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# Logarithmic Decay of a Wave Equation with Kelvin-Voigt Damping \*

Luc Robbiano <sup>†</sup> and Qiong Zhang <sup>‡§</sup>

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## Abstract

In this paper we analyze the long time behavior of a wave equation with local Kelvin-Voigt Damping. Through introducing proper class symbol and pseudo-differential calculus, we obtain a Carleman estimate, and then establish an estimate on the corresponding resolvent operator. As a result, we show the logarithmic decay rate for energy of the system without any geometric assumption on the subdomain on which the damping is effective.

**keywords.** Carleman estimate; wave equation; Kelvin-Voigt damping; logarithmic stability

**AMS 2010 subject classification:** Primary 93B05; Secondary 93B07, 35B37

## 1 Introduction

In this paper, we consider a wave equation with local Kelvin-Voigt damping and analyze long time behaviour for the -solution of the system. Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with smooth boundary  $\Gamma = \partial\Omega$ . Denote by  $\partial_n$  the unit outward normal vector on boundary  $\Gamma$ . The PDE model is as follows.

$$\begin{cases} y_{tt}(t, x) - \operatorname{div} [\nabla y(t, x) + a(x)\nabla y_t(t, x)] = 0 & \text{in } (0, \infty) \times \Omega, \\ y(t, x) = 0 & \text{on } (0, \infty) \times \Gamma, \\ y(0, x) = y^0, \quad y_t(0, x) = y^1 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where the coefficient function  $a(\cdot) \in L^1(\Omega)$  is nonnegative and not identically null.

The natural energy of system (1.1) is

$$E(t) = \frac{1}{2} \left[ \int_{\Omega} (|\nabla y(t)|^2 + |y_t(t)|^2) dx \right]. \quad (1.2)$$

A direct computation gives that

$$\frac{d}{dt} E(t) = - \int_{\operatorname{supp} a} a(x) |\nabla y_t(t)|^2 dx. \quad (1.3)$$

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Formula (1.3) shows that the only dissipative mechanism acting on the system is the viscoelastic damping  $\operatorname{div} [a \nabla y_t]$ , which is only effective on  $\operatorname{supp} a$ .

To rewrite the system as an evolution equation, we set the energy space as

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega), \quad (1.4)$$

with norm

$$\|Y\|_{\mathcal{H}} = \sqrt{\|\nabla y_1\|_{L^2(\Omega)}^2 + \|y_2\|_{L^2(\Omega)}^2}, \quad \forall Y = (y_1, y_2) \in \mathcal{H}. \quad (1.5)$$

Define an unbounded operator  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  by

$$\mathcal{A}Y = (y_2, \operatorname{div}(\nabla y_1 + a \nabla y_2)), \quad \forall Y = (y_1, y_2) \in D(\mathcal{A}),$$

and

$$D(\mathcal{A}) = \left\{ (y_1, y_2) \in \mathcal{H} : y_2 \in H_0^1(\Omega), \operatorname{div}(\nabla y_1 + a \nabla y_2) \in L^2(\Omega) \right\}.$$

Let  $Y(t) = (y(t), y_t(t))$ . Then system (1.1) can be written as

$$\frac{d}{dt} Y(t) = \mathcal{A}Y(t), \quad \forall t > 0, \quad Y(0) = (y^0, y^1). \quad (1.6)$$

It is known from [20] that if  $\operatorname{supp} a$  is non-empty, the operator  $\mathcal{A}$  generates a contractive  $C_0$  semigroup  $e^{t\mathcal{A}}$  on  $\mathcal{H}$  and  $i\mathbb{R} \subset \rho(\mathcal{A})$ , the resolvent of  $\mathcal{A}$ . Consequently, the semigroup  $e^{t\mathcal{A}}$  is strongly stable. Moreover, if the entire medium is of the viscoelastic type (i.e.  $\operatorname{supp} a = \bar{\Omega}$ ), the damping for the wave equation not only induces exponential energy decay, but also restricts the spectrum of the associated semigroup generator to a sector in the left half plane, and the associated semigroup is analytic ([13]). When the Kelvin-Voigt damping is localized on a subdomain of  $\Omega$ , the properties of system is quite complicated. First, it has been proved that properties of regularity and stability of 1-d system (1.1) depend on the continuousness of coefficient function  $a(\cdot)$ . More precisely, assume that  $\Omega = (-1, 1)$  and  $a(x)$  behaviours like  $x^\alpha$  with  $\alpha > 0$  in  $\operatorname{supp} a = [0, 1]$ . Then the solution of system (1.1) is eventually differentiable for  $\alpha > 1$ , exponentially stable for  $\alpha \geq 1$ , polynomially stable of order  $\frac{1}{1-\alpha}$  for  $0 < \alpha < 1$ , and polynomially stable of optimal decay rate 2 for  $\alpha = 0$  (see [19, 22, 25, 27]). For the higher dimensional system, the corresponding semigroup is exponentially stable when  $a(\cdot) \in \mathcal{C}^2(\Omega)$  and  $\operatorname{supp} a \supset \Gamma$  ([21]). However, when the Kelvin-Voigt damping is local and the material coefficient  $a(\cdot)$  is a positive constant on  $\operatorname{supp} a$ , the energy of system (1.1) does not decay exponentially for any geometry of  $\Omega$  and  $\operatorname{supp} a$  ([9, 28]). The reason is that the strong damping and non-continuousness of the coefficient function lead to reflection of waves at the interface  $\gamma \doteq \partial(\operatorname{supp} a) \setminus \Gamma$ , which then fails to be effectively damped because they do not enter the region of damping. It turns out that the Kelvin-Voigt damping does not follow the principle that "geometric optics" condition implies exponential stability, which is true for the wave equation with local viscous damping ([2]).

Recently, [29] proves the polynomial stability of system (1.1) when  $a(\cdot) \equiv a_0 > 0$  on  $\operatorname{supp} a$  and  $\operatorname{supp} a$  satisfies certain geometry conditions. Then, a natural problem is: how about the decay rate if  $\operatorname{supp} a \neq \emptyset$  is arbitrary? In [1],  $a$  is assumed discontinuous along a  $(d-1)$ -manifold,  $\operatorname{supp} a$  is arbitrary and the rate of the decay of semi-group is estimated by  $(\log t)^{-k}$  for a data in  $D(\mathcal{A}^k)$ . In this paper, we analyze the logarithmic decay properties of the solution to (1.1) when  $a$  is smooth and  $\operatorname{supp} a$  is arbitrary. The main result reads as follows.

**Theorem 1.1.** *Suppose that the coefficient function  $a(\cdot) \in \mathcal{C}_0^\infty(\Omega)$  is nonnegative and  $\operatorname{supp} a \subset \Omega$  is non-empty. Then the energy of the solution of (1.1) decays at logarithmic speed. More precisely,*

one has that there exists a positive constant  $C$  such that

$$\|e^{t\mathcal{A}}Y_0\|_{\mathcal{H}} \leq \frac{C}{[\log(t+2)]^{\frac{4k}{5}}} \|Y_0\|_{D(\mathcal{A}^k)}, \quad \forall t > 0, \quad Y_0 = (y^0, y^1) \in D(\mathcal{A}^k). \quad (1.7)$$

Our approach is based on the results duo to [7], which reduced the problem of determining the rate of energy decay to estimating the norm of the resolvent operator along the imaginary axis, see also [10, 18], etc. Our argument divides naturally three steps. First, in Section 2, we show some preliminaries including definitions and classical results about symbol, pseudo-differential calculus, and commutator estimate, etc. Then in Section 3, we prove corresponding Carleman estimates. Finally, in Section 4, we present a resolvent estimate and obtain Theorem 1.1. This theorem is a consequence of a resolvent estimate. The proof is given in Section 4. The method was developed in ([4, 5, 6, 16, 18, 26] and the references cited therein).

Throughout this paper, we use  $\|\cdot\|_V$  and  $(\cdot|\cdot)_V$  to denote the norm and inner product on  $L^2(V)$ , where  $V \subset \mathbb{R}^d$  if there is no further comments. When writing  $f \lesssim g$  (or  $f \gtrsim g$ ), we mean that there exists a positive constant  $C$  such that  $f \leq Cg$  (or  $f \geq Cg$ ). For  $j = 1, 2, \dots$ , define operators  $D_j = -i\partial_{x_j}$ ,  $D = (D_1, \dots, D_d)$ ,  $D^2 = \sum_{j=1}^d D_j^2$  and  $Da(x)D = \sum_{j=1}^d D_j a(x) D_j$ .

## 2 Preliminaries

We shall prove Theorem 1.1 by Weyl-Hörmander calculus, which was introduced Hörmander ([12, 15]). In this section, some definitions and results on the class of symbol and pseudo-differential calculus are given.

### 2.1 Symbol and Symbolic calculus

For any  $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$ ,  $\lambda \in \mathbb{R}$  and  $\tau > 0$ , we introduce the metric

$$g = g_{x,\xi} = \lambda dx^2 + \mu^{-2} d\xi^2, \quad \text{where } \mu^2 = \mu(\tau, \xi)^2 = \tau^2 + |\xi|^2, \quad (2.1)$$

and the weight

$$\nu = \nu(x, \lambda) = \sqrt{1 + \lambda^2 a(x)^2}. \quad (2.2)$$

Note that and  $g_{x,\xi}(X, \Xi) = \lambda |X|^2 + \mu^{-2}(\tau, \xi) |\Xi|^2$  for all  $X, \Xi \in \mathbb{R}^d$ . Then we have the following results.

**Lemma 2.1.** *Assume that there exist positive constants  $C$  and  $\lambda_0$  such that  $\lambda \geq \lambda_0$  and  $\tau \geq \max\{C\lambda, 1\}$ . It holds*

- (i) *The metric  $g = g_{x,\xi}$  defined by (2.1) is admissible, i.e., it is slowly varying and temperate.*
- (ii) *The weight  $\nu = \nu(x, \lambda)$  defined by (2.2) is admissible, i.e., it is  $g$ -continuous and  $g$ -temperate.*

*Proof.* (i) From Definition 18.4.1 in [12], the metric  $g_{x,\xi}$  defined by (2.1) is **slowly varying** if there exist  $\delta > 0$  and  $C > 0$  such that

$$g_{x,\xi}(y - x, \eta - \xi) \leq \delta \text{ implies } g_{y,\eta}(X, \Xi) \leq C g_{x,\xi}(X, \Xi), \quad \forall x, y, \xi, \eta, X, \Xi \in \mathbb{R}^d,$$

where the constants  $\delta$  and  $C$  are independent on the parameters  $\lambda$  and  $\tau$ .

Suppose  $0 < \delta \leq 1/4$  and

$$g_{x,\xi}(y - x, \eta - \xi) = \lambda |y - x|^2 + (\tau^2 + |\xi|^2)^{-1} |\eta - \xi|^2 \leq \delta.$$

Then, we have

$$\begin{aligned}\tau^2 + |\xi|^2 &\leq \tau^2 + 2|\xi - \eta|^2 + 2|\eta|^2 \\ &\leq \tau^2 + 2\delta(\tau^2 + |\xi|^2) + 2|\eta|^2.\end{aligned}$$

This implies that  $\tau^2 + |\xi|^2 \leq 4(\tau^2 + |\eta|^2)$ . Consequently,

$$g_{y,\eta}(X, \Xi) = \lambda|X|^2 + (\tau^2 + |\eta|^2)^{-1}|\Xi|^2 \leq \lambda|X|^2 + \frac{1}{4}(\tau^2 + |\xi|^2)^{-1}|\Xi|^2 \leq g_{x,\xi}(X, \Xi).$$

Therefore,  $g$  is slowly varying.

For a given metric  $g_{x,\xi}$ , the associated metric  $g_{x,\xi}^\sigma$  is defined by  $g_{x,\xi}^\sigma = (\tau^2 + |\xi|^2)dx^2 + \lambda^{-1}d\xi^2$ . The metric  $g_{x,\xi}$  is **temperate** if there exist  $C > 0$  and  $N > 0$ , such that

$$g_{x,\xi}(X, \Xi) \leq Cg_{y,\eta}(X, \Xi)(1 + g_{x,\xi}^\sigma(x - y, \xi - \eta))^N, \quad \forall x, y, \xi, \eta, X, \Xi \in \mathbb{R}^d, \quad (2.3)$$

where the constants  $C$  and  $N$  are independent on the parameters  $\lambda$  and  $\tau$  (Definition 18.5.1 in [12]).

For the metric  $g = g_{x,\xi}$  defined by (2.1), (2.3) is equivalent to

$$\begin{aligned}&\lambda|X|^2 + (\tau^2 + |\xi|^2)^{-1}|\Xi|^2 \\ &\leq C(\lambda|X|^2 + (\tau^2 + |\eta|^2)^{-1}|\Xi|^2)(1 + (\tau^2 + |\xi|^2)|x - y|^2 + \lambda^{-1}|\xi - \eta|^2)^N.\end{aligned} \quad (2.4)$$

First, assume that  $\tau^2 + |\eta|^2 \leq 4(\tau^2 + |\xi|^2)$ . It follows that

$$(\tau^2 + |\eta|^2) \leq C(\tau^2 + |\xi|^2)(1 + \lambda^{-1}|\xi - \eta|^2)^N, \quad C > 0, N > 0. \quad (2.5)$$

Then it is easy to obtain (2.4) from (2.5).

Secondly, consider the case  $\tau^2 + |\eta|^2 > 4(\tau^2 + |\xi|^2)$ . Then

$$|\eta| > 2|\xi|, \quad |\eta| > \sqrt{3}\tau, \quad (2.6)$$

and

$$|\xi - \eta| > \frac{1}{2}|\eta| > \frac{\sqrt{3}}{2}\tau > \frac{\sqrt{3}}{2}C\lambda. \quad (2.7)$$

It follows from (2.6) and (2.7) that

$$\lambda^{-1}|\xi - \eta|^2 > \frac{\sqrt{3}}{2}C|\xi - \eta| > \frac{\sqrt{3}}{4}C|\eta|.$$

Consequently,

$$(1 + \lambda^{-1}|\xi - \eta|^2)^2 > \frac{3}{16}C^2|\eta|^2 > \frac{3}{32}C^2(|\eta|^2 + 3\tau^2).$$

This together with  $\tau^2 + |\xi|^2 \geq 1$  yields that there exists a positive constant  $C$  such that (2.5) holds with  $N = 2$ .

(ii) It is known from Definition 18.4.2 in [12] that a weight  $\nu(x)$  is  **$g$ -continuous** if there exist  $\delta > 0$  and  $C > 0$  such that

$$g_{x,\xi}(y - x, \eta - \xi) \leq \delta \text{ implies } C^{-1}\nu(x) \leq \nu(y) \leq C\nu(x), \quad \forall x, y, \xi, \eta \in \mathbb{R}^d.$$

where the constants  $\delta$  and  $C$  are independent on the parameters  $\lambda$  and  $\tau$ . Since the weight  $\nu(x)$  defined by (2.2) does not depend on  $\xi$ , the above condition is reduced to

$$\lambda|x - y|^2 \leq \delta \text{ implies } C^{-1}\nu(x) \leq \nu(y) \leq C\nu(x), \quad \forall x, y \in \mathbb{R}^d.$$

The weight  $\nu(x)$  is  **$g$ -temperate** if there exist  $C > 0$  and  $N > 0$  such that

$$\nu(y) \leq C\nu(x)(1 + g_{y,\eta}^\sigma(x - y, \xi - \eta))^N, \quad \forall x, y, \xi, \eta \in \mathbb{R}^d, \quad (2.8)$$

where the constants  $C$  and  $N$  do not depend on the parameters  $\lambda$  and  $\tau$  (Definition 18.5.1 in [12]). The weight  $\nu(x)$  is admissible if it is  $g$ -continuous and  $g$ -temperate. When a weight is admissible, all the powers of this weight are  $g$ -continuous and  $g$ -temperate. Therefore, it suffice to prove that  $1 + \lambda a(x)$  is admissible.

Let  $s \in [0, t]$  and  $t \in [0, 1]$ . Define  $f(s) = \lambda a(x + s(y - x))$  and  $F(t) = \sup_{s \in [0, t]} f(s)$  where  $x, y \in \Omega$  satisfying  $\lambda |x - y|^2 \leq \delta$ . It is clear that  $f'(s) = \lambda a'(x + s(y - x))(y - x)$ . Combining this with the following inequality

$$|a'(x)|^2 \leq 2a(x)\|a''\|_{L^\infty(\Omega)}, \quad \forall x \in \Omega, \quad (2.9)$$

we obtain

$$|f'(s)| \leq \lambda |a'(x + s(y - x))| |y - x| \leq 2\lambda \|a''\|_{L^\infty(\Omega)}^{1/2} [a(x + s(y - x))]^{1/2} |y - x|.$$

The proof of (2.9) will be given later. Consequently,

$$\sup_{s \in [0, t]} |f'(s)| \leq 2\lambda^{1/2} \|a''\|_{L^\infty(\Omega)}^{1/2} F(t)^{1/2} |y - x|.$$

Since  $f(t) \leq f(0) + t \sup_{s \in [0, t]} |f'(s)|$ ,  $F$  is non-decreasing and  $\lambda |x - y|^2 \leq \delta$ , we obtain that for all  $t \in [0, 1]$ ,

$$f(t) \leq f(0) + C\lambda^{1/2} F(t)^{1/2} |y - x| \leq f(0) + C\sqrt{\delta} F(t)^{1/2} \leq f(0) + C\sqrt{\delta} F(\alpha)^{1/2},$$

where  $C = 2\|a''\|_{L^\infty(\Omega)}^{1/2}$  and  $\alpha \in [t, 1]$ . Note that  $f(0) = F(0)$ . It follows that

$$F(\alpha) = \sup_{t \in [0, \alpha]} f(t) \leq F(0) + C\sqrt{\delta} F(\alpha)^{1/2}.$$

This yields

$$1 + F(\alpha) \leq 1 + F(0) + C\sqrt{\delta}(1 + F(\alpha))^{1/2} \leq 1 + F(0) + C\sqrt{\delta}(1 + F(\alpha)). \quad (2.10)$$

By choosing  $\delta$  sufficiently small such that  $C\sqrt{\delta} \leq 1/2$ , one can deduce from (2.10) that

$$1 + F(\alpha) \leq 2(1 + F(0)), \quad \forall \alpha \in [t, 1].$$

In particular, we have

$$1 + \lambda a(y) \leq 2(1 + \lambda a(x)).$$

The above inequality remains true if we exchange  $x$  and  $y$ . Therefore, the weight  $1 + \lambda a(x)$  is  $g$ -continuous.

On the other hand, note that  $1 + \lambda a(x)$  is independent to  $\xi$ . Then, to obtain the weight  $1 + \lambda a(x)$  is  $\sigma$ -temperate, it is sufficient to prove that

$$1 + \lambda a(y) \leq C(1 + \lambda a(x))(1 + \tau^2 |x - y|^2)^N. \quad (2.11)$$

In fact, it is clear that  $1 + \lambda a(y) \leq 1 + \lambda(a(x) + C|x - y|)$  where  $C = \|a'\|_{L^\infty(\Omega)}$ . Therefore, there exists positive constant  $C' = CC^{-1}$  such that

$$1 + \lambda a(y) \leq (1 + \lambda a(x))(1 + C'\tau |x - y|) \leq (1 + \lambda a(x))(2 + 2(C'\tau |x - y|)^2)^{1/2}.$$

Thus, we obtain (2.11) with  $N = \frac{1}{2}$ ,  $C = 2 \max\{1, C'\}$ .  $\square$

**Remark 2.1.** We claim that (2.9) holds for any compactly supported and nonnegative function  $a \in \mathcal{C}^2(\Omega)$ . In fact, from the following identity

$$a(x+h) = a(x) + a'(x)h + \int_0^1 (1-t)a''(x+th)h^2 dt, \quad \forall h \in \mathbb{R},$$

one can get

$$a(x) + a'(x)h + \frac{1}{2}\|a''\|_{L^\infty(\Omega)}|h|^2 \geq 0.$$

Let  $h = y a'(x)$ , where  $y \in \mathbb{R}$  and  $x \in \Omega$  are arbitrary. It follows from the above inequality that

$$a(x) + |a'(x)|^2 y + \frac{1}{2}\|a''\|_{L^\infty(\Omega)}|a'(x)|^2 y^2 \geq 0, \quad \forall x \in \Omega, y \in \mathbb{R}.$$

Then,

$$|a'(x)|^4 - 2a(x)\|a''\|_{L^\infty(\Omega)}|a'(x)|^2 \leq 0,$$

and (2.9) is proved.

**Definition 2.1.** (Section 18.4.2 in [12]) Assume the weight  $m(x, \xi)$  is admissible and the metric  $g$  is defined by (2.1). Let  $q(x, \xi, \lambda, \tau)$  be a  $\mathcal{C}^\infty$  function with respect to  $(x, \xi)$  and  $\lambda, \tau$  be parameters satisfying conditions in Lemma 2.1. The symbol  $q(x, \xi, \lambda, \tau)$  is in **class**  $S(m, g)$  if for all  $\alpha, \beta \in \mathbb{N}^d$  there exists  $C_{\alpha, \beta}$  independent of  $\tau$  and  $\lambda$  such that

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi, \lambda, \tau)| \leq C_{\alpha, \beta} m(x, \xi) \lambda^{|\alpha|/2} \mu(\tau, \xi)^{-|\beta|}.$$

**Remark 2.2.** (i) It is clear that  $\mu = \sqrt{\tau^2 + |\xi|^2} \in S(\mu, g)$  since  $|\partial_\xi^\beta \mu(\tau, \xi)| \lesssim \mu^{1-|\beta|}$  for all  $\beta \in \mathbb{R}^d$ .

(ii) Let  $\nu$  be the weight defined by (2.2). It is easy to get that  $\lambda a \in S(\nu, g)$ . In fact, if  $|\alpha| \geq 2$ , it holds that  $|\partial_x^\alpha(\lambda a(x))| \leq C_\alpha \lambda \leq C_\alpha \lambda^{|\alpha|/2} \nu(x)$ , where  $C_\alpha > 0$ . For the case  $|\alpha| = 1$ , it follows from (2.9) that

$$|\partial_x^\alpha(\lambda a(x))| \leq \sqrt{2} \|a''\|_{L^\infty(\Omega)}^{\frac{1}{2}} \lambda^{\frac{1}{2}} (\lambda a(x))^{\frac{1}{2}}$$

Note that  $|\lambda a(x)| < C\nu^2(x)$  for some  $C > 0$ . This together with the above inequality, we have that

$$|\partial_x^\alpha(\lambda a(x))| < \sqrt{2C} \|a''\|_{L^\infty(\Omega)}^{\frac{1}{2}} \lambda^{\frac{1}{2}} \nu(x).$$

(iii) It is known from Lemma 18.4.3 of [12] that if the metric  $g$  and weights  $m_1, m_2$  are admissible, symbols  $a \in S(m_1, g)$  and  $b \in S(m_2, g)$ , then  $ab \in S(m_1 m_2, g)$ . In particular,  $(\lambda a)^j \mu^k \in S(\nu^j \mu^k, g)$  for all  $j, k \in \mathbb{N} \cup \{0\}$ .

**Definition 2.2.** Let  $b \in S(m, g)$  be a symbol and  $u \in \mathcal{S}(\mathbb{R}^d)$ , we set

$$b(x, D, \tau)u(x) = \text{Op}(b)u(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} b(x, \xi, \tau) \widehat{u}(\xi) d\xi.$$

It is known that  $\text{Op}(b) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$  is continuous and  $\text{Op}(b)$  can be uniquely extended to  $\mathcal{S}'(\mathbb{R}^d)$  continuously. The following two lemmas are consequences of Theorem 18.5.4 and 18.5.10 in [12].

**Lemma 2.2.** *Let  $b \in S(m, g)$  where  $m$  is an admissible weight and  $g$  is defined by (2.1). Then there exists  $c \in S(m, g)$  such that  $\text{Op}(b)^* = \text{Op}(c)$  and  $c(x, \xi) = \overline{b(x, \xi)} + r(x, \xi)$  where the remainder  $r \in S(\lambda^{\frac{1}{2}}\mu^{-1}m, g)$ .*

**Lemma 2.3.** *Let  $b \in S(m_1, g)$  and  $c \in S(m_2, g)$  where  $m_j$  are admissible weights for  $j = 1, 2$  and  $g$  is defined by (2.1). Denote by  $[\text{Op}(b), \text{Op}(c)] = \text{Op}(b) \circ \text{Op}(c) - \text{Op}(c) \circ \text{Op}(b)$  and Poisson bracket  $\{b, c\}(x, \xi, \tau) = \sum_{1 \leq j \leq d} (\partial_{\xi_j} b \partial_{x_j} c - \partial_{x_j} b \partial_{\xi_j} c)(x, \xi, \tau)$ . Then,*

(i) *there exists  $d \in S(m_1 m_2, g)$  such that  $\text{Op}(b) \text{Op}(c) = \text{Op}(d)$  and  $d(x, \xi) = b(x, \xi)c(x, \xi) + r(x, \xi)$  where  $r \in S(\lambda^{\frac{1}{2}}\mu^{-1}m_1 m_2, g)$ .*

(ii) *for commutator  $i[\text{Op}(b), \text{Op}(c)] = \text{Op}(f)$ , it holds that  $f \in S(\lambda^{\frac{1}{2}}\mu^{-1}m_1 m_2, g)$  and  $f(x, \xi) = \{b, c\}(x, \xi) + r(x, \xi)$  where  $r \in S(\lambda\mu^{-2}m_1 m_2, g)$ .*

The operators in  $S(\nu^j \mu^k, g)$  act on Sobolev spaces adapted to the class of symbol. Let  $b \in S(\nu^j \mu^k, g)$ , where  $\mu$  and  $g$  are defined by (2.1). Then there exists  $C > 0$  such that

$$\|\text{Op}(b)u\|_{\mathbb{R}^d} \leq C \|\nu^j \text{Op}(\mu^k)u\|_{\mathbb{R}^d}, \quad \forall u \in \mathcal{S}(\mathbb{R}^d).$$

By symbolic calculus, the above estimate is equivalent to  $\text{Op}(\mu^{-k}\nu^{-j}) \text{Op}(b)$  acts on  $L^2(\mathbb{R}^d)$  since the operators associated with symbol in  $S(1, g)$  act on  $L^2(\mathbb{R}^d)$ . In particular, if  $b \in S(\nu^j \mu, g)$ , then for any  $\lambda \geq \lambda_0$ ,  $\tau \geq \max\{\mathcal{C}\lambda, 1\}$  and  $u \in \mathcal{S}(\mathbb{R}^d)$ , it holds

$$\|\text{Op}(b)u\|_{\mathbb{R}^d} \leq C\tau \|\nu^j u\|_{\mathbb{R}^d} + C \|\nu^j Du\|_{\mathbb{R}^d},$$

where  $C > 0$  depends on positive constants  $\lambda_0$  and  $\mathcal{C}$ .

## 2.2 Commutator estimate

In this subsection, we suppose that  $\lambda = 1$  since the symbol does not depend on  $\lambda$ . The metric in (2.1) becomes

$$\tilde{g} = dx^2 + \mu^{-2} d\xi^2, \quad \text{where } \mu \text{ is defined by (2.1).} \quad (2.12)$$

To get the commutator estimate, we shall use the following Gårding inequality ([12, Theorem 18.6.7]).

**Lemma 2.4.** *Let  $b \in S(\mu^{2k}, \tilde{g})$  be real valued.  $\mu$  and  $\tilde{g}$  are defined by (2.12). We assume there exists  $C > 0$  such that  $b(x, \xi, \tau) \geq C\mu^{2k}$ . Then there exist  $\tilde{C} > 0$  and  $\tau_0 > 0$  such that*

$$\text{Re}(\text{Op}(b)w | w)_{\mathbb{R}^d} \geq \tilde{C} \|\text{Op}(\mu^k)w\|_{\mathbb{R}^d}^2, \quad \forall w \in \mathcal{S}(\mathbb{R}^d) \text{ and } \tau \geq \tau_0. \quad (2.13)$$

**Definition 2.3.** *Let  $V$  be a bounded open set in  $\mathbb{R}^d$ . We say that the weight function  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R})$  satisfies the **sub-ellipticity** condition in  $\overline{V}$  if  $|\nabla\varphi| > 0$  in  $\overline{V}$  and there exists constant  $C > 0$ ,*

$$\mathbb{P}_\varphi(x, \xi, \tau) = 0, \quad \forall (x, \xi) \in \overline{V} \times \mathbb{R}^d, \quad \tau > 0 \quad \Rightarrow \quad \{\mathfrak{q}_2, \mathfrak{q}_1\}(x, \xi, \tau) \geq C(|\xi|^2 + \tau^2)^{3/2}, \quad (2.14)$$

where  $\mathbb{P}_\varphi(x, \xi, \tau) = |\xi + i\tau\nabla\varphi(x)|^2 = \mathfrak{q}_2(x, \xi, \tau) + i\mathfrak{q}_1(x, \xi, \tau)$  and  $\mathfrak{q}_1, \mathfrak{q}_2$  are real valued.

**Lemma 2.5.** ([12]) *Let  $V$  be a bounded open set in  $\mathbb{R}^d$  and  $\psi \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R})$  be such that  $|\nabla\psi| > 0$  in  $\overline{V}$ . Then, for  $\gamma > 0$  sufficiently large,  $\varphi = e^{\gamma\psi}$  fulfills the sub-ellipticity property in  $V$ .*



**Lemma 2.6.** *Assume that  $\varphi$  satisfies the sub-ellipticity in Definition 2.3. For all  $w \in \mathcal{C}_0^\infty(V)$ , there exist  $C_1, C_2 > 0$  and  $\tau_0 > 0$  such that the following inequality holds for all  $\tau \geq \tau_0$ ,*

$$\begin{aligned} C_1 \tau^3 \|w\|_V^2 + C_1 \tau \|Dw\|_V^2 &\leq \operatorname{Re} \left( \operatorname{Op}(\{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\tau\xi \cdot \nabla\varphi(x)\})w \mid w \right)_V \\ &\quad + C_2 \tau^{-1} \| \operatorname{Op}(|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)w \|_V^2 \\ &\quad + C_2 \tau^{-1} \| \operatorname{Op}(2\tau\xi \cdot \nabla\varphi(x))w \|_V^2. \end{aligned} \quad (2.15)$$

*Proof.* First, by homogeneity in  $(\xi, \tau)$ , compactness arguments and sub-ellipticity condition, we claim that there exist constants  $C, \delta > 0$  such that

$$C \left[ |2\xi \cdot \nabla\varphi(x)|^2 + \mu^{-2} (|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)^2 \right] + \{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\xi \cdot \nabla\varphi(x)\} \geq \delta \mu^2. \quad (2.16)$$

The proof of (2.16) is classical. In fact, set

$$\mathcal{K} = \{(x, \xi, \tau) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} : x \in \overline{V}, |\xi|^2 + \tau^2 = 1, \tau \geq 0\},$$

and for  $(x, \xi, \tau) \in \mathcal{K}$ ,  $\kappa > 0$ ,

$$G(x, \xi, \tau, \kappa) = \kappa \left[ |2\xi \cdot \nabla\varphi(x)|^2 + \mu^{-2} (|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)^2 \right] + \{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\xi \cdot \nabla\varphi(x)\}.$$

If  $|2\xi \cdot \nabla\varphi(x)|^2 + \mu^{-2} (|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)^2 = 0$  for  $(x, \xi, \tau) \in \mathcal{K}$ , it is clear that there exists a positive constant  $\delta$  such that (2.16) holds due to the fact that  $\phi$  is sub-elliptic. When  $|2\xi \cdot \nabla\varphi(x)|^2 + \mu^{-2} (|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)^2 > 0$ , there exists a positive constant  $\kappa_{x, \xi, \tau}$  such that  $G(x, \xi, \tau, \kappa) > 0$  for every  $\kappa \geq \kappa_{x, \xi, \tau}$  since  $\{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\xi \cdot \nabla\varphi(x)\}$  is bounded on  $\mathcal{K}$ . By continuity of  $G(x, \xi, \tau, \kappa)$ , there exists a neighborhood of  $(x, \xi, \tau)$ , denoted by  $V_{x, \xi, \tau}$ , such that  $G(x, \xi, \tau, \kappa) > 0$  for all  $(x, \xi, \tau) \in V_{x, \xi, \tau}$  and  $\kappa \geq \kappa_{x, \xi, \tau}$ . Since  $\mathcal{K}$  is compact, there exist finite sets  $V_j = V_{x_j, \xi_j, \tau_j}$  and corresponding constants  $\kappa_j = \kappa_{x_j, \xi_j, \tau_j}$  ( $j = 1, 2, \dots, n$ ), such that  $\mathcal{K} \subset \cup_{j=1}^n V_j$  and  $G(x, \xi, \tau, \kappa) > 0$  for all  $(x, \xi, \tau) \in V_j$  and  $\kappa > \kappa_j$ . Let  $\tilde{\kappa} = \max\{\kappa_j : j = 1, 2, \dots, n\}$ . It follows that  $G(x, \xi, \tau, \tilde{\kappa}) > 0$  for all  $(x, \xi, \tau) \in \mathcal{K}$  and  $\kappa \geq \tilde{\kappa}$ . Finally, using the compactness of  $\mathcal{K}$  again, we conclude that there exists  $\delta > 0$  such that  $G(x, \xi, \tau, \tilde{\kappa}) \geq \delta$ . Thus, (2.16) is reached since  $g$  is a homogeneous function of degree 2 with respect to variables  $(\xi, \tau)$ .

By Gårding inequality (2.13), there exists a constant  $C > 0$  such that, for  $\tau \geq \tau_0$  with  $\tau_0$  sufficiently large,

$$\begin{aligned} C \| \operatorname{Op}(\mu)w \|_V^2 &\leq \operatorname{Re} \left( \operatorname{Op} \left( |2\xi \cdot \nabla\varphi(x)|^2 + \mu^{-2} (|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)^2 \right. \right. \\ &\quad \left. \left. + \{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\xi \cdot \nabla\varphi(x)\} \right)w \mid w \right)_V. \end{aligned} \quad (2.17)$$

Now we are going to estimate the terms  $\tau \operatorname{Op}(|2\xi \cdot \nabla\varphi(x)|^2)$  and  $\tau \operatorname{Op}(\mu^{-2} (|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)^2)$ . Firstly, it follows from Lemma 2.3 that

$$\tau^{-1} \operatorname{Op}(|2\tau\xi \cdot \nabla\varphi(x)|^2) = \tau^{-1} \operatorname{Op}(2\tau\xi \cdot \nabla\varphi(x))^* \operatorname{Op}(2\tau\xi \cdot \nabla\varphi(x)) + \tau \operatorname{Op}(r_1), \quad (2.18)$$

where  $r_1 \in S(\mu, \tilde{g})$  and  $\tilde{g}$  is defined by (2.12). Therefore, for any  $\varepsilon > 0$ , there exists a positive constant  $C_\varepsilon$  such that

$$\begin{aligned} &|(\tau^{-1} \operatorname{Op}(|2\tau\xi \cdot \nabla\varphi(x)|^2)w \mid w)_V| \\ &\leq \tau^{-1} \| \operatorname{Op}(2\tau\xi \cdot \nabla\varphi(x))w \|_V^2 + \tau |(\operatorname{Op}(r_1)w \mid w)_V| \\ &\leq \tau^{-1} \| \operatorname{Op}(2\tau\xi \cdot \nabla\varphi(x))w \|_V^2 + \varepsilon \tau \| \operatorname{Op}(\mu)w \|_V^2 + C_\varepsilon \tau \|w\|_V^2. \end{aligned} \quad (2.19)$$

Substituting (2.19) into (2.17) and choosing  $\varepsilon$  small enough, we have

$$\begin{aligned} & C\tau \|\text{Op}(\mu)w\|_V^2 \\ \leq & \text{Re} \left( \text{Op} \left( \tau\mu^{-2} (|\xi|^2 - \tau^2|\nabla\varphi(x)|^2)^2 + \tau\{|\xi|^2 - \tau^2|\nabla\varphi(x)|^2, 2\xi \cdot \nabla\varphi(x)\} \right) w \mid w \right)_V \\ & + \tau^{-1} \|\text{Op}(2\tau\xi \cdot \nabla\varphi(x))w\|_V^2 + C_\varepsilon\tau \|w\|_V^2. \end{aligned} \quad (2.20)$$

Secondly, by symbolic calculus, we have that

$$\tau \text{Op}(\mu^{-2} (|\xi|^2 - \tau^2|\nabla\varphi(x)|^2)^2) = \tau \text{Op}(r_0) \text{Op}(|\xi|^2 - \tau^2|\nabla\varphi(x)|^2) + \tau \text{Op}(r_2), \quad (2.21)$$

where  $r_0(x, \xi) = \mu^{-2}(|\xi|^2 - \tau^2|\nabla\varphi(x)|^2) \in S(1, \tilde{g})$  and  $r_2 \in S(\mu, \tilde{g})$ . Therefore, for all  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that

$$\begin{aligned} & |(\tau \text{Op}(r_0) \text{Op}(|\xi|^2 - \tau^2|\nabla\varphi(x)|^2)w \mid w)_V| \\ \leq & C_\varepsilon\tau^{-1} \|\text{Op}(|\xi|^2 - \tau^2|\nabla\varphi(x)|^2)w\|_V^2 + \varepsilon\tau^3 \|w\|_V^2. \end{aligned} \quad (2.22)$$

We choose  $\varepsilon$  small enough and combine (2.21)-(2.22) with (2.20) to get

$$\begin{aligned} C\tau \|\text{Op}(\mu)w\|_V^2 \leq & \text{Re} \left( \tau \text{Op} \{|\xi|^2 - \tau^2|\nabla\varphi(x)|^2, 2\xi \cdot \nabla\varphi(x)\} w \mid w \right)_V \\ & + C_\varepsilon\tau^{-1} \|\text{Op}(|\xi|^2 - \tau^2|\nabla\varphi(x)|^2)w\|_V^2 \\ & + \tau^{-1} \|\text{Op}(2\tau\xi \cdot \nabla\varphi(x))w\|_V^2 + C_\varepsilon(\tau + \varepsilon\tau^3) \|w\|_V^2. \end{aligned} \quad (2.23)$$

Finally, it is clear that there exist positive constant  $C$  such that

$$\tau^3 \|w\|_V^2 + \tau \|Dw\|_V^2 \leq C\tau \|\text{Op}(\mu)w\|_V^2. \quad (2.24)$$

Thus, we obtain (2.15) by using (2.23)-(2.24), choosing  $\varepsilon$  small enough and letting  $\tau > \tau_0$  large enough.  $\square$

### 3 Carleman Estimate

In this section, we shall prove several Carleman inequalities. Define the operator

$$P(x, D, \lambda) = D^2 + i\lambda Da(x)D - \lambda^2.$$

Let the weight function  $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d; \mathbb{R})$ . The associated conjugate operator of  $P(x, D, \lambda)$  is  $P_\varphi(x, D, \lambda) = e^{\tau\varphi} P(x, D, \lambda) e^{-\tau\varphi}$ . Then,

$$P_\varphi = (D + i\tau\nabla\varphi(x))^2 + i\lambda(D + i\tau\nabla\varphi(x))a(x)(D + i\tau\nabla\varphi(x)) - \lambda^2.$$

By setting  $Q_2 = \frac{1}{2}(P_\varphi + P_\varphi^*)$  and  $Q_1 = \frac{1}{2i}(P_\varphi - P_\varphi^*)$ , we have  $P_\varphi = Q_2 + iQ_1$ . We denote by  $p(x, \xi, \lambda)$ ,  $p_\varphi(x, \xi, \lambda)$  the associated symbol of  $P(x, D, \lambda)$ ,  $P_\varphi(x, D, \lambda)$ , respectively.

Let the metric  $g$  and weight  $\nu$  be defined by (2.1) and (2.2). Due to the results Remark 2.2, we know that  $D + i\tau\nabla\varphi(x)$  is an operator with symbol in  $S(\mu, g)$  class,  $\lambda a$  is in  $S(\nu, g)$ , and  $(1 + i\lambda a(x))|\xi + i\tau\nabla\varphi(x)|^2$ , the principal symbol of  $P_\varphi$  belongs to  $S(\nu\mu^2, g)$ . Furthermore, from Lemma 2.2 and 2.3, one has that

$$\begin{aligned} P_\varphi &= \text{Op} \left( (1 + i\lambda a(x))|\xi + i\tau\nabla\varphi(x)|^2 \right) - \lambda^2 + R_3, \\ Q_2 &= \text{Op}(q_2) - \lambda^2 + R_2, \\ Q_1 &= \text{Op}(q_1) + R_1, \end{aligned} \quad (3.1)$$

where  $q_2 = |\xi|^2 - \tau^2 |\nabla\varphi(x)|^2 - 2\lambda\tau a(x)\xi \cdot \nabla\varphi(x)$ ,  $q_1 = 2\tau\xi \cdot \nabla\varphi(x) + \lambda a(x)(|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2)$  belong to  $S(\nu\mu^2, g)$  and the symbols of  $R_j$  is in  $S(\lambda^{\frac{1}{2}}\nu\mu, g)$  for  $j = 1, 2, 3$ . It is clear that

$$\|P_\varphi v\|_V^2 = \|Q_2 v\|_V^2 + \|Q_1 v\|_V^2 + 2\operatorname{Re}(Q_2 v | iQ_1 v)_V, \quad \forall v \in \mathcal{C}^\infty(V). \quad (3.2)$$

In what follows, several Carleman estimates are introduced. First, we give an estimation on the subdomain which is far away from the boundary  $\Gamma$ .

**Theorem 3.1.** *Suppose  $\varphi$  satisfies sub-ellipticity condition in  $V \subset \Omega$ . Then, there exist positive constants  $C$ ,  $\tilde{K}$  and  $\lambda_0$ , such that for every  $v \in \mathcal{C}_0^\infty(V)$ , it holds*

$$\tau^3 \|\nu(x)v\|_V^2 + \tau \|\nu(x)Dv\|_V^2 + \|Q_1 v\|_V^2 + \|Q_2 v\|_V^2 \lesssim \|P_\varphi v\|_V^2, \quad (3.3)$$

where  $\lambda \geq \lambda_0$  and  $\tau \geq \max\{\tilde{K}|\lambda|^{\frac{5}{4}}, 1\}$ .

*Proof.* Let  $w = \nu(x)v$  in (2.15). We obtain

$$\begin{aligned} & C_1 \tau^3 \|\nu(x)v\|_V^2 + C_1 \tau \|D(\nu(x)v)\|_V^2 \\ & \leq \operatorname{Re} \left( \operatorname{Op}(\{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\tau\xi \cdot \nabla\varphi(x)\}) \nu(x)v | \nu(x)v \right)_V \\ & \quad + C_2 \tau^{-1} \|\operatorname{Op}(|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2) \nu(x)v\|_V^2 \\ & \quad + C_2 \tau^{-1} \|\operatorname{Op}(2\tau\xi \cdot \nabla\varphi(x)) \nu(x)v\|_V^2, \quad C_1, C_2 > 0. \end{aligned} \quad (3.4)$$

Since the symbol of  $[D, \nu(x)]$  is in  $S(\lambda^{\frac{1}{2}}\nu, g)$ , we have

$$\|\nu(x)Dv\|_V^2 \leq \|D(\nu(x)v)\|_V^2 + \|[D, \nu(x)]v\|_V^2 \leq \|D(\nu(x)v)\|_V^2 + C\lambda \|\nu(x)v\|_V^2.$$

Consequently, for  $\tau \geq C\lambda$  with  $C > 0$  sufficiently large, it holds

$$\tau^3 \|\nu(x)v\|_V^2 + \tau \|\nu(x)Dv\|_V^2 \lesssim \tau^3 \|\nu(x)v\|_V^2 + \tau \|D(\nu(x)v)\|_V^2. \quad (3.5)$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} & C'_1 (\tau^3 \|\nu(x)v\|_V^2 + \tau \|\nu(x)Dv\|_V^2) \\ & \leq \operatorname{Re} \left( \operatorname{Op}(\{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\tau\xi \cdot \nabla\varphi(x)\}) \nu(x)v | \nu(x)v \right)_V \\ & \quad + C_2 \tau^{-1} \|\operatorname{Op}(|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2) \nu(x)v\|_V^2 \\ & \quad + C_2 \tau^{-1} \|\operatorname{Op}(2\tau\xi \cdot \nabla\varphi(x)) \nu(x)v\|_V^2, \quad C'_1, C_2 > 0. \end{aligned} \quad (3.6)$$

Now, we estimate the first term on the right side hand of (3.6). Note that

$$2\operatorname{Re}(Q_2 v | iQ_1 v)_V = (Q_2 v | iQ_1 v)_V + (iQ_1 v | Q_2 v)_V = (i[Q_2, Q_1]v | v)_V, \quad (3.7)$$

where the principal symbol of  $i[Q_2, Q_1]$  is  $\{q_2, q_1\}$ . Due to Lemma 2.3, we obtain that

$$i[Q_2, Q_1] = \operatorname{Op}(\{q_2, q_1\}) + R_4, \quad (3.8)$$

where  $\{q_2, q_1\} \in S(\lambda^{\frac{1}{2}}\nu^2\mu^3, g)$  and the symbol of  $R_4$  is in  $S(\lambda\nu^2\mu^2, g)$ . A direct computation gives that

$$\begin{aligned} \{q_2, q_1\} &= (1 + a^2(x)\lambda^2) \{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, 2\tau\xi \cdot \nabla\varphi(x)\} \\ & \quad + (\{|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2, \lambda a(x)\} - \lambda a(x) \{2\tau\xi \cdot \nabla\varphi(x), \lambda a(x)\}) (|\xi|^2 - \tau^2 |\nabla\varphi(x)|^2) \\ & \quad - 2\tau\xi \cdot \nabla\varphi(x) (\lambda a(x) \{ \lambda a(x), |\xi|^2 - \tau^2 |\nabla\varphi(x)|^2 \} + \{ \lambda a(x), 2\tau\xi \cdot \nabla\varphi(x) \}). \end{aligned}$$

From the definition of  $q_1$  and  $q_2$ , we have

$$\begin{aligned} q_2(x, \xi) + \lambda a(x)q_1(x, \xi) &= (1 + \lambda^2 a(x)^2)(|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2), \\ q_1(x, \xi) - \lambda a(x)q_2(x, \xi) &= 2\tau \xi \cdot \nabla \varphi(x)(1 + \lambda^2 a^2(x)). \end{aligned} \quad (3.9)$$

Consequently,

$$\begin{aligned} &\{q_2, q_1\} \\ &= \nu^2(x) \{|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2, 2\tau \xi \cdot \nabla \varphi(x)\} \\ &\quad + \nu^{-2}(x)(q_2(x, \xi) + \lambda a(x)q_1(x, \xi))(\{|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2, \lambda a(x)\} - \lambda a(x)\{2\tau \xi \cdot \nabla \varphi(x), \lambda a(x)\}) \\ &\quad - \nu^{-2}(x)(q_1(x, \xi) - \lambda a(x)q_2(x, \xi))(\lambda a(x)\{\lambda a(x), |\xi|^2 - \tau^2 |\nabla \varphi(x)|^2\} + \{\lambda a(x), 2\tau \xi \cdot \nabla \varphi(x)\}). \end{aligned}$$

Then, it follows from (3.8) and the above equation that

$$\begin{aligned} i[Q_2, Q_1] &= \nu(x) \text{Op}(\{|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2, 2\tau \xi \cdot \nabla \varphi(x)\})\nu(x) \\ &\quad + B_1 \nu^{-1}(x) \text{Op}(q_2(x, \xi) + \lambda a(x)q_1(x, \xi)) \\ &\quad - B_2 \nu^{-1}(x) \text{Op}(q_1(x, \xi) - \lambda a(x)q_2(x, \xi)) + R_5, \end{aligned}$$

where

$$\begin{aligned} B_1 &= \text{Op}(\nu^{-1}(x)(\{|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2, \lambda a(x)\} - \lambda a(x)\{2\tau \xi \cdot \nabla \varphi(x), \lambda a(x)\})), \\ B_2 &= \text{Op}(\nu^{-1}(x)(\lambda a(x)\{\lambda a(x), |\xi|^2 - \tau^2 |\nabla \varphi(x)|^2\} + \{\lambda a(x), 2\tau \xi \cdot \nabla \varphi(x)\})), \end{aligned}$$

symbols of  $B_1$  and  $B_2$  belong to  $S(\lambda^{\frac{1}{2}}\nu\mu, g)$  and the symbol of  $R_5$  is in  $S(\lambda\nu^2\mu^2, g)$ . Combining this with (3.1) yields

$$\begin{aligned} i[Q_2, Q_1] &= \nu(x) \text{Op}(\{|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2, 2\tau \xi \cdot \nabla \varphi(x)\})\nu(x) \\ &\quad + B_1 \nu^{-1}(x)[Q_2(x, \xi) + \lambda a(x)Q_1(x, \xi) + \lambda^2] \\ &\quad - B_2 \nu^{-1}(x)[Q_1(x, \xi) - \lambda a(x)Q_2(x, \xi) - \lambda^3 a(x)] + R_5, \end{aligned} \quad (3.10)$$

We refer to Section 2 where the rules on symbolic calculus are given and precise. Therefore, by the continuity of pseudo-differential operator, we have that for  $j = 1, 2$ ,  $k = 0, 1$  and  $\ell = 1, 2$ ,

$$\begin{aligned} &|(B_j \nu(x)^{-1}(\lambda a(x))^k Q_\ell v | v)_V| \\ &= |(\nu(x)^{-1}(\lambda a(x))^k Q_\ell v | B_j^* v)_V| \lesssim \|Q_\ell v\|_V \|B_j^* v\|_V \\ &\leq \frac{1}{10} \|Q_\ell v\|_V^2 + C\lambda\tau^2 \|\nu(x)v\|_V^2 + C\lambda \|\nu(x)Dv\|_V^2, \quad C > 0, \end{aligned} \quad (3.11)$$

$$\begin{aligned} |(B_j \nu^{-1}(x)\lambda^2(\lambda a(x))^k v | v)_V| &\lesssim \lambda^{\frac{5}{2}}(\tau \|\nu(x)v\|_V + \|\nu(x)Dv\|_V) \|\nu(x)v\|_V \\ &\lesssim \lambda^{\frac{5}{2}}\tau \|\nu(x)v\|_V^2 + \lambda^{\frac{5}{2}}\tau^{-1} \|\nu(x)Dv\|_V^2, \end{aligned} \quad (3.12)$$

and

$$|(R_5 v | v)_V| \lesssim \lambda\tau^2 \|\nu(x)v\|_V^2 + \lambda \|\nu(x)Dv\|_V^2. \quad (3.13)$$

Due to (3.10)–(3.13), there exists a positive constant  $C$  such that

$$\begin{aligned} &(\text{Op}(\{|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2, 2\tau \xi \cdot \nabla \varphi(x)\})\nu(x)v | \nu(x)v)_V \\ &\leq (i[Q_2, Q_1]v | v)_V + \frac{1}{4} \sum_{\ell=1,2} \|Q_\ell v\|_V^2 + C(\lambda\tau^2 \|\nu(x)v\|_V^2 \\ &\quad + \lambda \|\nu(x)Dv\|_V^2 + \lambda^{\frac{5}{2}}\tau \|\nu(x)v\|_V^2 + \lambda^{\frac{5}{2}}\tau^{-1} \|\nu(x)Dv\|_V^2). \end{aligned} \quad (3.14)$$

Next, we are going to estimate the last two terms on the right side hand of (3.6). It follows from (3.1) and (3.9) that

$$\begin{aligned}
& \| \text{Op}(|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2) \nu(x) v \|_V^2 + \| \text{Op}(2\tau \xi \cdot \nabla \varphi(x)) \nu(x) v \|_V^2 \\
= & \| \text{Op}(\nu(x)^{-2} (q_2 + \lambda a(x) q_1)) \nu(x) v \|_V^2 + \| \text{Op}(\nu(x)^{-2} (q_1 - \lambda a(x) q_2)) \nu(x) v \|_V^2 \\
= & \| \nu(x)^{-1} [Q_2 + \lambda^2 - R_2 + \lambda a(x) (Q_1 - R_1)] v \|_V^2 \\
& + \| \nu(x)^{-1} [Q_1 - R_1 - \lambda a(x) (Q_2 + \lambda^2 - R_2)] v \|_V^2.
\end{aligned} \tag{3.15}$$

Combining this with the fact that the symbols of  $R_j$  are in  $S(\lambda^{\frac{1}{2}} \nu \mu, g)$  for  $j = 1, 2$ , we have

$$\begin{aligned}
& \| \text{Op}(|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2) \nu(x) v \|_V^2 + \| \text{Op}(2\tau \xi \cdot \nabla \varphi(x)) \nu(x) v \|_V^2 \\
\lesssim & \sum_{\ell=1,2} (\|Q_\ell v\|_V^2 + \|R_\ell v\|_V^2) + \lambda^4 \|v\|_V^2 \\
\lesssim & \sum_{\ell=1,2} \|Q_\ell v\|_V^2 + \lambda \tau^2 \|\nu(x) v\|_V^2 + \lambda \|\nu(x) Dv\|_V^2 + \lambda^4 \|v\|_V^2.
\end{aligned} \tag{3.16}$$

Consequently, for  $\tau \geq C\lambda$  with  $C$  large enough, it holds

$$\begin{aligned}
& \tau^{-1} \| \text{Op}(|\xi|^2 - \tau^2 |\nabla \varphi(x)|^2) \nu(x) v \|_V^2 + \tau^{-1} \| \text{Op}(2\tau \xi \cdot \nabla \varphi(x)) \nu(x) v \|_V^2 \\
\lesssim & \tau^{-1} \sum_{\ell=1,2} \|Q_\ell v\|_V^2 + \lambda \tau \|\nu(x) v\|_V^2 + \|\nu(x) Dv\|_V^2 + \lambda^3 \|v\|_V^2.
\end{aligned} \tag{3.17}$$

Finally, by (3.6), (3.14) and (3.17), one can choose  $\tau \geq \max\{\tilde{K}|\lambda|^{\frac{5}{4}}, 1\}$  with  $\tilde{K}$  sufficiently large such that for some  $C > 0$ ,

$$\begin{aligned}
C(\tau^3 \|\nu(x) v\|_V^2 + \tau \|\nu(x) Dv\|_V^2) \leq & (i[Q_2, Q_1]v | v)_V + \frac{1}{2} \sum_{\ell=1,2} \|Q_\ell v\|_V^2 \\
& + \varepsilon \tau^3 \|\nu(x) v\|_V^2 + \varepsilon \tau \|\nu(x) Dv\|_V^2,
\end{aligned} \tag{3.18}$$

where  $\varepsilon > 0$  is arbitrary. Choosing  $\varepsilon$  small with respect to  $C$ , using (3.2), (3.7) and (3.18), we obtain

$$\begin{aligned}
& C_1(\tau^3 \|\nu(x) v\|_V^2 + \tau \|\nu(x) Dv\|_V^2) + \frac{1}{2} (\|Q_1 v\|_V^2 + \|Q_2 v\|_V^2) \\
\leq & (i[Q_2, Q_1]v, v)_V + \|Q_1 v\|_V^2 + \|Q_2 v\|_V^2 = \|P_\varphi v\|_V^2,
\end{aligned}$$

which implies (3.3).  $\square$

**Remark 3.1.** *The estimates in (3.12) impose the assumption  $\tau \geq \tilde{K}|\lambda|^{\frac{5}{4}}$ . The other remainder terms only impose the condition  $\tau \geq C\lambda$ . This condition is related with the principal normal condition. Indeed for a complex operator, with symbol  $p_1 + ip_2$  where  $p_1, p_2$  are both real valued, the Carleman estimate is only true if  $\{p_1, p_2\} = 0$  on  $p_1 = p_2 = 0$ . Here the symbol of operator before conjugation by weight is  $|\xi|^2 - \lambda^2 + i\lambda a(x)|\xi|^2$ , and the Poisson bracket is  $\{|\xi|^2 - \lambda^2, \lambda a(x)|\xi|^2\} = 2\lambda(\xi \cdot \nabla a(x))|\xi|^2$ . We can estimate this term, uniformly in a neighborhood of  $a(x) = 0$ , by  $C\lambda a^{\frac{1}{2}}(x)|\xi|^3$ . This explanation does not justify the power  $|\lambda|^{\frac{5}{4}}$  found at the end of computations but shows the difficulties.*

**Theorem 3.2.** *Suppose  $\varphi$  satisfies sub-ellipticity condition in  $V \subset \Omega$ . Then, there exist positive constants  $\tilde{K}$  and  $\lambda_0$ , such that for every  $u \in \mathcal{C}_0^\infty(V)$ , it holds*

$$\tau^3 \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau \varphi} u\|_V^2 + \tau \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau \varphi} Du\|_V^2 \lesssim \|e^{\tau \varphi} Pu\|_V^2, \tag{3.19}$$

where  $\lambda \geq \lambda_0$  and  $\tau \geq \max\{\tilde{K}|\lambda|^{\frac{5}{4}}, 1\}$ .

*Proof.* Set  $v = e^{\tau\varphi}u$ . From Theorem 3.1, it suffices to prove that (3.19) is equivalent to

$$\tau^3\|\nu(x)v\|_V^2 + \tau\|\nu(x)Dv\|_V^2 \lesssim \|P_\varphi v\|_V^2. \quad (3.20)$$

First, assume (3.20) holds. Then,  $Dv = e^{\tau\varphi}(Du - i\tau\nabla\varphi u)$  and  $e^{\tau\varphi}Du = Dv + i\tau\nabla\varphi v$ . Then there exist positive constants  $c_1, c_2$  such that

$$\begin{aligned} c_1 (\|\nu(x)Dv\|_V + \tau\|\nu(x)v\|_V) &\leq \|\nu(x)e^{\tau\varphi}Du\|_V + \tau\|\nu(x)e^{\tau\varphi}u\|_V \\ &\leq c_2 (\|\nu(x)Dv\|_V + \tau\|\nu(x)v\|_V). \end{aligned} \quad (3.21)$$

Combining this with (3.20), we conclude that

$$\tau^3\|\nu(x)e^{\tau\varphi}u\|_V^2 + \tau\|\nu(x)e^{\tau\varphi}Du\|_V^2 \lesssim (\tau\|\nu(x)Dv\|_V^2 + \tau^3\|\nu(x)v\|_V^2) \lesssim \|e^{\tau\varphi}Pu\|_V^2.$$

On the other hand, (3.19) implies that

$$\tau^3\|\nu(x)e^{\tau\varphi}u\|_V^2 + \tau\|\nu(x)e^{\tau\varphi}Du\|_V^2 \lesssim \|P_\varphi v\|_V^2.$$

Then, we proved (3.20) from the above estimate and (3.21).  $\square$

Since there is higher order term  $\operatorname{div}(a(x)\nabla y_t)$  in system (1.1), it is necessary to deal with the term  $\operatorname{div}(a(x)\nabla f)$  for  $f \in H^1(\Omega)$  when proving the resolvent estimate. The following result is analogue to the work by Imanuvilov and Puel ([14]).

**Theorem 3.3.** *Suppose  $\varphi$  satisfies sub-ellipticity condition on  $V \subset \Omega$ . Then, there exist  $C, \tilde{K}, \lambda_0 > 0$ , such that for all  $u \in \mathcal{C}_0^\infty(V)$  satisfying*

$$P(x, D, \lambda)u = g_0 + \sum_{j=1}^d \partial_{x_j} g_j, \quad \text{where } g_j \in L^2(V), \quad j = 0, 1, \dots, d, \quad (3.22)$$

it holds

$$\tau\|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi}u\|_V^2 + \tau^{-1}\|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi}Du\|_V^2 \leq C \sum_{j=0}^d \|e^{\tau\varphi}g_j\|_V^2, \quad (3.23)$$

where  $\lambda \geq \lambda_0$  and  $\tau \geq \max\{\tilde{K}|\lambda|^{\frac{5}{4}}, 1\}$ .

*Proof.* First, from (3.1), we have  $D^2 = Q_2 + S_2$  and  $\lambda a(x)D^2 = Q_1 + S_1$  where  $S_1$  and  $S_2$  have symbols in  $S(\tau\nu\mu, g)$  if  $\tau \gtrsim \lambda$ . It follows that for any  $v \in \mathcal{C}_0^\infty(V)$ ,

$$\begin{aligned} \|D^2v\|_V^2 &\lesssim \|Q_2v\|_V^2 + \tau^2(\tau^2\|\nu(x)v\|_V^2 + \|\nu(x)Dv\|_V^2), \\ \|\lambda aD^2v\|_V^2 &\lesssim \|Q_1v\|_V^2 + \tau^2(\tau^2\|\nu(x)v\|_V^2 + \|\nu(x)Dv\|_V^2). \end{aligned} \quad (3.24)$$

Using (3.3), (3.24) and the fact that  $\|\nu(x)D^2v\|_V^2 \leq 2(\|D^2v\|_V^2 + \|\lambda a(x)D^2v\|_V^2)$ , we obtain

$$\tau^3\|\nu(x)v\|_V^2 + \tau\|\nu(x)Dv\|_V^2 + \tau^{-1}\|\nu(x)D^2v\|_V^2 \lesssim \|P_\varphi v\|_V^2. \quad (3.25)$$

Let  $\tilde{u}$  and  $\chi$  be in  $\mathcal{C}_0^\infty(\Omega)$  such that  $\chi = 1$  on a neighborhood of  $\operatorname{supp} \tilde{u}$ . Similarly to (3.21), we obtain

$$\tau\|\nu(x)\tilde{u}\|_V^2 + \tau^{-1}\|\nu(x)D\tilde{u}\|_V^2 \lesssim \tau\|\nu(x)\tilde{u}\|_V^2 + \tau^{-1}\|D(\nu(x)\tilde{u})\|_V^2.$$

Then, combining this with Fourier transform and the following inequality

$$\tau + \frac{|\xi|^2}{\tau} \lesssim \frac{\tau^3}{\tau^2 + |\xi|^2} + \frac{|\xi|^4}{\tau(\tau^2 + |\xi|^2)},$$

we conclude that

$$\tau \|\nu(x)\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D\tilde{u}\|_V^2 \lesssim \tau^3 \|\text{Op}(\mu^{-1})\nu(x)\chi\tilde{u}\|_V^2 + \tau^{-1} \|D^2 \text{Op}(\mu^{-1})\nu(x)\chi w\|_V^2. \quad (3.26)$$

From Lemma 2.3, we have  $\text{Op}(\mu^{-1})\nu\chi = \nu\chi \text{Op}(\mu^{-1}) + R_1$ , where  $R_1$  has a symbol in  $S(\mu^{-2}\nu\lambda^{\frac{1}{2}}, g)$ , and  $D^2 \text{Op}(\mu^{-1})\nu\chi = \nu D^2 \chi \text{Op}(\mu^{-1}) + R_2$ , where  $R_2$  has a symbol in  $S(\nu\lambda^{\frac{1}{2}}, g)$ . Then, it follows from (3.26) that

$$\tau \|\nu(x)\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D\tilde{u}\|_V^2 \lesssim \tau^3 \|\nu(x)\chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D^2 \chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2 + \lambda \|\nu(x)\tilde{u}\|_V^2.$$

For  $\tau \geq \max\{\mathcal{C}\lambda, 1\}$  with  $\mathcal{C}$  large enough, one has the following result from the above inequality.

$$\tau \|\nu(x)\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D\tilde{u}\|_V^2 \lesssim \tau^3 \|\nu(x)\chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2 + \tau^{-1} \|\nu D^2 \chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2. \quad (3.27)$$

Now, we apply (3.25) to  $v = \chi \text{Op}(\mu^{-1})\tilde{u}$  to have

$$\begin{aligned} & \tau^3 \|\nu(x)\chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2 + \tau \|\nu(x)D\chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D^2 \chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2 \\ & \lesssim \|P_\varphi \chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2. \end{aligned}$$

Thus, combining this with (3.27) yields

$$\tau \|\nu(x)\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D\tilde{u}\|_V^2 \lesssim \|P_\varphi \chi \text{Op}(\mu^{-1})\tilde{u}\|_V^2. \quad (3.28)$$

Finally, note that  $P_\varphi$  has a symbol in  $S(\nu\mu^2, g)$ . Consequently,  $P_\varphi \chi \text{Op}(\mu^{-1}) = \text{Op}(\mu^{-1})P_\varphi \chi + R$ , where  $R$  has a symbol in  $S(\nu\lambda^{\frac{1}{2}}, g)$ . Then, we can deduce from (3.28) that

$$\tau \|\nu(x)\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D\tilde{u}\|_V^2 \lesssim \|\text{Op}(\mu^{-1})P_\varphi \tilde{u}\|_V^2 + \lambda \|\nu\tilde{u}\|_V^2.$$

When  $\tau \geq \mathcal{C}\lambda$  with  $\mathcal{C}$  large enough, the error term  $\lambda \|\nu\tilde{u}\|_V^2$  can be absorbed by the left hand side. Consequently,

$$\tau \|\nu(x)\tilde{u}\|_V^2 + \tau^{-1} \|\nu(x)D\tilde{u}\|_V^2 \lesssim \|\text{Op}(\mu^{-1})P_\varphi \tilde{u}\|_V^2. \quad (3.29)$$

For  $\tilde{u} = e^{\tau\varphi}u$ , it follows from (3.29) and similar argument as (3.21) that

$$\begin{aligned} \tau \|\nu(x)e^{\tau\varphi}u\|_V^2 + \tau^{-1} \|\nu(x)e^{\tau\varphi}Du\|_V^2 & \lesssim \tau \|\nu(x)e^{\tau\varphi}u\|_V^2 + \tau^{-1} \|\nu(x)De^{\tau\varphi}u\|_V^2 \\ & \lesssim \|\text{Op}(\mu^{-1})P_\varphi e^{\tau\varphi}u\|_V^2. \end{aligned} \quad (3.30)$$

Obviously, one has that

$$P_\varphi \tilde{u} = e^{\tau\varphi}Pu = e^{\tau\varphi}g_0 + \sum_{j=1}^d e^{\tau\varphi}\partial_{x_j}g_j = e^{\tau\varphi}g_0 + \sum_{j=1}^d (\partial_{x_j}(e^{\tau\varphi}g_j) - \tau e^{\tau\varphi}g_j\partial_{x_j}\varphi),$$

which yields

$$\|\text{Op}(\mu^{-1})P_\varphi \tilde{u}\|_V^2 \lesssim \sum_{j=0}^d \|e^{\tau\varphi}g_j\|_V^2. \quad (3.31)$$

Hence, we obtain Theorem 3.3 from (3.30) and (3.31).  $\square$

**Remark 3.2.** Since  $a(\cdot)$  is nonnegative and not identically null, there exists  $\delta > 0$  such that  $\{x \in \Omega : a(x) > \delta\} \neq \emptyset$ . We introduce several sets as follows.

$$\begin{aligned} W_0 &= \Omega \setminus \text{supp } a, \\ W_1 &= \overline{\Omega} \setminus \mathcal{O}(\text{supp } a), \\ W_2 &= \Omega \setminus (\{x \in \Omega : a(x) \geq \delta\} \cup \mathcal{O}(\Gamma)), \\ W_3 &= \Omega \setminus \{x \in \Omega : a(x) \geq \frac{\delta}{2}\}, \end{aligned}$$

where  $\mathcal{O}(\Gamma)$  means the neighborhood of  $\Gamma$ .

It is known that there exists a function  $\psi \in \mathcal{C}^\infty(\Omega)$  such that ([11])

- 1)  $\psi(x) = 0$  for  $x \in \partial\Omega$ .
- 2)  $\partial_n \psi(x) < 0$  for  $x \in \partial\Omega$ .
- 3)  $\nabla \psi(x) \neq 0$  for  $x \in \overline{\Omega \setminus \{x \in \Omega : a(x) \geq \delta\}}$ .

Let  $\varphi = e^{\gamma\psi}$ . It follows from Lemma 2.5 that  $\varphi$  satisfies the sub-ellipticity condition on  $x \in \Omega \setminus \{x \in \Omega : a(x) \geq \delta\}$  if  $\gamma > 0$  is sufficiently large. Then, in Theorem 3.2 and 3.3, one can choose  $V$  as  $\Omega \setminus \{x \in \Omega : a(x) \geq \delta\}$ .

The following result is a classical Carleman estimate and corresponding to the Laplacian with Dirichlet boundary condition ([17], Proposition 2). We shall use it to deal with the terms on  $\Omega \setminus \text{supp } a$ .

**Lemma 3.1.** Suppose  $\varphi$  is chosen as in Remark 3.2. Then, there exist  $C > 0$ ,  $\lambda_0 > 0$ , such that for all  $u \in \mathcal{C}^\infty(\overline{\Omega})$  satisfying  $\text{supp } u \subset W_1$  and  $u = 0$  on  $\Gamma$ , it holds

$$\tau^3 \|e^{\tau\varphi} u\|_\Omega^2 + \tau \|e^{\tau\varphi} Du\|_\Omega^2 \lesssim \|e^{\tau\varphi} (D^2 - \lambda^2) u\|_\Omega^2, \quad (3.32)$$

where  $\lambda \geq \lambda_0$  and  $\tau \geq \max\{C\lambda, 1\}$ .

**Theorem 3.4.** Suppose  $\varphi$  is chosen as in Remark 3.2. Let  $u \in \mathcal{C}^\infty(\overline{\Omega})$  and satisfy

$$\begin{aligned} P(x, D, \lambda)u &= f_0 + \sum_{j=1}^d \partial_{x_j} f_j \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where  $f_j \in L^2(\Omega)$ ,  $\text{supp } f_0 \subset \Omega$  and  $\text{supp } f_j \subset \mathcal{O}(\text{supp } a)$  for  $j = 1, \dots, d$ . Then, there exist  $\tilde{K} > 0$ ,  $\lambda_0 > 0$ , such that for all  $\lambda \geq \lambda_0$  and  $\tau \geq \max\{\tilde{K}|\lambda|^{\frac{5}{4}}, 1\}$ , it holds

$$\tau \|e^{\tau\varphi} u\|_{W_3}^2 + \tau^{-1} \|e^{\tau\varphi} Du\|_{W_3}^2 \lesssim \sum_{j=0}^d \|e^{\tau\varphi} f_j\|_\Omega^2 + \lambda \|e^{\tau\varphi} u\|_{\{x \in \Omega : a(x) \geq \delta/2\}}^2,$$

where the positive constant  $\delta$  is defined as in Remark 3.2.

*Proof.* Let  $\chi_1, \chi_2 \in \mathcal{C}^\infty(\Omega)$  be non-negative and satisfy the following assumption

- (i)  $0 \leq \chi_1, \chi_2 \leq 1$ ;  $\chi_1$  and  $\chi_2$  are supported on  $W_1$  and  $W_2$ , respectively.
- (ii)  $\chi_1 + \chi_2 \geq 1$  in  $W_3$ . In particular,  $\chi_1 \equiv 1$  on  $[\mathcal{O}(\partial\Omega) \cap \Omega] \setminus \mathcal{O}(\text{supp } a)$ , and  $\chi_2 \equiv 1$  on  $\mathcal{O}(\text{supp } a) \setminus \{x \in \Omega : a(x) \geq \frac{\delta}{2}\}$ .



First, it is clear that

$$P\chi_2 u = \chi_2 f_0 + \sum_{j=1}^d (\partial_{x_j}(\chi_2 f_j) - f_j \partial_{x_j} \chi_2) + [P, \chi_2]u.$$

Since  $[P, \chi_2]$  is a first order operator, we have there exist  $a_0, a_1, \dots, a_d$  and  $b_0, b_1, \dots, b_d$  such that

$$[P, \chi_2]u = \sum_{j=0}^d a_j \partial_{x_j} u + \lambda \sum_{j=0}^d b_j \partial_{x_j} u,$$

where  $\text{supp } a_j \subset \{x \in \Omega : \frac{\delta}{2} < a(x) < \delta\} \cup [\Omega \setminus (\mathcal{O}(\text{supp } a) \cup \mathcal{O}(\text{supp } \partial\Omega))]$ ,  $\text{supp } b_j \subset \{x \in \Omega : \frac{\delta}{2} < a(x) < \delta\}$  for  $j = 0, 1, \dots, d$ . Then, applying Theorem 3.3 with  $\chi_2 u$  instead of  $u$ ,  $\Omega \setminus \{x \in \Omega : a(x) \geq \delta\}$  instead of  $V$ , we obtain

$$\begin{aligned} & \tau \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} \chi_2 u\|_{\Omega}^2 + \tau^{-1} \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} D\chi_2 u\|_{\Omega}^2 \\ & \lesssim \sum_{j=0}^d \|e^{\tau\varphi} f_j\|_{\Omega}^2 + \|e^{\tau\varphi} u\|_{\{x \in \Omega : \frac{\delta}{2} < a(x) < \delta\} \cup (\Omega \setminus \text{supp } a)}^2 + \lambda \|e^{\tau\varphi} u\|_{\{x \in \Omega : \frac{\delta}{2} < a(x) < \delta\}}^2. \end{aligned}$$

Consequently, due to  $(1 + \lambda^2 a(x)^2) \geq 1$ ,  $\tau \geq \{\mathcal{C}\lambda, 1\}$  and  $\lambda \geq \lambda_0$ , we obtain

$$\begin{aligned} & \tau \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} \chi_2 u\|_{\Omega}^2 + \tau^{-1} \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} D\chi_2 u\|_{\Omega}^2 \\ & \lesssim \sum_{j=0}^d \|e^{\tau\varphi} f_j\|_{\Omega}^2 + \|e^{\tau\varphi} u\|_{\Omega \setminus \text{supp } a}^2 + \lambda \|e^{\tau\varphi} u\|_{\{x \in \Omega : a(x) \geq \frac{\delta}{2}\}}^2. \end{aligned} \quad (3.33)$$

On the other hand, by using  $\chi_1 u$  instead of  $u$  in Lemma 3.1, we obtain

$$\tau^3 \|e^{\tau\varphi} \chi_1 u\|_{\Omega}^2 + \tau \|e^{\tau\varphi} D\chi_1 u\|_{\Omega}^2 \lesssim \|e^{\tau\varphi} (P - i\lambda D a(x) D) \chi_1 u\|_{\Omega}^2. \quad (3.34)$$

Since  $\chi_1 \partial_{x_j} f_j = 0$  for  $j = 1, \dots, d$ , we have

$$P\chi_1 u = \chi_1 f_0 + [P, \chi_1]u.$$

Therefore, combining these with (3.34) and  $\mathcal{O}(\text{supp } a) \cap \text{supp } \chi_1 = \emptyset$  yields

$$\tau \|e^{\tau\varphi} \chi_1 u\|_{\Omega}^2 + \tau^{-1} \|e^{\tau\varphi} D\chi_1 u\|_{\Omega}^2 \lesssim \tau^{-2} (\|e^{\tau\varphi} f_0\|_{\Omega}^2 + \|e^{\tau\varphi} [P, \chi_1]u\|_{\Omega}^2). \quad (3.35)$$

Adding up (3.35) and (3.33), using  $(1 + \lambda^2 a(x)^2) \geq 1$  and  $\tau \geq 1$ , we obtain

$$\begin{aligned} & \tau \|e^{\tau\varphi} (\chi_1 + \chi_2)u\|_{\Omega}^2 + \tau^{-1} \|e^{\tau\varphi} D(\chi_1 + \chi_2)u\|_{\Omega}^2 \\ & \lesssim \sum_{j=0}^d \|e^{\tau\varphi} f_j\|_{\Omega}^2 + \|e^{\tau\varphi} u\|_{\Omega \setminus \text{supp } a}^2 + \lambda \|e^{\tau\varphi} u\|_{\{x \in \Omega : a(x) \geq \frac{\delta}{2}\}}^2 + \tau^{-2} \|e^{\tau\varphi} [P, \chi_1]u\|_{\Omega}^2. \end{aligned}$$

Note that  $[P, \chi_1]$  is a first order operator and supported on  $\text{supp } \chi_1 \setminus \{x \in \Omega : \chi_1 \equiv 1\}$ , which is an subset of  $\{x \in \Omega : \chi_1 + \chi_2 = 1\}$ . Then, for  $\tau$  sufficiently large, we have

$$\tau \|e^{\tau\varphi} (\chi_1 + \chi_2)u\|_{\Omega}^2 + \tau^{-1} \|e^{\tau\varphi} (\chi_1 + \chi_2)Du\|_{\Omega}^2 \lesssim \sum_{j=0}^d \|e^{\tau\varphi} f_j\|_{\Omega}^2 + \lambda \|e^{\tau\varphi} u\|_{\{x \in \Omega : a(x) \geq \frac{\delta}{2}\}}^2.$$

Then as  $1 \leq \chi_1 + \chi_2 \leq 2$  on  $W_3$ , we obtain the statement of Theorem 3.4.  $\square$

## 4 Resolvent estimate

In this section, we shall prove the main result. From the results in [3, 7, 10], the logarithmic decay of the energy in Theorem 1.1 can be obtained through the following theorem.

**Theorem 4.1.** *Suppose conditions in Theorem 1.1 hold. Then, for every  $\lambda \in \mathbb{R}$  with  $|\lambda|$  large enough, there exists  $C > 0$  such that*

$$\left\| (\mathcal{A} - i\lambda)^{-1} \right\|_{\mathcal{L}(\mathcal{H})} \lesssim e^{C|\lambda|^{\frac{5}{4}}}. \quad (4.1)$$

Let  $\lambda$  be a real number such that  $|\lambda|$  is large enough. Consider the resolvent equation:

$$F = (\mathcal{A} - i\lambda)Y, \quad \text{where } Y = (y_1, y_2) \in D(\mathcal{A}), \quad F = (f_1, f_2) \in \mathcal{H}, \quad (4.2)$$

which yields

$$\begin{cases} \operatorname{div}(\nabla y_1 + a\nabla y_2) + \lambda^2 y_1 = i\lambda f_1 + f_2, & \text{in } \Omega, \\ y_2 = i\lambda y_1 + f_1 & \text{in } \Omega, \\ y_1|_{\Gamma} = 0 \end{cases} \quad (4.3)$$

In what follows, we shall prove that there exists a positive constant  $C$  such that

$$\|(y_1, y_2)\|_{\mathcal{H}} \lesssim e^{C|\lambda|^{\frac{5}{4}}} \|(f_1, f_2)\|_{\mathcal{H}}.$$

First, let  $\eta > 0$  and  $\chi \in \mathcal{C}_0^\infty(\Omega)$  be real valued function such that  $\operatorname{supp} \chi = \{x \in \Omega : a(x) > \eta\}$  and  $\chi \equiv 1$  in  $\{x \in \Omega : a(x) > 2\eta\}$ . The following lemma is helpful.

**Lemma 4.1.** *For  $y_1, f_1, f_2$  satisfying (4.3) and  $|\lambda|$  large enough, it holds*

$$(i) \quad \|\chi y_1\|_{\Omega} \lesssim \|\nabla y_1\|_{\{x \in \Omega : a(x) > \eta\}} + \|f_1\|_{H^1(\Omega)} + \|f_2\|_{\Omega}.$$

$$(ii) \quad \int_{\Omega} a(x) |\nabla y_1|^2 dx \lesssim \|f_1\|_{H^1(\Omega)}^2 + (\|f_1\|_{\Omega} + \|f_2\|_{\Omega}) \|y_1\|_{\Omega}.$$

*Proof.* (i) Multiplying the first equation in (4.3) by  $\chi^2 \bar{y}_1$  and using  $y_2 = i\lambda y_1 + f_1$ , we obtain,

$$\begin{aligned} \lambda^2 \|\chi y_1\|_{\Omega}^2 &= \int_{\Omega} \nabla y_1 \cdot \nabla(\chi^2 \bar{y}_1) dx + i\lambda \int_{\Omega} a(x) \nabla y_1 \cdot \nabla(\chi^2 \bar{y}_1) dx \\ &\quad + i\lambda \int_{\Omega} f_1 \chi^2 \bar{y}_1 dx + \int_{\Omega} f_2 \chi^2 \bar{y}_1 dx + \int_{\Omega} a(x) \nabla f_1 \cdot \nabla(\chi^2 \bar{y}_1) dx. \end{aligned} \quad (4.4)$$

Since  $\nabla(\chi^2 \bar{y}_1) = \chi^2 \nabla \bar{y}_1 + 2\bar{y}_1 \chi \nabla \chi$ , we have

$$\begin{aligned} \left| \int_{\Omega} a(x) \nabla y_1 \cdot \nabla(\chi^2 \bar{y}_1) dx \right| &\lesssim \|\nabla y_1\|_{\{x \in \Omega : a(x) > \eta\}}^2 + \|\chi y_1\|_{\Omega} \|\nabla y_1\|_{\{x \in \Omega : a(x) > \eta\}} \\ &\leq 2\|\nabla y_1\|_{\{x \in \Omega : a(x) > \eta\}}^2 + \frac{1}{4} \|\chi y_1\|_{\Omega}^2. \end{aligned} \quad (4.5)$$

By the similar argument, we can deal with the rest terms on the right hand side of (4.4). Combining these with (4.4), (4.5) yields

$$\begin{aligned} \lambda^2 \|\chi y_1\|_{\Omega}^2 &\lesssim \frac{1}{2} \lambda^2 \|\chi y_1\|_{\Omega}^2 + (|\lambda| + 1) \|\nabla y_1\|_{\{x \in \Omega : a(x) > \eta\}}^2 \\ &\quad + \|f_1\|_{\Omega}^2 + \lambda^{-2} \|f_2\|_{\Omega}^2 + (\lambda^{-2} + 1) \|\nabla f_1\|_{H^1(\Omega)}^2. \end{aligned} \quad (4.6)$$

Then, (i) is reached.

(ii) Multiplying the first equation in (4.3) by  $\bar{y}_1$  and using  $y_2 = i\lambda y_1 + f_1$ , we obtain,

$$\int_{\Omega} [-(i\lambda a(x) + 1)|\nabla y_1|^2 + \lambda^2 |y_1|^2] dx = \int_{\Omega} [a(x)\nabla f_1 \cdot \nabla \bar{y}_1 + (i\lambda f_1 + f_2)\bar{y}_1] dx. \quad (4.7)$$

Taking the imaginary part of (4.7) yields

$$\int_{\Omega} a(x)|\nabla y_1|^2 dx = -\lambda^{-1} \operatorname{Im} \int_{\Omega} [a(x)\nabla f_1 \cdot \nabla \bar{y}_1 + (i\lambda f_1 + f_2)\bar{y}_1] dx. \quad (4.8)$$

Thus, by the Cauchy-Schwarz inequality, one can conclude from (4.8) that

$$\begin{aligned} \int_{\Omega} a(x)|\nabla y_1|^2 dx &\leq |\lambda|^{-1} \left( \int_{\Omega} a(x)|\nabla y_1|^2 dx \right)^{\frac{1}{2}} \|f_1\|_{H^1(\Omega)} + (\|f_1\|_{\Omega} + |\lambda|^{-1} \|f_2\|_{\Omega}) \|y_1\|_{\Omega} \\ &\leq \frac{1}{4} \int_{\Omega} a(x)|\nabla y_1|^2 dx + |\lambda|^{-2} \|f_1\|_{H^1(\Omega)}^2 + (\|f_1\|_{\Omega} + |\lambda|^{-1} \|f_2\|_{\Omega}) \|y_1\|_{\Omega}. \end{aligned}$$

The proof of lemma is finished.  $\square$

**Proof of Theorem 4.1.** Due to Lemma 4.1 (i) and (ii), we have

$$\|y_1\|_{\{x \in \Omega : a(x) > 2\eta\}}^2 \lesssim \|\nabla y_1\|_{\{x \in \Omega : a(x) > \eta\}}^2 + \|f_1\|_{H^1(\Omega)}^2 + \|f_2\|_{\Omega}^2, \quad (4.9)$$

and

$$\|\nabla y_1\|_{\{x \in \Omega : a(x) > \eta\}}^2 \lesssim \|f_1\|_{H^1(\Omega)}^2 + (\|f_1\|_{\Omega} + \|f_2\|_{\Omega}) \|y_1\|_{\Omega}. \quad (4.10)$$

Combining (4.9) and (4.10) yields

$$\|y_1\|_{H^1(\{x \in \Omega : a(x) > 2\eta\})}^2 \lesssim \|f_1\|_{H^1(\Omega)}^2 + \|f_2\|_{L^2(\Omega)}^2 + (\|f_1\|_{\Omega} + \|f_2\|_{\Omega}) \|y_1\|_{\Omega}. \quad (4.11)$$

On the other hand, by (4.3), one has that  $y_1$  satisfies

$$\operatorname{div}(\nabla y_1 + i\lambda a(x)\nabla y_1) + \lambda^2 y_1 = i\lambda f_1 + f_2 - \operatorname{div}(a(x)\nabla f_1). \quad (4.12)$$

Then, applying Theorem 3.4 to  $y_1$  satisfying (4.12), we obtain

$$\begin{aligned} &\tau \|e^{\tau\varphi} y_1\|_{\Omega}^2 + \tau^{-1} \|e^{\tau\varphi} \nabla y_1\|_{\Omega}^2 \\ &\lesssim \lambda^2 \|e^{\tau\varphi} f_1\|_{\Omega}^2 + \|e^{\tau\varphi} f_2\|_{\Omega}^2 + \sum_{j=1}^d \|e^{\tau\varphi} a \partial_{x_j} f_1\|_{\Omega}^2 \\ &\quad + (\lambda + \tau) \|e^{\tau\varphi} y_1\|_{\{x \in \Omega : a(x) \geq \delta/2\}}^2 + \tau^{-1} \|e^{\tau\varphi} \nabla y_1\|_{\{x \in \Omega : a(x) \geq \delta/2\}}^2. \end{aligned}$$

Let  $c_1 = \min_{x \in \Omega} \varphi(x)$  and  $c_2 = \max_{x \in \Omega} \varphi(x)$ . We conclude from the above inequality and  $\tau \geq \max\{\tilde{K}|\lambda|^{\frac{d}{4}}, 1\}$  that

$$\begin{aligned} &\tau e^{2c_1\tau} \|y_1\|_{\Omega}^2 + \tau^{-1} e^{2c_1\tau} \|\nabla y_1\|_{\Omega}^2 \\ &\lesssim \lambda^2 e^{2c_2\tau} \|f_1\|_{H^1(\Omega)}^2 + e^{2c_2\tau} \|f_2\|_{\Omega}^2 + \tau e^{2c_2\tau} \|y_1\|_{H^1(\{x \in \Omega : a(x) \geq \delta/2\})}^2. \end{aligned} \quad (4.13)$$

Setting  $\eta = \frac{\delta}{8}$  and substituting (4.11) into (4.13), we obtain

$$\tau e^{2c_1\tau} \|y_1\|_{\Omega}^2 + \tau^{-1} e^{2c_1\tau} \|\nabla y_1\|_{\Omega}^2 \lesssim \tau^2 e^{2c_2\tau} \|f_1\|_{H^1(\Omega)}^2 + \tau e^{2c_2\tau} \|f_2\|_{\Omega}^2 + \tau e^{2c_2\tau} (\|f_1\|_{\Omega} + \|f_2\|_{\Omega}) \|y_1\|_{\Omega}.$$

Let  $c_3 = 2(c_2 - c_1) + 1$ . For  $\tau \geq \max\{\tilde{K}|\lambda|^{\frac{5}{4}}, 1\}$ , one has

$$\|y_1\|_{\Omega}^2 + \|\nabla y_1\|_{\Omega}^2 \lesssim e^{c_3\tau} \|f_1\|_{H^1(\Omega)}^2 + e^{c_3\tau} \|f_2\|_{\Omega}^2 + e^{c_3\tau} (\|f_1\|_{\Omega} + \|f_2\|_{\Omega}) \|y_1\|_{\Omega}.$$

For any  $\varepsilon > 0$ , using  $e^{c_3\tau} (\|f_1\|_{\Omega} + \|f_2\|_{\Omega}) \|y_1\|_{\Omega} \leq \varepsilon \|y_1\|_{\Omega}^2 + \frac{1}{4}\varepsilon^{-1} e^{2c_3\tau} (\|f_1\|_{\Omega} + \|f_2\|_{\Omega})^2$  in the above estimate, we conclude that

$$\|y_1\|_{\Omega}^2 + \|\nabla y_1\|_{\Omega}^2 \lesssim e^{2c_3\tau} (\|f_1\|_{H^1(\Omega)}^2 + \|f_2\|_{\Omega}^2),$$

which gives the desired result.  $\square$

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