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# Deriving multiple-input production and utility functions from elasticities of substitution functions\*

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## Abstract

For each production or utility function, we can define the corresponding elasticities of substitution functions; but is the reverse true? This paper shows that yes, and that this link is fruitful. By inverting the system of partial differential equations defining the elasticities of substitution functions, we uncover an analytical formula which encompasses *all* production and utility functions that are admissible in Arrow-Debreu equilibria. We highlight the “Constant Elasticities of Substitution Matrix” (CESM) class of functions which, unlike the CES functions, does not assume uniform substitutability among all pairs of goods. A shortcoming of our method is that it permits only to control for local concavity while it is difficult to control for global concavity.

**Keywords:** Production functions, utility functions, elasticity of substitution, marginal productivity, marginal utility, factor shares.

**JEL codes:** C60, D11, D20, D24, D33, E23, E25.

## Introduction

Production and utility functions are the tools that neoclassical economics use to model aggregation, respectively from the technical and subjective points of view. They constitute the most important building blocks of theory and empirics in several subfields of economics: economic growth, macroeconomics, international trade, industrial organization, etc.. In general equilibrium theory, the set of admissible intensive-form production functions over  $n \geq 1$  inputs is the set of functions of  $(\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^*$  of class  $\mathcal{C}^2$  that are strictly increasing with respect to each variable and strictly concave. The same set describes admissible utility functions.<sup>1</sup> We call these functions ‘aggregation functions’ (AFs).

Despite their importance in serving theoretical and empirical representations, only a handful of functional forms of AFs are actually used by researchers and practitioners. Cobb-Douglas and

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<sup>1</sup>Of course, unlike production functions, utility functions need not be strictly positive. However, restricting to strictly positive utility functions is not restrictive since preferences are not altered by strictly increasing transformations.

CES<sup>2</sup> functions vastly dominate, while the use of other specifications has remained marginal.<sup>3</sup> As a matter of fact, the modeler’s toolbox becomes more restrictive as the number of inputs increase.<sup>4</sup> For instance CES production and utility functions rest on the assumption that all pairs of goods have the same substitutability parameter, which can be a costly assumption even at moderate disaggregation levels.<sup>5</sup>

AFs are most often classified according to the associated elasticity of substitution functions (ESFs). ESFs are defined through the second-order derivatives of the AF and govern the evolution of first-order quantities as inputs vary.<sup>6</sup> As such, ES functions carry a central information on the associated AF.

In this paper, we show that the link between AFs and the associated ESFs is so tight that it permits to derive an analytical formula encompassing all AFs over a finite number of arguments ( $n \geq 1$ ). The method that we use consists in inverting the system of partial differential equations defining the ESFs.<sup>7</sup> We hinge on the method developed by Moysan and Senouci (2016) for one-input AFs and extend it to any number of inputs. Doing so, we uncover a new – to the best of our knowledge – duality result between AFs and ESFs, in the form of an analytical expression.<sup>8</sup>

All functions derived through our formula, however, are not AFs. Our formula generates functions that are strictly increasing w.r.t each variable, strictly concave w.r.t. each variable, but which might not be strictly concave. Still, we show how to control for local strict concavity around some reference input vector, which proves enough for local uses of our formula.

The rest of the paper is organized as follows. In section 1, we recall the theorem stated in Moysan and Senouci (2016) on one-input AFs, in section 2 we treat the case of two-input AFs before turning to the general  $n$ -input AFs in section 3. We investigate concavity issues in section 4 and conclude in section 5.

## 1 One-input aggregation functions

Let’s call  $\mathcal{N}_1$  the set of on-input AFs :

$$\mathcal{N}_1 = \{f \in \mathcal{C}^2 : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^* \mid f' > 0; f'' < 0\}$$

All functions  $f \in \mathcal{N}_1$  verify:  $\forall x > 0, f(x) > x f'(x)$ , we prove this in appendix A.1.

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<sup>2</sup>Solow (1956, p. 77), Arrow et al. (1961).

<sup>3</sup>Uzawa (1962) and McFadden (1963) extended the definition of the CES function to more than two inputs. Lu and Fletcher (1968), Revankar (1971) and Kadiyala (1972) defined some two-input, Variable Elasticity of Substitution (VES) production functions. Other existing functions include the translog class (Christensen et al. (1973, 1975)), the LINEX class (Kümmel et al. (1985), Ayres and Warr (2005)) and the Isoelastic Elasticity of Substitution class (Growiec and Mućk (2016)). See Mishra (2007) on the history of production functions.

<sup>4</sup>The usual procedure of nesting CES production functions has the drawbacks that the parameters lose their meaning of elasticities of substitution and that the analytical expression depends on the nesting order.

<sup>5</sup>Uzawa (1962, section VI) extended the multiple-input CES formula to allow for different elasticities of substitution among pairs of goods. Zhelobodko et al. (2012) and Parenti et al. (2017) have extended the Dixit-Stiglitz monopolistic competition framework to allow for more flexible preferences.

<sup>6</sup>Quantities governed by the ES include relative marginal rates of substitution, relative marginal costs, competitive factor shares, etc.. See Hicks (1932 [1963]), Robinson (1933 [1969]), Allen (1938) and Samuelson (1968).

<sup>7</sup>Lau (1976) attempted to link ESFs to cost functions.

<sup>8</sup>Our results can be seen as an extension of Arrow et al. (1961) result to any number of inputs and ES functions that are not necessarily constant nor equal across pairs of inputs.

For any  $f \in \mathcal{N}_1$ , we can define functions  $\alpha$ ,  $\Pi$  and  $\sigma$ :

$$\alpha(x) = \frac{xf'(x)}{f(x)} \in (0, 1) \quad (1)$$

$$\Pi(x) = \frac{\alpha(x)}{1-\alpha(x)} \in (0, +\infty) \quad (2)$$

$$\sigma(x) = -\frac{f'(x)(f(x)-xf'(x))}{xf(x)f''(x)} \in (0, +\infty) \quad (3)$$

For a production function,  $\alpha(x)$  is the competitive  $x$  share,  $\Pi(x)$  is the ratio of the competitive  $x$  share over the other factor competitive share, and  $\sigma$  is the ESF associated to  $f$ .  $\sigma$  is linked to  $\Pi$  through the equation:

$$\frac{\Pi'(x)}{\Pi(x)} = \frac{\sigma(x)-1}{\sigma(x)} \frac{1}{x} \quad (4)$$

By inverting equation (3), Moysan and Senouci (2016) have shown the following result:

**Theorem 1.** — Let  $f \in \mathcal{N}_1$  and let  $\sigma \in \{\sigma \in \mathcal{C}^0 : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*\}$  be the associated ESF. Let  $\bar{x} > 0$ . Then, there exists two constants  $\bar{A} > 0$  and  $\bar{\Pi} > 0$  such that:

$$\forall x > 0, \quad f(x) = \bar{A} \cdot \exp \left( \int_{\bar{x}}^x \frac{dx'}{x' \left( 1 + \frac{\exp \left( -\int_{\bar{x}}^{x'} \frac{\sigma-1}{\sigma} (x'') \frac{dx''}{x''} \right)}{\bar{\Pi}} \right)} \right) \quad (5)$$

Besides,  $\bar{A} = f(\bar{x})$ ,  $\bar{\Pi} = \Pi(\bar{x})$ .

— Conversely, let  $\sigma \in \{\sigma \in \mathcal{C}^0 : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*\}$  and let  $\bar{A} > 0$  and  $\bar{\Pi} > 0$ . Then, function  $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  defined by equation (5) is in  $\mathcal{N}_1$ .

*Proof.* See Moysan and Senouci (2016, p. 81). □

Theorem 1 shows that to design a function  $f \in \mathcal{N}_1$  amounts to (a) choose an ESF  $\sigma$  from the admissible set which is non other than  $\{\sigma \in \mathcal{C}^0 : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*\}$ , and (b) choose the value taken by  $f$  ( $= \bar{A} > 0$ ) and the value taken by  $\Pi$  ( $= \bar{\Pi} > 0$ ) at some specific input level  $\bar{x}$ . Consequently, (5) constitute a ‘recipe’ for all one-input AFs.

## 2 Two-input aggregation functions

For the ease of understanding of the  $n$ -input case in the next section, we first treat the two-input case in this section. We call  $\mathcal{N}_2$  the set of two-input AFs:

$$\mathcal{N}_2 = \left\{ f \in \mathcal{C}^2 : (\mathbb{R}_+^*)^2 \rightarrow \mathbb{R}_+^* \mid f_1 > 0, f_2 > 0, f \text{ strictly concave} \right\}$$

where  $f_i$  denotes the partial derivative of function  $f$  w.r.t. the  $i^{\text{th}}$  variable, for  $i \in \{1, 2\}$ .

We denote by  $\mathcal{N}'_2$  the set of functions of class  $\mathcal{C}^2$  that are strictly increasing and strictly concave w.r.t. each variable:

$$\mathcal{N}'_2 = \left\{ f \in \mathcal{C}^2 : (\mathbb{R}_+^*)^2 \rightarrow \mathbb{R}_+^* \mid f_1 > 0, f_2 > 0, f_{11} < 0, f_{22} < 0 \right\}$$

Clearly,  $\mathcal{N}_2 \subset \mathcal{N}'_2$  and  $\mathcal{N}'_2 \neq \mathcal{N}_2$ . We show now how theorem 1 can be extended to write functions belonging to  $\mathcal{N}'_2$  in terms of the associated ESFs as well as some constants.

Let  $f \in \mathcal{N}'_2$ . Then for any  $(x_1, x_2) \in (\mathbb{R}_+^*)^2$ , functions  $f(\cdot, x_2)$  and  $f(x_1, \cdot)$  belong to  $\mathcal{N}_1$ . Consequently, for any  $(x_1, x_2) \in (\mathbb{R}_+^*)^2$ ,  $f(x_1, x_2) > x_1 f_1(x_1, x_2)$  and  $f(x_1, x_2) > x_2 f_2(x_1, x_2)$ .

Let  $(\bar{x}_1, \bar{x}_2) \in (\mathbb{R}_+^*)^2$ . Applying the first part of theorem 1 to  $f(x_1, \cdot)$  yields:

$$f(x_1, x_2) = f(x_1, \bar{x}_2) \cdot \exp \left( \int_{\bar{x}_2}^{x_2} \frac{dx'_2}{x'_2 \left( 1 + \frac{\exp\left(-\int_{x'_2}^{x_2} \frac{\sigma_2 - 1}{\sigma_2} (x_1, x''_2) \frac{dx''_2}{x''_2}\right)}{\Pi_2(x_1, \bar{x}_2)} \right)} \right) \quad (6)$$

with:

$$\forall (x_1, x_2) \in (\mathbb{R}_+^*)^2, \quad \sigma_2(x_1, x_2) = - \frac{f_2(f - x_2 f_2)}{x_2 f f_{22}} \Big|_{(x_1, x_2)}$$

and:

$$\forall (x_1, x_2) \in (\mathbb{R}_+^*)^2, \quad \Pi_2(x_1, x_2) = \frac{x_2 f_2}{f - x_2 f_2} \Big|_{(x_1, x_2)}$$

Remark that, by the definition of  $\sigma_2$  and like in equation (4):

$$\Pi_1(x_1, x_2) = \Pi_1(\bar{x}_1, x_2) \cdot \exp \left( \int_{\bar{x}_1}^{x_1} \frac{\sigma_1 - 1}{\sigma_1} (x'_1, x_2) \frac{dx'_1}{x'_1} \right) \quad (7)$$

Furthermore,  $f(\cdot, \bar{x}_2) \in \mathcal{N}_1$ , so we can apply theorem 1 to function  $f(\cdot, \bar{x}_2)$  in equation (6), which yields:

$$f(x_1, x_2) = f(\bar{x}_1, \bar{x}_2) \cdot \exp \left( \int_{\bar{x}_1}^{x_1} \frac{dx'_1}{x'_1 \left( 1 + \frac{\exp\left(-\int_{x'_1}^{x_1} \frac{\sigma_1 - 1}{\sigma_1} (x''_1, \bar{x}_2) \frac{dx''_1}{x''_1}\right)}{\Pi_1(\bar{x}_1, \bar{x}_2)} \right)} \right) \cdot \exp \left( \int_{\bar{x}_2}^{x_2} \frac{dx'_2}{x'_2 \left( 1 + \frac{\exp\left(-\int_{x'_2}^{x_2} \frac{\sigma_2 - 1}{\sigma_2} (x_1, x''_2) \frac{dx''_2}{x''_2}\right)}{\Pi_2(x_1, \bar{x}_2)} \right)} \right) \quad (8)$$

Equation (8) contains the quantity  $\Pi_2(x_1, \bar{x}_2)$  which remains unknown.

**Lemma 1.** Let  $f \in \mathcal{N}_2$  and let functions  $\alpha_1$  and  $\alpha_2$  be defined by:  $\forall (x_1, x_2) \in (\mathbb{R}_+^*)^2, \alpha_i(x_1, x_2) = \frac{x_i f_i}{f} (x_1, x_2) = \frac{\Pi_i}{1 + \Pi_i} (x_1, x_2)$ . Then,  $\forall (x_1, x_2) \in (\mathbb{R}_+^*)^2$ :

$$\frac{x_2}{\alpha_1} \frac{\partial \alpha_1}{\partial x_2} = \alpha_2(x_1, x_2) (\sigma_{12}(x_1, x_2) - 1) \quad (9)$$

$$\frac{x_1}{\alpha_2} \frac{\partial \alpha_2}{\partial x_1} = \alpha_1(x_1, x_2) (\sigma_{12}(x_1, x_2) - 1) \quad (10)$$

where  $\sigma_{12}(x_1, x_2) = \frac{f f_{12}}{f_1 f_2} (x_1, x_2)$ .

*Proof.* Immediate from log-differentiation of  $\alpha_1$  and  $\alpha_2$ . □

For convenience, let's call  $\beta(x_1) = \alpha_2(x_1, \bar{x}_2)$ . From equations (7) at  $(x_1, \bar{x}_2)$  and (10), function  $\beta(\cdot)$  is a solution to the differential equation:

$$\beta'(x_1) = \beta(x_1) \frac{\sigma_{12}(x_1, \bar{x}_2) - 1}{x_1 \left( 1 + \frac{\exp\left(-\int_{\bar{x}_1}^{x_1} \frac{\sigma_1 - 1}{\sigma_1} (x'_1, \bar{x}_2) \frac{dx'_1}{x'_1}\right)}{\Pi_1(\bar{x}_1, \bar{x}_2)} \right)}$$

which easily integrates into:

$$\beta(x_1) = \beta(\bar{x}_1) \exp \left( \int_{\bar{x}_1}^{x_1} \frac{\sigma_{12}(x'_1, \bar{x}_2) - 1}{1 + \frac{\exp\left(-\int_{\bar{x}_1}^{x'_1} \frac{\sigma_1 - 1}{\sigma_1}(x''_1, \bar{x}_2) \frac{dx''_1}{x''_1}\right)}{\Pi_1(\bar{x}_1, \bar{x}_2)}} \frac{dx'_1}{x'_1} \right) \quad (11)$$

We can now state the central result of this section:

**Theorem 2.** Let  $f \in \mathcal{N}'_2$  and let  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_{12}$  be the elasticity of substitution functions associated to  $f$ , defined by:

$$\forall (x_1, x_2) \in (\mathbb{R}_+^*)^2, \begin{cases} \sigma_1(x_1, x_2) = \left. -\frac{f_1(f - x_1 f_1)}{x_1 f f_{11}} \right|_{(x_1, x_2)} \in \mathbb{R}_+^* \\ \sigma_2(x_1, x_2) = \left. -\frac{f_2(f - x_2 f_2)}{x_2 f f_{22}} \right|_{(x_1, x_2)} \in \mathbb{R}_+^* \\ \sigma_{12}(x_1, x_2) = \left. -\frac{f f_{12}}{f_1 f_2} \right|_{(x_1, x_2)} \in \mathbb{R} \end{cases}$$

Let  $(\bar{x}_1, \bar{x}_2) \in (\mathbb{R}_+^*)^2$ , and let  $\bar{A} = f(\bar{x}_1, \bar{x}_2) \in \mathbb{R}_+^*$ ,  $\bar{\Pi}_1 = \left. \frac{x_1 f_1}{f - x_1 f_1} \right|_{(\bar{x}_1, \bar{x}_2)} \in \mathbb{R}_+^*$ ,  $\bar{\Pi}_2 = \left. \frac{x_2 f_2}{f - x_2 f_2} \right|_{(\bar{x}_1, \bar{x}_2)} \in \mathbb{R}_+^*$ ,  $\bar{\alpha}_1 = \frac{\bar{\Pi}_1}{1 + \bar{\Pi}_1} \in (0, 1)$ , and  $\bar{\alpha}_2 = \frac{\bar{\Pi}_2}{1 + \bar{\Pi}_2} \in (0, 1)$ . Then, for all  $(x_1, x_2) \in (\mathbb{R}_+^*)^2$ :

$$f(x_1, x_2) = \bar{A} \cdot \exp \left( \int_{\bar{x}_1}^{x_1} \frac{dx'_1}{x'_1 \left( 1 + \frac{\exp\left(-\int_{\bar{x}_1}^{x'_1} \frac{\sigma_1 - 1}{\sigma_1}(x''_1, \bar{x}_2) \frac{dx''_1}{x''_1}\right)}{\bar{\Pi}_1} \right)} + \int_{\bar{x}_2}^{x_2} \frac{dx'_2}{x'_2 \left( 1 + \frac{\exp\left(-\int_{\bar{x}_2}^{x'_2} \frac{\sigma_2 - 1}{\sigma_2}(x_1, x''_2) \frac{dx''_2}{x''_2}\right)}{\bar{\Pi}_2(x_1, \bar{x}_2)} \right)} \right) \quad (12)$$

with:

$$\Pi_2(x_1, \bar{x}_2) = 1 / \left( \frac{\exp \left( - \int_{\bar{x}_1}^{x_1} \frac{\sigma_{12}(x'_1, \bar{x}_2) - 1}{1 + \frac{\exp\left(-\int_{\bar{x}_1}^{x'_1} \frac{\sigma_1 - 1}{\sigma_1}(x''_1, \bar{x}_2) \frac{dx''_1}{x''_1}\right)}{\bar{\Pi}_1}} \frac{dx'_1}{x'_1} \right)}{\bar{\alpha}_2} - 1 \right) \quad (13)$$

It also holds that, for all  $(x_1, x_2) \in (\mathbb{R}_+^*)^2$ :

$$f(x_1, x_2) = \bar{A} \cdot \exp \left( \int_{\bar{x}_1}^{x_1} \frac{dx'_1}{x'_1 \left( 1 + \frac{\exp\left(-\int_{\bar{x}_1}^{x'_1} \frac{\sigma_1 - 1}{\sigma_1}(x''_1, x_2) \frac{dx''_1}{x''_1}\right)}{\Pi_1(\bar{x}_1, x_2)} \right)} + \int_{\bar{x}_2}^{x_2} \frac{dx'_2}{x'_2 \left( 1 + \frac{\exp\left(-\int_{\bar{x}_2}^{x'_2} \frac{\sigma_2 - 1}{\sigma_2}(\bar{x}_1, x''_2) \frac{dx''_2}{x''_2}\right)}{\bar{\Pi}_2} \right)} \right) \quad (14)$$

with:

$$\Pi_1(\bar{x}_1, x_2) = 1 / \left( \frac{\exp \left( - \int_{\bar{x}_2}^{x_2} \frac{\sigma_{12}(\bar{x}_1, x'_2) - 1}{1 + \frac{\exp\left(-\int_{\bar{x}_2}^{x'_2} \frac{\sigma_2 - 1}{\sigma_2}(\bar{x}_1, x''_2) \frac{dx''_2}{x''_2}\right)}{\bar{\Pi}_2}} \frac{dx'_2}{x'_2} \right)}{\bar{\alpha}_1} - 1 \right) \quad (15)$$

*Proof.* We get equations (12)-(13) directly from equations (8) and (11). Equations (14)-(15) are obtained by symmetry in the arguments  $x_1$  and  $x_2$  – equivalently, equations (14)-(15) are obtained by applying the same method to  $f(\cdot, x_2)$  and then to  $f(\bar{x}_1, \cdot)$ .  $\square$

So all two-input AFs admit a representation of the form of equations (12)-(13) or, equivalently, of the form of equations (14)-(15) which, despite appearances, are equivalent.

However, the reciprocal is wrong: not all functions of the form of (12)-(13) or (14)-(15) are AFs. For instance, differentiating equation (12) with respect to  $x_2$  we get:

$$f_2(x_1, x_2) = \frac{f(x_1, x_2)}{x_2 \left( 1 + \frac{\exp\left(-\int_{\bar{x}_2}^{x_2} \frac{\sigma_2-1}{\sigma_2} (x_1, x_2') \frac{dx_2'}{x_2'}\right)}{\Pi_2(x_1, \bar{x}_2)} \right)}$$

$$f_{22}(x_1, x_2) = \frac{1}{\sigma_2(x_1, \bar{x}_2)} \frac{f_2(x_1, x_2) \frac{\exp\left(-\int_{\bar{x}_2}^{x_2} \frac{\sigma_2-1}{\sigma_2} (x_1, x_2') \frac{dx_2'}{x_2'}\right)}{\Pi_2(x_1, \bar{x}_2)}}{x_2 \left( 1 + \frac{\exp\left(-\int_{\bar{x}_2}^{x_2} \frac{\sigma_2-1}{\sigma_2} (x_1, x_2') \frac{dx_2'}{x_2'}\right)}{\Pi_2(x_1, \bar{x}_2)} \right)}$$

with  $\Pi_2(x_1, \bar{x}_2)$  like defined in equation (13). While  $f(x_1, x_2) > 0$  by construction, it appears that  $f_2(x_1, x_2)$  might be negative since nothing ensures that  $1 + \exp\left(-\int_{\bar{x}_2}^{x_2} \frac{\sigma_2-1}{\sigma_2} (x_1, x_2') \frac{dx_2'}{x_2'}\right) / \Pi_2(x_1, \bar{x}_2)$  is strictly positive, since  $\Pi_2(x_1, \bar{x}_2)$  in equation (13) might be negative, or even undefined. The sign of  $f_{22}$  is also undefined in this expression.

All in all, this means that equations (12)-(13) or (14)-(15) do a poor job in providing a ‘general recipe’ for all two-input AFs.

For instance, for constant  $\sigma_1 \neq 1$ ,  $\sigma_2 \neq 1$ ,  $\sigma_{12} \in \mathbb{R}$ , our method yields:

$$\left\{ \begin{array}{l} f(x_1, x_2) = \bar{A} \cdot \left( 1 - \bar{\alpha}_1 + \bar{\alpha}_1 \left( \frac{x_1}{\bar{x}_1} \right)^{\frac{\sigma_1-1}{\sigma_1}} \right)^{\frac{\sigma_1}{\sigma_1-1}} \cdot \left( 1 - \alpha_2(x_1, \bar{x}_2) + \alpha_2(x_1, \bar{x}_2) \left( \frac{x_2}{\bar{x}_2} \right)^{\frac{\sigma_2-1}{\sigma_2}} \right)^{\frac{\sigma_2}{\sigma_2-1}} \\ \text{with:} \quad \alpha_2(x_1, \bar{x}_2) = \bar{\alpha}_2 \left( 1 - \bar{\alpha}_1 + \bar{\alpha}_1 \left( \frac{x_1}{\bar{x}_1} \right)^{\frac{\sigma_1-1}{\sigma_1}} \right)^{(\sigma_{12}-1) \frac{\sigma_1}{\sigma_1-1}} \end{array} \right. \quad (16)$$

or, equivalently:

$$\left\{ \begin{array}{l} f(x_1, x_2) = \bar{A} \cdot \left( 1 - \alpha_1(\bar{x}_1, x_2) + \alpha_1(\bar{x}_1, x_2) \left( \frac{x_1}{\bar{x}_1} \right)^{\frac{\sigma_1-1}{\sigma_1}} \right)^{\frac{\sigma_1}{\sigma_1-1}} \cdot \left( 1 - \bar{\alpha}_2 + \bar{\alpha}_2 \left( \frac{x_2}{\bar{x}_2} \right)^{\frac{\sigma_2-1}{\sigma_2}} \right)^{\frac{\sigma_2}{\sigma_2-1}} \\ \text{with:} \quad \alpha_1(\bar{x}_1, x_2) = \bar{\alpha}_1 \left( 1 - \bar{\alpha}_2 + \bar{\alpha}_2 \left( \frac{x_2}{\bar{x}_2} \right)^{\frac{\sigma_2-1}{\sigma_2}} \right)^{(\sigma_{12}-1) \frac{\sigma_2}{\sigma_2-1}} \end{array} \right. \quad (17)$$

It is apparent from (16) or (17) that  $\alpha_1(x_1, x_2)$  and  $\alpha_2(x_1, x_2)$  might not be confined to the  $(0, 1)$  interval, implying that functions  $f$  defined through (16) or (17) might not be an AF over  $(\mathbb{R}_+^*)^2$ . However, as we will see in section 4, it is possible to use formulas (16) or (17) together with some criterion over  $(\sigma_1, \sigma_2, \sigma_{12}, \bar{\alpha}_1, \text{ and } \bar{\alpha}_2)$  to characterize functions that are ‘local AFs’, i.e. functions which satisfy the definition of AFs in the neighborhood of  $(\bar{x}_1, \bar{x}_2)$ .

But first, we replicate in the next section the result of theorem 2 for  $n$ -input AFs.

### 3 $n$ -input aggregation functions

Let  $n \geq 2$ . We call  $\mathcal{N}_n$  the set of two-input AFs:

$$\mathcal{N}_n = \{f \in \mathcal{C}^2 : (\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^* \mid \forall i \in \{1, \dots, n\}, f_i > 0, f \text{ strictly concave}\}$$

We denote by  $\mathcal{N}'_n$  the set of functions of class  $\mathcal{C}^2$  that are strictly increasing and strictly concave w.r.t. each variable:

$$\mathcal{N}'_n = \{f \in \mathcal{C}^2 : (\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^* \mid \forall i \in \{1, \dots, n\}, f_i > 0, f_{ii} < 0\}$$

Again,  $\mathcal{N}_n \subset \mathcal{N}'_n$  and  $\mathcal{N}_n \neq \mathcal{N}'_n$ .

For any function  $f \in \mathcal{N}_n$ , we define the elasticities of substitution functions:

$$\forall i \in \{1, \dots, n\}, \forall (x_1, \dots, x_n), \quad \sigma_i(x_1, \dots, x_n) = \frac{f_i(f - x_i f_i)}{-x_i f f_{ii}} \Big|_{(x_1, \dots, x_n)} \in \mathbb{R}_+^* \quad (18)$$

$$\forall i, j \in \{1, \dots, n\}, i \neq j, \forall (x_1, \dots, x_n), \quad \sigma_{ij}(x_1, \dots, x_n) = \frac{f f_{ij}}{f_i f_j} \Big|_{(x_1, \dots, x_n)} \in \mathbb{R} \quad (19)$$

as well as each factor's competitive share function  $\alpha_i$  and the corresponding ratios:

$$\forall i \in \{1, \dots, n\}, \begin{cases} \alpha_i = \frac{x_i f_i}{f} \in (0, 1) \\ \Pi_i = \frac{\alpha_i}{1 - \alpha_i} \in \mathbb{R} \end{cases} \quad (20)$$

By definition, ESFs  $\sigma_i$  and  $\sigma_{ij}$  verify:

$$\forall i \in \{1, \dots, n\}, \forall (x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n, \quad \frac{\partial \Pi_i}{\partial x_i}(x_1, \dots, x_n) = \frac{\sigma_i - 1}{\sigma_i}(x_1, \dots, x_n) \frac{\Pi_i(x_1, \dots, x_n)}{x_i} \quad (21)$$

$$\forall i, j \in \{1, \dots, n\}, i \neq j, \forall (x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n, \quad \frac{\partial \alpha_i}{\partial x_j}(x_1, \dots, x_n) = (\sigma_{ij}(x_1, \dots, x_n) - 1) \frac{\alpha_j(x_1, \dots, x_n)}{x_j} \quad (22)$$

Finally, let  $(\bar{x}_1, \dots, \bar{x}_n) \in (\mathbb{R}_+^*)^n$ . We first present a simple lemma that shall be useful in this section.

**Lemma 2.** *Let  $p \in \mathbb{N}^*$  and let  $h$  be a function of  $(\mathbb{R}_+^*)^p$  into  $\mathbb{R}_+^*$  of class  $\mathcal{C}^1$ . For any  $i \in \{1, \dots, p\}$ , define function  $m_i$  by  $m_i = h_i/h$ , where  $h_i$  is the first derivative of  $h$  w.r.t. its  $i^{\text{th}}$  variable. Then:*

$$\forall (x_1, \dots, x_p) \in (\mathbb{R}_+^*)^p, \quad h(x_1, \dots, x_p) = h(\bar{x}_1, \dots, \bar{x}_p) \cdot \exp\left(\sum_{i=1}^p \int_{\bar{x}_i}^{x_i} m_i(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_p)\right)$$

*Proof.* See appendix A.2. □

First, let's apply lemma 2 to  $f$ . Let  $(x_1, \dots, x_n) \in (\mathbb{R}_+^*)^n$ . By the definition of  $\alpha_i$ , for all  $i \in \{1, \dots, n\}$ , it holds that  $(f_i/f)(x_1, \dots, x_n) = \alpha_i(x_1, \dots, x_n)/x_i$ . Then, for all :

$$f(x_1, \dots, x_n) = f(\bar{x}_1, \dots, \bar{x}_n) \cdot \exp\left(\sum_{i=1}^n \int_{\bar{x}_i}^{x_i} \alpha_i(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right) \quad (23)$$

The next step is then to find a formula for  $\alpha_i(x_1, \dots, x_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n)$ . To do so, we use equations (21)-(22).



**Lemma 3.** For all  $i \in \{2, \dots, n\}$ :

$$\alpha_i(x_1, x_{i-1}, \bar{x}_i, \dots, \bar{x}_n) = \alpha_i(\bar{x}_1, \dots, \bar{x}_n) \cdot \exp\left(\sum_{j=1}^{i-1} \int_{\bar{x}_j}^{x_j} ((\sigma_{ij} - 1) \alpha_j)(x_1, \dots, x_{j-1}, x'_j, \bar{x}_{j+1}, \dots, \bar{x}_n) \frac{dx'_j}{x'_j}\right)$$

*Proof.* Let  $i \in \{2, \dots, n\}$ . Then, for all  $j \in \{1, \dots, i-1\}$ , equation (22) together with lemma 2 leads to the stated formula.  $\square$

**Lemma 4.** For all  $i \in \{2, \dots, n\}$ :

$$\alpha_i(x_1, \dots, x_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n) = \frac{1}{1 + \left( \frac{\exp\left(-\sum_{j=1}^{i-1} \int_{\bar{x}_j}^{x_j} (\sigma_{ij} - 1) \alpha_j(x_1, \dots, x_{j-1}, x'_j, \bar{x}_{j+1}, \dots, \bar{x}_n) \frac{dx'_j}{x'_j}\right)}{\alpha_i(\bar{x}_1, \dots, \bar{x}_n)} - 1 \right)} \exp\left(-\int_{\bar{x}_i}^{x_i} \frac{\sigma_{ii} - 1}{\sigma_i}(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right)$$

*Proof.* See appendix A.3.  $\square$

Lemma 4 links  $\alpha_i$  to the  $\alpha_j$ 's, with  $j < i$ . Again, a reasoning by induction allows us to derive all  $\alpha_i$ 's.

For the ease of notation, we call  $\bar{\alpha}_i = \alpha_i(\bar{x}_1, \dots, \bar{x}_n)$ .

We also define for all  $i \in \{1, \dots, n\}$  the operator  $\Gamma_i$ :

$$\Gamma_i : \mathcal{C}^1\left(\left(\mathbb{R}_+^*\right)^i, \mathbb{R}_+^*\right) \times \mathcal{C}^1\left(\left(\mathbb{R}_+^*\right)^i, \mathbb{R}_+^*\right) \times \mathcal{C}^1\left(\left(\mathbb{R}_+^*\right)^{i-1}, \mathbb{R}_+^*\right) \times \mathbb{R}_+^* \times \mathbb{R}_+^* \rightarrow \mathcal{C}^1\left(\left(\mathbb{R}_+^*\right)^i, \mathbb{R}_+^*\right)$$

by:

$$\Gamma_i(\varphi, \gamma, \eta, x_0, \delta) : \begin{cases} \left(\mathbb{R}_+^*\right)^i & \rightarrow \mathbb{R}_+^* \\ (x_1, \dots, x_i) & \mapsto \exp\left(-\int_{x_0}^{x_i} \frac{\gamma(x_1, \dots, x_{i-1}, x'_i) - 1}{1 + \exp\left(-\int_{x_0}^{x'_i} \frac{\varphi(x_1, \dots, x_{i-1}, x''_i)}{x''_i} dx''_i\right) \cdot \left(\frac{\eta(x_1, \dots, x_{i-1})}{\delta} - 1\right)} \cdot \frac{dx'_i}{x'_i}\right) \end{cases} \quad (24)$$

with the convention  $\left(\mathbb{R}_+^*\right)^0 = \emptyset$  and that all functions of  $\emptyset$  into  $\mathbb{R}_+^*$  are constant equal to 1.

We also define, for  $i \in \{1, \dots, n\}$  functions  $\psi(i) \in \mathcal{C}^1\left(\left(\mathbb{R}_+^*\right)^{i-1}, \mathbb{R}_+^*\right)$  by induction:

$$\begin{cases} \psi(1) = 1 \\ \forall i \in \{2, \dots, n\}, \forall (x_1, \dots, x_{i-1}) \in \left(\mathbb{R}_+^*\right)^{i-1}, \psi(i)(x_1, \dots, x_{i-1}) = \prod_{j=1}^{i-1} \Gamma_j\left(\frac{\sigma_{j-1}}{\sigma_j}, \sigma_{ij}, \psi(j), \bar{x}_j, \bar{\alpha}_j\right) \end{cases} \quad (25)$$

**Lemma 5.** For all  $i \in \{1, \dots, n\}$ :

$$\alpha_i(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n) = \frac{1}{1 + \exp\left(-\int_{\bar{x}_i}^{x_i} \frac{\sigma_{ii} - 1}{\sigma_i}(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right)} \left(\frac{\psi(i)(x_1, \dots, x_{i-1})}{\bar{\alpha}_i} - 1\right)$$

*Proof.* See appendix A.4.  $\square$

We can now state the most important result of this section:

**Theorem 3.** Let  $f \in \mathcal{N}_n$ . Then:

$$f(x_1, \dots, x_n) = \bar{A} \cdot \exp \left( \sum_{i=1}^n \int_{\bar{x}_i}^{x_i} \frac{1}{1 + \exp \left( - \int_{\bar{x}_i}^{x_i} \frac{\sigma_i - 1}{\sigma_i} (x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i} \right) \left( \frac{\psi(i)(x_1, \dots, x_{i-1})}{\bar{\alpha}_i} - 1 \right) \frac{dx'_i}{x'_i} \right)} \right) \quad (26)$$

where  $\bar{A} = f(\bar{x}_1, \dots, \bar{x}_n)$ .

*Proof.* Immediate from equation (23) and lemma 5.  $\square$

Theorem 3 states that for any AF  $f$ ,  $f$  is representable in terms of its  $n(n+1)/2$  ESFs  $\sigma_i$  and  $\sigma_{ij}$  as well as  $n+1$  constants, out of which  $n$  are linked to the first derivatives of  $f$  – i.e.  $(\bar{\alpha}_1, \dots, \bar{\alpha}_n)$  – and one is linked to the level of  $f$  – i.e.  $\bar{A} = f(\bar{x}_1, \dots, \bar{x}_n)$ . Like in theorem 2, the  $i$ 's can be permuted to yield other formula – so there exists  $n!$  formulas equivalent to (26).

However, like for theorem 2, the reciprocal of theorem 3 does not hold. It is not enough to take  $n(n+1)/2$  functions plus  $n+1$  constants and apply formula (26) to get a function  $f \in \mathcal{N}'_n$ . The resulting function might not belong to  $\mathcal{N}_n$  neither, and formula (26) might even be undefined.

In the next section, we show how formula (26) can be used to design functions that *are* AFs near  $(\bar{x}_1, \dots, \bar{x}_n)$ .

#### 4 A local characterization of $n$ -input aggregation functions

As we already highlighted in the  $n=2$  case, formula (26) does not yield a ‘general recipe’ for all AFs. More precisely: if one takes some ESFs  $(\sigma_i)_i$  and  $(\sigma_{ij})_{i \neq j}$  and some constants  $\bar{A}, (\bar{\alpha}_i)_i$ , and consider function  $f$  defined in (26); then function  $f$  is not necessarily an AF. However, although it is difficult to use equation (26) to design functions that are global AFs, we now show that it is straightforward to use equation (26) to get functions that are AFs around  $(\bar{x}_1, \dots, \bar{x}_n)$ .

**Theorem 4.** Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Let  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in (\mathbb{R}_+^*)^n$ .

For all  $i \in \{1, \dots, n\}$ , let  $\sigma_i \in \mathcal{C}^0((\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^*)$ . For all  $i \neq j \in \{1, \dots, n\}$ , let  $\sigma_{ij} \in \mathcal{C}^0((\mathbb{R}_+^*)^n \rightarrow \mathbb{R})$  with  $\sigma_{ij} = \sigma_{ji}$ . Let  $\bar{A} > 0$  and for all  $i \in \{1, \dots, n\}$ , let  $\bar{\alpha}_i \in (0, 1)$ , with  $\sum_{i=1}^n \bar{\alpha}_i < 1$ . Define for all  $i$ ,  $\bar{\Pi}_i = \bar{\alpha}_i / (1 - \bar{\alpha}_i)$ .

We call  $\mathcal{M}(\bar{x})$  the following symmetric matrix:

$$\mathcal{M}(\bar{x}) = \begin{pmatrix} -\frac{1}{\sigma_1(\bar{x})\bar{\Pi}_1} & \sigma_{12}(\bar{x}) & \cdots & \sigma_{1n}(\bar{x}) \\ \sigma_{12}(\bar{x}) & -\frac{1}{\sigma_2(\bar{x})\bar{\Pi}_2} & \cdots & \sigma_{2n}(\bar{x}) \\ \vdots & & \ddots & \vdots \\ \sigma_{1n}(\bar{x}) & \sigma_{2n}(\bar{x}) & \cdots & -\frac{1}{\sigma_n(\bar{x})\bar{\Pi}_n} \end{pmatrix}$$

Let  $f$  be the function  $(\mathbb{R}_+^*)^n \rightarrow \mathbb{R}_+^*$  defined in equation (26).

Assume that matrix  $\mathcal{M}(\bar{x})$  is negative definite. Then, function  $f$  is (i) well-defined around  $\bar{x}$ ; (ii) strictly increasing w.r.t. each variable  $x_1, \dots, x_n$  and (iii) strictly concave over some neighborhood of  $\bar{x}$ .

*Proof.* See appendix A.5.  $\square$

The practical use of theorem 4 is straightforward. Imagine a modeler willing to model an AF  $f \in \mathcal{N}_n$ . Suppose that the modeler is interested in the AF only around some vector of reference  $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$  – maybe because theory or real data points to this vector of reference. Assume that the modeler has some a priori assumptions on the ESFs  $((\sigma_i)_i$  and  $(\sigma_{ij})_{i \neq j}$ ) as well as on the values of the constants  $(\bar{A}$  and  $(\bar{\alpha}_i)_i$ ). Finally, assume that the matrix  $\mathcal{M}(\bar{x})$  is negative definite for such assumptions on ESFs and constants.

Then, theorem 4 states that the modeler should consider  $f$  as being perfectly described by equation (26). As long as the modeler makes only a *local* use of this formula – meaning that the modeler abstains from applying formula (26) to vectors  $x \in (\mathbb{R}_+^*)^n$  that do not lie in the neighborhood of  $\bar{x}$  – function  $f$  yielded by formula (26) is an AF. Also, by this method, all ‘local AFs’ can be generated with the proper choice of ESFs and constants.

To illustrate the method we have in mind, we lay down a simple model of production with three inputs: physical capital ( $X_1$ ), human capital ( $X_2$ ), raw labor ( $L$ ):  $Y = F(X_1, X_2, L)$ . We assume that  $F$  has constant-returns-to-scale and we normalize by  $L$ :

$$y = \frac{Y}{L} = f(x_1, x_2) \quad \text{with } x_1 = \frac{X_1}{L}, \quad x_2 = \frac{X_2}{L} \quad \text{and } f(\cdot, \cdot) = F(\cdot, \cdot, 1)$$

The competitive input shares are  $\alpha_1$  for physical capital,  $\alpha_2$  for human capital and  $1 - \alpha_1 - \alpha_2$  for raw labor. For simplicity, let’s assume that  $\alpha_1$  does not depend on  $x_1$  and that  $\alpha_2$  does not depend on  $x_2$ , which amounts to assume that  $\sigma_1 \equiv 1$  and  $\sigma_2 \equiv 1$ . For the crossed ESFs, the traditional assumption in this three-input framework is that physical capital is more complementary with human capital than with raw labor. We take  $\sigma_{12} > 1$ , so that when  $x_1$  (resp.  $x_2$ ) increases,  $\alpha_2$  (resp.  $\alpha_1$ ) increases and  $1 - \alpha_1 - \alpha_2$  increases (idem). If in real-world data  $y = \bar{A}$ ,  $x_i = \bar{x}_i$  and  $\alpha_i = \bar{\alpha}_i$  for  $i = 1, 2$ ; then from equation (16) the production function near  $(\bar{x}_1, \bar{x}_2)$  has to be:

$$f(x_1, x_2) = \bar{A} \left( \frac{x_1}{\bar{x}_1} \right)^{\bar{\alpha}_1} \left( \frac{x_2}{\bar{x}_2} \right)^{\bar{\alpha}_2} \left( \frac{x_1}{\bar{x}_1} \right)^{\bar{\alpha}_1(\sigma_{12}-1)} \quad (27)$$

Function  $f$  is a local AF around  $(\bar{x}_1, \bar{x}_2)$  if and only if the eigenvalues of the corresponding matrix  $\mathcal{M}(\bar{x})$  are strictly negative:

$$\mathcal{M}(\bar{x}) = \begin{pmatrix} -\frac{1-\bar{\alpha}_1}{\bar{\alpha}_1} & \sigma_{12} \\ \sigma_{12} & -\frac{1-\bar{\alpha}_2}{\bar{\alpha}_2} \end{pmatrix}$$

The trace of  $\mathcal{M}(\bar{x})$  is strictly negative, and so  $\mathcal{M}(\bar{x})$  is negative definite if and only if:

$$\sigma_{12}^2 < \frac{1-\bar{\alpha}_1}{\bar{\alpha}_1} \frac{1-\bar{\alpha}_2}{\bar{\alpha}_2} \quad (28)$$

To sum up, for  $\sigma_{12}$ ,  $\bar{\alpha}_1$  and  $\bar{\alpha}_2$  that satisfy constraint (28), function  $f$  defined by equation (27) is strictly increasing in each argument and strictly concave near  $(\bar{x}_1, \bar{x}_2)$ .

## 5 Conclusion

This paper has uncovered the link between production and utility functions on the one hand, and the elasticities of substitution functions on the other. We brought new duality results which lead us to a formula encompassing all production and utility functions that are admissible in the literature.

This approach also provides a general recipe for all *local* production and utility functions. We hope that modeling, especially over quite large numbers of inputs, will benefit from the results presented here. We also keep in mind that other definitions could be taken for the elasticities of substitution functions, which we keep in mind for future work.

## A Appendix

### A.1 Proof that $f \in \mathcal{N}_1 \implies \forall x > 0, f(x) > x f'(x)$

Let  $f \in \mathcal{N}_1$  and let  $x > \bar{x} > 0$ . Then:  $f(x) = f(\bar{x}) + \int_{\bar{x}}^x f'(x') dx'$ . Since  $f \in \mathcal{N}_1$ ,  $f'$  is strictly decreasing, and so:

$$f(x) = f(\bar{x}) + \int_{\bar{x}}^x f'(x') dx' > f(\bar{x}) + (x - \bar{x})f'(\bar{x}) \quad (29)$$

Since  $f$  is strictly increasing and strictly positive, it admits a limit near  $0^+$ . Let's call this limit  $l_0 \geq 0$ . As  $\bar{x}$  tends to  $0^+$ , inequality (29) yields:

$$f(x) \geq l_0 + x f'(\bar{x}) \geq x f'(x)$$

The passage to the limit makes the inequality wide rather than strict. We now show that it is, indeed, strict.

Let  $x > 0$  and assume  $f(x) = x f'(x)$ . Then, for all  $\bar{x} < x$ , inequality (29) implies:  $0 > f(\bar{x}) - \bar{x} f'(\bar{x})$ . But we also know that  $f(\bar{x}) \geq \bar{x} f'(\bar{x})$ . So  $f'(\bar{x}) < f'(x)$ , which constitutes a contradiction. Consequently,  $\forall x > 0, f(x) > x f'(x)$ .

### A.2 Proof of lemma 2

We prove lemma 2 by induction on  $p$ .

- If  $p = 1$ , then  $h'(x) = m(x)h(x)$  is integrable into  $h(x) = h(\bar{x}) \cdot \int_{\bar{x}}^x m(x') dx'$ .
- Assume that lemma 2 is valid for some  $p \in \mathbb{N}^*$ . Let  $(\bar{x}_1, \dots, \bar{x}_{p+1}) \in (\mathbb{R}_+^*)^{p+1}$ . Let then  $h$  be a function of  $(\mathbb{R}_+^*)^{p+1}$  into  $\mathbb{R}_+^*$  and for all  $i$ , define  $m_i = h_i/h$  where  $h_i$  is the first derivative of  $h$  w.r.t. the  $i^{\text{th}}$  variable. In particular,  $m_{p+1} = h_{p+1}/h$ . Then, according to the bullet point above, for all  $(x_1, \dots, x_{p+1}) \in (\mathbb{R}_+^*)^{p+1}$ :

$$h(x_1, \dots, x_{p+1}) = h(x_1, \dots, x_p, \bar{x}_{p+1}) \cdot \exp\left(\int_{\bar{x}_{p+1}}^{x_{p+1}} m_{p+1}(x_1, \dots, x_p, x'_{p+1}) dx'_{p+1}\right)$$

Function  $g : (x_1, \dots, x_p) \mapsto h(x_1, \dots, x_p, \bar{x}_{p+1})$  is a function of  $(\mathbb{R}_+^*)^p \rightarrow \mathbb{R}_+^*$  of class  $\mathcal{C}^1$  and such that  $\forall i \in \{1, \dots, p\}$ ,  $g_i = m_i h$ . Then, by the assumption of induction, for all  $(x_1, \dots, x_p) \in (\mathbb{R}_+^*)^p$ ,

$$g(x_1, \dots, x_p) = h(x_1, \dots, x_p, \bar{x}_{p+1}) = h(\bar{x}_1, \dots, \bar{x}_p, \bar{x}_{p+1}) \cdot \exp\left(\sum_{i=1}^p \int_{\bar{x}_i}^{x_i} m_i(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_{p+1}) dx'_i\right)$$

Hence, for all  $(x_1, \dots, x_{p+1}) \in (\mathbb{R}_+^*)^{p+1}$ :

$$h(x_1, \dots, x_{p+1}) = h(\bar{x}_1, \dots, \bar{x}_p, \bar{x}_{p+1}) \cdot \exp\left(\sum_{i=1}^{p+1} \int_{\bar{x}_i}^{x_i} m_i(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_{p+1}) dx'_i\right)$$

### A.3 Proof of lemma 4

By equation (21):

$$\begin{aligned} \Pi_i(x_1, \dots, x_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n) = \\ \Pi_i(x_1, \dots, x_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \cdot \exp\left(\int_{\bar{x}_i}^{x_i} \frac{\sigma_i - 1}{\sigma_i}(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right) \end{aligned}$$

Since  $\alpha_i = \Pi_i / (1 + \Pi_i)$ , the above equation leads to:

$$\alpha_i(x_1, \dots, x_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n) = \frac{1}{1 + \frac{\exp\left(-\int_{\bar{x}_i}^{x_i} \frac{\sigma_i - 1}{\sigma_i}(x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right)}{\Pi_i(x_1, \dots, x_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n)}}$$

From lemma 3, and since  $1/\Pi_i = 1/\alpha_i - 1$ , it holds that:

$$\begin{aligned} \alpha_i(x_1, \dots, x_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n) &= \frac{1}{1 + \left( \frac{1}{\alpha_i(x_1, \dots, x_{i-1}, \bar{x}_i, \bar{x}_{i+1}, \dots, \bar{x}_n)} - 1 \right) \exp\left(-\int_{\bar{x}_i}^{x_i} \frac{\sigma_i - 1}{\sigma_i} (x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right)} \\ &= \frac{1}{1 + \left( \frac{\exp\left(-\sum_{j=1}^{i-1} \int_{\bar{x}_j}^{x_j} (\sigma_j - 1) \alpha_j(x_1, \dots, x_{j-1}, x'_j, \bar{x}_{j+1}, \dots, \bar{x}_n) \frac{dx'_j}{x'_j}\right)}{\alpha_i(\bar{x}_1, \dots, \bar{x}_n)} - 1 \right) \exp\left(-\int_{\bar{x}_i}^{x_i} \frac{\sigma_i - 1}{\sigma_i} (x_1, \dots, x_{i-1}, x'_i, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right)} \end{aligned}$$

which completes the proof.

#### A.4 Proof of lemma 5

For simplicity, let's call  $A_i(x_1, \dots, x_i) = \alpha_i(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n)$ .

We show by induction the following proposition  $H(i)$ , for  $i \in \{1, \dots, n\}$ :

$$H(i): A_i(x_1, \dots, x_i) = \frac{1}{1 + \exp\left(-\int_{\bar{x}_i}^{x_i} \frac{\sigma_i - 1}{\sigma_i} (x_1, \dots, x_{i-1}, x'_i, \bar{x}_i, \dots, \bar{x}_n) \frac{dx'_i}{x'_i}\right) \left(\frac{\psi(i)(x_1, \dots, x_{i-1})}{\bar{\alpha}_i} - 1\right)}$$

- Equation (21) for  $i = 1$  implies  $\Pi_1(x_1, \bar{x}_2, \dots, \bar{x}_n) = \Pi_1(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n) \cdot \exp\left(\int_{\bar{x}_1}^{x_1} \frac{\sigma_1 - 1}{\sigma_1} (x'_1, \bar{x}_2, \dots, \bar{x}_n) \frac{dx'_1}{x'_1}\right)$ , which leads to:

$$A_1(x_1) = \alpha_1(x_1, \bar{x}_2, \dots, \bar{x}_n) = \frac{1}{1 + \exp\left(-\int_{\bar{x}_1}^{x_1} \frac{\sigma_1 - 1}{\sigma_1} (x'_1, \bar{x}_2, \dots, \bar{x}_n) \frac{dx'_1}{x'_1}\right) \left(\frac{1}{\bar{\alpha}_1} - 1\right)} \implies H(1) \text{ is true.}$$

- Let  $i \in \{1, \dots, n-1\}$  such that for all  $j \leq i$ ,  $H(j)$  is true:

$$\forall j \leq i, A_j(x_1, \dots, x_j) = \frac{1}{1 + \exp\left(-\int_{\bar{x}_j}^{x_j} \frac{\sigma_j - 1}{\sigma_j} (x_1, \dots, x_{j-1}, x'_j, \bar{x}_{j+1}, \dots, \bar{x}_n) \frac{dx'_j}{x'_j}\right) \left(\frac{\psi(j)(x_1, \dots, x_{j-1})}{\bar{\alpha}_j} - 1\right)} \quad (30)$$

Using lemma 3:

$$A_{i+1}(x_1, \dots, x_{i+1}) = \frac{1}{1 + \left(\frac{1}{\bar{\alpha}_{i+1}} \exp\left(-\sum_{j=1}^i ((\sigma_j - 1) A_j)(x_1, \dots, x_{j-1}, x'_j, \bar{x}_{j+1}, \dots, \bar{x}_n) \frac{dx'_j}{x'_j}\right) - 1\right) \cdot \exp\left(-\int_{\bar{x}_{i+1}}^{x_{i+1}} \frac{\sigma_{i+1} - 1}{\sigma_{i+1}} (x_1, \dots, x_i, x'_{i+1}, \bar{x}_{i+2}, \dots, \bar{x}_n) \frac{dx'_{i+1}}{x'_{i+1}}\right)}$$

Using equation (30), the above equation leads to:

$$A_{i+1}(x_1, \dots, x_{i+1}) = \frac{1}{1 + \exp\left(\int_{\bar{x}_{i+1}}^{x_{i+1}} \frac{\sigma_{i+1} - 1}{\sigma_{i+1}} (x_1, \dots, x_i, x'_{i+1}, \bar{x}_{i+2}, \dots, \bar{x}_n) \frac{dx'_{i+1}}{x'_{i+1}}\right) \cdot \left(\frac{1}{\bar{\alpha}_{i+1}} \prod_{j=1}^i \exp\left(-\int_{\bar{x}_j}^{x_j} \frac{\sigma_j - 1}{\sigma_j} (x_1, \dots, x_{j-1}, x'_j, \bar{x}_{j+1}, \dots, \bar{x}_n) \frac{dx'_j}{x'_j}\right) \left(\frac{\psi(j)(x_1, \dots, x_{j-1})}{\bar{\alpha}_j} - 1\right)\right) - 1}$$

Therefore:

$$A_{i+1}(x_1, \dots, x_{i+1}) = \frac{1}{1 + \exp\left(\int_{\bar{x}_{i+1}}^{x_{i+1}} \frac{\sigma_{i+1} - 1}{\sigma_{i+1}} (x_1, \dots, x_i, x'_{i+1}, \bar{x}_{i+1}, \dots, \bar{x}_n) \frac{dx'_{i+1}}{x'_{i+1}}\right) \left(\frac{\psi(i+1)(x_1, \dots, x_i)}{\bar{\alpha}_i} - 1\right)}$$

which means that  $H(n+1)$  is true.

#### A.5 Proof of theorem 4

- By construction,  $f$  is well-defined around  $\bar{x}$ , strictly positive and of class  $\mathcal{C}^2$  around  $\bar{x}$ .
- It is also obvious by the construction of  $f$  that:

$$\forall i \in \{1, \dots, n\}, f_i(\bar{x}) = \frac{f(\bar{x})}{\bar{x}_i} \bar{\alpha}_i > 0$$

By continuity of  $f_i$  we conclude that  $f_i(x) > 0$  for all  $x$  in some neighborhood of  $\bar{x}$ .

- The Hessian matrix of  $f$  around  $\bar{x}$  is  $\mathcal{S}(\bar{x}) = (f_{ij}(\bar{x}))_{i,j \in \{1, \dots, n\}}$ . We now show that  $\mathcal{S}$  is negative definite if and only if  $\mathcal{M}(\bar{x})$  is negative definite.

**Lemma 6.** Let  $S$  be a symmetric  $n \times n$  matrix and let  $\lambda > 0$ . Let  $S'$  be the square matrix obtained by multiplying one column of  $S$  by  $\lambda$ , and let  $S''$  be the square matrix obtained by multiplying one line of  $S$  by  $\lambda$ .

Then,  $S$  is negative definite if and only if  $S'$  is negative definite, and if and only if  $S''$  is negative definite.

*Proof.* Let  $S$  be a symmetric  $n \times n$  matrix and let  $\lambda > 0$ . Without loss of generality, let's assume that  $S'$  is obtained by multiplying the first column of  $S$  by  $\lambda$ , while  $S''$  is obtained by multiplying the first row of  $S$  by  $\lambda$ :

$$S = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad S' = \begin{pmatrix} \lambda a_{11} & a_{12} & \cdots & a_{1n} \\ \lambda a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ \lambda a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad S'' = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

According to Sylvester's criterion,  $S$  is negative definite if and only if its leading principal minors alternate in signs:

$$S \text{ is negative definite} \iff \left\{ \mu_1 = a_{11} < 0, \mu_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \dots \right\}$$

For  $i \in \{1, \dots, n\}$  Let's call respectively  $\mu'_i$  and  $\mu''_i$  the  $i^{\text{th}}$  leading principal minor of  $S'$  and  $S''$ . It is obvious that:

$$\forall i \in \{1, \dots, n\}, \mu'_i = \mu''_i = \lambda \mu_i$$

$\mu_i$  is the same sign as  $\mu'_i$  as well as  $\mu''_i$ . Consequently,  $S$  is negative definite if and only if  $S'$  is negative definite, and if and only if  $S''$  is negative definite.  $\square$

In the spirit of lemma 6, for each  $i, j \in \{1, \dots, n\}$  we multiply the  $i^{\text{th}}$  row of  $\mathcal{S}(\bar{x})$  by  $1/f_i(\bar{x})$  and the  $j^{\text{th}}$  column of  $\mathcal{S}(\bar{x})$  by  $1/f_j(\bar{x})$ . We also multiply all rows and columns by  $f(\bar{x})$ . This proves that:  $\mathcal{S}(\bar{x})$  is negative definite  $\iff \mathcal{M}(\bar{x}) = \left( \frac{f \cdot f_{ij}}{f_i \cdot f_j}(\bar{x}) \right)_{i,j \in \{1, \dots, n\}}$  is negative definite.

The non-diagonal elements of  $\mathcal{M}(\bar{x})$  are  $(\sigma_{ij}(\bar{x}))_{i \neq j}$ . The diagonal elements of  $\mathcal{M}(\bar{x})$  are  $\frac{f f_{ii}}{f_i^2}(\bar{x}) = -\frac{1}{\sigma_i} \frac{f - \bar{x}_i f_i}{\bar{x}_i f_i}(\bar{x}) = -\frac{1}{\sigma_i \Pi_i}(\bar{x})$ .

If  $\mathcal{M}(\bar{x})$  is definite negative, then by continuity  $f$  is strictly concave over some neighborhood of  $\bar{x}$ .

## References

- [1] R. G. D. ALLEN (1938) *Mathematical Analysis for Economists*, Macmillan: London.
- [2] K. J. ARROW, H. B. CHENERY, B. S. MINHAS and R. M. SOLOW (1961) "Capital-Labor Substitution and Economic Efficiency," *Review of Economics and Statistics*, Vol. 43, No. 3 (Aug.), pp. 225–250.
- [3] L. R. CHRISTENSEN, D. W. JORGENSON and L. J. LAU (1973) "Transcendental Logarithmic Production Frontiers," *Review of Economics and Statistics*, Vol. 55, No. 1 (Feb.), pp. 28–45.
- [4] L. R. CHRISTENSEN, D. W. JORGENSON and L. J. LAU (1975) "Transcendental Logarithmic Utility Functions," *American Economic Review*, Vol. 65, No. 3 (Jun.), pp. 367–383.
- [5] J. GROWIEC and J. MUĆK (2016) "Isoelastic Elasticity of Substitution Production Functions," *Warsaw School of Economics, Collegium of Economic Analysis Working Paper No. 2016-001*.
- [6] J. R. HICKS (1932 [2<sup>nd</sup> edition 1963]) *The Theory of Wages*, London: Macmillan.
- [7] K. R. KADIYALA (1972) "Production Functions and Elasticity of Substitution," *Southern Economic Journal*, Vol. 38, No. 3 (Jan.), pp. 281–284.
- [8] R. KÜMMEL, W. STRASSL, A. GOSSNER and W. EICHHORN (1985) "Technical Progress and Energy Dependent Production Functions," *Zeitschrift für Nationalökonomie / Journal of Economics*, Vol. 45, No. 3 (Sep.), pp. 285–311.
- [9] L. J. LAU (1976) "A Note on Elasticity of Substitution Functions," *Review of Economic Studies*, Vol. 43, No. 2 (Jun.), pp. 353–358.
- [10] Y. LU and L. B. FLETCHER (1968) "A Generalization of the CES Production Function," *Review of Economics and Statistics*, Vol. 50, No. 4 (Nov.), pp. 449–452.
- [11] Z. GRILICHES (1969) "Capital-Skill Complementarity," *Review of Economics and Statistics*, Vol. 51, No. 4 (Nov.), pp. 465–468.
- [12] D. MCFADDEN (1963) "Constant Elasticity of Substitution Production Functions," *Review of Economic Studies*, Vol. 30, No. 2 (Jun.), pp. 73–83.
- [13] S. K. MISHRA (2007) "A Brief History of Production Functions," *MPRA Working Paper No. 5254*, available at [mpra.ub.uni-muenchen.de/5254](http://mpra.ub.uni-muenchen.de/5254).
- [14] G. MOYSAN and M. SENOUCI (2016) "A Note on 2-Input Neoclassical Production Functions," *Journal of Mathematical Economics*, Vol. 67 (Dec.), pp. 80–86.
- [15] M. PARENTI, P. USHCHEV and J.-F. THISSE (2017) "Toward a Theory of Monopolistic Competition," *Journal of Economic Theory*, Vol. 67 (Jan.), pp. 86–115.
- [16] N. S. REVANKAR (1971) "A Class of Variable Elasticity of Substitution Production Functions," *Econometrica*, Vol. 39, No. 1 (Jan.), pp. 61–71.
- [17] J. ROBINSON (1933 [2<sup>nd</sup> edition 1969]) *The Economics of Imperfect Competition*, London: Macmillan.

- [18] P. A. SAMUELSON (1968) "Two Generalizations of the Elasticity of Substitution," in *Value, Capital and Growth* (J.N. Wolfe ed.); Edinburgh:Edinburgh University Press.
- [19] R. M. SOLOW (1956) "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, Vol. 70, No. 1. (Feb.), pp. 65–94.
- [20] H. UZAWA (1962) "Production Functions with Constant Elasticities of Substitution," *Review of Economic Studies*, Vol. 29, No. 4 (Oct.), pp. 291–299.
- [21] E. ZHELOBODKO, S. KOKOVIN, M. PARENTI and J.-F. THISSE (2012) "Monopolistic Competition: Beyond the Constant Elasticity of Substitution," *Econometrica*, Vol. 80, No. 6 (Nov.), pp. 2765–2784.