3)
\end{aligned}$$

The second term in the right hand side of the above inequality can be handled by using the mean value theorem as follows:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n \left|\widetilde{G}_n(X_i^n+\mu) - \widetilde{G}_n(X_i+\mu)\right| &\le \frac{1}{n}\sum_{i=1}^n (\|\widetilde{g}_n - g\|_\infty + \|g\|_\infty)|X_i^n - X_i| \\
&\le \delta_n(o_{a.s.}(1) + ||g||_\infty) \times \left(\frac{\sum_{i=1}^n |\varepsilon_i|}{n}\right),
\end{aligned}$$

where according to Silverman (1978) $\|\widetilde{g}_n - g\|_\infty = o_{a.s.}(1)$. Similarly to (4.2), using the Strong of Large Numbers on the $|\varepsilon_i|$'s, we get that this second term is a $O_{a.s.}(\delta_n)$. Since the first term in the right hand side of (4.3) is a $O_{a.s.}(\delta_n/h_n)$ according to Lemma 1 (i), we obtain the wanted result.

For the proof of result (ii), let proceed similarly to Bordes and Vandekerkhove (2010) and investigate the partial derivative of $\dot{d}_n^{(n)}(\vartheta_0)$ with respect to μ (more complicated case). Consider for any cdf G, the generic expression $\mathcal{H}(x,\vartheta_0,G) := H(x;\vartheta_0,G)\frac{\partial H}{\partial \mu}(x;\vartheta_0,G)$, $x\in\mathbb{R}$. According to (2.4) in Bordes and Vandekerkhove (2010), we have at point ϑ_0:

$$\left|\frac{\partial d_n^{(n)}}{\partial \mu}(\vartheta_0)-\frac{\partial d_n}{\partial \mu}(\vartheta_0)\right| \le \Delta_1(\mathbf{X}^{(n)},\widetilde{G}_n^{(n)},\widetilde{G}_n)+\Delta_2(\mathbf{X}^{(n)},\mathbf{X}^n,\widetilde{G}_n),$$

$$\text{where}\quad \Delta_1(\mathbf{X}^{(n)},\widetilde{G}_n^{(n)},\widetilde{G}_n)=\frac{2}{n}\sum_{i=1}^n\left|\mathcal{H}(X_i^n;\vartheta_0,\widetilde{G}_n^{(n)})-\mathcal{H}(X_i^n;\vartheta_0,\widetilde{G}_n)\right|,$$

$$\Delta_2(\mathbf{X}^{(n)},\mathbf{X}^n,\widetilde{G}_n)=\frac{2}{n}\sum_{i=1}^n\left|\mathcal{H}(X_i^n;\vartheta_0,\widetilde{G}_n)-\mathcal{H}(X_i;\vartheta_0,\widetilde{G}_n)\right|.$$

For $\Delta_1(\mathbf{X}^{(n)},\widetilde{G}_n^{(n)},\widetilde{G}_n)$, since $H(\cdot;\vartheta_0,G)=0$ and $\frac{\partial H}{\partial\mu}(\cdot;\vartheta_0,G)=2f(\cdot)$, we can write:

$$\begin{aligned}
\Delta_1(\mathbf{X}^{(n)},\widetilde{G}_n^{(n)},\widetilde{G}_n) \le{}& \frac{2}{n}\sum_{i=1}^n\left|H(X_i^n;\vartheta_0,\widetilde{G}_n^{(n)})-H(X_i^n;\vartheta_0,\widetilde{G}_n)\right|\\
&\times\left|\frac{\partial H}{\partial\mu}(X_i^n;\vartheta_0,\widetilde{G}_n^{(n)})-\frac{\partial H}{\partial\mu}(X_i^n;\vartheta_0,\widetilde{G}_n)\right|\\
&+\frac{2}{n}\sum_{i=1}^n\left|H(X_i^n;\vartheta_0,\widetilde{G}_n)-H(X_i^n;\vartheta_0,G)\right|\\
&\times\left|\frac{\partial H}{\partial\mu}(X_i^n;\vartheta_0,\widetilde{G}_n^{(n)})-\frac{\partial H}{\partial\mu}(X_i^n;\vartheta_0,\widetilde{G}_n)\right|\\
&+\frac{2}{n}\sum_{i=1}^n\left|\frac{\partial H}{\partial\mu}(X_i^n;\vartheta_0,\widetilde{G}_n)-2f(X_i^n)\right|\\
&\times\left|H(X_i^n;\vartheta_0,\widetilde{G}_n^{(n)})-H(X_i^n;\vartheta_0,\widetilde{G}_n)\right|\\
&+\frac{4}{n}\sum_{i=1}^n|f(X_i^n)|\times\left|H(X_i^n;\vartheta_0,\widetilde{G}_n^{(n)})-H(X_i^n;\vartheta_0,\widetilde{G}_n)\right|\\
\le{}& c_1\left(\|\widetilde{G}_n^{(n)}-\widetilde{G}_n\|_\infty+\|\widetilde{G}_n-G\|_\infty\right)\|\widetilde{g}_n^{(n)}-\widetilde{g}_n\|_\infty\\
&+c_2\left(\|\widetilde{g}_n-g\|_\infty+\|f\|_\infty\right)\|\widetilde{G}_n^{(n)}-\widetilde{G}_n\|_\infty\\
={}& O_{a.s.}\left(\frac{\delta_n^2}{h_n^3}\right)+O_{a.s.}\left(\frac{\delta_n}{h_n}\right).
\end{aligned}$$

For $\Delta_2(\mathbf{X}^{(n)}, \mathbf{X}^n, \widetilde{G}_n)$ let us notice first that for any $(x, x') \in \mathbb{R}^2$ we have:

$$\begin{aligned}
&\left|H(x;\vartheta_0,\widetilde{G}_n) - H(x';\vartheta_0,\widetilde{G}_n)\right| \\
&\leq \frac{1}{p_0}\left|[\widetilde{G}(x+\mu) - \widetilde{G}(x'-\mu)] + [\widetilde{G}(-x+\mu) - \widetilde{G}(-x'-\mu)]\right| \\
&+ \frac{1-p_0}{p_0}\left|[F_0(x+\mu) - F_0(x'-\mu)] + [F_0(-x+\mu) - F_0(-x'-\mu)]\right| \\
&\leq \frac{2}{p_0}(\|\widetilde{g}_n - g\|_\infty + \|g\|_\infty)|x-x'| + \frac{2(1-p_0)}{p_0}\|f_0\|_\infty|x-x'|, \qquad (4.4)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial H}{\partial \mu}(x;\vartheta_0,\widetilde{G}_n) - \frac{\partial H}{\partial \mu}(x';\vartheta_0,\widetilde{G}_n)\right| \\
&\leq \frac{1}{p_0}\left|[\widetilde{g}_n(x+\mu) - \widetilde{g}_n(x'-\mu)] + [\widetilde{g}_n(-x+\mu) - \widetilde{g}_n(-x'-\mu)]\right| \\
&+ \frac{1-p_0}{p_0}\left|[f_0(x+\mu) - f_0(x'-\mu)] + [f_0(-x+\mu) - f_0(-x'-\mu)]\right| \\
&\leq \frac{2}{p_0}(\|\widetilde{g}'_n - g'\|_\infty + \|g'\|_\infty)|x-x'| + \frac{2(1-p_0)}{p_0}\|f'_0\|_\infty|x-x'|. \qquad (4.5)
\end{aligned}$$

Using (4.4) and (4.5) we obtain

$$\begin{aligned}
\Delta_2(\mathbf{X}^{(n)}, \mathbf{X}^n, \widetilde{G}_n) \quad \leq \quad & \frac{2}{n}\sum_{i=1}^n \left|H(X_i^n;\vartheta_0,\widetilde{G}_n) - H(X_i;\vartheta_0,\widetilde{G}_n)\right| \\
& \times \left|\frac{\partial H}{\partial \mu}(X_i^n;\vartheta_0,\widetilde{G}_n) - \frac{\partial H}{\partial \mu}(X_i;\vartheta_0,\widetilde{G}_n)\right| \\
& + \frac{2}{n}\sum_{i=1}^n \left|H(X_i;\vartheta_0,\widetilde{G}_n) - H(X_i;\vartheta_0,G)\right| \\
& \times \left|\frac{\partial H}{\partial \mu}(X_i^n;\vartheta_0,\widetilde{G}_n) - \frac{\partial H}{\partial \mu}(X_i;\vartheta_0,\widetilde{G}_n)\right| \\
& + \frac{2}{n}\sum_{i=1}^n \left|\frac{\partial H}{\partial \mu}(X_i;\vartheta_0,\widetilde{G}_n) - 2f(X_i)\right| \\
& \times \left|H(X_i^n;\vartheta_0,\widetilde{G}_n) - H(X_i;\vartheta_0,\widetilde{G}_n)\right| \\
& + \frac{4}{n}\sum_{i=1}^n |f(X_i)| \times \left|H(X_i^n;\vartheta_0,\widetilde{G}_n) - H(X_i;\vartheta_0,\widetilde{G}_n)\right| \\
= \quad & O_{a.s.}(\delta_n^2) + O_{a.s.}(\delta_n^2) + O_{a.s.}(\delta_n||\widetilde{G}_n - G\|_\infty) + O_{a.s.}(\delta_n),
\end{aligned}$$

which by (4.1) concludes the proof for (ii). For the proof of result (iii) we use

the following decomposition at any point $\vartheta \in \Theta$:

$$\begin{aligned}
&\left\|\ddot{d}_n^{(n)}(\vartheta) - \ddot{d}_n(\vartheta)\right\| \\
&\leq \frac{2}{n}\sum_{k=1}^{n}\left\|H(X_i^{(n)};\vartheta,\widetilde{G}_n^{(n)})\ddot{H}(X_i^n;\vartheta,\widetilde{G}_n^{(n)}) - H(X_i;\vartheta,\widetilde{G}_n)\ddot{H}(X_i;\vartheta,\widetilde{G}_n)\right\| \\
&+ \frac{2}{n}\sum_{k=1}^{n}\left\|\dot{H}(X_i^{(n)};\vartheta,\widetilde{G}_n^{(n)})\dot{H}^T(X_i^{(n)};\vartheta,\widetilde{G}_n^{(n)}) - \dot{H}(X_i;\vartheta,\widetilde{G}_n)\dot{H}^T(X_i;\vartheta,\widetilde{G}_n)\right\| \\
&\leq \sum_{j=1}^{4} T_{j,1} + T_{j,2},
\end{aligned}$$

where for $j = 1,\ldots,4$, $T_{j,1}$ and $T_{j,2}$ are alternatively equal to

$$\begin{aligned}
&\frac{2}{n}\sum_{k=1}^{n}|H(X_i^{(n)};\vartheta,\widetilde{G}_n^{(n)})|\left\|\ddot{H}(X_i^n;\vartheta,\widetilde{G}_n^{(n)}) - \ddot{H}(X_i^n;\vartheta,\widetilde{G}_n)\right\| = O_{a.s.}\left(\frac{\delta_n}{h_n^3}\right) \\
&\frac{2}{n}\sum_{k=1}^{n}|H(X_i^{(n)};\vartheta,\widetilde{G}_n^{(n)})|\left\|\ddot{H}(X_i^n;\vartheta,\widetilde{G}_n) - \ddot{H}(X_i;\vartheta,\widetilde{G}_n)\right\| = O_{a.s.}\left(\delta_n\right) \\
&\frac{2}{n}\sum_{k=1}^{n}\left\|\ddot{H}(X_i;\vartheta,\widetilde{G}_n)\right\|\left|H(X_i^n\vartheta,\widetilde{G}_n^{(n)}) - H(X_i^n;\vartheta,\widetilde{G}_n)\right| = O_{a.s.}\left(\frac{\delta_n}{h_n}\right) \\
&\frac{2}{n}\sum_{k=1}^{n}\left\|\ddot{H}(X_i;\vartheta,\widetilde{G}_n)\right\|\left|H(X_i^n;\vartheta,\widetilde{G}_n) - H(X_i;\vartheta,\widetilde{G}_n)\right| = O_{a.s.}\left(\delta_n\right) \\
&\frac{2}{n}\sum_{k=1}^{n}\left\|\dot{H}(X_i^n;\vartheta,\widetilde{G}_n^{(n)})\right\|\left\|\dot{H}(X_i^n;\vartheta,\widetilde{G}_n^{(n)}) - \dot{H}(X_i^n;\vartheta,\widetilde{G}_n)\right\| = O_{a.s.}\left(\frac{\delta_n}{h_n^2}\right) \\
&\frac{2}{n}\sum_{k=1}^{n}\left\|\dot{H}(X_i;\vartheta,\widetilde{G}_n^{(n)})\right\|\left\|\dot{H}(X_i^n;\vartheta,\widetilde{G}_n) - \dot{H}(X_i;\vartheta,\widetilde{G}_n)\right\| = O_{a.s.}\left(\frac{\delta_n}{h_n^2}\right) \\
&\frac{2}{n}\sum_{k=1}^{n}\left\|\dot{H}(X_i^n;\vartheta,\widetilde{G}_n)\right\|\left\|\dot{H}(X_i^n;\vartheta,\widetilde{G}_n^{(n)}) - \dot{H}(X_i^n;\vartheta,\widetilde{G}_n)\right\| = O_{a.s.}\left(\frac{\delta_n}{h_n^2}\right) \\
&\frac{2}{n}\sum_{k=1}^{n}\left\|\dot{H}(X_i;\vartheta,\widetilde{G}_n)\right\|\left\|\dot{H}(X_i^n;\vartheta,\widetilde{G}_n) - \dot{H}(X_i;\vartheta,\widetilde{G}_n)\right\| = O_{a.s.}\left(\delta_n\right).
\end{aligned}$$

The above results come from painful but straightforward calculations. To explain briefly how we get these rates we can basically say that the first factors after the sum sign are always $O_{a.s.}(1)$ due to Silverman (1978) if they are $\widetilde{G}_n$ dependent and $O_{a.s.}(1+\delta_n/h_n^{1+k})$, where $k = 0,1,2$ denotes the order of derivation of H, if they are $\widetilde{G}_n^{(n)}$ dependent. Next, due to the mean value theorem, Silverman (1978) uniform consistency result on the kernel estimator and its derivatives and (2.3), the difference terms involving X_i^n and X_i based on $\widetilde{G}_n$ are all $O_{a.s.}(\delta_n)$. On the other hand due to approximation Lemma 2, the difference terms involving $\widetilde{G}_n^{(n)}$

and $\widetilde{G}_n$ located at the same argument value X_i^n are all $O_{a.s.}(\delta_n/h_n^{1+k})$ where $k = 0, 1, 2$ denotes the order of derivation of H. □

Theorem 2.
(i) Suppose that Conditions **(K)**, **(B)** *and* **(I)** *are satisfied,* Θ *is a compact subset of* $(0,1) \times \Phi_c$, G *is strictly increasing on* $\mathbb{R}$, F_0 *and* F *are twice continuously differentiable with second derivatives in* $L^1(\mathbb{R})$, *then we have* $\|\bar{\vartheta}_n - \vartheta_0\| = o_{a.s.}(n^{-1/4+\alpha})$ *for all* $\alpha > 0$.

(ii) Suppose in addition that Condition **(A)** *is satisfied, then we have*

$$\|\bar{\vartheta}_n^{(n)} - \vartheta_0\| = O_{a.s.}\left(\left(n^{-1/2+\alpha} + \delta_n/h_n^2\right)^{1/2-\delta}\right),$$

for all $\alpha > 0$ *and* $0 < \delta < 1/2$.

(iii) Under the conditions of (i), the estimator $\bar{\vartheta}_n = (\bar{p}_n, \bar{\mu}_n)$ *is asymptotically normally distributed:*

$$\sqrt{n}(\bar{p}_n - p_0, \bar{\mu}_n - \mu_0) \longrightarrow \mathcal{N}(0, \Sigma), \quad as \quad n \to +\infty,$$

where $\Sigma = \mathcal{I}(\vartheta_0)^{-1} J(\theta_0) \mathcal{I}(\vartheta_0)^{-1}$, *with*

$$\mathcal{I}(\vartheta_0) = \int_{\mathbb{R}} \dot{H}(x; \vartheta_0, G) \dot{H}^T(x; \vartheta_0, G) dG(x) > 0$$

and $J(\theta_0) = \mathbb{V}(H(X_1, \vartheta_0, G)\dot{H}(X_1, \vartheta_0, G))$.

(iv) Under the conditions of (ii), and if

$$\sqrt{n}\left(\frac{\delta_n^2}{h_n^3} + \frac{\delta_n}{h_n}\right) \longrightarrow 0, \quad and \quad \frac{\delta_n}{h_n^3} \longrightarrow 0, \quad as \quad n \to +\infty, \tag{4.6}$$

the estimator $\bar{\vartheta}_n^{(n)} = (\bar{p}_n^{(n)}, \bar{\mu}_n^{(n)})$ *associated with the triangular array* $(\mathbf{X}^{(n)})_{n\geq 1}$ *defined in (2.2) is asymptotically normally distributed:*

$$\sqrt{n}(\bar{p}_n^{(n)} - p_0, \bar{\mu}_n^{(n)} - \mu_0) \longrightarrow \mathcal{N}(0, \Sigma), \quad as \quad n \to +\infty.$$

Proof. The proofs of (i) and (iii) are detailed in Bordes and Vandekerkhove (2010). For the proof of result (ii) it is enough to notice that

$$\sup_{\vartheta\in\Theta} |d_n^{(n)} - d| \leq \sup_{\vartheta\in\Theta} |d_n^{(n)} - d_n| + \sup_{\vartheta\in\Theta} |d_n - d| = O_{a.s.}(\delta_n/h_n + n^{-1/2+\alpha}),$$

with $\alpha > 0$, and consider $\gamma_n = n^{-1/2+\alpha} + \delta_n/h_n$ along with $\eta_n = (n^{-1/2+\alpha} + \delta_n/h_n)^{1/2-\delta}$, with $\delta > 0$ in the proof of Theorem 3.1 of Bordes and Vandekerkhove (2010) . Doing so we insure that $\gamma_n = o(\eta_n^2)$ which concludes the proof of (ii).

For the proof of (iv) we consider the Taylor expansion of $\dot{d}_n^{(n)}$ around ϑ_0:

$$\ddot{d}_n^{(n)}(\vartheta^{*(n)}_n)\sqrt{n}(\bar{\vartheta}_n^{(n)} - \vartheta_0) = -\sqrt{n}\dot{d}_n^{(n)}(\vartheta_0) = -\sqrt{n}\dot{d}_n(\vartheta_0) + o_{a.s.}(1),$$

where $\vartheta^{*(n)}_n$ lies in the line segment with extremities $\bar{\vartheta}_n^{(n)}$ and ϑ_0, and $o_{a.s.}(1) = -\sqrt{n}(\dot{d}_n^{(n)}(\vartheta_0) - \dot{d}_n(\vartheta_0))$ according to Lemma 2 if $\sqrt{n}(\delta^2/h_n^3 + \delta/h_n) \to 0$ as $n \to +\infty$. Noticing now that:

$$\begin{aligned}\|\ddot{d}_n^{(n)}(\vartheta^{*(n)}_n) - \mathcal{I}(\vartheta_0)\| &\leq \|\ddot{d}_n^{(n)}(\vartheta^{*(n)}_n) - \ddot{d}_n(\vartheta^{*(n)}_n)\| + \|\ddot{d}(\vartheta^{*(n)}_n) - \mathcal{I}(\vartheta_0)\| \\ &\leq \sup_{\Theta}\|\ddot{d}_n^{(n)} - \ddot{d}_n\| + \|\ddot{d}(\vartheta^{*(n)}_n) - \mathcal{I}(\vartheta_0)\|,\end{aligned}$$

where the first term in the right hand side is a $o_{a.s.}(1)$ if $\delta_n/h_n^3 \to 0$ as $n \to +\infty$ according to Lemma 2 (iii) and the second term is also a $o_{a.s.}(1)$ according to (3.16) in the proof of Theorem 3.2 in Bordes and Vandekerkhove (2010). □

Remark 3. *Since the bandwidth rate recommended in Bordes and Vandekerkhove (2010, Remark 3.1) to satisfy Condition (B) is $n^{-1/4-\gamma}$, with $\gamma \in (0, 1/8)$ we observe that for this range of rates condition (4.6) is satisfied if:*

$$\frac{\delta_n^2}{n^{-5/4-3\gamma}} + \frac{\delta_n}{n^{-3/4-\gamma}} \longrightarrow 0, \quad \textit{and} \quad \frac{\delta_n}{n^{-3/4-3\gamma}} \longrightarrow 0, \quad \textit{as} \quad n \to +\infty,$$

which leads to consider $\delta_n = n^{-3/4-\xi}$ with $\xi > 3\gamma$.

Remark 4. *The conditions imposed in (4.6) do not look optimal to us but they provide for the first time, to the best of our knowledge, a framework for nonparametric contiguous alternatives in the parametric family testing problem. To improve these rates in the future we plan to carefully investigate the Donsker theorem associated with the empirical process $\mathbb{G}_n = \sqrt{n}(\widehat{G}_n^{(n)} - G^{(n)})$, where $\widehat{G}_n^{(n)}$ denotes the empirical cdf of a $G^{(n)}$-distributed generic triangular array $(X_1^n, \ldots, X_n^n)$, where $G^{(n)}$ converges "smoothly enough" towards a given cdf G and revisit the uniform almost sure convergence results of the kernel density estimate and its derivatives in Silverman (1978).*

5. Asymptotic behavior of the MLE

In this section we propose to derive the asymptotic covariance matrix involved in the Central Limit Theorem associated with maximum likelihood estimator for the Gaussian case, that is when f belongs to $\mathcal{G}$ the set of normal densities $f_{(\mu,s)}$ with mean μ and variance $\theta = s$,. Let us denote by $g_\phi(x) = (1-p)f_{(0,1)}(x) + pf_{(\mu,s)}(x)$ where $\phi = (\phi_1, \phi_2, \phi_3) = (p, \mu, s) \in (0,1) \times \Lambda$ and $\ell_\phi(x) := \ln(g_\phi(x))$. We now define the gradient of $\ell_\phi(x)$:

$$\dot{\ell}_\phi(x) = \left(\frac{\partial}{\partial\phi_1}\ell_\phi(x), \frac{\partial}{\partial\phi_2}\ell_\phi(x), \frac{\partial}{\partial\phi_3}\ell_\phi(x)\right)^T.$$

For simplicity matters we denote $\dot{f}^{\phi_i}_{(\mu,s)}(x) := \frac{\partial}{\partial\phi_i} f_{(\mu,s)}(x)$, $i = 1, 2, 3$. We then obtain

$$\frac{\partial}{\partial\phi_1}\ell_\phi(x) = \frac{-f_{(0,1)}(x) + f_{(\mu,s)}(x)}{g_\phi(x)}$$
$$\frac{\partial}{\partial\phi_2}\ell_\phi(x) = \frac{p\dot{f}^{\mu}_{(\mu,s)}(x)}{g_\phi(x)}, \quad \text{with} \quad \dot{f}^{\mu}_{(\mu,s)}(x) = \frac{x-\mu}{s} f_{(\mu,s)}(x)$$
$$\frac{\partial}{\partial\phi_3}\ell_\phi(x) = \frac{p\dot{f}^{s}_{(\mu,s)}(x)}{g_\phi(x)}, \quad \text{with} \quad \dot{f}^{s}_{(\mu,s)}(x) = \left[-\frac{1}{2s} + \frac{(x-\mu)^2}{2s^2}\right] f_{(\mu,s)}(x).$$

The Hessian matrix of $\ell_\phi(x)$ is denoted $\ddot{\ell}_\phi(x) = \left(\frac{\partial^2}{\partial\phi_i\partial\phi_j}\ell_\phi(x)\right)_{1\le i\le j\le 3}$ with:

$$\frac{\partial^2}{\partial^2\phi_1}\ell_\phi(x) = -\frac{(-f_{(0,1)}(x) + f_{(\mu,s)})^2(x)}{g^2_\phi(x)}$$
$$\frac{\partial^2}{\partial^2\phi_2}\ell_\phi(x) = p\frac{\ddot{f}^{\mu}_{(\mu,s)}(x)}{g_\phi(x)} - \left(p\frac{\dot{f}^{\mu}_{(\mu,s)}(x)}{g_\phi(x)}\right)^2$$
$$\frac{\partial^2}{\partial^2\phi_3}\ell_\phi(x) = p\frac{\ddot{f}^{s}_{(\mu,s)}(x)}{g_\phi(x)} - \left(p\frac{\dot{f}^{s}_{(\mu,s)}(x)}{g_\phi(x)}\right)^2,$$

and

$$\ddot{f}^{\mu}_{(\mu,s)}(x) = -\frac{1}{s}f_{(\mu,s)}(x) + \frac{x-\mu}{s}\dot{f}^{\mu}_{(\mu,s)}(x) = -\frac{1}{s}f_{(\mu,s)}(x) + \left(\frac{x-\mu}{s}\right)^2 f_{(\mu,s)}(x)$$
$$\ddot{f}^{s}_{(\mu,s)}(x) = \left[\frac{1}{2s^2} - \frac{(x-\mu)^2}{s^3}\right] f_{(\mu,s)}(x) + \left[-\frac{1}{2s} + \frac{(x-\mu)^2}{2s^2}\right]\dot{f}^{s}_{(\mu,s)}(x)$$

$$\begin{aligned}\frac{\partial^2}{\partial\phi_1\partial\phi_2}\ell_\phi(x) &= \frac{\partial^2}{\partial\phi_2\partial\phi_1}\ell_\phi(x)\\ &= \frac{\dot{f}^{\mu}_{(\mu,s)}(x)}{g_\phi(x)} - p\frac{(-f_{(0,1)}(x) + f_{(\mu,s)}(x))\dot{f}^{\mu}_{(\mu,s)}(x)}{g^2_\phi(x)}\\ \frac{\partial^2}{\partial\phi_1\partial\phi_3}\ell_\phi(x) &= \frac{\partial^2}{\partial\phi_3\partial\phi_1}\ell_\phi(x)\\ &= \frac{\dot{f}^{s}_{(\mu,s)}(x)}{g_\phi(x)} - p\frac{(-f_{(0,1)}(x) + f_{(\mu,s)}(x))\dot{f}^{s}_{(\mu,s)}(x)}{g^2_\phi(x)}\end{aligned}$$

$$\frac{\partial^2}{\partial\phi_2\partial\phi_3}\ell_\phi(x) = \frac{\partial^2}{\partial\phi_3\partial\phi_2}\ell_\phi(x) = \frac{p(x-\mu)}{s^2} \times \frac{\left[-f_{(\mu,s)}(x) + s\dot{f}^s_{(\mu,s)}(x)\right]}{g_\phi(x)} - \frac{p^2(x-\mu)}{s} \times \frac{f_{(\mu,s)}(x)\dot{f}^s_{(\mu,s)}(x)}{g^2_\phi(x)}.$$

Given the above expressions we can derive under standard conditions, see van der Vaart (1998, p.63), the basic asymptotic normality of the MLE:

$$\sqrt{n}(\widehat{p}_n - p_0, \widehat{\mu}_n - \mu_0, \widehat{s}_n - s_0) \longrightarrow \mathcal{N}(0_{\mathbb{R}^3}, A(\phi_0)^{-1}B(\phi_0)A(\phi_0)^{-1}),$$

as $n \to +\infty$, where

$$A(\phi_0) = \mathbb{E}\left(\ddot{\ell}_{\phi_0}(X_1)\right) \quad \text{and} \quad B(\phi_0) = \mathbb{E}\left(\dot{\ell}_{\phi_0}(X_1)\dot{\ell}^T_{\phi_0}(X_1)\right)$$

are respectively consistently estimated by

$$\widehat{A}_n = \frac{1}{n}\sum_{i=1}^n \ddot{\ell}_{\widehat{\phi}_n}(X_i) \quad \text{and} \quad \widehat{B}_n = \frac{1}{n}\sum_{i=1}^n \dot{\ell}_{\widehat{\phi}_n}(X_i)\dot{\ell}^T_{\widehat{\phi}_n}(X_i).$$

6. Graph for comparison between the maximum likelihood and the semiparametric estimator

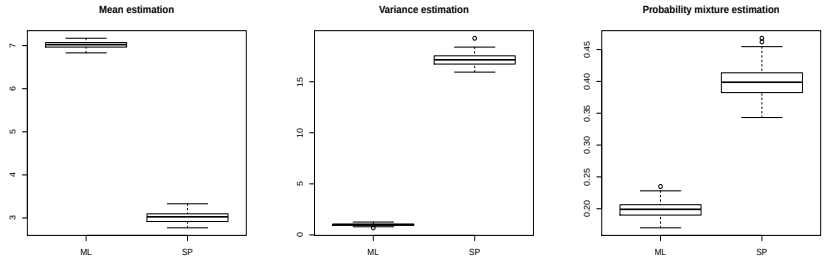

Fig 1: Boxplot for maximum likelihood and semiparametric estimators of m, s, p when $n = 1000$, under the mean deviation trap effect for $\mu = 3$ and $a = 4$, based on 200 repetitions.

7. Graph of the empirical Level

8. Graphs of the alternatives considered in Section 8.2

Row 1: 1-shifted Student $t(3)$ alternative distribution (plain) and a null-type Gaussian distribution with similar parameters $\mathcal{N}(1, 3)$ (dashed).

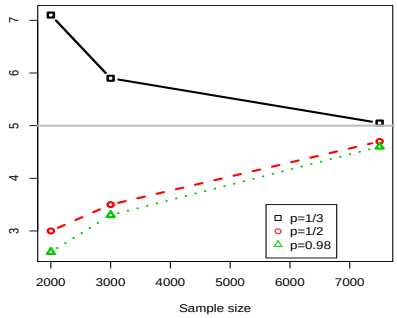

Fig 2: Empirical levels for parameter values $(p, \mu, s) = (1/3, 1.5, 1)$ ($\square$), $(p, \mu, s) = (0.5, 1.5, 2)$ ($\circ$) and $(p, \mu, s) = (0.98, -0.15, 0.8)$ ($\triangle$) with sample sizes $n = 2000, 3000, 7500$.

Row 2: 1-shifted Student $t(10)$ alternative distribution (plain) and a null-type Gaussian distribution with similar parameters $\mathcal{N}(1, 1.25)$ (dashed).
Row 3: $\mathcal{L}(1, 1)$ Laplace distribution (plain) and a null-type Gaussian component with similar parameters $\mathcal{N}(1, 2)$ (dashed).
Row 4: $\mathcal{L}(1, 2)$ Laplace distribution (plain) and a null-type Gaussian component with similar parameters $\mathcal{N}(1, 8)$ (dashed).
Columns 1, 2, 3 correspond respectively to $p = 1/3,\ 1/2,\ 0.98$.

References

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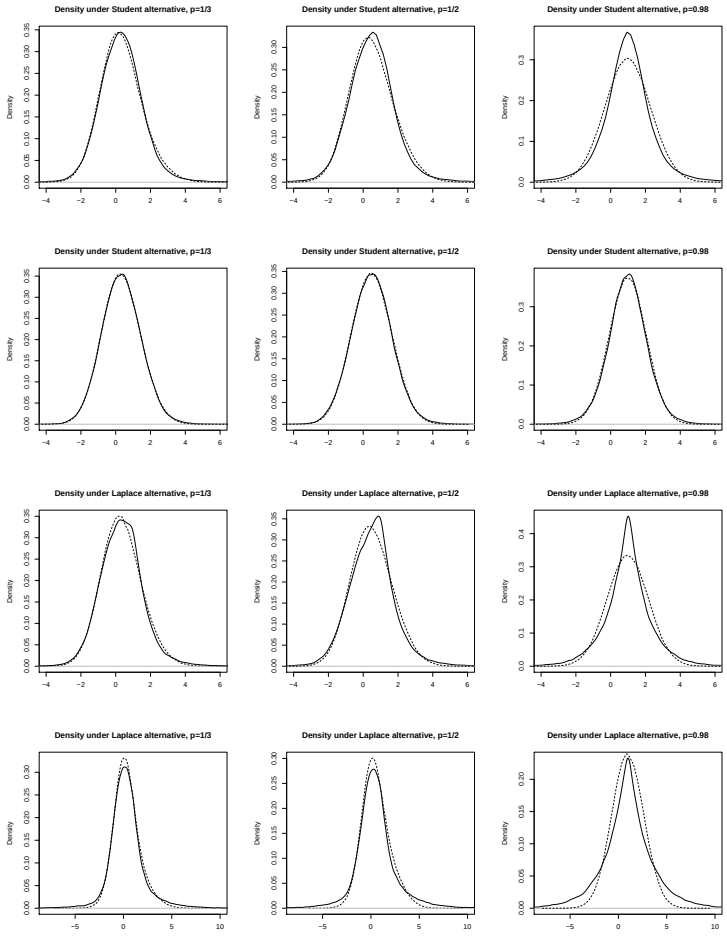

Fig 3: Plot of the graph of g in model (1.1) under several conditions.