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# Conformal higher-spin gravity: linearized spectrum = symmetry algebra

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**ABSTRACT:** The linearized spectrum and the algebra of global symmetries of conformal higher-spin gravity decompose into infinitely many representations of the conformal algebra. Their characters involve divergent sums over spins. We propose a suitable regularization adapted to their evaluation and observe that their characters are actually equal. This result holds in the case of type-A and type-B (and their higher-depth generalizations) theories and confirms previous observations on a remarkable rearrangement of dynamical degrees of freedom in conformal higher-spin gravity after regularization.

**KEYWORDS:** Conformal Field Theory, Higher Spin Gravity, Higher Spin Symmetry

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## 1 Introduction

Whether Einstein’s gravity can be extended to a theory with larger symmetries than the usual diffeomorphisms is a challenging question and attempts at answering it have brought many interesting results not only to the physics of gravitation but also to mathematics. An old example is provided by the fruitful exchanges between Weyl’s theory of gravity [1] and conformal geometry. Higher-spin gravity is a much younger example and its underlying geometry remains somewhat elusive. However, its deep connections with important topics

of modern mathematics (such as deformation quantization,<sup>1</sup> jet bundles, tractors, etc) might keep some surprises in store.

What we shall study in this paper is the theory which combines the conformal and higher-spin extensions of gravity, namely conformal higher-spin (CHS) gravity. The study of CHS gauge fields was started by Fradkin and Tseytlin in [10], where the higher-spin generalization of the linearized conformal gravity action was obtained. Subsequently, the cubic interactions of CHS fields were analyzed by Fradkin and Linetsky in a frame-like formulation [11, 12] (see also [13, 14] for early works on the CHS superalgebra) where the quartic order analysis nevertheless becomes difficult, similarly to the Fradkin-Vasiliev construction of the massless higher-spin (MHS) interactions [15, 16]. In three dimensions however, the full non-linear action could be obtained in the Chern-Simons formulation [17–19]. The unfolded formulation on which Vasiliev’s equations are based can be also applied to free CHS gauge fields (see e.g. [20–26]).

In fact, one of the simplest ways to understand CHS gravity in even dimensions is viewing it as the logarithmically divergent part of the effective action of a conformal scalar field in the background of all higher-spin fields [27–29], which can be computed perturbatively in the weak field expansion. In the context of higher-spin holography, the CHS action can be again related to (the logarithmically divergent part of) the on-shell bulk action of the MHS gravity in one higher dimension with standard boundary conditions. The CHS field equations can also be obtained as the obstruction in the power series expansion (à la Fefferman-Graham) of the solutions to the bulk MHS field equations [30, 31]. This holographic picture makes it clear that the global symmetry of the CHS theory matches that of MHS theory (see e.g. [29]). This scenario actually extends beyond the case of the conformal scalar field, the construction of CHS theories via the effective action ensures that there is a one-to-one correspondence between free vector model conformal field theories (CFTs) and CHS theories,<sup>2</sup> while the higher-spin holographic duality ensures that there is a one-to-one correspondence between them and MHS theories. Accordingly, we will denote the zoo of CHS theories following the by-now standard nomenclature (type A, B, etc) for MHS theories.<sup>3</sup> The type-A, -B and -C higher-spin gravities on  $\text{AdS}_{d+1}$  are dual to the (singlet sector of) vector model  $\text{CFT}_d$  of free scalars, spinors and  $\frac{d-2}{2}$ -forms, respectively.

CHS gravity shows several interesting properties. In particular, it shares the main drawback and virtues of Weyl’s gravity. On the one hand, it is a higher-derivative theory so it is non-unitary. On the other hand, it is an interacting theory with a massless

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<sup>1</sup>The close relationships between higher-spin gravity and deformation quantization was observed in various instances over the last three decades. It was first observed by Vasiliev in his realisation of higher-spin algebras in terms of star products [2]. Aside from this seminal paper, other examples are the relation between the quantized hyperboloid and the deformed oscillator algebra which is instrumental in Vasiliev nonlinear equations [3, 4], the connection of the latter equations with Fedosov quantization [5–7] and with some formality theorems [8, 9], etc.

<sup>2</sup>See e.g. [32, 33], as well as [34, 35] where the effective actions have been calculated in the world-line formulation.

<sup>3</sup>In the sequel, the somewhat ambiguous terms “CHS theory” or “CHS gravity” will stand either for the specific example of type-A CHS gravity or, on the contrary, for generic CHS theories. The context should make it clear.

spin-two field in its spectrum whose interactions with the other fields include the minimal coupling, so it is a theory of gravity. Moreover, its global symmetries include conformal symmetry thus it is scale-invariant at the classical level. Consequently, the absence of gauge anomalies (in particular, Weyl anomalies and their higher-spin versions) would ensure that CHS gravity is UV-finite (see e.g. [10] for the original arguments). In fact, an advantage of CHS gravity over its spin-two counterpart is that it might be anomaly-free, as suggested by some preliminary tests. From a quantum gravity perspective, this motivates the study of CHS gravity as an interesting toy model of a UV-finite (albeit non-unitary) theory of quantum gravity. The spectra of higher-spin gravity theories (CHS and MHS) involve infinite towers of gauge fields. The corresponding infinite number of contributions require some regularization which opens the possibility that the collective behaviour of such infinite collection of fields may be much softer than their individual behaviour. In fact, due to the presence of huge symmetries, such systems have been observed to exhibit remarkable cancellation properties in their scattering amplitudes, one-loop anomalies, partition functions, etc. From a higher-spin theory perspective, CHS gravity can be viewed as a theory of interacting massless and partially-massless fields of all depths and all spins. It has the nice feature of being perturbatively local and of admitting a relatively simple metric-like formulation. Moreover, it appears to possess similar features to MHS theory. In fact, non-standard MHS gravity could possibly arise from CHS gravity at tree level by imposing suitable boundary conditions around an anti de Sitter (AdS) background, in the same way that Einstein's gravity is related to Weyl's gravity [36, 37].

The linear equation for the conformal spin- $s$  field is local in even  $d$  dimensions and has  $2s + d - 4$  derivatives. Generically, the corresponding kinetic operator is higher-derivative and can be factorized into the ones for massless and partially-massless fields of the same spin (around  $\text{AdS}_d$ ) as was suggested in [38–41]<sup>4</sup> (see also [45] for related discussions). This factorization allows us to compute easily its Weyl anomaly and it was found that the  $a$ -anomaly coefficient vanishes [38, 46]. The  $c$ -anomaly would also vanish assuming a factorized form of the equation in the presence of non-vanishing curvature [38]. Unfortunately, such a factorization does not actually happen in general, as was shown in [40]. As a consequence, to determine whether the  $c$ -anomaly vanishes or not, one needs the linear conformal spin- $s$  equation in an arbitrary gravitational background, which has been the subject of several recent papers [47, 48] (see also [49, 50] for other strategies).

Scattering amplitudes of CHS gravity also exhibit surprising features: four-point scattering amplitudes of external scalar fields, Maxwell fields and Weyl gravitons mediated by the infinite tower of CHS fields vanish [51, 52]. There exists yet another approach to CHS theory using the twistor formalism [53]. The twistor CHS theory has the same linear action as the conventional CHS, but it is not clear that the two CHS theories are the same at the full interacting level. Recently, scattering amplitudes of twistor CHS theory have been studied in [54, 55] (see also [56] in the context of conformal gravity).

In this paper, we will focus on the spectrum and symmetry algebra of CHS gravity. More precisely, we will show that its total space of on-shell one-particle states in even

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<sup>4</sup>The factorization in the spin-two case was observed already in [42–44].

dimension  $d$  carries the same (reducible) representation of the conformal algebra  $\mathfrak{so}(2, d)$  as its global symmetry algebra (spanning the adjoint representation of the higher-spin algebra), i.e.

$$\text{On-shell CHS} = \text{Higher-spin Algebra} . \quad (1.1)$$

Strictly speaking, we will sum the characters of the CHS fields making up the spectrum of CHS gravity and compare them to the character of the corresponding conformal higher-spin algebra, which is isomorphic to the higher-spin algebra of the associated MHS gravity. In such derivation, it is crucial to use the identity

$$\chi_{\text{KT}}(q, \mathbf{x}) = \chi_{\text{CHS}}(q, \mathbf{x}) + \chi_{(\text{P})\text{M}}(q, \mathbf{x}) - \chi_{(\text{P})\text{M}}(q^{-1}, \mathbf{x}) , \quad (1.2)$$

where  $\chi_{\text{KT}}$ ,  $\chi_{\text{CHS}}$  and  $\chi_{(\text{P})\text{M}}$  designate the  $\mathfrak{so}(2, d)$  characters of a Killing tensor and its associated CHS and (partially-)massless field, respectively. Here,  $q$  and  $\mathbf{x} = (x_1, \dots, x_{\frac{d}{2}})$  are related to the temperature and the chemical potentials for the angular momenta in the usual way (see (2.11) in the next section). The identity (1.2) relates the characters of a given CHS gauge field on flat spacetime  $M_d$ , of the associated partially-massless field in  $\text{AdS}_{d+1}$  as well as the Killing tensor of the latter.<sup>5</sup> This identity was first derived for the massless totally symmetric case in [64] (see also [46] for related discussion), then further explored in [65]. The generalized version of the identity to partially-massless as well as mixed-symmetry cases is derived in appendix A. The idea of the proof of (1.1) is to perform the sum of (1.2) over all fields in the spectrum of CHS gravity. After the sum, the terms corresponding to the last two terms in (1.2) cancel each other because of parity properties of the corresponding characters. A possible physical interpretation of the result (1.1) might be that the dynamical degrees of freedom of CHS gravity reorganize into its asymptotic symmetries. In this sense, CHS gravity is somehow “topological”. This interpretation resonates with similar observations on the scattering amplitudes [51, 52] and the one-loop partition functions on various backgrounds [65, 66] of type-A CHS gravity.

The paper is organized as follows. In section 2, we start by reviewing the field theoretical definition of type-A CHS theory as spelled out in [29, 67], then move on to the derivation of the property (1.1). In section 3, we extend property (1.1) to a couple of classes of higher-depth CHS theories, whose spectrum are made up of the CHS fields associated to the partially-massless fields making the spectrum of the type- $A_\ell$  [31] and type- $B_\ell$  theories. We start by describing the field theoretical realization of the type- $A_\ell$  CHS theory in section 3.1 and move on to show that (1.1) holds for the type- $A_\ell$  in section 3.2 and for the type- $B_\ell$  in section 3.3. We conclude the paper in section 4 by discussing our main result and commenting on its implication for the computation of the thermal partition function and the free energy on AdS background of this theory. In particular, we point out that turning on chemical potentials for the  $\mathfrak{so}(d)$  angular momenta provides an alternative

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<sup>5</sup>Let us recall that any partially-massless field  $\varphi_{\mathbb{Y}}$  in  $\text{AdS}_{d+1}$  with minimal energy  $\Delta_{\text{PM}}$  and spin  $\mathbb{Y}$  is associated with a conformal field  $\phi_{\mathbb{Y}}$  on  $M_d$  with conformal weight  $d - \Delta_{\text{PM}}$  and same spin  $\mathbb{Y}$  (which is used as a source for the dual operator  $J_{\Delta, \mathbb{Y}}$  in the generating functional of correlation functions of the dual CFT). Moreover, the Killing tensors of  $\varphi_{\mathbb{Y}}$  in  $\text{AdS}_{d+1}$  are isomorphic to the conformal Killing tensors of  $\phi_{\mathbb{Y}}$  on  $M_d$ . See [21, 57] for the analysis in full generality, as well as [39, 58–63].

regularization of the sum over the infinite tower of higher-spin fields, which we also compare to the previously used regularization in the literature. Finally, appendix A contains the derivation of identity (1.2), together with a more detailed description of the various modules of importance and their field theoretical interpretations in  $\text{AdS}_{d+1}/\text{CFT}_d$ . In appendix B we detail the  $d = 2$  case as a toy model, while appendix C contains the branching rule of the on-shell (totally-symmetric) CHS field module. In appendix D, we provide a short review of nonlinear CHS theory. In particular, we reexamine the formal operator approach of Segal [28] and provide a heuristic argument supporting the “topological” nature of CHS gravity.

## 2 Type-A conformal higher-spin gravity

### 2.1 Field theory of type-A conformal higher-spin gravity

The free theory of the conformal spin- $s$  field in even  $d$ -dimensional Minkowski<sup>6</sup> spacetime  $M_d$ , is described by the local action

$$S_{\text{FT}}[h_s] = \int_{M_d} d^d x h_s \mathbb{P}_{\text{TT}}^s \square^{s+\frac{d-4}{2}} h_s, \quad (2.1)$$

where  $h_s$  is a totally symmetric rank- $s$  tensor and  $\mathbb{P}_{\text{TT}}^s$  is the projector to transverse and traceless symmetric tensors of rank  $s$ . The differential operator  $\square^{s+\frac{d-4}{2}}$  compensates the non-locality of  $\mathbb{P}_{\text{TT}}^s$  so that the action is local and conformally invariant.

- The field  $h_s$ , referred to as *off-shell Fradkin-Tseytlin* (FT) field is a symmetric rank- $s$  field of conformal weight  $\Delta_{h_s} = 2 - s$ . It is also called as the *shadow* field. Due to the projector  $\mathbb{P}_{\text{TT}}^s$ , the action has the gauge symmetry

$$h_s \sim h_s + \partial \xi_{s-1} + \eta \sigma_{s-2}, \quad (2.2)$$

with  $\xi_{s-1}$  a rank- $(s-1)$  symmetric tensor,  $\eta$  the Minkowski metric and  $\sigma_{s-2}$  a rank- $(s-2)$  symmetric tensor. Here, we used the schematic notation where all the indices are implicit.

- The *conformal Killing tensor* is the set of parameters  $(\xi_{s-1}, \sigma_{s-2})$  satisfying the conformal Killing equation

$$\partial \xi_{s-1} + \eta \sigma_{s-2} = 0, \quad (2.3)$$

i.e. the gauge parameters leaving the FT field inert under (2.2).

- The equation of motion of the action (2.1) reads

$$\mathbb{P}_{\text{TT}}^s \square^{s+\frac{d-4}{2}} h_s \approx 0. \quad (2.4)$$

This equation is referred to as the spin- $s$  *Bach equation* where equalities that only hold on-shell will be denoted by the weak equality symbol  $\approx$  hereafter. An equivalence class (2.2) of fields  $h_s$  obeying this equation will be referred to as *on-shell Fradkin-Tseytlin* field.

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<sup>6</sup>Whenever global issues would be relevant,  $M_d$  should stand for its conformal compactification  $S^1 \times S^{d-1}$ .

- The action (2.1) can be also rewritten, after integrating by part, as

$$S_{\text{FT}}[h_s] = (-1)^s \int_{M_d} d^d x \, C_{s,s} \square^{\frac{d-4}{2}} C_{s,s}, \quad (2.5)$$

where  $C_{s,s} = \mathbb{P}_T^{s,s} \partial^s h_s$  is the (generalized) *Weyl tensor* of the FT field (here  $\mathbb{P}_T^{s,s}$  denotes the traceless projector onto the two-row Young diagram displayed in (2.6) below). It is a traceless tensor with the symmetry of a rectangular two-row Young diagram,

$$C_{s,s} \underset{\mathfrak{so}(1,d-1)}{\sim} \begin{array}{|c|} \hline s \\ \hline s \\ \hline \end{array}. \quad (2.6)$$

The Weyl tensor  $C_{s,s}$  is a primary field with conformal weight  $\Delta_{C_{s,s}} = 2$  and is invariant under the gauge transformations (2.2). In particular, for  $s = 2$  it corresponds to the linearized Weyl tensor.

So far we have considered the free theory of conformal spin- $s$  gauge fields. An interacting theory of CHS gauge fields can be constructed from the effective (also called “induced”) action of a free scalar field [27, 29, 34, 67] in a higher-spin background. Starting from the action of a free complex scalar field  $\phi$  coupled to higher-spin sources  $h_s$  via traceless conserved currents  $J_s = \bar{\phi} \partial^s \phi$  (where  $\bar{\phi}$  denotes the complex conjugate of  $\phi$ ):

$$S[\phi; \{h_s\}_{s \in \mathbb{N}}] = \int_{M_d} d^d x \left( \bar{\phi} \square \phi + \sum_{s=0}^{\infty} J_s h_s \right), \quad (2.7)$$

we obtain the effective action

$$e^{-W_{\Lambda}[\{h_s\}_{s \in \mathbb{N}}]} = \int_{\Lambda} \mathcal{D}\phi e^{-S[\phi; \{h_s\}_{s \in \mathbb{N}}]}, \quad (2.8)$$

where  $\Lambda$  is the ultraviolet (UV) cut-off. The logarithmically divergent part  $W_{\log}$  of the effective action  $W_{\Lambda}$  is a local and nonlinear functional of the shadow fields  $h_s$ . Moreover it can be shown (see e.g. [67]) to reproduce the free action (2.1) at the quadratic order:

$$W_{\log}[\{h_s\}_{s \in \mathbb{N}}] = \sum_{s=0}^{\infty} S_{\text{FT}}[h_s] + \mathcal{O}(h_s^3), \quad (2.9)$$

and contains also the interaction terms  $\mathcal{O}(h_s^3)$ , which can be perturbatively calculated — see for instance [29, 48, 67]. Therefore,  $W_{\log}$  can be regarded as an action of interacting FT fields, up to the introduction of a dimensionless coupling constant  $\kappa$ :  $S_{\text{CHS}} = \kappa W_{\log}$ . Note that the  $s = 0$  term in (2.9) corresponds to the conformal scalar with  $d - 4$  derivatives. Hence, for  $d = 4$  the scalar field becomes auxiliary and drops out from the spectrum of CHS.

According to the AdS/CFT correspondence,  $W_{\log}$  should be equal to its AdS counterpart, that is, the logarithmically divergent part of the on-shell AdS action in the limit where the location of the boundary is pushed to infinity. In this way, the type-A CHS theory is linked to the type-A MHS theory. For instance, the free action  $S_{\text{FT}}[h_s]$  is related to the Fronsdal action in  $\text{AdS}_{d+1}$ . Schematically, we have

$$S_{\text{Fronsdal}}[\Phi_s = \mathcal{K} h_s] = \log R \, S_{\text{FT}}[h_s] + \text{regular or polynomially divergent terms}, \quad (2.10)$$



where  $\mathcal{K}$  is the boundary-to-bulk propagator of Fronsdal field  $\Phi_s$  and  $R$  is the distance from the center to the boundary of the regularized AdS space (see e.g. [68–71]). The interacting theory of CHS fields enjoys a non-Abelian global symmetry generated by the conformal Killing tensors. From the effective action point of view, this CHS symmetry is nothing but the maximal symmetry of the free scalar field. Hence, it coincides with the type-A MHS symmetry in  $\text{AdS}_{d+1}$ .

Suppose that we are interested in the quantum properties of CHS theory such as the one-loop free energy. Then, this quantity is closely related to the  $\mathfrak{so}(2, d)$  character of the free CHS theory. The latter character can be viewed as a single-particle partition function on  $S^1 \times S^{d-1}$  (the conformal boundary of thermal  $\text{AdS}_{d+1}$ ) where we turn on, besides the temperature  $\beta^{-1}$ , the chemical potentials  $\Omega_i$  corresponding to the angular momenta [72, 73], upon the identification,

$$q = e^{-\beta}, \quad x_i = e^{\beta \Omega_i}. \quad (2.11)$$

In order to compute the character of the free CHS theory, we need to determine first the characters of the individual  $\mathfrak{so}(2, d)$ -modules relevant in CHS theory.

## 2.2 Relevant modules

Let us start by reviewing the relevant modules in the free CHS theory (see appendix F of [64]). As usual in a conformal field theory, a primary field (together with its descendants) is described<sup>7</sup> by a (generalized) Verma module  $\mathcal{V}(\Delta; \mathbb{Y})$ , which is induced from the  $\mathfrak{so}(2) \oplus \mathfrak{so}(d)$  module with lowest weight  $[\Delta; \mathbb{Y}]$ . Here  $\Delta$  is a real number corresponding to the conformal weight, and  $\mathbb{Y} := (s_1, \dots, s_r)$  is an integral dominant  $\mathfrak{so}(d)$ -weight, i.e.

$$\begin{aligned} s_1 \geq s_2 \geq \dots \geq s_{r-1} \geq |s_r|, & \quad [d = 2r] \\ s_1 \geq s_2 \geq \dots \geq s_r \geq 0, & \quad [d = 2r + 1] \end{aligned} \quad (2.12)$$

where  $s_1, \dots, s_r$  are either all integers or all half-integers, corresponding to the spin, and  $r$  is the rank of  $\mathfrak{so}(d)$ . The irreducible module obtained as a quotient of the Verma module  $\mathcal{V}(\Delta; \mathbb{Y})$  by its maximal submodule will be denoted  $\mathcal{D}(\Delta; \mathbb{Y})$ .

**Rac.** We introduce first the  $\mathfrak{so}(2, d)$ -module called Rac of order- $\ell$  (or  $\ell$ -lineton) describing the conformal scalar field with  $\ell$ th power of the wave operator as kinetic operator. Off-shell, a scalar field  $\phi$  of conformal weight  $\frac{d-2\ell}{2}$  is described by the Verma module  $\mathcal{V}(\frac{d-2\ell}{2}, \mathbf{0})$  where  $\mathbf{0}$  stands for the trivial  $\mathfrak{so}(d)$ -weight. On-shell, such a scalar field obeying to the polywave equation,

$$\square^\ell \phi \approx 0, \quad (2.13)$$

is described by the irreducible<sup>8</sup> module

$$\text{Rac}_\ell = \frac{\mathcal{V}(\frac{d-2\ell}{2}, \mathbf{0})}{\mathcal{V}(\frac{d+2\ell}{2}, \mathbf{0})}. \quad (2.14)$$

<sup>7</sup>Strictly speaking, a generalized Verma module is the algebraic dual of the infinite jet space at a point of a primary field [74].

<sup>8</sup>Notice that, strictly speaking, when  $d$  is even the module (2.14) is irreducible if only if  $\ell < \frac{d}{2}$ .

This module is unitarizable for  $\ell = 1$ , in which case it is simply called “Rac” (or scalar singleton). The value  $\ell = \frac{d-4}{2}$  for the higher-order Rac module gives precisely the  $s = 0$  part of the CHS theory spectrum. Notice that the scalar FT field is absent in  $d = 4$  and is unitary only for  $d = 6$ , in which case it corresponds to the usual Rac singleton as  $\ell = \frac{d-4}{2} = 1$ . When  $d \geq 8$  however, the order of the scalar singleton is greater than 1 and therefore this field is non-unitary, as the on-shell FT fields.

**Conserved current.** Let us now introduce the module describing the conserved spin- $s$  current  $J_s$ :

$$\partial \cdot J_s \approx 0. \quad (2.15)$$

i.e. a totally symmetric, traceless and divergenceless rank- $s$  tensor. This current corresponds to the module  $\mathcal{D}(s + d - 2; (s))$  defined as the quotient

$$\mathcal{D}(s + d - 2; (s)) = \frac{\mathcal{V}(s + d - 2; (s))}{\mathcal{V}(s + d - 1; (s - 1))}. \quad (2.16)$$

The module  $\mathcal{V}(s + d - 2; (s))$  contains symmetric and traceless rank- $s$  tensors with conformal weight  $\Delta = s + d - 2$  whereas the module  $\mathcal{V}(s + d - 1; (s - 1))$  is isomorphic to the divergence of such tensors. As a consequence, modding out this submodule is equivalent to imposing the conservation law (2.15). From the  $\text{AdS}_{d+1}$  perspectives, the *unitarizable* module (2.16) corresponds to the Hilbert space of the massless spin- $s$  field with Dirichlet boundary conditions, i.e. the *normalizable* solutions thereof.

**Off-shell Fradkin-Tseytlin field.** The *off-shell* FT (or shadow) field corresponds to the module  $\mathcal{S}(2 - s; (s))$  whose field-theoretical realization is a totally symmetric rank- $s$  tensor field  $h_s$  quotiented by the gauge symmetries (2.2). The case  $s = 0$  is somewhat degenerate: the off-shell scalar FT field is simply  $\mathcal{S}(2; \mathbf{0}) = \mathcal{V}(2; \mathbf{0})$ . The precise group-theoretical description of this definition of the shadow field in terms of Verma modules remains somewhat elusive. Notice that this module is not a (quotient of) Verma module(s) but is rather related to the contragredient thereof. Indeed, classical field-theoretical terms it is the dual of the conserved current  $J_s$  with respect to the inner product  $\int d^d x h_s J_s$  in the path integral. Equivalently, in CFT terms it is the algebraic dual of the conserved current  $J_s$  with respect to the identity two-point function  $\langle h_s J_{s'} \rangle = \delta_{ss'}$ .<sup>9</sup>

Fortunately, the equivalent field-theoretical description of the off-shell FT field  $h_s$  in terms of the gauge-invariant spin- $s$  Weyl tensor  $C_{s,s} = \partial^s h_s + \dots$  has a transparent group-theoretical description as a submodule of the corresponding Verma module, i.e.  $\mathcal{S}(2 - s; (s)) \subset \mathcal{V}(2; (s, s))$ . In this case, the previous inner product becomes  $\int d^d x h_s J_s = (-1)^s \int d^d x C_{s,s} k_{s,s}$  where  $k_{s,s}$  is the (non-local) prepotential of the conserved current:  $J_s \approx (\partial \cdot)^s k_{s,s}$ . Let us stress that this description applies to any dimension  $d$ . However, for  $d$  odd, the module  $\mathcal{S}(2 - s; (s))$  for the off-shell FT field is irreducible and isomorphic to the module

$$\mathcal{D}(2; (s, s)) \cong \frac{\mathcal{V}(2; (s, s))}{\mathcal{D}(3; (s, s, 1))}, \quad (2.17)$$

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<sup>9</sup>More precisely, it is the “shadow operator” of  $J_s$  in the sense of [75].

describing the spin- $s$  Weyl tensor  $C_{s,s}$ , where the quotient by the irreducible submodule  $\mathcal{D}(3; (s, s, 1))$  implements the generalized Bianchi identities.

**On-shell Fradkin-Tseytlin field.** For  $d$  even, the module  $\mathcal{S}(2-s; (s))$  of the *off-shell* FT field is reducible, corresponding to the possibility of imposing the (local) Bach equation. Interestingly, the module (2.16) may be interpreted as the left-hand-side of (2.4), i.e. the *Bach tensor*. Accordingly, the module corresponding to the *on-shell* FT field is in fact given by the following quotient:

$$\mathcal{D}(2; (s, s)) \cong \frac{\mathcal{S}(2-s; (s))}{\mathcal{D}(s+d-2; (s))}, \quad (2.18)$$

where the left-hand-side can be interpreted as the spin- $s$  on-shell Weyl tensor  $C_{s,s}$  given in (2.6). Again  $\mathcal{S}(2-s; (s))$  corresponds to the off-shell FT field whereas the quotient by  $\mathcal{D}(s+d-2; (s))$  has the interpretation of imposing the Bach equation (2.4). At first glance, it might be curious why the above quotient module is  $\mathcal{D}(2; (s, s))$  which has the lowest weight 2. While the quotient (2.18) corresponds to the direct translation of the field-theoretical definition of the on-shell FT field, the module  $\mathcal{D}(2; (s, s))$  can be expressed as another quotient using its Bernstein-Gel'fand-Gel'fand (BGG) sequence (see e.g. [57]). The latter gives

$$\mathcal{D}(2; (s, s)) \cong \frac{\mathcal{V}(2; (s, s))}{\mathcal{U}(3; (s, s, 1))}, \quad (2.19)$$

where  $\mathcal{U}(3; (s, s, 1))$  implements the identities à la Bianchi obeyed by the on-shell Weyl tensor. Notice that the degenerate case  $s = 0$  is consistent with (2.18) in the sense that the on-shell scalar FT field, which is an order  $\ell = \frac{d-4}{2}$  scalar singleton, is

$$\mathcal{D}(2; \mathbf{0}) = \frac{\mathcal{V}(2; \mathbf{0})}{\mathcal{V}(d-2; \mathbf{0})} = \text{Rac}_{\frac{d-4}{2}}. \quad (2.20)$$

**Conformal Killing tensors.** The last module we shall introduce is the one corresponding to conformal Killing tensors,

$$\mathcal{D}(1-s; (s-1)) \cong \frac{\mathcal{V}(1-s; (s-1))}{\mathcal{U}(2-s; (s))}. \quad (2.21)$$

Here  $\mathcal{V}(1-s; (s-1))$  corresponds to the (large) gauge parameters of FT field  $h_s$  and  $\mathcal{U}(2-s; (s))$  has the interpretation of pure gauge shadow fields, thus quotienting by this submodule corresponds to imposing the conformal Killing equation (2.3). Notice that there is a one-to-one correspondence between Killing tensors on  $\text{AdS}_{d+1}$  and conformal Killing tensors on  $M_d$ , so the finite-dimensional irreducible  $\mathfrak{so}(d, 2)$ -module  $\mathcal{D}(1-s; (s-1))$  has clear bulk and boundary interpretations.

### 2.3 Character of the Fradkin-Tseytlin module

The characters of the modules presented in the previous section can be related to the characters of the Verma module,

$$\chi_{\mathcal{V}(\Delta, \mathbb{Y})}(q, \mathbf{x}) = q^\Delta \mathcal{P}_d(q, \mathbf{x}) \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}), \quad (2.22)$$

where  $\chi_{\mathbb{Y}}^{\mathfrak{so}(d)}$  is the character of the subalgebra  $\mathfrak{so}(d)$  of  $\mathfrak{so}(2, d)$  and the function  $\mathcal{P}_d$  is given, both for even and odd  $d$ , by

$$\mathcal{P}_d(q, \mathbf{x}) = \frac{1}{(1-q)^{d-2r}} \prod_{k=1}^r \frac{1}{(1-qx_k)(1-qx_k^{-1})}. \quad (2.23)$$

Let us point out one important property of the above character,

$$\chi_{\mathcal{V}(\Delta, \mathbb{Y})}(q^{-1}, \mathbf{x}) = (-1)^d \chi_{\mathcal{V}(d-\Delta, \mathbb{Y})}(q, \mathbf{x}), \quad (2.24)$$

which is simply a consequence of the behaviour of  $\mathcal{P}_d$  under  $q \rightarrow q^{-1}$ .

In this paper, we aim to find the character of the total linearized spectrum of CHS theory. For that, we need first the character of the spin- $s$  on-shell FT field — which had been obtained in [64] using the BGG resolution of finite-dimensional  $\mathfrak{so}(2, d)$ -modules [57] — then sum over all the spins. In order to avoid the technicalities, we shall present only key steps of the derivation, but interested readers can find more details in appendix A.

Let us begin with the character of the spin- $s$  on-shell FT field. From the definition (2.18), we first find

$$\chi_{\mathcal{D}(2;(s,s))} = \chi_{\mathcal{S}(2-s;(s))} - \chi_{\mathcal{D}(s+d-2;(s))}, \quad (2.25)$$

where the off-shell FT field character  $\chi_{\mathcal{S}(2-s;(s))}$  and the Bach tensor (or conserved current) character  $\chi_{\mathcal{D}(s+d-2;(s))}$  are given by

$$\chi_{\mathcal{S}(2-s;(s))} = \chi_{\mathcal{V}(2-s;(s))} - \chi_{\mathcal{V}(1-s;(s-1))} + \chi_{\mathcal{D}(1-s;(s-1))}, \quad (2.26)$$

$$\chi_{\mathcal{D}(s+d-2;(s))} = \chi_{\mathcal{V}(s+d-2;(s))} - \chi_{\mathcal{V}(s+d-1;(s-1))}. \quad (2.27)$$

The equation (2.27) follows directly from the definition (2.16). The heuristic behind (2.26) is that one should subtract from the character of the module  $\mathcal{V}(2-s;(s))$  describing  $h_s$  the character of the pure gauge modes. This can be done by subtracting the character of the module  $\mathcal{V}(1-s;(s-1))$  describing the gauge parameters. However, this removes too much. Indeed, when the gauge parameters are equal to conformal Killing tensors, they leave  $h_s$  inert (by definition). For this reason one has to correct by adding the character of  $\mathcal{D}(1-s;(s-1))$ . In more physical terms, the module  $\mathcal{V}(1-s;(s-1))$  of gauge parameters contains large gauge transformations — which are physical — associated with the conformal Killing tensor module  $\mathcal{D}(1-s;(s-1))$ . Inserting (2.26) and (2.27) in (2.25), we obtain

$$\chi_{\mathcal{D}(2;(s,s))} = \chi_{\mathcal{D}(1-s;(s-1))} + \chi_{\mathcal{V}(2-s;(s))} - \chi_{\mathcal{V}(1-s;(s-1))} - \chi_{\mathcal{V}(s+d-2;(s))} + \chi_{\mathcal{V}(s+d-1;(s-1))}, \quad (2.28)$$

which relates the spin- $s$  on-shell FT field module  $\chi_{\mathcal{D}(2;(s,s))}$  to the character of the conformal Killing tensor module  $\chi_{\mathcal{D}(1-s;(s-1))}$  up to the characters of a few Verma modules. The relation (2.28) does not yet express the character  $\chi_{\mathcal{D}(2;(s,s))}$  in terms of Verma module characters  $\chi_{\mathcal{V}(\Delta, \mathbb{Y})}$  alone due to the presence of  $\chi_{\mathcal{D}(1-s;(s-1))}$  on the right-hand-side. In principle, we could further work out to get rid of the latter module using another relation for the modules but it will turn out to be useful to do the opposite. In fact, the expression (2.28)

naturally leads to an interesting and suggestive expression of the character of the full CHS theory. This is thanks to the special property that the Verma module part of (2.28) enjoys:

$$\chi_{\mathcal{V}(2-s;(s))}(q, \mathbf{x}) - \chi_{\mathcal{V}(1-s;(s-1))}(q, \mathbf{x}) = (-)^d \chi_{\mathcal{D}(s+d-2;(s))}(q^{-1}, \mathbf{x}). \quad (2.29)$$

Note that the above property is a simple consequence of (2.24) and (2.27). This leads to the relation [64]

$$\chi_{\mathcal{S}(2-s;(s))}(q, \mathbf{x}) = \chi_{\mathcal{D}(1-s;(s-1))}(q, \mathbf{x}) + (-)^d \chi_{\mathcal{D}(s+d-2;(s))}(q^{-1}, \mathbf{x}), \quad (2.30)$$

which is valid in any dimension  $d > 2$  (even or odd). Finally for even  $d$ , the character of the spin- $s$  on-shell FT field module coincides with that of the conformal Killing tensor module up to just two additional terms:

$$\chi_{\mathcal{D}(2;(s,s))}(q, \mathbf{x}) = \chi_{\mathcal{D}(1-s;(s-1))}(q, \mathbf{x}) + \chi_{\mathcal{D}(s+d-2;(s))}(q^{-1}, \mathbf{x}) - \chi_{\mathcal{D}(s+d-2;(s))}(q, \mathbf{x}), \quad (2.31)$$

as follows from (2.25) and (2.29). Interestingly, both of these terms are given by the characters of the conserved current module, but one is with  $q$  while the other is with  $q^{-1}$ . The formula (2.31) is the instance of the identity (1.2) which is relevant for type-A CHS gravity. It applies to the degenerate case  $s = 0$  as well, except that the first term of the right-hand-side is absent in this case.

## 2.4 Character of type-A on-shell conformal higher-spin gravity

We shall use the relation (2.31) to derive the character of the CHS theory linearized spectrum. The theory contains the FT fields of spin 1 to  $\infty$  and the scalar field with a kinetic operator containing  $d - 4$  derivatives. Focusing first on the FT fields, we consider

$$\begin{aligned} \sum_{s=0}^{\infty} \chi_{\mathcal{D}(2;(s,s))}(q, \mathbf{x}) &= \sum_{s=1}^{\infty} \chi_{\mathcal{D}(1-s;(s-1))}(q, \mathbf{x}) \\ &+ \sum_{s=0}^{\infty} \chi_{\mathcal{D}(s+d-2;(s))}(q^{-1}, \mathbf{x}) - \sum_{s=0}^{\infty} \chi_{\mathcal{D}(s+d-2;(s))}(q, \mathbf{x}). \end{aligned} \quad (2.32)$$

The first term in the right-hand-side of the equality is nothing but the character of the adjoint module of the CHS symmetry algebra. Re-expressing the two series in the second line using the Flato-Fronsdal theorem [76–78]:

$$\left( \chi_{\text{Rac}}(q, \mathbf{x}) \right)^2 = \sum_{s=0}^{\infty} \chi_{\mathcal{D}(s+d-2;(s))}(q, \mathbf{x}), \quad (2.33)$$

the series (2.32) becomes

$$\sum_{s=0}^{\infty} \chi_{\mathcal{D}(2;(s,s))}(q, \mathbf{x}) = \sum_{s=0}^{\infty} \chi_{\mathcal{D}(1-s;(s-1))}(q, \mathbf{x}) + \left( \chi_{\text{Rac}}(q^{-1}, \mathbf{x}) \right)^2 - \left( \chi_{\text{Rac}}(q, \mathbf{x}) \right)^2. \quad (2.34)$$

The second line of the above formula vanishes because the character of the Rac singleton obeys the property

$$\chi_{\text{Rac}}(q^{-1}, \mathbf{x}) = (-1)^{d+1} \chi_{\text{Rac}}(q, \mathbf{x}). \quad (2.35)$$

Finally, we find that the character of all the on-shell fields in the free CHS theory coincides with that of the global symmetry of CHS theory:

$$\sum_{s=0}^{\infty} \chi_{\mathcal{D}(2;(s,s))} = \sum_{s=1}^{\infty} \chi_{\mathcal{D}(1-s;(s-1))} . \quad (2.36)$$

This result can be understood as the equality (1.1) for type-A CHS gravity.<sup>10</sup> Actually, both sides of (2.36) involve divergent series which require some regularization. However, the equality (2.36) itself only assumed the validity of the Flato-Fronsdal theorem which does not need any regularization.<sup>11</sup> Consequently, confident in the validity of (2.36) one might somehow reduce the issue of regularizing the character of the CHS spectrum (the left-hand-side) to the one of the higher-spin algebra<sup>12</sup> (the right-hand-side). By construction, the corresponding regularization of CHS theory would preserve higher-spin symmetries, an important requirement of a sensible regularization but which is usually not guaranteed.

There is another virtuous corollary of the relation (1.1): the Casimir energy of free CHS theory on the Einstein static universe  $\mathbb{R} \times S^{d-1}$  is ensured to vanish in any regularization consistent with (1.1). In fact, the vanishing of the Casimir energy is ensured when the partition function is invariant under the map  $q \rightarrow 1/q$  (see [80]) and this property automatically holds for the right-hand-side of (1.1), since the character of each Killing tensor module obeys this property. Actually, the  $a$ -anomaly of CHS gravity is also guaranteed to vanish by virtue of this property of the character. Indeed, the  $a$ -anomaly of a  $d$ -dimensional FT field coincides with the difference of the free energy of the associated massless field in Euclidean  $\text{AdS}_{d+1}$  with Neumann boundary condition and the same with Dirichlet condition [46]. Since the free energy with Neumann boundary condition is simply minus that with Dirichlet condition (the contribution of Killing tensor module simply vanishes), the  $a$ -anomaly of the  $d$ -dimensional CHS gravity is just minus two times the free energy of MHS gravity in Euclidean  $\text{AdS}_{d+1}$ . The cancellation of the latter can be shown using the method of character integral representation of zeta function [81–83].

## 2.5 Character of type-A off-shell conformal higher-spin gravity

We can also derive an off-shell version of the identity (1.1). From (2.18), the spin- $s$  off-shell FT field module is related to the on-shell one by

$$\chi_{\mathcal{S}(2-s;(s))} = \chi_{\mathcal{D}(2;(s,s))} + \chi_{\mathcal{D}(s+d-2;(s))} , \quad (2.39)$$

---

<sup>10</sup>Let us illustrate why (1.1) does not hold for *minimal* CHS gravity with only *even* spins in the spectrum. In such case, the Flato-Fronsdal theorem involves a symmetric plethysm of the Rac module:

$$\frac{1}{2} \left[ \left( \chi_{\text{Rac}}(q, x_i) \right)^2 - \chi_{\text{Rac}}(q^2, x_i^2) \right] = \sum_{s \in 2\mathbb{N}} \chi_{\mathcal{D}(s+d-2;(s))}(q, \mathbf{x}) . \quad (2.37)$$

Then (2.31) and the analogue of (2.35) imply the relation

$$\sum_{s \in 2\mathbb{N}} \chi_{\mathcal{D}(2;(s,s))}(q, x_i) = \sum_{s \in 2\mathbb{N}_0} \chi_{\mathcal{D}(1-s;(s-1))}(q, x_i) + \chi_{\text{Rac}}(q^2, x_i^2) , \quad (2.38)$$

where the extra term on the right-hand-side has no clear interpretation in this context (as it decomposes into an *alternating* sum of the characters of massless  $\text{AdS}_{d+1}$  fields of all integer spin).

<sup>11</sup>The only assumption is that it holds for each sum of characters (in  $q$  and in  $1/q$ ) separately, despite the fact that the corresponding power series have distinct region of convergence ( $|q| < 1$  and  $|q| > 1$ ).

<sup>12</sup>Some convergence and regularization issues of the latter were addressed in [79].

Bulk MHS theory	Bulk field	$\Delta_s$	Boundary operator	Boundary theory
Dirichlet (standard)	Normalizable solution	$\Delta_+ = s + d - 2$	Conserved current	Free CFT
Neumann (exotic)	Non-normalizable solution	$\Delta_- = 2 - s$	Shadow field	CHS theory

**Table 1.** List of relevant fields in Dirichlet vs Neumann MHS theories

whereas the off-shell FT scalar  $\mathcal{S}(2;0) = \mathcal{V}(2;0)$  is related to the on-shell one by

$$\chi_{\mathcal{S}(2;0)} = \chi_{\text{Rac}_{\frac{d-4}{2}}} + \chi_{\mathcal{V}(d-2;0)}, \quad (2.40)$$

where the first term on the right-hand-side is absent in  $d = 4$ . In fact, the above off-shell scalar becomes an auxiliary field in four dimensions. In odd dimension  $d$ , one can make use of (2.30). Summing over the characters of these off-shell field modules, we arrive at

$$\sum_{s=0}^{\infty} \chi_{\mathcal{S}(2-s;(s))} = \sum_{s=1}^{\infty} \chi_{\mathcal{D}(1-s;(s-1))} + (-)^d \sum_{s=0}^{\infty} \chi_{\mathcal{D}(s+d-2;(s))}, \quad (2.41)$$

which holds for any dimension. For  $d$  even, this result can be viewed as

$$d \text{ even:} \quad \text{Off-Shell CHS} = \text{Higher-spin Algebra} \oplus \text{Dirichlet MHS}, \quad (2.42)$$

where the last term on the right-hand-side, the linearized spectrum of MHS gravity around  $\text{AdS}_{d+1}$  with Dirichlet boundary conditions, corresponds to the last series in (2.41). The two terms on the right-hand-side of (2.42) have natural interpretations in terms of the higher-spin algebra: they are important modules of the latter. The first term is the adjoint module while the second term is the so-called twisted adjoint module of the higher-spin algebra.

From a holographic perspective, another interpretation of this last result is possible, purely in terms of bulk fields. For a spin- $s$  massless bulk field, two boundary conditions are available: either the standard (“Dirichlet”) boundary condition which allows normalizable bulk solutions corresponding to a conserved current  $J_s$  with conformal weight  $\Delta_+ = s + d - 2$ , or the exotic (“Neumann”) boundary condition which allows non-normalizable bulk solutions corresponding to a shadow field  $h_s$  with conformal weight  $\Delta_- = 2 - s$ . Accordingly, the bulk theory with standard (respectively, exotic) boundary condition for *all* fields will be referred to as Dirichlet (respectively, Neumann) MHS theory. They are summarized and compared in table 1. The holographic dual of Dirichlet MHS theory is a free scalar CFT [84, 85]. Following the usual considerations on double-trace deformations and holographic degeneracy [86–88], applied for all spin- $s$  conserved currents, one is lead to the conclusion that the holographic dual of Neumann MHS theory is CHS gravity. This scenario has been extensively discussed for the MHS theory around  $\text{AdS}_4$  and Chern-Simons CHS theory around  $\text{M}_3$  (see e.g. [46, 89, 90]), but the logic works for any dimension (see e.g. [60] for any spin at free level and [91] for spin-two at interacting level).



This exotic type of holographic dualities where both sides can be gravity theories (though of Einstein vs Weyl type) has been denoted “AdS/IGT” in [46] — where IGT stands for induced gauge theory — in order to distinguish it from standard AdS/CFT correspondence. This type of holographic duality is somewhat less familiar and is a subtle one, so let us expand a little bit and present some details. One starts by adding double-trace deformations for all  $U(N)$ -singlet primary operators  $J_s$  (with  $s = 0, 1, 2, \dots$ ) bilinear in the  $N$  complex scalar  $\phi^i$ :

$$S_{\{\lambda_s\}_{s \in \mathbb{N}}}[\{\phi^i\}_{i=1, \dots, N}, \{h_s\}_{s \in \mathbb{N}}] = S[\{\phi^i\}_{i=1, \dots, N}, \{h_s\}_{s \in \mathbb{N}}] + \sum_{s=0}^{\infty} \frac{\lambda_s}{2N} \int d^d x J_s^2. \quad (2.43)$$

These deformations explicitly break all gauge symmetries of the background fields  $h_s$ . Notice that essentially all<sup>13</sup> these deformations are irrelevant in the infrared (IR). The corresponding effective action is defined as

$$e^{-N W_{\Lambda}^{\{\lambda_s\}_{s \in \mathbb{N}}}[\{h_s\}_{s \in \mathbb{N}}]} = \int_{\Lambda} \left[ \prod_{i=1}^N \mathcal{D}\phi^i \right] e^{-S_{\{\lambda_s\}_{s \in \mathbb{N}}}[\{\phi^i\}_{i=1, \dots, N}, \{h_s\}_{s \in \mathbb{N}}]}. \quad (2.44)$$

The standard Hubbard-Stratonovich trick corresponds to the introduction of auxiliary fields  $\sigma_s$  through Gaussian integrals as follows:

$$e^{-\frac{\lambda_s}{2N} \int d^d x J_s^2} = \int \mathcal{D}\sigma e^{\int d^d x \left( (\sigma_s - h_s) J_s - \frac{N}{2\lambda_s} (\sigma_s - h_s)^2 \right)}, \quad (2.45)$$

where we neglected an infinite prefactor. Note that  $\sigma_s$  does not have any gauge symmetries. The integration over the dynamical scalar field  $\phi$  is now again a Gaussian integral which can now be performed leading to

$$e^{-N W_{\Lambda}^{\{\lambda_s\}_{s \in \mathbb{N}}}[\{h_s\}_{s \in \mathbb{N}}]} = \int_{\Lambda} \mathcal{D}\sigma e^{-N \left( W_{\Lambda}[\{\sigma_s\}_{s \in \mathbb{N}}] - \sum_{s=0}^{\infty} \frac{1}{2\lambda_s} \int d^d x (\sigma_s - h_s)^2 \right)}. \quad (2.46)$$

One may then perform the field redefinition  $h_s = \lambda_s j_s$  such that the new background field  $j_s$  has the bare scaling dimension of a conserved current,

$$\begin{aligned} e^{-N W_{\Lambda}^{\{\lambda_s\}_{s \in \mathbb{N}}}[\{\lambda_s j_s\}_{s \in \mathbb{N}}]} \\ = e^{N \sum_{s=0}^{\infty} \frac{\lambda_s}{2} \int d^d x j_s^2} \int_{\Lambda} \mathcal{D}\sigma e^{-N \left( W_{\Lambda}[\{\sigma_s\}_{s \in \mathbb{N}}] + \sum_{s=0}^{\infty} \int d^d x \left( \sigma_s j_s - \frac{1}{2\lambda_s} \sigma_s^2 \right) \right)}. \end{aligned} \quad (2.47)$$

Considering the vicinity of the UV fixed point where<sup>14</sup>  $\lambda_s \rightarrow \infty$ , one gets

$$e^{-N W_{\Lambda}^{\text{UV}}[\{j_s\}_{s \in \mathbb{N}}]} = \int_{\Lambda} \mathcal{D}h e^{-N \left( W_{\Lambda}[\{h_s\}_{s \in \mathbb{N}}] + \sum_{s=0}^{\infty} h_s j_s \right)}, \quad (2.48)$$

<sup>13</sup>Except possibly for very low spin  $s$  and dimension  $d$  which require a separate discussion (see e.g. [92] for detailed discussion of the double-trace deformation and its holographic interpretation in the  $s = 0$  case). More precisely, for  $d = 4$  the  $s = 0$  term is marginal while for  $d = 2$  the  $s = 0$  term is relevant and the  $s = 1$  is marginal.

<sup>14</sup>Except possibly for low spin  $s$  and dimension  $d$  which might require a separate discussion which will be avoided here for the sake of simplicity.



where  $W_{\Lambda}^{\text{UV}}[\{j_s\}_{s \in \mathbb{N}}]$  stands for the UV-divergent part of the functional  $W_{\Lambda}^{\{\lambda_s\}_{s \in \mathbb{N}}}[\{\lambda_s j_s\}_{s \in \mathbb{N}}] + \sum_{s=0}^{\infty} \frac{\lambda_s}{2} \int d^d x j_s^2$ . In (2.48), the Hubbard-Stratonovitch field  $\sigma_s$  has been denoted  $h_s$  (consistently with its bare scaling dimension) in order to stress that gauge symmetry is restored in the limit  $\lambda_s \rightarrow \infty$  due to the disappearance of the quadratic term.<sup>15</sup> Focusing on the logarithmically divergent piece, the left-hand-side in (2.48) can be interpreted as the generating functional of the correlators in CHS gravity. In particular, in the large- $N$  (i.e. semiclassical) limit one has that the logarithmically divergent piece of  $W_{\Lambda}^{\text{UV}}[\{j_s\}_{s \in \mathbb{N}}]$  is the Legendre transform of  $W_{\log}[\{h_s\}_{s \in \mathbb{N}}]$ .

The holographic dual to a Legendre transform on the boundary is a change of boundary conditions from Dirichlet to Neumann on the bulk fields. In fact, the solution space of the spin- $s$  MHS fields with Neumann boundary condition is an  $\mathfrak{so}(2, d)$  module in one-to-one correspondence with the module of a spin- $s$  shadow field (see e.g. [30] for a manifestly conformal and gauge invariant field-theoretical description). Consequently, the linearized spectra of off-shell CHS theory and Neumann MHS are in one-to-one correspondence. Therefore the result (2.42) can be rephrased purely in bulk terms as follows:

$$d \text{ even:} \quad \text{Neumann MHS} = \text{Higher-spin Algebra} \oplus \text{Dirichlet MHS} . \quad (2.49)$$

In other words, our character computation suggests that asymptotic charges account for all extra dynamical degrees of freedom in MHS theory when all boundary conditions are modified from Dirichlet to Neumann ones. Let us stress that both holographic duals to Dirichlet and Neumann MHS theories are (respectively, infrared and ultraviolet) fixed points with *unbroken* conformal higher-spin symmetries (rigid symmetries in the former case, gauge symmetries in the latter) since all spins are on the same footing.

As a side remark, one may observe that, the opposite sign in the last term on the right-hand-side of (2.30) for  $d$  odd leads after summation over all spins to a relation between characters which one can rewrite as

$$d \text{ odd:} \quad \text{Dirichlet MHS} \oplus \text{Neumann MHS} = \text{Higher-spin Algebra} . \quad (2.50)$$

This relation suggests that the linearized MHS theory on  $\text{AdS}_{d+1}$  spacetime of even dimension *without* imposing any boundary condition (i.e. considering both normalizable *and* non-normalizable solutions) might also be somewhat “topological” in the sense that its dynamical degrees of freedom reorganize into its asymptotic symmetries.

### 3 Extensions to type- $A_{\ell}$ and type- $B_{\ell}$ theories

In the previous section, we have shown that the character of the type-A CHS theory coincides with the character of the adjoint module of the type-A higher-spin algebra. In this section, we provide more non-trivial evidences of this intriguing observation by generalizing the result to the type- $A_{\ell}$  and type- $B_{\ell}$  theories.

<sup>15</sup>Therefore (2.48) would require a careful discussion of the ghost contribution in the measure. This will not be performed here because we only intend to sketch the logic of the proof.

### 3.1 Field theory of type- $A_\ell$ theory

Let us introduce another class of conformal gauge fields which are cousins of the spin- $s$  FT field. They are described by totally-symmetric rank- $s$  tensor  $h_s^{(t)}$  like the usual FT field, but they have weaker gauge symmetry [31]

$$\delta_{\xi,\sigma} h_s^{(t)} = \partial^t \xi_{s-t} + \eta \sigma_{s-2}, \quad (3.1)$$

with  $\xi_{s-t}$  a rank- $(s-t)$  symmetric tensor,  $\eta$  the Minkowski metric and  $\sigma_{s-2}$  a rank- $(s-2)$  symmetric tensor. The integer  $t$  takes value inside the range  $1 \leq t \leq s$ , parameterizes these class of fields and will be referred to as the *depth* (so that the usual FT field corresponds to  $t = 1$ ). As in the usual FT field case, we can define a gauge-invariant field-strength, that is, a Weyl-like tensor as

$$C_{s,s-t+1}^{(t)} = \mathbb{P}_T^{s,s-t+1} \partial^{s-t+1} h_s^{(t)} \underset{\mathfrak{so}(1,d-1)}{\sim} \boxed{\frac{s}{s-t+1}}, \quad (3.2)$$

which also has conformal weight  $\Delta_{C_{s,s-t+1}^{(t)}} = 2$ . In even  $d$  dimensions, the action for the depth- $t$  and spin- $s$  FT field is then given by

$$S_{\text{FT}(t)}[h_s^{(t)}] = (-1)^{s-t+1} \int_{M_d} d^d x C_{s,s-t+1}^{(t)} \square^{\frac{d-4}{2}} C_{s,s-t+1}^{(t)}. \quad (3.3)$$

After integrating by part, the action takes the form of

$$S_{\text{FT}(t)}[h_s^{(t)}] = \int_{M_d} d^d x h_s^{(t)} \mathbb{P}_{T^{tT}}^s \square^{s-t+\frac{d-2}{2}} h_s^{(t)}, \quad (3.4)$$

where  $\mathbb{P}_{T^{tT}}^s$  is the  $t$ -ple transverse and traceless projector<sup>16</sup> which becomes local after multiplying by the factor  $\square^{s-t+1}$ . The condition  $\delta_{\xi,\sigma} h_s^{(t)} = 0$  defines now the depth- $t$  conformal Killing tensors.

An interacting theory of the depth- $t$  FT fields can be obtained as an effective action, similarly to the  $t = 1$  case. We replace the free scalar action by its analog of order  $2\ell$  in the derivatives and couple the system to the higher-spin sources  $h_s^{(t)}$  via a set of currents  $J_s^{(t)}$  which are traceless and  $t$ -ple divergenceless:

$$\eta \cdot J_s^{(t)} \approx 0, \quad (\partial \cdot)^t J_s^{(t)} \approx 0. \quad (3.5)$$

One can show that for a given free scalar action with a fixed  $\ell$ , we can find currents of all integers spin with  $t = 1, 3, \dots, 2\ell - 1$  [31, 93, 94]. These currents take the form

$$J_s^{(2k-1)} = \bar{\phi} \partial^s \square^{\ell-k} \phi + \dots, \quad (k = 1, 2, \dots, \ell) \quad (3.6)$$

where the “...” stands for additional terms ensuring (3.5), and it has the conformal weight

$$\Delta_{s,k} = s + d - 2k. \quad (3.7)$$

<sup>16</sup>Note that the condition of the  $t$ -ple transversality and tracelessness does not fix the projector uniquely. We need to impose the locality condition on  $\mathbb{P}_{T^{tT}}^s \square^{s-t+1}$  to determine the action uniquely.

The tensors  $J_s^{(\ell)}$  with  $t > s$  do not satisfy any (partial-)conservation condition since (3.5) is not defined in such case. Still, these tensors can be used as part of the basis operators for the space of operators bilinear in the order- $\ell$  scalar singleton. When  $\ell \leq \frac{d}{4}$ , all these operators are primary, and the space of operators with dimension  $s + d - 2k$  and spin  $s$  is spanned by the basis

$$\{J_s^{(2k-1)}, \square J_s^{(2k+1)}, \dots, \square^\ell J_s^{(2k+2\ell-1)}\}. \quad (3.8)$$

Starting from the  $2\ell$ -derivative scalar field action in the background of higher-spin fields of depths  $t = 1, 3, \dots, 2\ell - 1$ ,

$$S[\phi; \{h_s^{(2k-1)}\}_{s \in \mathbb{N}, k \in \{1, 2, \dots, \ell\}}] = \int_{M_d} d^d x \left( \bar{\phi} \square^\ell \phi + \sum_{s=0}^{\infty} \sum_{k=1}^{\ell} J_s^{(2k-1)} h_s^{(2k-1)} \right), \quad (3.9)$$

and integrating out the scalar field  $\phi$ , we obtain an effective action. Again, the logarithmically divergent part of the effective action is a local functional of higher-spin fields, and for  $\ell \leq \frac{d}{4}$ , it has the structure

$$W_{\log}^{(\ell)}[\{h_s^{(2k-1)}\}_{s \in \mathbb{N}, k \in \{1, 2, \dots, \ell\}}] = \sum_{s=0}^{\infty} \sum_{k=1}^{\ell} S_{\text{FT}}^{(2k-1)}[h_s^{(2k-1)}] + \mathcal{O}(h^3). \quad (3.10)$$

The functional  $W_{\log}^{(\ell)}$  can be regarded as an action of interacting higher-depth FT fields up to the introduction of a dimensionless coupling constant  $\kappa$ :  $S_{\text{CHS}}^{(\ell)} = \kappa W_{\log}^{(\ell)}$ . This interacting theory contains not only the higher-depth FT fields but also other non-gauge conformal fields, referred to as *special* in [95]. We will refer to this class of fields as “special FT” for the sake of uniformity in the terminology. They correspond to the fields of spin- $s$  and conformal weight  $\Delta = 1 - s + t$  with  $t \geq s + 1$ . Although they do not enjoy any gauge symmetry, we will keep referring to the parameter  $t$  defining those fields as their depth. In the quadratic part (3.10), the fields with  $0 \leq s \leq 2(\ell - 1)$  and  $\frac{s+2}{2} \leq k \leq \ell$  correspond to this class. The free Lagrangians of these fields still has  $2(s - t) + d - 2$  derivatives but do not have any gauge symmetry. A trivial but important restriction to these fields is that the number of derivatives of their free Lagrangian cannot be negative. This gives a dimension dependent upper bound for  $t$ , namely  $t \leq s + \frac{d-2}{2}$ , and hence  $k \leq \frac{d+2s}{4}$ . The latter bound is irrelevant when it is not smaller than  $\ell$ . The  $s = 0$  case gives the lowest bound, and hence the condition that this be not smaller than  $\ell$  imposes  $\ell \leq \frac{d}{4}$ . From the CFT point of view, the bound implies that there is no operators with conformal weights lower than  $\frac{d}{2}$ . This bound can actually be relaxed without encountering an inconsistency due to a subtle phenomenon discussed below. The origin of the quadratic Lagrangian in (3.10) is the local contact terms hidden in the two point functions of  $J_s^{(2k-1)}$ :

$$\langle J_s^{(2k-1)}(x) J_s^{(2k-1)}(0) \rangle \propto \frac{\eta^s}{|x|^{2s+2d-4k+\epsilon}} + \dots \xrightarrow{\epsilon \sim 0} \frac{\rho_{\Delta_{s,k}}}{\epsilon} \left( \eta^s \square^{s+\frac{d}{2}-2k} + \dots \right) \delta^{(d)}(x), \quad (3.11)$$

where  $\rho_{\Delta} = \pi^{\frac{d}{2}} / [2^{2\Delta-d} \Gamma(\Delta - \frac{d-2}{2}) \Gamma(\Delta)]$ . One can note here that the contact terms are absent for  $k > \frac{d+2s}{4}$ .

This higher-depth CHS theory appears from the on-shell action of type- $A_\ell$  MHS theory in  $\text{AdS}_{d+1}$ . Analogously to the usual FT/massless case, the depth- $t$  FT fields in  $M_d$  can be related to the depth- $t$  partially-massless field in  $\text{AdS}_{d+1}$ :

$$S_{\text{PM}(t)}[\Phi_s^{(t)} = \mathcal{K}^{(t)} h_s^{(t)}] = \log R S_{\text{FT}(t)}[h_s^{(t)}] + (\text{regular or polynomially divergent terms}). \quad (3.12)$$

Here,  $\Phi_s^{(t)}$  is the spin- $s$  and depth- $t$  partially-massless field and  $\mathcal{K}^{(t)}$  is the corresponding boundary-to-bulk propagator. Both type- $A_\ell$  theories (CHS gravity in  $M_d$  and partially-massless HS gravity in  $\text{AdS}_{d+1}$ ) have the same global symmetries: the type- $A_\ell$  HS symmetry algebra generated by higher-depth conformal Killing tensors.

When  $\ell > \frac{d}{4}$ , we face an interesting phenomenon referred to as *extension* on the CFT side in [93]. Let us briefly review this extension hereafter. For  $\ell > \frac{d}{4}$ , the operator spectrum contains pairs of currents  $J_s^{(2k-1)}$  and  $J_s^{(2k'-1)}$  with the same spin but with the respectively dual conformal weights,  $\Delta_{s,k} + \Delta_{s,k'} = d$ , in other words,

$$k + k' = s + \frac{d}{2}, \quad k \leq k'. \quad (3.13)$$

For  $\ell < \frac{d}{2}$ , none of these currents are (partially-)conserved (see figure 1), and the current operator with higher conformal dimensions becomes a descendent of the other:

$$J_s^{(2k-1)} \propto \square^{k'-k} J_s^{(2k'-1)}. \quad (3.14)$$

Hence, the operators  $J_s^{(2k-1)}$  are both primary and descendent. Because of the degeneracy (3.14), we loose one basis operator from (3.8). As a consequence, we can find a new basis operator  $\tilde{J}_s^{(2k-1)}$  which is neither primary nor descendent.<sup>17</sup> Interestingly, the cross two point function of  $J_s^{(2k'-1)}$  and  $\tilde{J}_s^{(2k-1)}$  does not vanish but gives

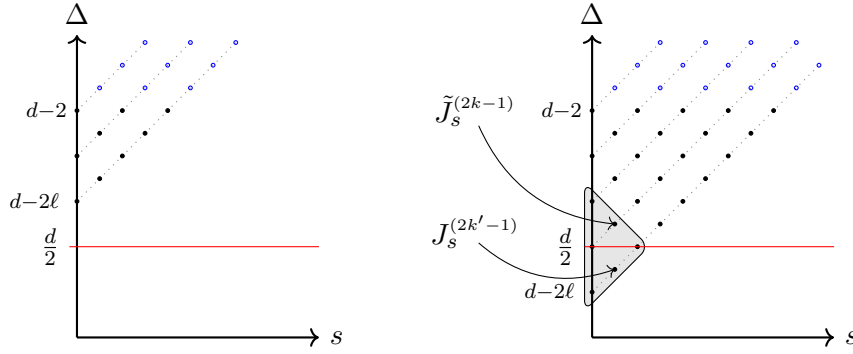
$$\langle J_s^{(2k'-1)}(x) \tilde{J}_s^{(2k-1)}(0) \rangle \propto \frac{\eta^s}{|x|^{d+\epsilon}} + \dots \xrightarrow{\epsilon \sim 0} \frac{\rho_{\frac{d}{2}}}{\epsilon} \eta^s \delta^{(d)}(x). \quad (3.15)$$

This implies that the scalar field action (3.9) in higher-spin background, where all currents  $J_s^{(2k-1)}$  with the condition (3.13) are replaced by  $\tilde{J}_s^{(2k-1)}$  will lead to the effective action  $W_{\log}^{(\ell)}$  containing the quadratic terms

$$\tilde{h}_s^{(2k-1)} \left( \square^{s+\frac{d}{2}-2k} + \dots \right) \tilde{h}_s^{(2k-1)} + \tilde{h}_s^{(2k-1)} h_s^{(2k'-1)}, \quad (3.16)$$

where  $\tilde{h}_s^{(2k-1)}$  are the source fields of the operators  $\tilde{J}_s^{(2k-1)}$  and we dropped numerical coefficients in each terms. This shows that the FT fields  $h_s^{(2k'-1)}$  with  $k' > \frac{d+2s}{4}$  are in fact Lagrange multipliers enforcing  $\tilde{h}_s^{(2k-1)} \approx 0$ . Hence, both of  $h_s^{(2k'-1)}$  and  $\tilde{h}_s^{(2k-1)}$  with (3.13) are absent in the on-shell spectrum. It is shown in [96] that the AdS counterpart of this extension phenomenon is the mass term mixing between the fields  $\Phi_s^{(2k-1)}$  and  $\Phi_s^{(2k'-1)}$ .

<sup>17</sup>Notice that this phenomenon is a consequence of the fact that the tensor product of two order- $\ell$  scalar singletons contains reducible (though indecomposable) representations for  $\ell > \frac{d}{4}$ .



**Figure 1.** The spectrum of bilinear operators, labeled by  $(\Delta, s)$ , of the order- $\ell$  scalar CFT is depicted. The black solid circles and the blue empty circles designate respectively the operators without conservation condition and with (partial-)conservation condition. The number of the dotted lines is  $\ell$ . The left diagram is the case with  $\ell < \frac{d}{4}$ , whereas the right one with  $\frac{d}{4} \leq \ell < \frac{d}{2}$ . The operators in the shaded region do not give rise to on-shell FT fields.

Since the operators  $J_s^{(2k'-1)}$  with (3.13) have conformal weights lower than  $\frac{d}{2}$ , we can think of deforming the  $U(N)$  free CFT with the double trace deformations of such operators:

$$\int_{M_d} d^d x \left( \bar{\phi}^a \square^\ell \phi_a + \sum_{s,k'} g_{s,k'} J_s^{(2k'-1)} J_s^{(2k'-1)} \right). \quad (3.17)$$

This will lead to an interacting CFT in IR, where a new class of operators  $\mathcal{J}_s^{(2k'-1)}$  having the same conformal weight as  $J_s^{(2k'-1)}$  may arise. From the AdS point of view, the double-trace deformation corresponds to changing the boundary conditions of the relevant fields. It will be interesting to clarify how the “extension” phenomenon will affect the mechanism of double-trace deformation/changing boundary conditions.

When  $\ell \geq \frac{d}{2}$ , the solution space of  $\square^\ell \phi \approx 0$  contains a conformally-invariant subspace. Hence, we are lead to consider either the finite-dimensional quotient part, or the subspace part. In fact, the quotient space corresponds to the space of harmonic polynomials with maximum order  $\ell - \frac{d}{2}$ . The zero mode of the  $\ell = 1$  and  $d = 2$  case is the familiar example. The CFT based on the quotient part has been studied in [93]. We postpone the analysis of CHS theory in this case to a future work.

### 3.2 Character of type- $A_\ell$ theory

The modules relevant to the type- $A_\ell$  CHS theory are analogous to those of the  $\ell = 1$  theory: they are

- the module  $\mathcal{D}(s - t + d - 1; (s))$  of  $t$ -ple conserved current of spin- $s$ ,
- the modules  $\mathcal{D}(2, (s, s - t))$  and  $\mathcal{S}(1 + t - s, (s))$  of the on- and off-shell FT fields of spin- $s$  and depth- $t$  (introduced in [31]), and
- the module  $\mathcal{D}(1 - s; (s - t))$  of the conformal Killing tensor which is a finite-dimensional irrep corresponding to the  $\mathfrak{so}(2, d)$  two-row Young diagram  $\begin{array}{|c|} \hline s-1 \\ \hline s-t \\ \hline \end{array}$  [97, 98].

The character of the partially-conserved current module is simple:

$$\chi_{\mathcal{D}(s+d-t-1;(s))}(q, \mathbf{x}) = q^{s+d-t-1} \left( \chi_{(s)}^{\mathfrak{so}(d)}(\mathbf{x}) - q^t \chi_{(s-t)}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \mathcal{P}_d(q, \mathbf{x}), \quad (3.18)$$

whereas the characters of the other modules are more involved and their explicit forms are given in appendix A. What is important for our purpose is that these characters satisfy an analogous relation to (2.31),

$$\chi_{\mathcal{D}(2;(s,s-t+1))}(q, \mathbf{x}) = \chi_{\mathcal{D}(1-s;(s-t))}(q, \mathbf{x}) + \chi_{\mathcal{D}(s+d-t-1;(s))}(q^{-1}, \mathbf{x}) - \chi_{\mathcal{D}(s+d-t-1;(s))}(q, \mathbf{x}). \quad (3.19)$$

Using the above properties together with the type- $A_\ell$  Flato-Fronsdal theorem [31, 99]

$$\left( \chi_{\text{Rac}_\ell} \right)^2 = \sum_{t=1,3,\dots}^{2\ell-1} \left[ \sum_{s=0}^{t-1} \chi_{\mathcal{V}(s-t+d-1;(s))} + \sum_{s=t}^{\infty} \chi_{\mathcal{D}(s-t+d-1;(s))} \right], \quad (3.20)$$

we can handily compute the character of the entire type- $A_\ell$  CHS theory. Here, the module  $\mathcal{V}(s-t+d-1;(s))$  corresponds to a spin- $s$  operator without any conservation condition. The field content of the type- $A_\ell$  CHS theory consists of

- the FT fields with depth  $t = 1, 3, \dots, 2\ell - 1$  and spin  $s = t, t + 1, \dots$ , and
- the special FT fields with depth in the same range but spin  $s = 0, 1, \dots, t - 1$ .

We first sum the characters over the higher-depth FT field modules and obtain

$$\begin{aligned} \sum_{t=1,3,\dots}^{2\ell-1} \sum_{s=t}^{\infty} \chi_{\mathcal{D}(2;(s,s-t+1))}(q, \mathbf{x}) &= \sum_{t=1,3,\dots}^{2\ell-1} \sum_{s=t}^{\infty} \chi_{\mathcal{D}(1-s;(s-t))}(q, \mathbf{x}) \\ &+ \left( \chi_{\text{Rac}_\ell}(q^{-1}, \mathbf{x}) \right)^2 - \left( \chi_{\text{Rac}_\ell}(q, \mathbf{x}) \right)^2 \\ &- \sum_{t=1,3,\dots}^{2\ell-1} \sum_{s=0}^{t-1} \left[ \chi_{\mathcal{V}(s-t+d-1;(s))}(q^{-1}, \mathbf{x}) - \chi_{\mathcal{V}(s-t+d-1;(s))}(q, \mathbf{x}) \right]. \end{aligned} \quad (3.21)$$

Again the second line vanishes due to the property

$$\chi_{\text{Rac}_\ell}(q^{-1}, \mathbf{x}) = (-1)^{d+1} \chi_{\text{Rac}_\ell}(q, \mathbf{x}), \quad (3.22)$$

and in the third line we find the character corresponding to the module of the special FT fields:

$$\begin{aligned} \chi_{\mathcal{D}(1+t-s;(s))}(q, \mathbf{x}) &= \chi_{\mathcal{V}(1+t-s;(s))}(q, \mathbf{x}) - \chi_{\mathcal{V}(s-t+d-1;(s))}(q, \mathbf{x}) \\ &= \chi_{\mathcal{V}(s-t+d-1;(s))}(q^{-1}, \mathbf{x}) - \chi_{\mathcal{V}(s-t+d-1;(s))}(q, \mathbf{x}). \end{aligned} \quad (3.23)$$

Here, we used again the property (2.24) for even  $d$ . The modules  $\mathcal{D}(1+t-s;(s))$  with  $t = s, s+1, \dots$  arise from the “non-standard” BGG sequence of  $\mathfrak{so}(2, d)$ , and they are associated with the special FT fields. This module exists in fact only if  $s \geq t - \frac{d}{2} + 2$ ,

which is the same condition that the special FT field action has positive derivatives. This condition is satisfied if  $\ell < \frac{d}{4}$ , and we obtain

$$\sum_{t=1,3,\dots}^{2\ell-1} \left[ \sum_{s=0}^{t-1} \chi_{\mathcal{D}(1+t-s;(s))} + \sum_{s=t}^{\infty} \chi_{\mathcal{D}(2;(s,s-t+1))} \right] = \sum_{t=1,3,\dots}^{2\ell-1} \sum_{s=t}^{\infty} \chi_{\mathcal{D}(1-s;(s-t))}, \quad (3.24)$$

which confirms again the observation (1.1) in the type- $A_\ell$  cases with  $\ell < \frac{d}{4}$ .

Let us consider now the case  $\ell \geq \frac{d}{4}$  where the third line of (3.21) contains terms with  $s \leq t - \frac{d}{2} + 1$ . For  $s = t - \frac{d}{2} + 1$ , the term simply vanishes. For  $s \leq t - \frac{d}{2}$ , it becomes

$$\chi_{\mathcal{V}(s-t+d-1;(s))}(q^{-1}, \mathbf{x}) - \chi_{\mathcal{V}(s-t+d-1;(s))}(q, \mathbf{x}) = -\chi_{\mathcal{D}(s-t+d-1;(s))}(q, \mathbf{x}), \quad (3.25)$$

and cancels the characters  $\chi_{\mathcal{D}(1+t-s;(s))}$  with  $t - \frac{d}{2} + 2 \leq s \leq \frac{t-d+1}{2} + \ell$  if  $\ell < \frac{d}{2}$ . In the end, for  $\frac{d}{4} \leq \ell < \frac{d}{2}$  we find

$$\sum_{t=1,3,\dots}^{2\ell-1} \left[ \sum_{s=\max\{0, \frac{t-d+3}{2}+\ell\}}^{t-1} \chi_{\mathcal{D}(1+t-s;(s))} + \sum_{s=t}^{\infty} \chi_{\mathcal{D}(2;(s,s-t+1))} \right] = \sum_{t=1,3,\dots}^{2\ell-1} \sum_{s=t}^{\infty} \chi_{\mathcal{D}(1-s;(s-t))}. \quad (3.26)$$

The left-hand-side of the equality is precisely the linearized on-shell spectrum of type- $A_\ell$  theory for  $\frac{d}{4} \leq \ell < \frac{d}{2}$ , and therefore confirms once again (1.1).

**Going off-shell.** The character of the off-shell depth- $t$  FT field with  $1 \leq t \leq s$  is related to that of the on-shell one through

$$\chi_{\mathcal{S}(1+t-s;(s))} = \chi_{\mathcal{D}(2;(s,s-t+1))} + \chi_{\mathcal{D}(s+d-t-1;(s))}, \quad (3.27)$$

whereas that of the special FT field modules  $\mathcal{S}(1+t-s;(s)) = \mathcal{V}(1+t-s;(s))$  with  $t \geq s$  satisfy

$$\chi_{\mathcal{S}(1+t-s;(s))} = \chi_{\mathcal{D}(1+t-s;(s))} + \chi_{\mathcal{V}(s+d-t-1;(s))}. \quad (3.28)$$

Using (3.27) and (3.28), one can check that the relation (2.42) holds for any value of  $\ell$ , i.e. both of  $\ell < \frac{d}{4}$  and  $\frac{d}{4} \leq \ell < \frac{d}{2}$  cases.

### 3.3 Character of type- $B_\ell$ theory

Finally, let us consider the type- $B_\ell$  CHS theory based on the order- $\ell$  spinor singleton which will be denoted  $\text{Di}_\ell$ . It is related to the type- $B_\ell$  MHS theory in an analogous way to the type- $A_\ell$  case. Here, we focus on the case where the value of  $\ell$  is smaller enough than the dimension  $d$  so that we do not encounter any subtle issue analogous to what we found in the type- $A_\ell$  theory.

The field content of the type- $B_\ell$  CHS theory,

$$\begin{aligned}
 \text{Type-}B_\ell \text{ CHS} = & \bigoplus_{t=-2(\ell-1)}^{2(\ell-1)} 2 \mathcal{D}(1+t; \mathbf{0}) \\
 & \oplus \bigoplus_{t=1}^{2\ell-1} 2(2-\delta_{t,2\ell-1}) \left[ \bigoplus_{s=1}^{t-1} \mathcal{D}(1-s+t; (s)) \oplus \bigoplus_{s=t}^{\infty} \mathcal{D}(2; (s, s-t+1)) \right. \\
 & \left. \oplus \bigoplus_{m=1}^{r-1} \left\{ \bigoplus_{s=1}^t \mathcal{D}(1-s+t; (s, \mathbf{1}^m)) \oplus \bigoplus_{s=t+1}^{\infty} \mathcal{D}(1; (s, s-t+1, \mathbf{1}^{m-1})) \right\} \right],
 \end{aligned} \tag{3.29}$$

can be read off from that of the type- $B_\ell$  MHS theory, which in turn is given by the corresponding Flato-Fronsdal theorem [99] (see also [100] for the field-theoretical approach):

$$\chi_{\text{Di}_\ell}^2 = \sum_{t=-2(\ell-1)}^{2(\ell-1)} 2 \chi_{\mathcal{D}(d-t-1; \mathbf{0})} + \sum_{t=1}^{2\ell-1} 2(2-\delta_{t,2\ell-1}) \sum_{m=0}^{r-1} \sum_{s=1}^{\infty} \chi_{\mathcal{D}(s+d-t-1; (s, \mathbf{1}^m))}. \tag{3.30}$$

The universal factor 2 arises because we consider the parity-invariant Di module which possesses the pseudo higher-spin currents with  $\gamma_{d+1}$  insertion. Note that the character  $2 \chi_{\mathcal{D}(s+d-t-1; (s, \mathbf{1}^{r-1}))}$  should be understood as  $\chi_{\mathcal{D}(s+d-t-1; (s, \mathbf{1}_+^{r-1}))} + \chi_{\mathcal{D}(s+d-t-1; (s, \mathbf{1}_-^{r-1}))}$ . This theory contains infinitely many higher-spin gauge fields corresponding to the modules  $\mathcal{D}(s+d-t-1; (s, \mathbf{1}^m))$  with  $s \geq t+1-\delta_{m,0}$ , whose Killing tensors form the type- $B_\ell$  higher-spin algebra. Note that for  $s < t+1-\delta_{m,0}$ , the generalized Verma module is irreducible:  $\mathcal{D}(s+d-t-1; (s, \mathbf{1}^m)) = \mathcal{V}(s+d-t-1; (s, \mathbf{1}^m))$ . The vector space of the type- $B_\ell$  higher-spin algebra can be decomposed into  $\mathfrak{so}(2, d)$ -modules as [78]

$$\text{Type-}B_\ell \text{ Algebra} \simeq \bigoplus_{t=1}^{2\ell-1} (2-\delta_{t,2\ell-1}) \bigoplus_{m=0}^{r-1} \bigoplus_{s=t+1-\delta_{m,0}}^{\infty} m \left\{ \begin{array}{c} \boxed{\phantom{0}} \\ \boxed{s-1} \\ \boxed{s-t} \\ \vdots \\ \boxed{\phantom{0}} \end{array} \right\}. \tag{3.31}$$

The character of the conformal Killing tensors appearing above is related to the character of the corresponding FT field module in  $M_d$  through

$$\begin{aligned}
 \chi_{\mathcal{D}(1; (s, s-t+1, \mathbf{1}^{m-1}))}(q, \mathbf{x}) = & \chi_{\mathcal{D}(1-s; (s-t, \mathbf{1}^m))}(q, \mathbf{x}) \\
 & + \chi_{\mathcal{D}(s+d-t-1; (s, \mathbf{1}^m))}(q^{-1}, \mathbf{x}) - \chi_{\mathcal{D}(s+d-t-1; (s, \mathbf{1}^m))}(q, \mathbf{x}),
 \end{aligned} \tag{3.32}$$

where  $m \geq 1$  and  $s \geq t+1$ . If the latter condition is not satisfied, then we get the special FT field module:

$$\chi_{\mathcal{D}(1-s+t; (s, \mathbf{1}^m))}(q, \mathbf{x}) = \chi_{\mathcal{V}(s+d-t-1; (s, \mathbf{1}^m))}(q^{-1}, \mathbf{x}) - \chi_{\mathcal{V}(s+d-t-1; (s, \mathbf{1}^m))}(q, \mathbf{x}), \tag{3.33}$$

We sum the characters (3.32) and (3.33) together with the  $m=0$  counterparts (3.19) and (3.23) over the field content of type- $B_\ell$  CHS theory (3.29), and use the Flato-Fronsdal theorem (3.30) to simplify the series. In the end, we find the relations (1.1) and (2.42) hold also for the type- $B_\ell$  theory.



## 4 Discussion

In this paper, we calculated the  $\mathfrak{so}(2, d)$  character of CHS gravity spectrum and showed that it coincides with that of its global symmetry algebra, namely (conformal) higher-spin algebra. The evaluation of the full character requires the summation of the characters of each field appearing in the spectrum of CHS gravity. The character of each conformal gauge field satisfies the relation (1.2), which provides a simple link to its associated conformal Killing tensor, the collection of which span the CHS symmetry algebra. Our key observation is that, in the character of each gauge field, the contributions besides the Killing tensor cancel out after the summation. Below, let us make two remarks: first about the key relation (1.2), then about the summation.

The relation (1.2) — whose concrete versions for partially-massless fields of symmetric and hook-symmetry types are given respectively in (3.19) and in (3.32) — is in fact very general. In appendix A, we established the relation for partially-massless fields of any depth and any symmetry: see (A.44). Note that this relation is valid also for non-gauge fields — e.g. for totally-symmetric and hook-symmetry type fields we have (3.23) and (3.33) — where only the Killing tensor contribution is absent: see (A.40) for the most general case. Another way to state this property is that the character of any  $d$ -dimensional conformal field, irrespective of whether it has gauge symmetry or not, is given by the difference of the characters of the associated field in  $\text{AdS}_{d+1}$  with different boundary conditions. The character for the field with Neumann condition is the same as the character with Dirichlet condition but  $q$  replaced by  $q^{-1}$ , plus — if the field has gauge symmetries — the character of the Killing tensor. Therefore, the character of a theory composed of multiple conformal fields is always given by the character of its global symmetries and the rest which is the difference of the character of the  $\text{AdS}_{d+1}$  counterpart theory under the flip  $q \rightarrow q^{-1}$ . Hence, when the character of the  $\text{AdS}_{d+1}$  theory is symmetric under  $q \rightarrow q^{-1}$ , only the character of the global symmetries survives as is the case for type- $A_\ell$  and type- $B_\ell$ . For this reason, our observation extends neither to the type-C case nor to the minimal theories, but holds for the type-AB case [101] and type-AZ theory [102].

Let us turn to the issue of the summation over spins or field contents. Since we have just equated, as series, the character of CHS gravity and that of its symmetries, we did not elaborate on whether this series is convergent or not. In fact, there are several issues. First, the character of the  $\text{AdS}_{d+1}$  theory is given by a series which is convergent inside the disk  $|q| < 1$ . It shows the symmetry  $q \rightarrow q^{-1}$  only after an analytic continuation, and hence the cancellation of these terms already involves a regularization. Next, the character of the higher-spin symmetries is given by a series which does not have any region of convergence. Nevertheless, we can still evaluate this series dividing it into the pieces having different regions of convergence, and this procedure makes sense only in terms of distributions (see [79] for more discussions). After these regularizations, we obtain the character of CHS gravity as a function of  $q$  and  $\mathbf{x}$ . For instance, for the type-A CHS theory in  $d = 4$  we find

$$\chi_{\text{CHS}}^{\mathfrak{so}(2,4)}(q, x_1, x_2) = - \left( \frac{q(1-q^2)}{(1-x_1q)(1-x_1^{-1}q)(1-x_2q)(1-x_2^{-1}q)} \right)^2 + (q \leftrightarrow x_1) + (q \leftrightarrow x_2). \quad (4.1)$$

Since the one-loop partition function is the character evaluated at  $\mathbf{x} = \mathbf{1}$ , we can attempt to get the CHS partition function from this character. The limit  $x_1, x_2 \rightarrow 1$  of the last two terms of (4.1) is singular, so we would need to subtract the finite part in the limit. However, this procedure is somewhat ambiguous as it depends on how we take this limit. Moreover, the simplest trials (e.g. sending  $x_1$  and  $x_2$  consecutively or simultaneously to 1) do not easily reproduce the result found in [64]:

$$\mathcal{Z}_{\text{CHS}}(q) = -\frac{q^2 (11 + 26q + 11q^2)}{6(1-q)^6}. \quad (4.2)$$

The above was obtained by summing the partition functions of each FT field (i.e. their individual characters evaluated in  $\mathbf{x} = \mathbf{1}$ ) with a damping parameter  $\epsilon$ , i.e.

$$\mathcal{Z}_{\text{CHS}}(q) := \oint_{\mathcal{C}} \frac{d\epsilon}{2\pi i \epsilon} \sum_{s=0}^{\infty} e^{-\epsilon(s+\frac{d-3}{2})} \chi_{\mathcal{D}(2;(s,s)_0)}^{\mathfrak{so}(2,4)}(q, \mathbf{x})|_{\mathbf{x}=\mathbf{1}}. \quad (4.3)$$

Here  $\mathcal{C}$  is a contour encircling the origin of the complex  $\epsilon$ -plane, so the contour integral picks up the finite part of the integrand. The shift  $\frac{d-3}{2}$  in the  $e^{-\epsilon(s+\frac{d-3}{2})}$  regularization is introduced to ensure the vanishing of the  $a$ -anomaly coefficient and the Casimir energy of CHS gravity [38, 80]. As already detailed in [64], a closer investigation reveals that the subtleties lie in the first term of the on-shell FT field character,

$$\begin{aligned} \chi_{\mathcal{D}(2;(s,s))}^{\mathfrak{so}(2,4)}(q, x_1, x_2) &= \chi_{\mathcal{V}(2;(s,s)_0)}^{\mathfrak{so}(2,4)}(q, x_1, x_2) - 2\chi_{\mathcal{V}(s+2;(s))}^{\mathfrak{so}(2,4)}(q, x_1, x_2) \\ &\quad + 2\chi_{\mathcal{V}(s+3;(s-1))}^{\mathfrak{so}(2,4)}(q, x_1, x_2), \end{aligned} \quad (4.4)$$

because the second and third terms in the above expression lead to convergent series as we sum over the spins. The first term is the character of the generalized Verma module with lowest weight  $[2; (s, s)_0]$ , so the spin-dependent part is simply the  $\mathfrak{so}(4)$  character,

$$\chi_{(s,s)_0}^{\mathfrak{so}(4)}(x_1, x_2) = \chi_s^{\mathfrak{so}(3)}(x_1 x_2) + \chi_s^{\mathfrak{so}(3)}(x_1 x_2^{-1}). \quad (4.5)$$

Therefore, we see that the issue is in fact how to sum the above character evaluated at  $x_1 = x_2 = 1$  over all spins. For a concrete understanding, let us consider

$$\sum_{s=0}^{\infty} \chi_s^{\mathfrak{so}(3)}(x) e^{-\epsilon(s+\frac{1}{2})} = \frac{e^{-\frac{1}{2}\epsilon} (1 + e^{-\epsilon}) x}{(e^{-\epsilon} - x)(-1 + e^{-\epsilon} x)}. \quad (4.6)$$

The above function is regular in either limit  $x \rightarrow 1$  or  $\epsilon \rightarrow 0$  but not the simultaneous limit. And depending on the evaluation order, we indeed obtain different results. Setting  $x$  to 1 and then extracting the finite part in the  $\epsilon \rightarrow 0$  expansion leads to the partition function (4.2), whereas setting first  $\epsilon$  to 0 and then extracting the finite part in the  $x \rightarrow 1$  expansion leads to

$$\mathcal{Z}_{\text{CHS}}(q) = -\frac{2q^2(1 + 4q + q^2)}{(1-q)^6}. \quad (4.7)$$

It is interesting to note that the above partition function also gives vanishing Casimir energy. The cancellation of the  $a$ -anomaly of CHS gravity can be also shown using the characters (see subsection 2.4).

Let us conclude by a few comments on CHS theories around an AdS background. The kinetic operator of spin- $s$  Fradkin-Tseytlin fields around  $\text{AdS}_d$  factorize into a product of kinetic operators of spin- $s$  partially-massless fields with all possible depths (together with a finite number of spin- $s$  massive fields in dimensions  $d \geq 4$ ) [39, 40, 45]. Notice that this property was recently generalized to the case of higher-depth Fradkin-Tseytlin fields in [41]. From a group-theoretical perspective, the factorization of the kinetic operator of Fradkin-Tseytlin fields is related to the branching rule of the corresponding module: as detailed in appendix C, the on-shell Fradkin-Tseytlin module branches into the  $\mathfrak{so}(2, d-1)$  modules which correspond to the kernels of each factor of the Fradkin-Tseytlin kinetic operator.<sup>18</sup> Consequently, the CHS gravity around an AdS background can be viewed as a *local* interacting theory of massless, partially-massless and massive (with “the mass squared” smaller than that of massless fields) fields of arbitrary spins. Such a field content as an  $\text{AdS}_d$  theory can be obtained by simply branching the representation carried by the CHS gravity spectrum from  $\mathfrak{so}(2, d)$  onto  $\mathfrak{so}(2, d-1)$ . For instance, in  $d = 4$  dimensions the type-A CHS theory spectrum branches onto the direct sum of partially-massless fields in  $\text{AdS}_4$  with all integer spins and depths.<sup>19</sup> Taking a definite boundary condition<sup>20</sup> we find that the  $\mathfrak{so}(2, 3)$  character of this theory reads

$$\sum_{s=1}^{\infty} \sum_{t=1}^s \chi_{\mathcal{D}(s+2-t; (s))}^{\mathfrak{so}(2,3)}(\beta, \alpha) = \frac{\sinh^2 \frac{\beta}{2} \cos \alpha - \sin^2 \frac{\alpha}{2}}{8 (\cosh \beta - \cos \alpha)^2 \sin^2 \frac{\alpha}{2} \sinh^2 \frac{\beta}{2}}, \quad (4.8)$$

where we have written the character in terms of the temperature  $\beta$  and the variable  $\alpha$  defined through  $q = e^{-\beta}$  and  $x = e^{i\alpha}$ . Using the method introduced in [81, 82], we can compute the one-loop free energy of this theory in Euclidean  $\text{AdS}_4$  using the above character. To do so, one needs to evaluate the character (4.8) and its derivatives (with respect to the variable  $\alpha$ ) in  $\alpha \rightarrow 0$ , which is singular here (contrary to MHS theories and their partially-massless cousins). As a consequence, one has to introduce a new regularization scheme to cure this additional source of divergence. We can easily think of two possibilities:

- To extract only the finite part of (4.8) in its  $\alpha \rightarrow 0$  expansion which leads to the one-loop free energy

$$\Gamma_{\text{AdS}_4}^{(1)} = -\frac{\zeta(3)}{192\pi^2} - \frac{\zeta(5)}{48\pi^4} - \frac{\zeta(7)}{64\pi^6}. \quad (4.9)$$

- To introduce a damping factor in the summation of the character,

$$\sum_{s=1}^{\infty} \sum_{t=1}^s e^{-\epsilon(s+\gamma)} \chi_{\mathcal{D}(s+2-t; (s))}^{\mathfrak{so}(2,3)}(\beta, \alpha), \quad (4.10)$$

so that the resulting character has a finite limit in  $\alpha \rightarrow 0$ . One can then extract the finite part in the  $\epsilon \rightarrow 0$  expansion and compute the one-loop free energy using the

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<sup>18</sup>A noteworthy subtlety here is that here appear both of the modules corresponding to the solutions with Dirichlet and Neumann boundary conditions.

<sup>19</sup>When  $d \geq 6$ , the branching rule will also contain an infinite number of massive fields with arbitrarily high integer spin.

<sup>20</sup>When  $d = 4$ , the partially-massless fields with Dirichlet and Neumann boundary conditions correspond to isomorphic modules.

resulting expression. Using a shift  $\gamma = \frac{1}{2}$  as in the case of MHS or CHS theory in four dimensions, we end up with

$$\Gamma_{\text{AdS}_4}^{(1)} = -\frac{\zeta(3)}{7680\pi^2} - \frac{\zeta(5)}{384\pi^4}. \quad (4.11)$$

None of the results (4.10) and (4.11) is particularly convincing, and hence we would need clearer guidelines for the regularization from other inspections. This naturally calls for a better understanding of CHS gravity from the point of view of an “all-depth” partially-massless higher-spin theory in  $\text{AdS}_d$ . In fact the latter perspective is interesting to explore in its own right and there are several amusing questions. The first question is whether CHS gravity can afford a truncation like the one from Weyl gravity to Einstein gravity [36, 37]. The truncation to MHS fields would not be consistent since there is no subalgebra of the CHS algebra compatible with the truncated spectrum, but the truncation to all odd-depth fields may lead to a consistent theory [103]. The second question is what might be the  $\text{CFT}_{d-1}$  dual of CHS gravity viewed as an exotic partially-massless higher-spin theory in  $\text{AdS}_d$ . It cannot be one of the usual free CFTs as they do not exhibit such operator spectra, but their exotic cousins might be decent candidates: in [103] the non-local free scalar CFT with the kinetic operator  $\sqrt{\square}$  was proposed as the dual of the odd-depth truncation of CHS gravity in AdS. The third question is whether CHS gravity can be defined as a local theory in certain odd dimensions. It is well known that three-dimensional CHS gravity can be realized as a Chern-Simons theory. Therefore, it is natural to inquire whether the higher-dimensional Chern-Simons actions may prove useful for the local realization of some parity-breaking CHS gravity in odd dimensions. Lastly, assuming this works, one might then speculate whether one can iterate the construction [65]: MHS gravity in  $\text{AdS}_{d+1}$  leads to CHS gravity in  $\text{AdS}_d$ , which in turn could lead to another class of CHS gravity in  $\text{AdS}_{d-1}$ , etc.

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## A Zoo of modules

In this appendix, we review the description of the various fields of interest for this paper in terms of the corresponding  $\mathfrak{so}(2, d)$ -modules, as well as the associated characters. To

do so systematically, we go through the BGG resolution of these modules, following the discussions in [57, 64, 104] (see also [73, 105] for more details on the relevant characters from a different perspective).

- *AdS<sub>d+1</sub> gauge field module / CFT<sub>d</sub> current:* the module defined by the lowest weight

$$[s_p + d - p - t; (s_1, \dots, s_p, s_{p+1}, \dots, s_r)], \quad 1 \leq t \leq s_p - s_{p+1}, \quad (\text{A.1})$$

corresponds to a partially-massless mixed-symmetry field with spin  $\mathbb{Y} = (s_1, \dots, s_r)$  and minimal energy  $\Delta_p^{(t)} := s_p + d - p - t$ . This irreducible module is defined as the quotient of the generalized Verma module induced by the  $\mathfrak{so}(2) \oplus \mathfrak{so}(d)$  module of weight  $[\Delta_p^{(t)}; \mathbb{Y}]$ , by its maximal submodule. The latter is the module whose lowest weight reads

$$[s_p + d - p; (s_1, \dots, s_{p-1}, \underset{\substack{\uparrow \\ p\text{th}}}{s_p - t}, s_{p+1}, \dots, s_r)]. \quad (\text{A.2})$$

The presence of this submodule signals the fact that these partially massless fields are subject to gauge symmetries, namely the submodule of lowest weight  $[\Delta_p^{(t)} + t; \mathbb{Y}_{p,t}]$  with

$$\mathbb{Y}_{p,t} := (s_1, \dots, s_{p-1}, s_p - t, s_{p+1}, \dots, s_r). \quad (\text{A.3})$$

The resulting quotient,

$$\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y}) = \frac{\mathcal{V}(\Delta_p^{(t)}; \mathbb{Y})}{\mathcal{D}(\Delta_p^{(t)} + t; \mathbb{Y}_{p,t})}, \quad (\text{A.4})$$

can be realized as the space of traceless and divergenceless tensor with symmetry  $\mathbb{Y}$  and solution to the wave equation

$$(\nabla^2 - m_{p,t}^2) \varphi_{\mathbb{Y}} = 0, \quad m_{p,t}^2 := \Delta_p^{(t)}(\Delta_p^{(t)} - d) - \sum_{k=1}^r s_k, \quad (\text{A.5})$$

modulo the gauge transformations

$$\delta_{\xi} \varphi_{\mathbb{Y}} = \nabla^t \xi_{\mathbb{Y}_{p,t}}, \quad (\text{A.6})$$

where  $\xi_{\mathbb{Y}_{p,t}}$  is also a traceless and divergenceless tensor with symmetry  $\mathbb{Y}_{p,t}$  and subject to a wave equation similar that of  $\varphi_{\mathbb{Y}}$  (see [106–111] for more details). When  $p > 1$ , the gauge symmetry (A.6) is reducible, i.e. it is trivial for some particular class of gauge parameters. This implies that the module  $\mathcal{D}(\Delta_p^{(t)} + t; \mathbb{Y}_{p,t})$  of gauge parameters is itself a quotient, obtained by modding out of  $\mathcal{V}(\Delta_p^{(t)} + t; \mathbb{Y}_{p,t})$  the irreducible submodule describing the parameters leading to trivial gauge transformations. For a generic partially-massless as we are considering here, there are  $p$  classes of “gauge parameters”, the genuine one with diagram  $\mathbb{Y}_{p,t}$  and  $(p-1)$  reducibilities, which are obtained from the BGG resolution. Denoting  $\mathbb{Y}_{p,t}^{(k)}$  the Young diagram describing the  $k$ th of these reducibility parameters, they can be expressed in terms of  $\mathbb{Y}_{p,t}$  by

$$\mathbb{Y}_{p,t}^{(k)} = (s_1, \dots, s_{p-k-1}, s_{p-k} - \underset{\substack{\uparrow \\ (p-k)\text{th}}}{1} - n_{k,p}, \dots, s_{p-1} - 1 - n_{1,p}, s_p - t, s_{p+1}, \dots, s_r) \quad (\text{A.7})$$

$\uparrow$   $p$ th

with

$$n_{j,p} := s_{p-j} - s_{p-j+1}, \quad 1 \leq k \leq p-1 \quad (\text{A.8})$$

with by convention  $n_{0,p} = 0$ . Defining

$$\nu_{k,p} := \sum_{j=1}^k n_{k,p}, \quad 1 \leq k \leq p-1, \quad (\text{A.9})$$

with again the convention that  $\nu_{0,p} = 0$ , the minimal energy of these reducibility parameters reads

$$\Delta_p^{(t)} + t + k + \nu_{k,p}. \quad (\text{A.10})$$

Schematically, each reducibility parameter is obtained from the previous one by removing more and more boxes. Starting from the gauge parameter, where  $t$  boxes are removed from the  $p$ th row, one then obtain the first reducibility parameter by removing boxes from the row above (the  $(p-1)$ th one) until this row has one less box than the one below (the  $p$ th one) in the original Young diagram (i.e.  $s_p - 1$ ). Applying the same procedure to the obtained reducibility parameter, one can construct the next one and so on.

From the  $d$ -dimensional point of view, these modules describe (partially-)conserved currents of spin  $\mathbb{Y}$ , i.e. operators of conformal weight  $\Delta_p^{(t)}$  which are traceless tensors with the symmetry properties of the Young diagram  $\mathbb{Y}$ . The conservation law obeyed by these operators is of the form

$$\mathbb{P}_T^{\mathbb{Y}_{p,t}}(\partial^t J_{\mathbb{Y}}) \approx 0, \quad (\text{A.11})$$

where  $\mathbb{P}_T^{\mathbb{Y}_{p,t}}$  is a traceless projector onto  $\mathbb{Y}_{p,t}$ . In this context the submodules involved in the definition of this irreducible module are interpreted slightly differently than in the  $\text{AdS}_{d+1}$  case. The first submodule is spanned by the conservation law (A.11), i.e. a suitable projection of a (or several) divergence(s), and all its descendants. The remaining submodules correspond to descendants of the conservation law which are identically<sup>21</sup> vanishing (i.e. for symmetry reasons).

We can now write down (irrespective of the parity of  $d$ ) the character of the module gauge field module  $\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y})$  as follows:

$$\chi_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y})}^{\text{so}(2,d)}(q, \mathbf{x}) = \left( q^{\Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\text{so}(d)}(\mathbf{x}) + \sum_{k=0}^{p-1} (-1)^{k+1} q^{\Delta_p^{(t)} + t + k + \nu_{k,p}} \chi_{\mathbb{Y}_{p,t}^{(k)}}^{\text{so}(d)}(\mathbf{x}) \right) \mathcal{P}_d(q, \mathbf{x}), \quad (\text{A.12})$$

with the convention that  $\mathbb{Y}_{p,t}^{(0)} = \mathbb{Y}_{p,t}$ .

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<sup>21</sup>For instance, the module  $\mathcal{D}(d-3; (1,1))$  corresponds to an antisymmetric tensor  $F_{ab} = -F_{ba}$  obeying the conservation law  $\partial^b F_{ab} \approx 0$ . Its first submodule is  $\mathcal{D}(d-2; (1))$  correspond to the image of the divergence of  $F_{ab}$  in  $\mathcal{D}(d-3; (1,1))$ , while the second submodule  $\mathcal{D}(d-1; (0))$  is the image of the trivially conserved quantity  $\partial^a \partial^b F_{ab}$  in  $\mathcal{D}(d-2; (1))$ .

- *Curvature of the gauge field in  $AdS_{d+1}$* : the module whose lowest weight reads

$$[s_{p+1} + d - p - 1; (s_1, \dots, s_p, \overset{\uparrow}{s_p - t + 1}, s_{p+2}, \dots, s_r)] \quad (\text{A.13})$$

(p+1)th

corresponds to the module of the curvature  $R_{\mathbb{Y}_c}$  of the gauge field  $\varphi_{\mathbb{Y}}$ . It is obtained by taking  $s_p - s_{p+1} - t + 1$  derivative of  $\varphi_{\mathbb{Y}}$  and projecting them so that the resulting object as the symmetry property of the Young diagram  $\mathbb{Y}_c$  which is obtained from  $\mathbb{Y}$  by adding  $s_p - s_{p+1} - t + 1$  to the  $(p+1)$ th row

$$\mathbb{Y}_c = (s_1, \dots, s_p, \overset{\uparrow}{s_p - t + 1}, s_{p+2}, \dots, s_r). \quad (\text{A.14})$$

(p+1)th

Schematically, the curvature is given by

$$R_{\mathbb{Y}_c} = \mathbb{P}^{\mathbb{Y}_c} \left( \partial^{\Delta_p^{(t)} - \Delta_c} \phi_{\mathbb{Y}} \right), \quad (\text{A.15})$$

where  $\mathbb{P}_T^{\mathbb{Y}_c}$  is the projector onto the symmetry of the Young diagram  $\mathbb{Y}_c$  (without any tracelessness condition). Accordingly, the minimal energy of the module of the curvature tensor is given by

$$\Delta_c = s_{p+1} + d - p - 1 \equiv \Delta_p^{(t)} + s_{p+1} - s_p + t - 1, \quad (\text{A.16})$$

i.e. the minimal energy of the gauge field  $\varphi_{\mathbb{Y}}$  added by the number of derivative acting on it to make up the curvature. The character of this module reads (irrespectively of the parity of  $d$ )

$$\begin{aligned} \chi_{\mathcal{D}(\Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2, d)}(q, \mathbf{x}) = & \left( q^{\Delta_c} \chi_{\mathbb{Y}_c}^{\mathfrak{so}(d)}(\mathbf{x}) - q^{\Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) \right. \\ & \left. + \sum_{k=0}^{p-1} (-1)^k q^{\Delta_p^{(t)} + t + k + \nu_{k,p}} \chi_{\mathbb{Y}_{p,t}^{(k)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \mathcal{P}_d(q, \mathbf{x}). \end{aligned} \quad (\text{A.17})$$

- *Conformal Killing tensor*: the finite-dimensional module whose BGG resolution contains the above gauge field module corresponds to that of its conformal Killing tensor. In  $AdS_{d+1}$ , this corresponds to the space of solution of

$$\mathbb{P}^{\mathbb{Y}} \left( \nabla^t \xi_{\mathbb{Y}_{p,t}} \right) = 0, \quad (\text{A.18})$$

where  $\mathbb{P}^{\mathbb{Y}}$  projects onto the symmetry of the Young diagram  $\mathbb{Y}$  without any tracelessness condition, whereas in  $M_d$  this module corresponds to the space of solution of the conformal Killing equation

$$\mathbb{P}_T^{\mathbb{Y}} \left( \partial^t \xi_{\mathbb{Y}_{p,t}} \right) = 0. \quad (\text{A.19})$$

In terms of  $\mathfrak{so}(2, d)$ -weight, the conformal Killing tensor module has lowest weight

$$[1 - s_1; (s_2 - 1, \dots, s_p - 1, s_p - t, s_{p+1}, \dots, s_{r-1}, (-)^{d+1} s_r)]. \quad (\text{A.20})$$

Written in terms of its  $\mathfrak{so}(2) \oplus \mathfrak{so}(d)$  components, this character reads

$$\begin{aligned}
 \chi_{\mathcal{D}(1-s_1; (s_2-1, \dots, s_p-1, s_p-t, s_{p+1}, \dots, s_{r-1}, (-)^{d+1}s_r))}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = & \quad (\text{A.21}) \\
 \mathcal{P}_d(q, \mathbf{x}) \left[ \sum_{k=1}^p (-1)^{k-1} \left( q^{d-\Delta_p^{(t)}-t-p+k-\nu_{p-k,p}} \chi_{\mathbb{Y}_{p,t,-}^{(p-k)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right. \right. \\
 & \left. \left. + (-)^d q^{\Delta_p^{(t)}+t+p-k+\nu_{p-k,p}} \chi_{\mathbb{Y}_{p,t,+}^{(p-k)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \right. \\
 & + (-1)^p \left( q^{d-\Delta_p^{(t)}} \chi_{\mathbb{Y}_-}^{\mathfrak{so}(d)}(\mathbf{x}) + (-)^d q^{\Delta_p^{(t)}} \chi_{\mathbb{Y}_+}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \\
 & \left. + \sum_{m=0}^{r-p-1} (-1)^{m+p+1} \left( q^{d-\Delta_c+\bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,-}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) + (-)^d q^{\Delta_c-\bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,+}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \right],
 \end{aligned}$$

where

$$\mathbb{Y}_{c,\pm}^{(m)} = (s_1, \dots, s_p, \underset{\substack{\uparrow \\ (p+1)\text{th}}}{s_p-t+1}, s_{p+2} + \bar{n}_{1,p}, \dots, s_{p+m+1} + \underset{\substack{\uparrow \\ (p+m+1)\text{th}}}{\bar{n}_{m,p}}, s_{p+m+2}, \dots, \pm s_r) \quad (\text{A.22})$$

with

$$\bar{n}_{j,p} := s_{p+j} - s_{p+j+1} + 1, \quad \bar{\nu}_{m,p} := \sum_{j=1}^m \bar{n}_{j,p}, \quad 1 \leq j \leq r-p-1, \quad (\text{A.23})$$

and by convention  $\bar{\nu}_{0,p} = 0$ . Its character coincides with that of the finite-dimensional  $\mathfrak{so}(2+d)$  representation

$$(s_1 - 1, s_2 - 1, \dots, s_p - 1, s_p - t, s_{p+1}, \dots, s_{r-1}, (-)^{d+1}s_r), \quad (\text{A.24})$$

evaluated in  $(q^{-1}, \mathbf{x})$ , i.e.

$$\begin{aligned}
 \chi_{\mathcal{D}(1-s_1; (s_2-1, \dots, s_p-1, s_p-t, s_{p+1}, \dots, (-)^{d+1}s_r))}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) & \quad (\text{A.25}) \\
 = \chi_{(s_1-1, s_2-1, \dots, s_p-1, s_p-t, s_{p+1}, \dots, (-)^{d+1}s_r)}^{\mathfrak{so}(d+2)}(q^{-1}, \mathbf{x}).
 \end{aligned}$$

- *On-shell Fradkin-Tseytlin field in  $M_d$* : the module defined by the lowest weight

$$[p+1-s_{p+1}; (s_1, \dots, s_p, \underset{\substack{\uparrow \\ (p+1)\text{th}}}{s_p-t+1}, s_{p+2}, \dots, (-)^{d+1}s_r)] \quad (\text{A.26})$$

corresponds to the on-shell Weyl tensor  $C_{\mathbb{Y}_c}$  of the Fradkin-Tseytlin module, which is obtained by acting with  $\Delta_p^{(t)} - \Delta_c = s_p - s_{p+1} - t + 1$  derivatives on  $\phi_{\mathbb{Y}}$ , projected so that the resulting tensor has the symmetry of  $\mathbb{Y}_c$ . Schematically, we have

$$C_{\mathbb{Y}_c} = \mathbb{P}_{\text{T}}^{\mathbb{Y}_c} \left( \partial^{\Delta_p^{(t)} - \Delta_c} \phi_{\mathbb{Y}} \right), \quad (\text{A.27})$$



where  $\mathbb{P}_T^{\mathbb{Y}_c}$  is the traceless projector onto the symmetry of the Young diagram  $\mathbb{Y}_c$ . The character of this module reads, for  $d = 2r$ :

$$\chi_{\mathcal{D}(d-\Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = \mathcal{P}_d(q, \mathbf{x}) \left[ \sum_{m=0}^{r-p-1} (-1)^m \left( q^{d-\Delta_c+\bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,-}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) + q^{\Delta_c-\bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,+}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) - 2 \left( q^{\Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) + \sum_{k=0}^{p-1} (-1)^{k+1} q^{\Delta_p^{(t)}+t+k+\nu_{k,p}} \chi_{\mathbb{Y}_{p,t}^{(k)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \right], \quad (\text{A.28})$$

and for  $d = 2r + 1$ :

$$\chi_{\mathcal{D}(-\Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = \mathcal{P}_d(q, \mathbf{x}) \left[ \sum_{m=0}^{r-p-1} (-1)^m \left( q^{d-\Delta_c+\bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,-}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) - q^{\Delta_c-\bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,+}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) + q^{\Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) + \sum_{k=0}^{p-1} (-1)^{k+1} q^{\Delta_p^{(t)}+t+k+\nu_{k,p}} \chi_{\mathbb{Y}_{p,t}^{(k)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right]. \quad (\text{A.29})$$

Notice that in both cases, characters of modules labelled by the  $\mathfrak{so}(d)$ -weights  $\mathbb{Y}_c^{(m)}$  correspond to the generalized Bianchi identities verified by the Weyl tensor.

- *Off-shell Fradkin-Tseytlin field in  $M_d$* : the off-shell Fradkin-Tseytlin, or shadow, field  $\phi_{\mathbb{Y}}$  associated to the (partially) massless field  $\varphi_{\mathbb{Y}}$  is a field of conformal weight

$$\Delta_{\phi_{\mathbb{Y}}} = d - \Delta_p^{(t)} = p + t - s_p, \quad (\text{A.30})$$

and spin  $\mathbb{Y}$ , subject to the gauge transformation

$$\delta_{\xi, \sigma} = \mathbb{P}^{\mathbb{Y}} \left( \partial^t \xi_{\mathbb{Y}_{p,t}} + \eta \sigma_{\check{\mathbb{Y}}} \right), \quad (\text{A.31})$$

where  $\mathbb{P}^{\mathbb{Y}}$  project onto the symmetry of  $\mathbb{Y}$  whereas  $\check{\mathbb{Y}}$  denotes any Young diagram obtained by taking a trace of  $\mathbb{Y}$ , so that  $\sigma_{\check{\mathbb{Y}}}$  should be understood as a collection of Weyl gauge parameters with the symmetry of all possible traces of  $\mathbb{Y}$ . As already mentioned previously, this field does not correspond to a generalized Verma module and consequently is not found in the BGG resolution in even dimension. However, as proposed in [64], the corresponding character can still be computed by translating the field-theoretical definition of this conformal gauge field. In even  $d$  dimensions, we therefore define

$$\chi_{S(d-\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) := q^{d-\Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) \mathcal{P}_d(q, \mathbf{x}) - \chi_{\mathcal{U}(d-\Delta_p^{(t)}; \mathbb{Y}_-)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}), \quad (\text{A.32})$$

where:

- The first term, which is the character of the generalized module with lowest weight  $(d - \Delta_p^{(t)}; \mathbb{Y})$  is meant to represent the fact that the shadow field is a tensor of symmetry  $\mathbb{Y}$  and with conformal weight  $d - \Delta_p^{(t)}$ ;

- The second term is the character of the module  $\mathcal{U}(d - \Delta_p^{(t)}; \mathbb{Y}_-)$ . This module appears in the BGG resolution as the maximal submodule of  $\mathcal{V}(d - \Delta_p^{(t)} - t; \mathbb{Y}_{p,t,-})$ , and corresponds to the pure gauge modes of the shadow field associated to the gauge symmetry (A.31).

The character  $\chi_{\mathcal{U}(d - \Delta_p^{(t)}; \mathbb{Y}_-)}^{\mathfrak{so}(2,d)}(q, \mathbf{x})$  can be computed following the algorithm spelled out in [64], which yields

$$\begin{aligned} \chi_{\mathcal{U}(d - \Delta_p^{(t)}; \mathbb{Y}_-)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = & \mathcal{P}_d(q, \mathbf{x}) \left[ q^{d - \Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) \right. \\ & - \sum_{m=0}^{r-p-1} (-1)^m \left( q^{d - \Delta_c + \bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,-}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) + q^{\Delta_c - \bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,+}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \\ & \left. + q^{\Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) + \sum_{k=0}^{p-1} (-1)^{k+1} q^{\Delta_p^{(t)} + t + k + \nu_{k,p}} \chi_{\mathbb{Y}_{p,t}^{(k)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right]. \end{aligned} \quad (\text{A.33})$$

Upon using (A.33), we can now write explicitly the character of the shadow field  $\phi_{\mathbb{Y}}$  as

$$\begin{aligned} \chi_{\mathcal{S}(d - \Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = & \mathcal{P}_d(q, \mathbf{x}) \left[ \sum_{m=0}^{r-p-1} (-1)^m \left( q^{d - \Delta_c + \bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,-}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) + q^{\Delta_c - \bar{\nu}_{m,p}} \chi_{\mathbb{Y}_{c,+}^{(m)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \right. \\ & \left. - \left( q^{\Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) + \sum_{k=0}^{p-1} (-1)^{k+1} q^{\Delta_p^{(t)} + t + k + \nu_{k,p}} \chi_{\mathbb{Y}_{p,t}^{(k)}}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \right]. \end{aligned} \quad (\text{A.34})$$

As pointed out in section 2, the shadow field module in odd dimension is isomorphic to that of the off-shell Weyl tensor in odd dimensions, and hence their character are identical. It is worth noticing though that the  $d = 2r + 1$  case can be describe in a similar way to the  $d = 2r$  case, by defining

$$\chi_{\mathcal{S}(d - \Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) := q^{d - \Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) \mathcal{P}_d(q, \mathbf{x}) - \chi_{\mathcal{D}(d - \Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}). \quad (\text{A.35})$$

The two terms here have the same interpretation as in the  $d = 2r$  case, except the module of the pure gauge modes of the shadow field  $\mathcal{D}(d - \Delta_p^{(t)}; \mathbb{Y})$  is irreducible in  $d = 2r + 1$ . As a consequence, this module is now part of the BGG resolution in the sense that it is isomorphic to the quotient

$$\mathcal{D}(d - \Delta_p^{(t)}; \mathbb{Y}) \cong \frac{\mathcal{V}(d - \Delta_p^{(t)}; \mathbb{Y})}{\mathcal{D}(d - \Delta_c; \mathbb{Y}_c)}, \quad (\text{A.36})$$

where one can recognize the module corresponding to the off-shell Weyl tensor in the denominator of the above equation. Its character therefore reads

$$\chi_{\mathcal{D}(d - \Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = q^{d - \Delta_p^{(t)}} \chi_{\mathbb{Y}}^{\mathfrak{so}(d)}(\mathbf{x}) \mathcal{P}_d(q, \mathbf{x}) - \chi_{\mathcal{D}(d - \Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}), \quad (\text{A.37})$$

and hence using the definition (A.35) we recover the expected result

$$\chi_{\mathcal{S}(d-\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = \chi_{\mathcal{D}(d-\Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}), \quad (\text{A.38})$$

in accordance with the fact that the module of the shadow field is isomorphic to that of the off-shell Weyl tensor.

**Series of modules with non-integral weight.** Another class of irreducible modules, with non-integral weight, can exist and are defined by the quotient

$$\mathcal{D}(\Delta; \mathbb{Y}) \cong \frac{\mathcal{V}(\Delta; \mathbb{Y}_+)}{\mathcal{D}(d-\Delta; \mathbb{Y}_-)} \quad (\text{A.39})$$

with  $\mathbb{Y}_{\pm} = (s_1, \dots, s_{r-1}, (\pm)^{d+1} s_r)$  an arbitrary  $\mathfrak{so}(d)$  integral dominant weight and

- For  $d = 2r$ :  $\Delta = k - s_k$  for  $k = 1, \dots, r$  with  $k = r$  only if  $s_r \neq 0$ ;
- For  $d = 2r + 1$ :  $\Delta = \frac{d-2\ell}{2}$  with  $\ell = 1, \dots, r$  for bosonic irreps, or  $\Delta = \frac{d+1-2\ell}{2}$  for fermionic irreps (with the same condition on  $\ell$  as in the bosonic case).

The character of the module (A.39) is given by

$$\chi_{\mathcal{D}(\Delta; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = \left( q^{\Delta} \chi_{\mathbb{Y}_+}^{\mathfrak{so}(d)}(\mathbf{x}) - q^{d-\Delta} \chi_{\mathbb{Y}_-}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \mathcal{P}_d(q, \mathbf{x}). \quad (\text{A.40})$$

The scalar and the spinor order- $\ell$  singletons (respectively,  $\text{Rac}_{\ell}$  and  $\text{Di}_{\ell}$ ) are part of this class of module [31], as their conformal weight read

$$\Delta = \frac{d-2\ell}{2} + s, \quad (\text{A.41})$$

with  $s = 0$  or  $s = \frac{1}{2}$  (i.e. the spin of these fields). It is clear from the above conditions that in odd dimension  $d$ , these two families of singletons exhaust the fields described by this class of module. However, for  $d$  even one can consider more general conformal fields (in particular they can be of spin greater or equal to one). In general, the above class can be realized as conformal fields  $\phi_{\mathbb{Y}}$  of spin  $\mathbb{Y}$  obeying to a wave equation of the form:

$$\partial^{d-2\Delta} \phi_{\mathbb{Y}} \approx 0. \quad (\text{A.42})$$

Notice that even if  $\mathbb{Y} \neq \mathbf{0}$ , these fields do not enjoy any gauge symmetry (contrarily to, for instance, the usual CHS fields sourcing the conserved current in (2.7)).

**Character relations.** Now that we have written down the character of the modules involved in the description of CHS fields, we can derive the relation (1.2) used throughout the paper. Combining (A.12), (A.21) and (A.29), one can see that the three character, for  $d = 2r$ , are related by

$$\begin{aligned} & \chi_{\mathcal{D}(1-s_1; (s_2-1, \dots, s_p-1, s_p-t, s_{p+1}, \dots, -s_r))}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) \\ &= (-1)^{p+1} \left( \chi_{\mathcal{D}(d-\Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) + \chi_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y}_+)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) - \chi_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y}_-)}^{\mathfrak{so}(2,d)}(q^{-1}, \mathbf{x}) \right). \end{aligned} \quad (\text{A.43})$$

On top of that, one can derive the following relation between (A.12), (A.29) and (A.34)

$$\chi_{\mathcal{S}(d-\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = \chi_{\mathcal{D}(d-\Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) + \chi_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}), \quad (\text{A.44})$$

which, upon using the previously derived (A.43), also leads to

$$\chi_{\mathcal{S}(d-\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = (-1)^{p+1} \chi_{\mathcal{D}(1-s_1; (s_2-1, \dots, s_p-1, s_p-t, s_{p+1}, \dots, -s_r))}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) + \chi_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y}_-)}^{\mathfrak{so}(2,d)}(q^{-1}, \mathbf{x}). \quad (\text{A.45})$$

For  $d = 2r + 1$  however, the relation reads

$$\chi_{\mathcal{D}(1-s_1; (s_2-1, \dots, s_p-1, s_p-t, s_{p+1}, \dots, s_r))}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) = (-1)^{p+1} \left( \chi_{\mathcal{D}(d-\Delta_c; \mathbb{Y}_c)}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) + \chi_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q^{-1}, \mathbf{x}) \right). \quad (\text{A.46})$$

Recalling that for odd  $d$  the character of the Weyl module is identical to that of the shadow field, we can write the following relation valid in any dimension:

$$\begin{aligned} \chi_{\mathcal{D}(1-s_1; (s_2-1, \dots, s_p-1, s_p-t, s_{p+1}, \dots, (-1)^{d+1}s_r))}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) \\ = (-1)^{p+1} \left( \mathcal{S}_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) + (-1)^{d+1} \chi_{\mathcal{D}(\Delta_p^{(t)}; \mathbb{Y})}^{\mathfrak{so}(2,d)}(q^{-1}, \mathbf{x}) \right). \end{aligned} \quad (\text{A.47})$$

The above relation allows to express the character of the shadow field in terms of the character of the associated partially massless field, plus or minus the character of their conformal Killing tensor (depending on the symmetry of these fields).

**Action principles in the metric-like formulation.** Let us conclude this appendix by commenting on the action principle for a free conformal field of spin  $\mathbb{Y}$  and conformal weight  $d - \Delta_p^{(t)}$ . Schematically, it takes the form

$$S_{\text{CHS}}[\phi_{\mathbb{Y}}] = \int_{M_d} d^d x \, \phi_{\mathbb{Y}} \mathbb{P}_{\text{TT}}^{\mathbb{Y}}(\partial^{2\Delta_p^{(t)}-d} \phi_{\mathbb{Y}}) = (-1)^{\Delta_p^{(t)}-\Delta_c} \int_{M_d} d^d x \, C_{\mathbb{Y}_c} \partial^{2\Delta_c-d} C_{\mathbb{Y}_c}. \quad (\text{A.48})$$

Notice that the number of derivatives involved in this action can be constrained by requiring the latter to be quadratic in the shadow field with spin  $\mathbb{Y}$ , and that the integrand has conformal weight  $d$ . A careful analysis of such action principles can be found in [21], where in particular the form of the action (A.48) is derived by requiring it to be conformally invariant (see also [95] for the case of totally symmetric fields).

## B Two-dimensional case

The  $d = 2$  conformal algebra is a direct sum of two  $d = 1$  conformal algebras:  $\mathfrak{so}(2, 2) = \mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1)$  corresponding to left and right movers if we write the flat metric in light-cone coordinates  $x^\pm$  on  $M_2$  as  $ds^2 = 2dx^+ dx^-$ . Accordingly, the spin label can have positive/negative real values corresponding to left/right chirality. The  $\mathfrak{so}(2, 2)$  Verma modules will be denoted as  $\mathcal{V}(\Delta; s)$  and are related to the  $\mathfrak{so}(2, 1)$  Verma module  $\mathcal{V}_w$  with lowest weight  $w$  as

$$\mathcal{V}(\Delta; s) = \mathcal{V}_{\frac{\Delta+s}{2}} \otimes \mathcal{V}_{\frac{\Delta-s}{2}} \quad (\text{B.1})$$

Introducing the complex variables,

$$z = q x = e^{-\beta+i\alpha}, \quad \bar{z} = q x^{-1} = e^{-\beta-i\alpha}, \quad (\text{B.2})$$

for the  $\mathfrak{so}(2,2)$  weights, the character of  $\mathcal{V}(\Delta; s)$  is given by

$$\chi_{\mathcal{V}(\Delta; s)}^{\mathfrak{so}(2,2)}(q, x) = \chi_{\mathcal{V}_{\frac{\Delta+s}{2}}}^{\mathfrak{so}(2,1)}(z) \chi_{\mathcal{V}_{\frac{\Delta-s}{2}}}^{\mathfrak{so}(2,1)}(\bar{z}) = \frac{z^{\frac{\Delta+s}{2}} \bar{z}^{\frac{\Delta-s}{2}}}{(1-z)(1-\bar{z})}. \quad (\text{B.3})$$

The chirality-invariant modules will be denoted with 0 as subscript, e.g.  $\mathcal{V}(\Delta; s)_0 := \mathcal{V}(\Delta; +s) \oplus \mathcal{V}(\Delta; -s)$ .

**Conserved currents.** The modules of conserved currents remain well-defined for  $d = 2$ . In particular, if one applies the definition (2.16) in each sector separately, then one gets

$$\mathcal{D}(s; +s) := \frac{\mathcal{V}(s; +s)}{\mathcal{V}(s+1; s-1)} \cong \mathcal{V}_s \otimes 1 \quad (\text{B.4})$$

$$\mathcal{D}(s; -s) := \frac{\mathcal{V}(s; -s)}{\mathcal{V}(s+1; 1-s)} \cong 1 \otimes \mathcal{V}_s \quad (\text{B.5})$$

for the two chiralities, where 1 stands for the trivial representation of  $\mathfrak{so}(2,1)$  and we made use of (B.1) and  $\mathcal{V}_0/\mathcal{V}_1 \cong 1$ . Accordingly, their respective characters are

$$\chi_{\mathcal{D}(s; +s)}^{\mathfrak{so}(2,2)}(q, x) = \frac{z^s}{1-z} = \chi_{\mathcal{V}_s}^{\mathfrak{so}(2,1)}(z), \quad (\text{B.6})$$

$$\chi_{\mathcal{D}(s; -s)}^{\mathfrak{so}(2,2)}(q, x) = \frac{\bar{z}^s}{1-\bar{z}} = \chi_{\mathcal{V}_s}^{\mathfrak{so}(2,1)}(\bar{z}). \quad (\text{B.7})$$

The above can be straightforwardly generalized to the partially-conserved currents: (see e.g. section 6 of [112]).

For  $s \geq 2$ , one can identify the parity-invariant modules  $\mathcal{D}(s; s)_0 := \mathcal{D}(s; +s) \oplus \mathcal{D}(s; -s)$  as describing the spin- $s$  traceless conserved currents. Indeed, the tracelessness implies that all components of the type  $J_{+ \dots +}$  vanish, in which case the conservation condition becomes either  $\partial_- J_{+ \dots +} = 0$  or  $\partial_- J_{- \dots -} = 0$ . This leads to  $J_{\pm \dots \pm} = \varphi_{\pm}^{(s)}(x^{\pm})$ , where  $\varphi_{\pm}^{(w)}$  are densities of weight  $w$  in one dimension described as the  $\mathfrak{so}(2,1)$  Verma module  $\mathcal{V}_w$ . The spins  $s < 2$  require a more careful discussion (see e.g. section 4.2 of [79]) so we will just mention that  $\text{Di} = \mathcal{D}(\frac{1}{2}; \frac{1}{2})_0$  describes the conformal spinor while  $\mathcal{D}(1; \pm 1)$  describe a left/right chiral boson  $J_{\pm} = \varphi_{\pm}^{(1)}(x^{\pm})$  which both solves the Klein-Gordon equation  $\partial_+ \partial_- J_{\pm} = 0$  (so it is a submodule of  $\text{Rac}$ ) and the conservation law  $\partial_- J_+ + \partial_+ J_- = 0$  (so it is a submodule of the complete module describing the spin-1 conserved current).

**Shadow fields.** The modules  $\mathcal{D}(2; (s, s))$  are ill-defined for  $d = 2$  and  $s > 1$ , in accordance with the fact that Weyl tensors are identically vanishing in two dimensions. In such case, one might replace the formula (2.39) for the character of the shadow field by the analytic continuation of the identity (2.30) for  $d = 2$ . By formally applying (2.30) one finds the intriguing relations

$$\chi_{\mathcal{S}(2-s; +s)}^{\mathfrak{so}(2,2)}(q, x) = \frac{\bar{z}^s}{\bar{z} - 1} = -\chi_{\mathcal{V}_s}^{\mathfrak{so}(2,1)}(\bar{z}), \quad (\text{B.8})$$

$$\chi_{\mathcal{S}(2-s; -s)}^{\mathfrak{so}(2,2)}(q, x) = \frac{z^s}{z - 1} = -\chi_{\mathcal{V}_s}^{\mathfrak{so}(2,1)}(z). \quad (\text{B.9})$$

Modules	AdS <sub>3</sub>	CFT <sub>2</sub>	Equivalent descriptions
$\mathcal{D}(\frac{1}{2}; \frac{1}{2})_0$	Di	Conformal spinor	$(\mathcal{V}_{\frac{1}{2}} \otimes 1) \oplus (1 \otimes \mathcal{V}_{\frac{1}{2}})$
$\mathcal{D}(1; 1)_0$	U(1) <sup>⊗2</sup> Chern-Simons with Dirichlet behavior	Chiral bosons	$(\mathcal{V}_1 \otimes 1) \oplus (1 \otimes \mathcal{V}_1)$
$\mathcal{S}(1; 1)_0$	U(1) <sup>⊗2</sup> Chern-Simons with Neumann behavior	Shadows of chiral bosons	$(\bar{\mathcal{V}}_0 \otimes 1) \oplus (1 \otimes \bar{\mathcal{V}}_0)$
$\mathcal{D}(s; s)_0$	Massless spin- $s$ field with Dirichlet behavior	Conserved spin- $s$ current	$(\mathcal{V}_s \otimes 1) \oplus (1 \otimes \mathcal{V}_s)$
$\mathcal{S}(s; s)_0$	Massless spin- $s$ field with Neumann behavior	Shadow spin- $s$ field	$(\bar{\mathcal{V}}_{1-s} \otimes 1) \oplus (1 \otimes \bar{\mathcal{V}}_{1-s})$
$\mathcal{D}(1-s; s-1)_0$	Killing tensor	Conformal Killing tensor	$(\mathcal{D}_{s-1} \otimes 1) \oplus (1 \otimes \mathcal{D}_{s-1})$

**Table 2.** List of relevant  $\mathfrak{so}(2, 2)$  modules and their field-theoretical interpretations

which one can summarize as

$$\chi_{\mathcal{S}(2-s; \pm s)}^{\mathfrak{so}(2, 2)} = -\chi_{\mathcal{D}(s; \mp s)}^{\mathfrak{so}(2, 2)}. \quad (\text{B.10})$$

In fact, the shadow fields are pure gauge in  $d = 2$  and the negative sign corresponds to the residual gauge symmetries.<sup>22</sup> Consider a shadow field  $h_s$  of spin  $s \geq 2$ , one can first reach the traceless gauge where all components of the type  $h_{+ \dots}$  vanish. The residual gauge symmetry reads  $\delta_\xi h_{\pm \dots \pm} = \partial_\pm \xi_{\pm \dots \pm}$  and allows to fix the reach the trivial gauge  $h_s = 0$ . Even after this complete gauge-fixing, the parameters of the residual gauge symmetry are functions of one variable:  $\xi_{\pm \dots \pm} = \varphi_{\mp}^{(1-s)}(x^\mp)$ . This is nothing but the higher-spin version of the usual infinite-dimensional enhancement of conformal symmetries in  $d = 2$ .

Using (2.24) for  $d = 1$ , one can obtain the relation

$$\chi_{\mathcal{V}_{1-s}}^{\mathfrak{so}(2, 1)}(q^{-1}) = -\chi_{\mathcal{V}_s}^{\mathfrak{so}(2, 1)}(q). \quad (\text{B.11})$$

The contragredient of the  $\mathfrak{so}(2, 1)$  lowest-weight Verma module  $\mathcal{V}_\Delta$  is the  $\mathfrak{so}(2, 1)$  highest-weight Verma module with highest weight  $-\Delta$ , which we will denote  $\bar{\mathcal{V}}_\Delta$ . Its character is  $\chi_{\bar{\mathcal{V}}_\Delta}^{\mathfrak{so}(2, 1)}(q) = \chi_{\mathcal{V}_\Delta}^{\mathfrak{so}(2, 1)}(q^{-1})$ . Hence, it may be tempting to reinterpret the character formulae for the spin- $s$  shadow fields in terms of the one for the contragredient  $\mathfrak{so}(2, 1)$  Verma module with highest weight  $s - 1$ . In this sense, one can write the formal equality:  $\mathcal{S}(s; +s) = 1 \otimes \bar{\mathcal{V}}_{-(s-1)}$  and  $\mathcal{S}(s; -s) = \bar{\mathcal{V}}_{-(s-1)} \otimes 1$ .

The list of the relevant parity-invariant  $\mathfrak{so}(2, 2)$ -modules and their field-theoretical interpretations are summarized in table 2.

**Conformal higher-spin gravity.** Conformal higher-spin gravity is highly degenerate in two dimensions since its action identically vanishes for  $s \geq 2$  in a topologically trivial background while the sector of low spin ( $s = 0, 1$ ) is non-local. Nevertheless, its partition function can be defined (see [64, 113]). As far as characters are concerned, one can write  $d = 2$  versions of (2.41). They rely on the  $d = 2$  Flato-Fronsdal theorems (cf. section 4.2

<sup>22</sup>This formula is in perfect agreement with the standard treatment of  $d = 2$  case (cf. the appendix of [113] for a detailed discussion) and in particular reproduces the partition function given in [64] by setting  $x = 1$ .

in [79]). For instance, in the simpler type-B case,

$$\left(\chi_{\mathcal{D}\mathcal{I}}^{\mathfrak{so}(2,2)}\right)^2 = 2\chi_{\mathcal{D}(1,0)}^{\mathfrak{so}(2,2)} + \sum_{s=1}^{\infty} \chi_{\mathcal{D}(s,s)_0}^{\mathfrak{so}(2,2)}. \quad (\text{B.12})$$

This implies the relation

$$\sum_{s=1}^{\infty} \chi_{\mathcal{S}(2-s,s)_0}^{\mathfrak{so}(2,2)} = \sum_{s=1}^{\infty} \chi_{\mathcal{D}(1-s,s-1)_0}^{\mathfrak{so}(2,2)} + \sum_{s=1}^{\infty} \chi_{\mathcal{D}(s,s)_0}^{\mathfrak{so}(2,2)}. \quad (\text{B.13})$$

Strictly speaking, the relation for the full spectrum of type-B off-shell CHS theory would involve a term  $\mathcal{D}(1,0)_0 = 2\mathcal{D}(1,0)$  on each side of (B.13) which we omitted for simplicity. Notice that (B.13) can also be derived from the fact that the finite-dimensional spin- $s$   $\mathfrak{so}(2,1)$ -module can be seen as the following quotient:  $\mathcal{D}_{s-1} \cong \bar{\mathcal{V}}_{1-s}/\bar{\mathcal{V}}_s$ . Using (B.10) and (B.13), one deduces

$$\sum_{s=1}^{\infty} \chi_{\mathcal{S}(2-s,s)_0}^{\mathfrak{so}(2,2)} = - \sum_{s=1}^{\infty} \chi_{\mathcal{D}(s,s)_0}^{\mathfrak{so}(2,2)} = \frac{1}{2} \sum_{s=1}^{\infty} \chi_{\mathcal{D}(1-s,s-1)_0}^{\mathfrak{so}(2,2)}. \quad (\text{B.14})$$

One may define formally the character of the parity-symmetric spin- $s$  on-shell FT module as the analytic continuation of the one for the quotient in the right-hand-side of (2.18) for  $d = 2$ , i.e. the difference  $\chi_{\mathcal{S}(2-s,s)_0}^{\mathfrak{so}(2,2)} - \chi_{\mathcal{D}(s,s)_0}^{\mathfrak{so}(2,2)}$ . By summing over all spins, one can say that (1.1) also holds in  $d = 2$ .

## C Branching rule for the Fradkin-Tseytlin module

In this appendix, we derive the branching rule for the  $\mathfrak{so}(2,d)$ -module of the on-shell FT field in even dimension  $d = 2r$ , i.e. we decompose this irreducible  $\mathfrak{so}(2,d)$ -module into a direct sum of  $\mathfrak{so}(2,d-1)$ -modules. The latter can naturally be interpreted as fields in  $\text{AdS}_d$ , thereby allowing us to recover the factorization property of the kinetic operator of CHS fields in  $\text{M}_d$  into a product of kinetic operators of partially-massless and massive fields in  $\text{AdS}_d$  obtained in [39, 40, 45].

### C.1 Branching rule for generalized Verma modules

In order to derive the branching rules for on-shell Fradkin-Tseytlin modules, we first need the following lemma [114, 115]:

**Lemma C.1.** *The  $\mathfrak{so}(2,d)$  generalized Verma module*

$$\mathcal{V}(\Delta; (s_1, \dots, s_r)) := \mathcal{U}(\mathfrak{so}(2,d)) \otimes_{\mathcal{U}((\mathfrak{so}(2) \oplus \mathfrak{so}(d)) \ltimes \mathbb{R}^d)} \mathbb{V}_{[\Delta; (s_1, \dots, s_r)]} \quad (\text{C.1})$$

where  $\mathbb{V}_{[\Delta; (s_1, \dots, s_r)]}$  is an  $\mathfrak{so}(2) \oplus \mathfrak{so}(d)$  module (on which  $\mathbb{R}^d$  acts trivially) of highest weight  $[\Delta; (s_1, \dots, s_r)]$  branches onto  $\mathfrak{so}(2,d-1)$  according to the following branching rule

- For  $d = 2r$

$$\mathcal{V}(\Delta; (s_1, \dots, s_r)) \xrightarrow[\mathfrak{so}(2,d-1)]{\mathfrak{so}(2,d)} \bigoplus_{n=0}^{\infty} \bigoplus_{\sigma_1=s_2}^{s_1} \cdots \bigoplus_{\sigma_{r-1}=s_r}^{s_{r-1}} \mathcal{V}(\Delta+n; (\sigma_1, \dots, \sigma_{r-1})); \quad (\text{C.2})$$

- For  $d = 2r + 1$

$$\mathcal{V}(\Delta; (s_1, \dots, s_r)) \xrightarrow[\mathfrak{so}(2, d-1)]{\mathfrak{so}(2, d)} \bigoplus_{n=0}^{\infty} \bigoplus_{\sigma_1=s_2}^{s_1} \dots \bigoplus_{\sigma_r=-s_r}^{s_r} \mathcal{V}(\Delta + n; (\sigma_1, \dots, \sigma_r)). \quad (\text{C.3})$$

The idea of the proof (for more technical details, see e.g. [114, 115]) is simple enough, when expressed in terms of the Poincaré-Birkhoff-Witt basis, so that we recall it here, before giving another proof in terms of characters.

**Poincaré-Birkhoff-Witt basis.** The generalized Verma module  $\mathcal{V}(\Delta; \mathbb{Y})$  defined by (C.1) is constructed as follows: one starts from the module  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$  which is an irreducible representation of the homothety subalgebra  $\mathfrak{p}_d := (\mathfrak{so}(2) \oplus \mathfrak{so}(d)) \in \mathbb{R}^d$  in which the action of  $\mathbb{R}^d$  is trivial. Concretely, the conformal algebra admits a three-grading decomposition so that (as a vector space),

$$\mathfrak{so}(2, d) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}, \quad (\text{C.4})$$

with

$$\mathfrak{g}_{-1} = \text{span}\{K_a\} \cong \mathbb{R}^d, \quad \mathfrak{g}_0 = \text{span}\{M_{ab}, D\}, \quad \mathfrak{g}_{+1} = \text{span}\{P_a\} \cong \mathbb{R}^d, \quad (\text{C.5})$$

where  $K_a$  are the special conformal transformation generators,  $P_a$  the translation generators,  $M_{ab}$  the  $\mathfrak{so}(d)$  generators and  $D$  the dilation generator (while  $a, b = 1, \dots, d$ ). The vector space  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$  carries a representation of  $\mathfrak{p}_d$  in the sense that  $\mathfrak{so}(2)$  acts diagonally with  $\Delta$  as eigenvalue,  $\mathfrak{so}(d)$  acts through the representation  $\mathbb{Y} = (s_1, \dots, s_r)$  while  $\mathbb{R}^d \cong \mathfrak{g}_{-1} \cong \text{span}(K_a)$  acts trivially. In other words, it defines a primary field, i.e. a field with fixed conformal weight  $\Delta$  and spin  $\mathbb{Y}$  annihilated by special conformal transformations at the origin. The generalized Verma module  $\mathcal{V}(\Delta; \mathbb{Y})$  is then constructed by acting with the whole universal enveloping algebra  $\mathcal{U}(\mathfrak{so}(2, d))$  on the finite-dimensional module  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$ , modulo the action of  $\mathfrak{p}_d$ . In practice, the action of (arbitrary powers of) the generators of  $\mathfrak{g}_{+1} \cong \mathbb{R}^d$  on  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$  defines new vectors (or states) in the generalized Verma module, so that

$$\mathcal{V}(\Delta; \mathbb{Y}) \cong \mathcal{U}(\mathfrak{g}_{+1}) \otimes \mathbb{V}_{[\Delta; \mathbb{Y}]}. \quad (\text{C.6})$$

The action of the conformal algebra on  $\mathcal{V}(\Delta; \mathbb{Y})$  is then defined by the action of the subalgebra  $\mathfrak{p}_d$  on  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$  together with the Lie bracket between generators of this subalgebra and of  $\mathfrak{g}_{+1}$ . In other words, the representation of  $\mathfrak{so}(2, d)$  is induced from representation of the subalgebra  $\mathfrak{p}_d$ . In more concrete terms, we can write a basis of  $\mathcal{V}(\Delta; \mathbb{Y})$  as

$$|\Delta + n; \mathbb{Y}\rangle_{a_1 \dots a_n} = P_{a_1} \dots P_{a_n} |\Delta; \mathbb{Y}\rangle, \quad (\text{C.7})$$

where  $|\Delta; \mathbb{Y}\rangle$  stands for a generic basis element of  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$ . In order to obtain the branching rule of  $\mathcal{V}(\Delta; \mathbb{Y})$ , the first task is to branch the representation  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$  of  $\mathfrak{p}_d$  to a direct sum of representation of its lower dimensional counterpart,  $\mathfrak{p}_{d-1}$ . Due to the fact that the  $\mathbb{R}^d$  part acts trivially and the  $\mathfrak{so}(2)$  is present in both  $\mathfrak{p}_d$  and  $\mathfrak{p}_{d-1}$ , we only need to consider



the branching of the  $\mathfrak{so}(d)$  component. For both  $\mathfrak{so}(2r)$  and  $\mathfrak{so}(2r+1)$ , the branching rules are well known and read respectively

$$(s_1, \dots, s_r) \xrightarrow[\mathfrak{so}(2r)]{\mathfrak{so}(2r+1)} \bigoplus_{\sigma_1=s_2}^{s_1} \cdots \bigoplus_{\sigma_{r-1}=s_{r-2}}^{s_{r-1}} \bigoplus_{\sigma_r=-s_r}^{s_r} (\sigma_1, \dots, \sigma_r), \quad (\text{C.8})$$

and

$$(s_1, \dots, s_r) \xrightarrow[\mathfrak{so}(2r-1)]{\mathfrak{so}(2r)} \bigoplus_{\sigma_1=s_2}^{s_1} \cdots \bigoplus_{\sigma_{r-2}=s_{r-1}}^{s_{r-2}} \bigoplus_{\sigma_{r-1}=s_r}^{s_{r-1}} (\sigma_1, \dots, \sigma_{r-1}). \quad (\text{C.9})$$

The next thing to do is to decompose the action of  $\mathcal{U}(\mathfrak{g}_{+1}) \cong \mathcal{U}(\mathbb{R}^d)$  on the direct sum of the representation of  $\mathfrak{p}_{d-1}$  into a direct sum of  $\mathfrak{so}(2, d-1)$  generalized Verma modules. To do so, let us single out one generator of  $\mathfrak{g}_{+1}$ , say  $P_d$ , so that the remaining generators  $P_i$  ( $i = 1, \dots, d-1$ ) span  $\mathbb{R}^{d-1}$ . The basis of  $\mathcal{V}(\Delta; \mathbb{Y})$  now reads

$$P_{i_1} \dots P_{i_p} (P_d)^n |\Delta; \sigma\rangle, \quad n, p \in \mathbb{N}, \quad \sigma \in \mathcal{B}(\mathbb{Y}), \quad (\text{C.10})$$

where the indices  $i_1, \dots, i_p$  run from 1 to  $d-1$ , while  $|\Delta; \sigma\rangle$  stands for a generic basis element of the finite-dimensional  $\mathfrak{p}_{d-1}$ -module  $\mathbb{V}_{[\Delta; \mathbb{Y}]}$  labeled by the  $\mathfrak{so}(2)$ -eigenvalue  $\Delta$  and  $\mathfrak{so}(d-1)$ -weight  $\sigma$ , and  $\mathcal{B}(\mathbb{Y})$  designates the set of  $\mathfrak{so}(d-1)$ -weights  $\sigma$  in the above branching rule of the  $\mathfrak{so}(d)$  irrep  $\mathbb{Y}$ . Due to the fact that  $P_d$  has weight  $+1$  under the adjoint action of the dilation generator (as any generator of  $\mathfrak{g}_{+1}$ ), and commutes with any element of  $\mathfrak{so}(d-1)$ , we have

$$(P_d)^n |\Delta; \sigma\rangle \cong |\Delta + n; \sigma\rangle, \quad (\text{C.11})$$

i.e. it defines an irrep of  $\mathfrak{p}_{d-1}$  with  $\mathfrak{so}(2)$ -weight  $\Delta + n$  and  $\mathfrak{so}(d-1)$ -weight  $\sigma$ . Then the branching rules (C.3)–(C.2) of the conformal algebra result from the branching rules (C.8)–(C.9) of the rotation subalgebra.

**Characters.** The branching rules (C.3)–(C.2) can also be recovered at the level of characters as follows:

- For  $d = 2r + 1$ , we have

$$\mathcal{P}_d(q, \mathbf{x}) = \frac{1}{1-q} \mathcal{P}_{d-1}(q, \mathbf{x}), \quad (\text{C.12})$$

and

$$\chi_{(s_1, \dots, s_r)}^{\mathfrak{so}(d)}(\mathbf{x}) = \sum_{\sigma=s_2}^{s_1} \cdots \sum_{\sigma_r=-s_r}^{s_r} \chi_{(\sigma_1, \dots, \sigma_r)}^{\mathfrak{so}(d-1)}(\mathbf{x}), \quad (\text{C.13})$$

so that we readily obtain

$$\chi_{\mathcal{V}(\Delta; (s_1, \dots, s_r))}^{\mathfrak{so}(2, d)}(q, \mathbf{x}) = \sum_{n=0}^{\infty} \sum_{\sigma=s_2}^{s_1} \cdots \sum_{\sigma_r=-s_r}^{s_r} \chi_{\mathcal{V}(\Delta+n; (\sigma_1, \dots, \sigma_r))}^{\mathfrak{so}(2, d-1)}(q, \mathbf{x}), \quad (\text{C.14})$$

in accordance with (C.3).

- For  $d = 2r$ , the rule (C.2) is less straightforward to recover at the level of character due to the fact that the rank  $\mathfrak{so}(2, d)$  is  $r + 1$  whereas the rank of  $\mathfrak{so}(2, d - 1)$  is  $r$ , and hence the characters of modules of the latter depend on one less variables than the characters of the former. This can be already observed for the branching of  $\mathfrak{so}(d)$  irreps (C.15) which at the level of characters reads

$$\chi_{(s_1, \dots, s_r)}^{\mathfrak{so}(d)}(\mathbf{x}) + \chi_{(s_1, \dots, -s_r)}^{\mathfrak{so}(d)}(\mathbf{x}) = \sum_{k=1}^r \mathcal{A}_k^{(r)}(\mathbf{x}) \sum_{\sigma=s_2}^{s_1} \cdots \sum_{\sigma_{r-1}=s_r}^{s_{r-1}} \chi_{(\sigma_1, \dots, \sigma_{r-1})}^{\mathfrak{so}(d-1)}(\mathbf{x}_k), \quad (\text{C.15})$$

where

$$\mathbf{x}_k = (x_1, \dots, x_{k-1}, x_k, \dots, x_r), \quad (\text{C.16})$$

and

$$\mathcal{A}_k^{(r)}(\mathbf{x}) = (x_k^{s_r} + x_k^{-s_r}) \frac{\delta(x_1 + x_1^{-1}, \dots, x_{k-1} + x_{k-1}^{-1}, 2, x_{k+1} + x_{k+1}^{-1}, \dots, x_r + x_r^{-1})}{\delta(x_1 + x_1^{-1}, \dots, x_r + x_r^{-1})}, \quad (\text{C.17})$$

where  $\delta(X_1, \dots, X_r)$  is the  $r \times r$  Vandermonde determinant. In order to simplify (C.15), one can set one of the variables  $\mathbf{x}$  of the  $\mathfrak{so}(d)$  character to 1, say  $x_r$ , so that the characters on both sides of this equations depend on the same number of variables. Due to the fact that

$$\mathcal{A}_k^{(r)}(\tilde{\mathbf{x}}) = 2 \delta_{k,r}, \quad \text{with} \quad \tilde{\mathbf{x}} := (x_1, \dots, x_{r-1}, 1), \quad (\text{C.18})$$

and

$$\chi_{(s_1, \dots, s_{r-1}, s_r)}^{\mathfrak{so}(d)}(\tilde{\mathbf{x}}) = \chi_{(s_1, \dots, s_{r-1}, -s_r)}^{\mathfrak{so}(d)}(\tilde{\mathbf{x}}), \quad (\text{C.19})$$

the branching rule (C.15) simplifies to

$$\chi_{(s_1, \dots, s_{r-1}, \pm s_r)}^{\mathfrak{so}(d)}(\tilde{\mathbf{x}}) = \sum_{\sigma=s_2}^{s_1} \cdots \sum_{\sigma_{r-1}=s_r}^{s_{r-1}} \chi_{(\sigma_1, \dots, \sigma_{r-1})}^{\mathfrak{so}(d-1)}(\mathbf{x}_r). \quad (\text{C.20})$$

Similarly, we have

$$\mathcal{P}_d(q, \tilde{\mathbf{x}}) = \frac{1}{1-q} \mathcal{P}_{d-1}(q, \mathbf{x}_r), \quad (\text{C.21})$$

and hence

$$\chi_{\mathcal{V}(\Delta; (s_1, \dots, s_r))}^{\mathfrak{so}(2, d)}(q, \tilde{\mathbf{x}}) = \sum_{n=0}^{\infty} \sum_{\sigma=s_2}^{s_1} \cdots \sum_{\sigma_r=s_r}^{s_{r-1}} \chi_{\mathcal{V}(\Delta+n; (\sigma_1, \dots, \sigma_{r-1}))}^{\mathfrak{so}(2, d-1)}(q, \mathbf{x}_r), \quad (\text{C.22})$$

in accordance with (C.2).

This concludes the alternative proof via the computation of characters.

## C.2 Branching rule for irreducible lowest-weight modules

Having the above Lemma C.1 at hand, we can then obtain the branching rule of an irreducible  $\mathfrak{so}(2, d)$ -module  $\mathcal{D}(\Delta; \mathbb{Y})$  defined in terms of quotients of generalized Verma modules. Indeed, as we have seen previously (in appendix A), the characters of such quotient are given by an alternating sum of characters of generalized modules  $\mathcal{V}(\Delta'; \mathbb{Y}')$  as defined in (C.1), while the branching rule of such a module translates straightforwardly at the level of its character. Hence, applying the identity (C.22) and (C.14) to each of the terms appearing in the character of an irreducible module  $\mathcal{D}(\Delta; \mathbb{Y})$ , and reinterpreting the resulting expression as a sum of characters of irreducible modules of  $\mathfrak{so}(2, d-1)$ , we can deduce the branching rule for  $\mathcal{D}(\Delta; \mathbb{Y})$ .

**Fradkin-Tseytlin modules.** As emphasized in the rest of the paper, the module of central interest in CHS gravity is that of the on-shell (totally-symmetric) FT field introduced in (2.18). Before deriving its branching, let us spell out its character in all dimensions (taking  $d = 2r$  in the rest of the section):

- The character of the  $\mathfrak{so}(2, d)$ -module of the *on-shell* FT field in even  $d$  dimensions reads

$$\begin{aligned} \chi_{\mathcal{D}(2;(s,s))}^{\mathfrak{so}(2,d)}(q, \mathbf{x}) &= \sum_{m=0}^{r-2} (-1)^m \left( q^{2+m} \chi_{(s,s,1^m_+)}^{\mathfrak{so}(d)}(\mathbf{x}) + q^{d-2-m} \chi_{(s,s,1^m_-)}^{\mathfrak{so}(d)}(\mathbf{x}) \right) \mathcal{P}_d(q, \mathbf{x}) \\ &\quad - 2(q^{s+d-2} \chi_{(s)}^{\mathfrak{so}(d)}(\mathbf{x}) - q^{s+d-1} \chi_{(s-1)}^{\mathfrak{so}(d)}(\mathbf{x})) \mathcal{P}_d(q, \mathbf{x}); \end{aligned} \quad (\text{C.23})$$

- The character of the  $\mathfrak{so}(2, d-1)$ -module of the *off-shell* FT field in odd  $d-1$  dimensions reads

$$\begin{aligned} \chi_{\mathcal{D}(2;(s,s-t+1))}^{\mathfrak{so}(2,d-1)}(q, \mathbf{x}) &= \sum_{m=0}^{r-3} (-1)^m (q^{2+m} - q^{d-3-m}) \chi_{(s,s-t+1,1^m)}^{\mathfrak{so}(d-1)}(\mathbf{x}) \mathcal{P}_{d-1}(q, \mathbf{x}) \\ &\quad + (q^{s+d-t-2} \chi_{(s)}^{\mathfrak{so}(d-1)}(\mathbf{x}) - q^{s+d-2} \chi_{(s-t)}^{\mathfrak{so}(d-1)}(\mathbf{x})) \mathcal{P}_{d-1}(q, \mathbf{x}). \end{aligned} \quad (\text{C.24})$$

Now applying Lemma C.1 as explained in the previous paragraph to the character of the Weyl module for  $d = 2r$  yields, after taking care of a few cancellations

$$\chi_{\mathcal{D}(2;(s,s))}^{\mathfrak{so}(2,d)}(q, \tilde{\mathbf{x}}) = \sum_{t=1}^{s+\frac{d-4}{2}} \chi_{\mathcal{D}(s+d-t-2;(s))}^{\mathfrak{so}(2,d-1)}(q, \mathbf{x}_r) + \chi_{\mathcal{S}(1+t-s;(s))}^{\mathfrak{so}(2,d-1)}(q, \mathbf{x}_r). \quad (\text{C.25})$$

The characters appearing on the right-hand-side of the above equation correspond to the following modules:

- For  $t = 1, \dots, s$ , the characters  $\chi_{\mathcal{D}(s+d-t-2;(s))}^{\mathfrak{so}(2,d-1)}(q, \mathbf{x}_r)$  are those of the modules describing partially-conserved currents of spin- $s$  and depth- $t$  in  $d-1$  dimensions (given in (3.18)), while  $\chi_{\mathcal{S}(1+t-s;(s))}^{\mathfrak{so}(2,d-1)}(q, \mathbf{x}_r)$  are the characters of the corresponding spin- $s$  and depth- $t$  shadow fields in  $M_{d-1}$ . Notice that the former/latter modules can also be interpreted as  $\text{AdS}_d$  fields, namely as spin- $s$  and depth- $t$  partially-massless fields with Dirichlet/Neumann boundary behavior.

- For  $t = s+1, \dots, s+\frac{d-4}{2}$ , the characters  $\chi_{\mathcal{D}(s+d-t-2;(s))}^{\mathfrak{so}(2,d-1)}(q, \mathbf{x}_r)$  are those of the modules describing massive fields on  $\text{AdS}_d$  with minimal energy  $\Delta_+ = d-3, \dots, \frac{d}{2}$ , while  $\chi_{\mathcal{S}(1+t-s;(s))}^{\mathfrak{so}(2,d-1)}(q, \mathbf{x}_r)$  are the characters of spin- $s$  massive fields on  $\text{AdS}_d$  with minimal energy  $\Delta_- = 2, \dots, \frac{d}{2}-1$ . These modules come by pair of conjugate dimensions  $\Delta_+$  and  $\Delta_- = d - \Delta_+$  corresponding to massive field with identical mass but Dirichlet vs Neumann boundary behavior.

Having in mind that the  $\mathfrak{so}(2, d-1)$ -modules  $\mathcal{S}(s+d-t-2;(s))$  and  $\mathcal{D}(2;(s, s-t+1))$  coincide for  $t = 1, \dots, s$  and correspond to the off-shell FT module either in terms of the shadow field  $h_s$  or in terms of its Weyl tensor  $C_{s,s}$ , we can write the branching rule:

$$\begin{aligned} \mathcal{D}(2;(s, s)) &\xrightarrow[\mathfrak{so}(2,d-1)]{\mathfrak{so}(2,d)} \bigoplus_{t=1}^s \mathcal{D}(2;(s, s-t+1)) \oplus \mathcal{D}(s+d-t-2;(s)) \\ &\oplus \bigoplus_{n=0}^{d-5} \mathcal{D}(d-3-n;(s)). \end{aligned} \quad (\text{C.26})$$

This decomposition is in accordance with the results of [39, 40, 45] (conjectured in [38]) where the decomposition of the CHS wave operator in  $\mathbb{R}^d$  into a product of wave operators in  $\text{AdS}_d$  for the partially massless fields of depths  $t = 1, \dots, s$ . Firstly, the left-hand-side describes the on-shell FT field on  $d$ -dimensional conformally flat space  $M_d$  as a module of the conformal algebra  $\mathfrak{so}(2, d)$ . Secondly, the right-hand-side correspond to its description in terms of partially massless fields on  $\text{AdS}_d$  as modules of the isometry algebra  $\mathfrak{so}(2, d-1)$ . Thirdly, each partially massless field gives rise to two modules: one for each possible boundary choice. Following similar observations in [65], let us rephrase the heuristic argument as follows: the conformal boundary of Euclidean<sup>23</sup>  $\text{AdS}_{d+1}$  is the conformally-flat sphere  $S^d$ . In turn, Euclidean  $\text{AdS}_d$  is conformal to one hemisphere of  $S^d$  with two possible choices of boundary conditions at the equator  $S^{d-1}$ . Therefore, defining a conformal field on  $S^d$  in terms of the one on Euclidean  $\text{AdS}_d$  we need to sum over the two boundary condition choices.

Notice that the case  $d = 4$  is special, as no massive nor shadow fields appears in the branching rule, so that

$$\mathcal{D}(2;(s, \pm s)) \xrightarrow[\mathfrak{so}(2,3)]{\mathfrak{so}(2,4)} \bigoplus_{t=1}^s \mathcal{D}(s+2-t;(s)). \quad (\text{C.27})$$

In fact, the modules  $\mathcal{D}(2;(s, s-t+1))$  are absent since finite-dimensional irreducible  $\mathfrak{so}(3)$ -modules labelled by two-row Young diagrams vanish (and hence these modules do not exist for  $\mathfrak{so}(2, 3)$ ).

**Higher-depth Fradkin-Tseytlin modules.** More generally, for a depth- $t$  FT field, the branching rule for  $d = 4$  reads

$$\mathcal{D}(2;(s, \pm(s-t+1))) \xrightarrow[\mathfrak{so}(2,3)]{\mathfrak{so}(2,4)} \bigoplus_{\sigma=s-t+1}^s \bigoplus_{\tau=\sigma-s+t}^{\sigma} \mathcal{D}(\sigma+2-\tau;(\sigma)), \quad (\text{C.28})$$

<sup>23</sup>We preferred the Euclidean signature because the geometric picture is more intuitive, but it holds in Lorentzian signature as well.

i.e. partially massless fields of different spins (as well as different depth) appear. Notice that this is in accordance with the factorization of the partition function of maximal depth FT fields derived in [66]. In higher dimensions ( $d = 2r \geq 6$ ) the branching rule also involves more modules (as in the  $t = 1$  case), namely

$$\begin{aligned}
 & \mathcal{D}(2; (s, s-t+1)) \\
 & \begin{array}{c} \mathfrak{so}(2,d) \\ \downarrow \\ \mathfrak{so}(2,d-1) \end{array} \quad \bigoplus_{\sigma=s-t+1}^s \bigoplus_{\tau=\sigma-s+t}^{\sigma} \mathcal{D}(2; (\sigma, \sigma-\tau+1)) \oplus \mathcal{D}(\sigma+d-\tau-2; (\sigma)) \\
 & \quad \oplus \bigoplus_{\sigma=s-t+1}^s \bigoplus_{n=0}^{d-5} \mathcal{D}(d-3-n; (\sigma)) \\
 & \cong \bigoplus_{\sigma=s-t+1}^s \bigoplus_{\tau=\sigma-s+t}^{\sigma+\frac{d-4}{2}} \mathcal{S}(1+\tau-\sigma; (\sigma)) \oplus \mathcal{D}(\sigma+d-\tau-2; (\sigma))
 \end{aligned} \tag{C.29}$$

Notice that this branching rule contains  $2t(s-t+\frac{d-2}{2})$  modules whereas the kinetic operator for the FT field contains  $s-t+\frac{d-2}{2}$  factors (see e.g. [41]). The kernel of each factor operator, labelled by  $k$  ranging from 0 to  $s-t+\frac{d-4}{2}$ , corresponds to the module  $\bigoplus_{\sigma=s-t+1}^s \mathcal{S}(1+t+k-s; (\sigma)) \oplus \mathcal{D}(s+d-t-k-2; (\sigma))$ . For instance, the  $d = 4$  maximal-depth ( $s = t$ ) FT fields have two-derivative kinetic operators but their spectrum is made of maximal-depth PM fields of spin 1 to  $s$  (see section of 3.4 of [116] for the concrete example of  $s = t = 2$  case).

## D A primer on nonlinear CHS gravity

The nonlinear action of type-A CHS gravity is defined as the logarithmically divergent part of the effective action of a conformal scalar field in the background of all shadow fields [27, 28]. As such, it remains a somewhat formal definition which becomes concrete only once it is computed perturbatively in the weak field expansion around the conformally-flat vacuum solution.

**Formal operator approach.** The tower of shadow fields  $h_s(x)$  is conveniently packed in a generating function over phase space:  $h(x, p) = \sum_s h^{\mu_1 \dots \mu_s}(x) p_{\mu_1} \dots p_{\mu_s}$ . Using Weyl calculus, the latter can be interpreted as the symbol of a Hermitian differential operator  $\hat{H}(\hat{x}, \hat{p}) = \sum_s h^{\mu_1 \dots \mu_s}(x) \hat{p}_{\mu_1} \dots \hat{p}_{\mu_s} + \dots$  and the CHS action is the Seeley-DeWitt coefficient  $a_{\frac{d}{2}}[\hat{G}]$  of the operator  $\hat{G} = \hat{p}^2 + \hat{H}$ . The weak field expansion of the latter coefficient can be computed via standard techniques in quantum mechanics [28, 29]. Several qualitative features of nonlinear CHS gravity are more easily understood within what Segal called the “formal operator approach” (which will be shortened here to “formal approach” for brevity) which consists in writing all formulae in terms of operators rather than their symbols and treating these operators as large  $n \times n$  matrices. This formalism is very useful for elucidating the structure of the theory but its field-theoretical interpretation is sometimes fragile. Only the perturbative formulation with symbols admitting power series expansion in momenta admits a clear interpretation as a local field theory. From the point of view

of the effective action, the formal approach amounts to treat the conformal scalar field  $\phi$  as if it were a large  $n$ -vector (for instance an element of the space  $\mathbb{C}^\infty$  of infinite sequences with finitely-many non-vanishing complex entries<sup>24</sup>). Therefore, the formal approach is essentially<sup>25</sup> equivalent to ignoring functional and locality issues, as well as polynomial UV-divergencies, in the field-theoretical discussions of section 2. In other words, “formal CHS gravity” should correspond to the limiting case  $d \rightarrow 0$  in a suitable dimensional regularization of  $d$ -dimensional CHS theory (and, similarly, of its related Neumann MHS theory around  $\text{AdS}_{d+1}$ ).

**Formal group of symmetries.** Using Weyl calculus, the action (2.7) which is quadratic in the conformal scalar field can be rewritten in a compact way as

$$S[\phi; \{h_s\}_{s \in \mathbb{N}}] = \langle \phi | \hat{G} | \phi \rangle. \quad (\text{D.1})$$

It is manifestly invariant under the gauge transformations

$$|\phi\rangle \rightarrow \hat{O}^{-1} |\phi\rangle, \quad (\text{D.2})$$

$$\hat{G} \rightarrow \hat{O}^\dagger \hat{G} \hat{O}, \quad (\text{D.3})$$

where  $\hat{O}$  is an invertible operator and  $\hat{O}^\dagger$  stands for its Hermitian conjugate. The spectrum of off-shell CHS gravity is spanned by Hermitian differential operators  $\hat{G}$  which are gauge fields with symmetries (D.3) and they couple to the free scalar field  $\phi$  as the kinetic operator in (D.1). In the formal approach, the gauge symmetries (D.3) form a group of invertible operators. These finite gauge symmetries (D.3) are actually generated by two Hermitian differential operators  $\hat{S}$  and  $\hat{X}$  as follows:

$$\hat{O} = e^{\hat{S} + i\hat{X}}. \quad (\text{D.4})$$

They correspond, respectively, to higher-spin Weyl transformations

$$\hat{G} \rightarrow \hat{O} \hat{G} \hat{O}, \quad \hat{O}^\dagger = \hat{O} \quad (\text{D.5})$$

for invertible and Hermitian operators  $\hat{O} = e^{\hat{S}}$ , and to higher-spin diffeomorphisms

$$\hat{G} \rightarrow \hat{O}^{-1} \hat{G} \hat{O}, \quad \hat{O}^\dagger = \hat{O}^{-1} \quad (\text{D.6})$$

for unitary operators  $\hat{O} = e^{i\hat{X}}$ .<sup>26</sup> On the one hand, it is not clear whether such invertible operators (D.4) form a group because the products of such operators may not be the expo-

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<sup>24</sup>In mathematical terms, the vector space  $\mathbb{C}^\infty$  is the direct limit of the  $\mathbb{N}$ -filtration of vector spaces:

$$\mathbb{C} \hookrightarrow \mathbb{C}^2 \hookrightarrow \dots \hookrightarrow \mathbb{C}^N \hookrightarrow \mathbb{C}^{N+1} \hookrightarrow \dots$$

The collection of the sequences with one entry equal to 1, and all other entries vanishing, provides a countably-infinite basis of the vector space  $\mathbb{C}^\infty$ .

<sup>25</sup>More precisely, the  $U(N)$ -vector model should be replaced with the large- $n$  limit of the  $U(N)$ -singlet sector of a  $U(nN)$ -vector model (see e.g. [92]).

<sup>26</sup>Notice that the space  $\mathbb{C}^\infty$  carries the fundamental representation of the infinite general linear group  $\text{GL}(\infty, \mathbb{C})$  which one can describe as the group of infinite invertible matrices with finitely-many non-vanishing complex entries outside the diagonal. Its Lie algebra is the Lie algebra  $\mathfrak{gl}(\infty, \mathbb{C})$  of infinite matrices with finitely-many non-vanishing complex entries. In particular, the higher-spin diffeomorphisms may correspond to the adjoint action of the real subgroup  $U(\infty) \subset \text{GL}(\infty, \mathbb{C})$  on its Lie algebra  $\mathfrak{u}(\infty)$  of infinite Hermitian matrices with finitely-many non-vanishing entries.

nential of some *differential* operators (i.e. of *finite* order in the derivatives). This subtlety somewhat precludes a mathematically rigorous definition of the group of higher-spin gauge symmetries. On the other hand, the infinitesimal transformations are perfectly well-defined because differential operators form an associative algebra under composition product.

**Algebra of gauge symmetries.** The infinitesimal form of the transformations (D.2)–(D.3) is

$$\delta|\phi\rangle = -(\hat{S} + i\hat{X})|\phi\rangle, \quad (\text{D.7})$$

$$\delta\hat{G} = [\hat{G}, \hat{S}]_+ + i[\hat{G}, \hat{X}]_-, \quad (\text{D.8})$$

where  $[\hat{A}, \hat{B}]_{\pm} = \hat{A}\hat{B} \pm \hat{B}\hat{A}$  denotes the (anti)commutator. The linearization of (D.8) is

$$\delta\hat{H} = [\hat{p}^2, \hat{S}]_+ + i[\hat{p}^2, \hat{X}]_- \quad (\text{D.9})$$

Let  $\sigma(x, p)$  and  $\xi(x, p)$  be the Weyl symbols of  $\hat{S}(\hat{x}, \hat{p})$  and  $\hat{X}(\hat{x}, \hat{p})$  with tensors  $\sigma_{s-2}(x)$  and  $\xi_{s-1}(x)$  as coefficients in the power series expansion in momenta. Then, the infinitesimal gauge transformation (D.9) reproduces the form of the gauge equivalence (2.2) of shadow fields (up to signs and factors which we omit for simplicity). In other words, the spectrum of *linearized* off-shell CHS gravity (i.e. the tower of all shadow fields) can be encoded in a single Hermitian operator  $\hat{H}$  modulo the gauge equivalence (D.9). In fact, the spectrum of *nonlinear* off-shell CHS gravity can be defined as the space of Hermitian differential operators  $\hat{G}$  which are (i) in the vicinity of the vacuum  $\hat{G}_0 = \hat{p}^2$  and (ii) given modulo the gauge symmetries (D.8).

**Algebra of global symmetries.** By definition, the global symmetries of CHS gravity preserve the vacuum, i.e. they correspond to operators  $\hat{O}_0$  such that

$$\hat{O}_0^\dagger \hat{G}_0 \hat{O}_0 = \hat{G}_0. \quad (\text{D.10})$$

The generators of such operators span the (C)HS algebra. The finite transformations generated by such operators  $\hat{O}_0$  are linear transformations of the perturbation

$$\hat{H} \rightarrow \hat{O}_0^\dagger \hat{H} \hat{O}_0 \quad (\text{D.11})$$

since they preserve the vacuum by definition. Its infinitesimal version defines the action of (C)HS algebra on the spectrum of free off-shell CHS gravity. In fact, one can check that the gauge equivalence relation is preserved by this action. Therefore, the spectrum of free *off-shell* CHS gravity is a module of the higher-spin algebra.<sup>27</sup> As a corollary, the spectrum of free *on-shell* CHS gravity is also a module of the higher-spin algebra (because the tower of Bach tensors span a submodule isomorphic to the twisted-adjoint module, thus the quotient is also a module).

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<sup>27</sup>Another simple way to reach this conclusion is to consider the pairing  $\int d^d x h_s j_s$  between shadow fields  $h_s$  and conserved currents  $j_s$ . The tower of currents spans the twisted-adjoint module. Each shadow field is the dual module of the current with respect to this pairing, therefore the tower of shadow fields is the dual of the twisted-adjoint module.

**Formal action.** In the formal approach, the action of CHS gravity is obtained by looking for a functional  $W_{\text{CHS}}[\hat{G}]$  which must be invariant under the gauge symmetries (D.3). The invariance under higher-spin diffeomorphisms (D.6), i.e. the adjoint action of unitary operators, implies that  $W_{\text{CHS}}[\hat{G}] = \text{Tr}[F(\hat{G})]$  for some function  $F$ , while the invariance under higher-spin Weyl transformations (D.5) was argued by Segal to imply that  $F(x) = C \Theta_0(x) + c \Delta(x)$  where  $\Theta_0$  is the left-continuous Heaviside step function (i.e. it vanishes at the origin) while  $\Delta(x)$  is the discontinuous function vanishing for all  $x$  except at the origin  $x = 0$  where it is equal to the unity. The heuristic argument is as follows: since  $\hat{G}$  is Hermitian, it can be diagonalized by a unitary transformation,

$$\hat{U}^{-1} \hat{G} \hat{U} = \sum_i \lambda_i |i\rangle\langle i| \quad (\text{D.12})$$

where  $\hat{U}^\dagger = \hat{U}^{-1}$  and the eigenvalues  $\lambda_i$  are real numbers. Therefore,  $\text{Tr}[F(\hat{G})] = \sum_i F(\lambda_i)$  for a function  $F$  to be determined. Transformations (D.5) can rescale the eigenvalues by positive factors, hence an invariant function  $F(\lambda)$  can only depend on  $\lambda$  through its sign. This essentially leaves the Heaviside step function as only possibility up to a normalization factor  $C$  and a constant  $c$  related to the value that one assigns at the origin to the Heaviside function. Segal considered the choice  $C = 1$  and  $c = \frac{1}{2}$ , i.e.

$$S_{\text{CHS}}^{\text{Segal}}[\hat{G}] := \text{Tr}[\Theta_{\frac{1}{2}}(\hat{G})]. \quad (\text{D.13})$$

where  $\Theta_{\frac{1}{2}} := \Theta_0 + \frac{1}{2}\Delta = \frac{1}{2} + \frac{1}{2}\text{sgn}$  (where “sgn” stands for the sign function).

**Heat kernel expansion.** A standard field-theoretical regularization of the one-loop effective action is the heat kernel prescription:

$$W_{\Lambda}^{\text{Heat}}[\hat{G}] := - \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{dt}{t} \text{Tr} \left[ \exp(-t\hat{G}) \right], \quad (\text{D.14})$$

if one ignores IR divergencies. For a semidefinite Hermitian differential operator of the form  $\hat{G} = \hat{p}^2 + \hat{H}$ , the heat kernel expansion takes the form:

$$\text{Tr}[\exp(-t\hat{G})] = t^{-\frac{d}{2}} \sum_{n=0}^{\infty} t^n a_n[\hat{G}] \quad (\text{D.15})$$

where  $a_n$  are the Seeley-DeWitt coefficients. For  $d$  even, one can say that the coefficient of the logarithmically divergent term identifies with the nonlinear CHS action:

$$S_{\text{CHS}}[\hat{G}] := W_{\log}^{\text{Heat}}[\hat{G}] = a_{\frac{d}{2}}[\hat{G}] = \frac{1}{2\pi i} \oint \frac{dz}{z} \text{Tr}[\exp(-z\hat{G})]. \quad (\text{D.16})$$

which can be seen as contour integral around the origin of the heat kernel. If the trace of heat kernel  $\text{Tr}[\exp(-z\hat{G})]$  is analytic in the region  $\text{Re}(z) > 0$  (except maybe for the neighborhood of the origin) and approaches to zero when  $|z| \rightarrow \infty$  in this region, then the contour integral in (D.16) can be modified as

$$a_{\frac{d}{2}}[\hat{G}] = \frac{1}{2\pi i} \int_{+i\infty-0^+}^{-i\infty-0^+} \frac{dz}{z} \text{Tr}[\exp(-z\hat{G})]. \quad (\text{D.17})$$



The Heaviside step function admits an integral representation of analogous form,

$$\Theta_{\frac{1}{2}}(\lambda) = \frac{1}{2\pi i} \int_{+i\infty-0^+}^{-i\infty-0^+} \frac{dz}{z} \exp(-z\lambda). \quad (\text{D.18})$$

Therefore, it is tempting to identify  $\text{Tr}[\Theta_{\frac{1}{2}}(\hat{G})]$  with  $a_{\frac{d}{2}}[\hat{G}]$ , the logarithmically divergent piece of the effective action (D.17), in agreement with Segal's formal action (D.13). However, one needs to be cautious here. For instance, if we boldly permute the contour integral with the trace in (D.16), then we would be lead to consider the identity function instead of the Heaviside:

$$1 = \frac{1}{2\pi i} \oint \frac{dz}{z} \exp(-z\lambda). \quad (\text{D.19})$$

Then, we would identify the logarithmically divergent term  $a_{\frac{d}{2}}[\hat{G}]$  with  $\text{Tr}(\hat{1})$ . Indeed, even in the formal viewpoint taken by Segal,  $\text{Tr}(\hat{1})$ , namely the number  $n$  of eigenvalues, is invariant under higher-spin Weyl transformations (D.5). One may still think that  $\text{Tr}(\hat{1})$  looks more weird than  $\text{Tr}(\Theta(\hat{G}))$ . But in fact both of them are very formal expressions as they give divergent series when summing over the eigenvalues. The corresponding regularization is invisible in the integral representation (D.18) and (D.19) for individual eigenvalues. Another way to regularize  $\text{Tr}(\hat{1})$  is by regarding it as  $\lim_{z \rightarrow 0} \text{Tr}(\hat{G}^{-z})$ , which is known to give the logarithmically divergent term  $a_{\frac{d}{2}}[\hat{G}]$  in the zeta function regularization method. The reason for the formal equivalence between  $\text{Tr}(\Theta(\hat{G}))$  and  $\text{Tr}(\hat{1})$  — under the aforementioned conditions imposed on  $\text{Tr}(\exp(-z\hat{G}))$  — is because the operator  $\hat{G}$  under consideration is positive semidefinite while the contribution of the zero modes is probably of zero measure.

**Formal equation of motion.** Let us consider the equation of motion of CHS gravity derived from the action principle. On the one hand, varying the operator  $\hat{G}$  in the action (D.16) as

$$\delta S_{\text{CHS}}(\hat{G}) = -\frac{1}{2\pi i} \oint dz \text{Tr}(e^{-z\hat{G}} \delta \hat{G}), \quad (\text{D.20})$$

we obtain the equation of motion as

$$\frac{1}{2\pi i} \oint dz \langle x_1 | e^{-z\hat{G}} | x_2 \rangle \approx 0. \quad (\text{D.21})$$

The above equation is bilocal depending on both  $x_1$  and  $x_2$ , but we can reorganize them into the center position  $x = \frac{1}{2}(x_1 + x_2)$  and the relative position  $q = x_2 - x_1$  taken as an auxiliary variable generating infinitely many higher-spin fields. On the other hand, varying the operator  $\hat{G}$  in the formal action (D.13) would lead to the operator equation

$$\langle i_1 | \delta(\hat{G}) | i_2 \rangle \approx 0. \quad (\text{D.22})$$

Using (D.12), one finds the collection of equations  $\delta(\lambda_i) \approx 0$  stating that the Hermitian operator  $\hat{G}$  is non-singular in the sense that it has no normalizable eigenvector with vanishing eigenvalue, as was argued by Segal. This is indeed the case for the vacuum solution

$\hat{G}_0 = \hat{p}^2$  in Euclidean signature since there are no normalizable solutions of Laplace equation. Therefore, one may also argue that the formal equation of motion is somewhat empty since all operators  $\hat{G} = \hat{G}_0 + \hat{H}$  in the close vicinity of the vacuum solution keep this property. This agrees with the alternative formal action  $\text{Tr}(\hat{1})$ . Their equivalence indeed relies on ignoring the contribution of zero modes.

**Formal gauge fixing.** In the formal approach, one may then use the Cholevsky decomposition of positive-definite Hermitian matrices

$$\hat{G} \approx \hat{L} \hat{D} \hat{L}^\dagger, \quad (\text{D.23})$$

where  $\hat{D}$  can be any chosen diagonal matrix with positive entries and  $\hat{L}$  is a lower-triangular invertible matrix with positive diagonal entries. Therefore, on-shell  $\hat{G}$  can be mapped to the vacuum solution in the formal approach (setting  $\hat{D} = \hat{G}_0$  and  $\hat{L}^\dagger = \hat{O}$ ) by a (large) gauge transformation.

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