



**HAL**  
open science

# GENERALIZATIONS OF SEVERAL INEQUALITIES RELATED TO MULTIVARIATE GEOMETRIC MEANS

Bo-Yan Xi, Ying Wu, Huan-Nan Shi, Feng Qi

► **To cite this version:**

Bo-Yan Xi, Ying Wu, Huan-Nan Shi, Feng Qi. GENERALIZATIONS OF SEVERAL INEQUALITIES RELATED TO MULTIVARIATE GEOMETRIC MEANS. 2018. hal-01865896v1

**HAL Id: hal-01865896**

**<https://hal.science/hal-01865896v1>**

Preprint submitted on 2 Sep 2018 (v1), last revised 19 Jan 2020 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# GENERALIZATIONS OF SEVERAL INEQUALITIES RELATED TO MULTIVARIATE GEOMETRIC MEANS

BO-YAN XI, YING WU, HUAN-NAN SHI, AND FENG QI

ABSTRACT. In the paper, by some methods in the theory of majorization, the authors generalize several inequalities related to multivariate geometric means.

## 1. INTRODUCTION

Let  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n = (0, \infty)^n$  and  $n \geq 2$ . Then the arithmetic and geometric means of  $n$  positive numbers  $x_1, x_2, \dots, x_n$  are defined by

$$A(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n x_i \quad \text{and} \quad G(\mathbf{x}) = \left( \prod_{i=1}^n x_i \right)^{1/n}.$$

In [12, p. 208, 3.2.34], it was collected that

$$\prod_{i=1}^n (x_i + 1) \geq [G(\mathbf{x}) + 1]^n. \quad (1.1)$$

In the paper [26], Wang and Chen established that the inequality

$$\prod_{i=1}^n [(x_i + 1)^p - 1] \geq [(G(\mathbf{x}) + 1)^p - 1]^n \quad (1.2)$$

is valid for  $p > 1$ . In the paper [24], Wang extended the inequality (1.2) as follows:

- (1) if  $p \geq 1$ , then the inequality (1.2) is valid;
- (2) if  $0 < p \leq 1$ , then the inequality (1.2) is reversed;
- (3) if  $p > 0$ , then

$$\prod_{i=1}^n [(x_i + 1)^p + 1] \geq [(G(\mathbf{x}) + 1)^p + 1]^n; \quad (1.3)$$

- (4) if  $p < 0$  and  $0 < x_i < \frac{1}{|p|}$  for  $1 \leq i \leq n$ , then the inequality (1.3) is reversed.

In [8, p. 11], it was collected and proved that

$$\sqrt[n]{\prod_{k=1}^n (a_k + b_k)} \geq \sqrt[n]{\prod_{k=1}^n a_k} + \sqrt[n]{\prod_{k=1}^n b_k}, \quad (1.4)$$

where  $a_k, b_k \geq 0$ . When  $a_k = x_k$  and  $b_k = 1$ , the inequality (1.4) becomes (1.1). The inequality (1.4) is the Minkowski inequality of product form.

We observe that inequalities in (1.1), (1.2), (1.3), and (1.4) can be rearranged as

$$G(\mathbf{x} + \mathbf{1}) \geq G(\mathbf{x}) + 1, \quad G((\mathbf{x} + \mathbf{1})^p \pm \mathbf{1}) \geq [G(\mathbf{x}) + 1]^p \pm 1, \quad (1.5)$$

and

$$G(\mathbf{a} + \mathbf{b}) > G(\mathbf{a}) + G(\mathbf{b}), \quad (1.6)$$

---

2010 *Mathematics Subject Classification.* Primary 26E60; Secondary 26D15, 26A51.

*Key words and phrases.* Schur-convex function; Schur-geometrically convex function; elementary symmetric function; inequality; multivariate geometric mean; majorization.

This paper was typeset using  $\mathcal{A}\mathcal{M}\mathcal{S}$ - $\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$ .

where

$$\mathbf{x} + \mathbf{1} = (x_1 + 1, x_2 + 1, \dots, x_n + 1)$$

and

$$(\mathbf{x} + \mathbf{1})^p \pm \mathbf{1} = ((x_1 + 1)^p \pm 1, (x_2 + 1)^p \pm 1, \dots, (x_n + 1)^p \pm 1).$$

The inequality (1.6) reveals that the geometric mean  $G(\mathbf{x})$  is sub-additive. For information about the sub-additivity, please refer to [2, 6, 7, 14, 15, 17, 9, 10] and closely related references therein. The sub-additive property of the geometric mean  $G(\mathbf{x})$  can also be derived from the property that the geometric mean  $G(\mathbf{x})$  is a Bernstein function, see [5, 16, 18, 19, 20, 21, 22] and closely related references therein.

In this paper, by some methods in the theory of majorization, we will generalize the above inequalities in (1.1), (1.2), (1.3), (1.5), and (1.6).

## 2. DEFINITIONS AND LEMMAS

Now we recall some definitions and lemmas.

It is well known that a function  $f(x_1, x_2, \dots, x_n)$  of  $n$  variables is said to be symmetric if its value is unchanged for any permutation of its  $n$  variables  $x_1, x_2, \dots, x_n$ .

**Definition 2.1** ([11, 25]). Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ .

(1) If

$$\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} \quad \text{and} \quad \sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$$

for  $1 \leq k \leq n-1$ , then  $\mathbf{x}$  is said to be majorized by  $\mathbf{y}$  (in symbol  $\mathbf{x} \prec \mathbf{y}$ ), where  $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$  and  $y_{[1]} \geq y_{[2]} \geq \dots \geq y_{[n]}$  are rearrangements of  $\mathbf{x}$  and  $\mathbf{y}$  in a descending order.

(2) For  $\mathbf{x}, \mathbf{y} \in \Omega$  and  $\lambda \in [0, 1]$ , if

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} = (\lambda x_1 + (1 - \lambda) y_1, \lambda x_2 + (1 - \lambda) y_2, \dots, \lambda x_n + (1 - \lambda) y_n) \in \Omega,$$

then  $\Omega \subseteq \mathbb{R}^n$  is said to be a convex set.

**Definition 2.2** ([11, 25]). Let  $\Omega \subseteq \mathbb{R}^n$  be a symmetric and convex set.

- (1) If  $\mathbf{x} \prec \mathbf{y}$  on  $\Omega$  implies  $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$ , then we say that the function  $\varphi : \Omega \rightarrow \mathbb{R}$  is Schur-convex on  $\Omega$ .
- (2) If  $-\varphi$  is a Schur-convex function on  $\Omega$ , then we say that  $\varphi$  is Schur-concave on  $\Omega$ .

**Lemma 2.1** ([11, 25]). *Let  $\Omega \subseteq \mathbb{R}^n$  be a symmetric and convex set with nonempty interior  $\Omega^\circ$  and let  $\varphi : \Omega \rightarrow \mathbb{R}$  is continuous on  $\Omega$  and continuously differentiable on  $\Omega^\circ$ . If and only if  $\varphi$  is symmetric on  $\Omega$  and*

$$(x_1 - x_2) \left( \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2} \right) \geq 0, \quad \mathbf{x} \in \Omega^\circ,$$

*the function  $\varphi$  is Schur-convex on  $\Omega$ .*

**Definition 2.3** ([27, pp. 64 and 107]). Let  $\mathbf{x}, \mathbf{y} \in \Omega \subseteq \mathbb{R}_+^n$ .

- (1) If  $(x_1^\lambda y_1^{1-\lambda}, x_2^\lambda y_2^{1-\lambda}, \dots, x_n^\lambda y_n^{1-\lambda}) \in \Omega$  for  $\mathbf{x}, \mathbf{y} \in \Omega$  and  $\lambda \in [0, 1]$ , then  $\Omega$  is called a geometrically convex set.
- (2) If  $\Omega$  is a geometrically convex set and

$$(\ln x_1, \ln x_2, \dots, \ln x_n) \prec (\ln y_1, \ln y_2, \dots, \ln y_n)$$

implies  $\varphi(\mathbf{x}) \leq \varphi(\mathbf{y})$  for any  $\mathbf{x}, \mathbf{y} \in \Omega$ , then  $\varphi : \Omega \rightarrow \mathbb{R}_+$  is said to be a Schur-geometrically convex function on  $\Omega$ .

- (3) If  $-\varphi$  is a Schur-geometrically convex function on  $\Omega$ , then  $\varphi$  is said to be a Schur-geometrically concave function on  $\Omega$ .

**Lemma 2.2** ([27, p. 108]). *Let  $\Omega \subseteq \mathbb{R}_+^n$  be a symmetric and geometrically convex set with a nonempty interior  $\Omega^\circ$  and let  $\varphi : \Omega \rightarrow \mathbb{R}_+$  be continuous on  $\Omega$  and differentiable in  $\Omega^\circ$ . If and only if  $\varphi$  is symmetric on  $\Omega$  and*

$$(\ln x_1 - \ln x_2) \left( x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2} \right) \geq 0, \quad \mathbf{x} \in \Omega^\circ,$$

*the function  $\varphi$  is Schur-geometrically convex on  $\Omega$ .*

For more information on the Schur-convexity and the Schur-geometric convexity, please refer to the papers [3, 4, 23] and the monographs [11, 27].

**Lemma 2.3** (Bernoulli's inequality [1, 13]). *For  $0 \neq x > -1$ , if  $\alpha > 1$  or  $\alpha < 0$ , then*

$$(1+x)^\alpha > 1 + \alpha x; \tag{2.1}$$

*if  $0 < \alpha < 1$ , then the inequality (2.1) is reversed.*

**Lemma 2.4.** *For  $p \in \mathbb{R} \setminus \{0\}$  and  $x \in \mathbb{R}_+$ , define*

$$\begin{aligned} h_1(x) &= \frac{p(x+1)^{p-1}}{(x+1)^p + 1}, & g_1(x) &= \frac{px(x+1)^{p-1}}{(x+1)^p + 1}, \\ h_2(x) &= \frac{p(x+1)^{p-1}}{(x+1)^p - 1}, & g_2(x) &= \frac{px(x+1)^{p-1}}{(x+1)^p - 1}. \end{aligned}$$

- (1) *If  $p < 0$ , the function  $h_1(x)$  is increasing on  $\mathbb{R}_+$ ; if  $0 < p \leq 2$ , the function  $h_1(x)$  is decreasing on  $\mathbb{R}_+$ ; if  $p > 2$ , the function  $h_1(x)$  is increasing on  $(0, (p-1)^{1/p} - 1)$  and on  $((p-1)^{1/p} - 1, \infty)$ .*
- (2) *If  $p > 0$ , the function  $g_1(x)$  is increasing on  $\mathbb{R}$ ; if  $p < 0$ , the function  $g_1(x)$  is decreasing on  $(0, \frac{1}{|p|}]$  and increasing on  $[\frac{2}{|p|}, \infty)$ .*
- (3) *For  $p \neq 0$ , the function  $h_2(x)$  is decreasing on  $\mathbb{R}_+$ ;*
- (4) *If  $p \geq 1$ , the function  $g_2(x)$  is increasing on  $\mathbb{R}$ ; if  $p < 1$  and  $p \neq 0$ , the function  $g_2(x)$  is decreasing on  $\mathbb{R}$ .*

*Proof.* The proof of Lemma 2.4 is straightforward.  $\square$

Let  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and  $i_1, \dots, i_k$  are positive integers. The elementary symmetric functions are defined by  $E_0(\mathbf{x}) = 1$ ,

$$E_k(\mathbf{x}) = E_k(x_1, x_2, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k x_{i_j}, \quad 1 \leq k \leq n,$$

and  $E_k(\mathbf{x}) = 0$  for  $k < 0$  or  $k > n$ .

**Lemma 2.5** (Newton's inequality [11, p. 134]). *For  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$  and  $n \geq 2$ , let  $\widehat{F}_k(\mathbf{x}) = \frac{1}{\binom{n}{k}} E_k(\mathbf{x})$  for  $1 \leq k \leq n-1$ . Then*

$$[\widehat{F}_k(\mathbf{x})]^2 \geq \widehat{F}_{k-1}(\mathbf{x}) \widehat{F}_{k+1}(\mathbf{x}).$$

### 3. MAIN RESULTS

In this section, we will make use of the Schur-convexity of the symmetric function

$$E_k((\mathbf{x} + \mathbf{1})^p + \boldsymbol{\xi}) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k [(x_{i_j} + 1)^p + \xi_j], \quad 2 \leq k \leq n, \quad n \geq 2$$

to generalize inequalities in (1.1), (1.2), (1.3), (1.5), and (1.6), where  $\mathbf{x} \in \mathbb{R}_+^n$ , the quantities  $i_1, i_2, \dots, i_k$  are positive integers,  $\xi = 0, \pm 1$ , and  $\boldsymbol{\xi} = \mathbf{0}, \pm \mathbf{1}$ .

Our main results are Theorems 3.1 to 3.3 below.

**Theorem 3.1.** Let  $2 \leq k \leq n$ ,  $p \in \mathbb{R} \setminus \{0\}$ ,  $i_1, \dots, i_k \in \mathbb{N}$ , and  $a = \frac{(n-k)(p-1)}{k-1}$  for  $p > 1$ .

- (1) If  $2 \leq k \leq n-1$ ,  $0 < p \leq 1$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , or if  $2 \leq k \leq n-1$ ,  $p > 1$  with  $a < 1$ , and  $\mathbf{x} \in (0, a^{-1/p} - 1]^n$ , or if  $k = n$ ,  $p > 0$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , then

$$\binom{n}{k} [G(\mathbf{x}) + 1]^{kp} \leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (x_{i_j} + 1)^p \leq \binom{n}{k} [A(\mathbf{x}) + 1]^{kp}; \quad (3.1)$$

if  $p < 0$  and  $\mathbf{x} \in (0, \frac{1}{|p|}]^n$ , or if  $k = n$ ,  $p < 0$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , then the double inequality (3.1) is reversed.

- (2) If  $2 \leq k \leq n-1$ ,  $p > 1$  with  $a > 1$ , and  $\mathbf{x} \in (0, a^{1/p} - 1]^n$ , then

$$\binom{n}{k} [A(\mathbf{x}) + 1]^{kp} \leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (x_{i_j} + 1)^p.$$

- (3) If  $2 \leq k \leq n-1$ ,  $p > 0$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , then

$$\binom{n}{k} [G(\mathbf{x}) + 1]^{kp} \leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k (x_{i_j} + 1)^p; \quad (3.2)$$

if  $p < 0$  and  $\mathbf{x} \in \mathbb{R}_+^n$ , then the inequality (3.2) is reversed.

*Proof.* When  $k = n$ , we have

$$\frac{\partial E_n((\mathbf{x} + \mathbf{1})^p)}{\partial x_1} = \frac{p}{(x_1 + 1)} E_n((\mathbf{x} + \mathbf{1})^p).$$

From this, we obtain

$$\begin{aligned} (x_1 - x_2) \left[ \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p)}{\partial x_1} - \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p)}{\partial x_2} \right] \\ = -\frac{p(x_1 - x_2)^2}{(x_1 + 1)(x_2 + 1)} E_n((\mathbf{x} + \mathbf{1})^p) \begin{cases} \leq 0, & p > 0; \\ \geq 0, & p < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} (x_1 - x_2) \left[ x_1 \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p)}{\partial x_1} - x_2 \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p)}{\partial x_2} \right] \\ = \frac{p(x_1 - x_2)^2}{(x_1 + 1)(x_2 + 1)} E_n((\mathbf{x} + \mathbf{1})^p) \begin{cases} \geq 0, & p > 0; \\ \leq 0, & p < 0. \end{cases} \end{aligned}$$

Using Lemmas 2.1 and 2.2, we arrive at

- (1) if  $p > 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p)$  is Schur-concave on  $\mathbb{R}_+^n$ ;
- (2) if  $p < 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p)$  is Schur-convex on  $\mathbb{R}_+^n$ ;
- (3) if  $p > 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p)$  is Schur-geometrically convex on  $\mathbb{R}^n$ ;
- (4) if  $p < 0$ , the  $E_n((\mathbf{x} + \mathbf{1})^p)$  is Schur-geometrically concave on  $\mathbb{R}^n$ .

Since

$$(A_n(\mathbf{x}), A_n(\mathbf{x}), \dots, A_n(\mathbf{x})) \prec \mathbf{x} \quad (3.3)$$

and

$$(\ln G_n(\mathbf{x}), \ln G_n(\mathbf{x}), \dots, \ln G_n(\mathbf{x})) \prec \ln \mathbf{x} \quad (3.4)$$

for  $\mathbf{x} \in \mathbb{R}_+^n$ , applying Definitions 2.2 and 2.3, we obtain the double inequality (3.1) for  $k = n$ .

When  $2 \leq k \leq n-1$ , a direct differentiation yields

$$\frac{\partial E_k((\mathbf{x} + \mathbf{1})^p)}{\partial x_1} = p(x_1 + 1)^{p-1} (x_2 + 1)^p E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p)$$

$$+ p(x_1 + 1)^{p-1} E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p),$$

where  $\widehat{\mathbf{x}} = (x_3, \dots, x_n)$ . We clearly see that

$$\begin{aligned} \Delta_A(E_k((\mathbf{x} + \mathbf{1})^p)) &\triangleq (x_1 - x_2) \left[ \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p)}{\partial x_1} - \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p)}{\partial x_2} \right] \\ &= -p(x_1 - x_2)^2 (x_1 + 1)^{p-1} (x_2 + 1)^{p-1} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad + p(x_1 - x_2) [(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p) \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} &(\ln x_1 - \ln x_2) \left[ x_1 \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p)}{\partial x_1} - x_2 \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p)}{\partial x_2} \right] \\ &= p(\ln x_1 - \ln x_2)(x_1 - x_2)(x_1 + 1)^{p-1} (x_2 + 1)^{p-1} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad + p(\ln x_1 - \ln x_2) [x_1(x_1 + 1)^{p-1} - x_2(x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p). \end{aligned} \quad (3.6)$$

Using the equation (3.6) and Lemma 2.2, we obtain that

- (1) if  $p > 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p)$  is Schur-geometrically convex on  $\mathbb{R}_+^n$ ;
- (2) if  $p < 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p)$  is Schur-geometrically concave on  $(0, \frac{1}{|p|}]^n$ ;
- (3) if  $p < 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p)$  is Schur-geometrically convex on  $(\frac{1}{|p|}, \infty)^n$ .

By the equation (3.5) and Lemma 2.1, we reveal that

- (1) if  $p < 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p)$  is Schur-convex on  $\mathbb{R}_+^n$ ;
- (2) if  $0 < p \leq 1$ , then  $E_k((\mathbf{x} + \mathbf{1})^p)$  is Schur-concave on  $\mathbb{R}_+^n$ ;
- (3) if  $p > 1$  and  $\mathbf{x} \in \mathbb{R}_+^n$ ,
  - (a) when  $x_1 = x_2$ , we have  $\Delta_A(E_k((\mathbf{x} + \mathbf{1})^p)) = 0$ ;
  - (b) when  $x_1 \neq x_2$ ,
    - (i) by Lagrange's mean value theorem, we have

$$\frac{1}{(x_2 + 1)^{p-1}} - \frac{1}{(x_1 + 1)^{p-1}} = \frac{(p-1)(x_1 - x_2)}{(\xi + 1)^p} \quad (3.7)$$

for at least one interior point  $\xi \in (\min\{x_1, x_2\}, \max\{x_1, x_2\})$ ;

- (ii) from Newton's inequality, we obtain

$$\begin{aligned} \frac{(n-k)(n-2)}{(k-1) \sum_{i=3}^n (x_i + 1)^{-p}} &= \frac{(n-k) \widehat{F}_{n-2}((\widehat{\mathbf{x}} + \mathbf{1})^p)}{(k-1) \widehat{F}_{n-3}((\widehat{\mathbf{x}} + \mathbf{1})^p)} \leq \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p)}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p)} \\ &\leq \frac{(n-k) \widehat{F}_1((\widehat{\mathbf{x}} + \mathbf{1})^p)}{(k-1) \widehat{F}_0((\widehat{\mathbf{x}} + \mathbf{1})^p)} = \frac{n-k}{(k-1)(n-2)} \sum_{i=3}^n (x_i + 1)^p. \end{aligned} \quad (3.8)$$

Substituting (3.7) and (3.8) into the inequality (3.5) yields

$$\begin{aligned} \Delta_A(E_k((\mathbf{x} + \mathbf{1})^p)) &= -p(x_1 - x_2)^2 (x_1 + 1)^{p-1} (x_2 + 1)^{p-1} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad + p(x_1 - x_2) [(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &= p(x_1 - x_2)(x_1 + 1)^{p-1} (x_2 + 1)^{p-1} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad \times \left[ -(x_1 - x_2) + \left( \frac{1}{(x_2 + 1)^{p-1}} - \frac{1}{(x_1 + 1)^{p-1}} \right) \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p)}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p)} \right] \\ &= p(x_1 - x_2)^2 (x_1 + 1)^{p-1} (x_2 + 1)^{p-1} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad \times \left( -1 + \frac{p-1}{(\xi + 1)^p} \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p)}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p)} \right). \end{aligned} \quad (3.9)$$

If  $a > 1$  and  $\mathbf{x} \in (0, a^{1/p} - 1]^n$ , using  $\min\{x_1, x_2\} < \xi < \max\{x_1, x_2\}$ , we have

$$\frac{n-2}{\sum_{i=3}^n (x_i + 1)^{-p}} > \frac{n-2}{\sum_{i=3}^n (0 + 1)^{-p}} = \frac{n-k}{k-1}$$

and

$$\frac{1}{(\xi + 1)^p} \geq \frac{k - 1}{(n - k)(p - 1)}. \quad (3.10)$$

Consequently, inequalities from (3.9) to (3.10) imply  $\Delta(E_k((\mathbf{x} + \mathbf{1})^p)) \leq 0$  for  $\mathbf{x} \in (0, a^{1/p} - 1]^n$ .

If  $a < 1$  and  $\mathbf{x} \in (0, a^{-1/p} - 1]^n$ , we obtain

$$\begin{aligned} \Delta_A(E_k((\mathbf{x} + \mathbf{1})^p)) &= p(x_1 - x_2)^2(x_1 + 1)^{p-1}(x_2 + 1)^{p-1}E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad \times \left[ -1 + \frac{p-1}{(\xi + 1)^p} \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p)}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p)} \right] \\ &\leq p(x_1 - x_2)^2(x_1 + 1)^{p-1}(x_2 + 1)^{p-1}E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad \times \left[ -1 + \frac{(n-k)(p-1)}{(k-1)(n-2)(\xi + 1)^p} \sum_{i=3}^n (x_i + 1)^p \right] \\ &\leq p(x_1 - x_2)^2(x_1 + 1)^{p-1}(x_2 + 1)^{p-1}E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \\ &\quad \times \left\{ -1 + \frac{(n-k)(p-1)}{(k-1)(\xi + 1)^p} \left[ \left( \left( \frac{k-1}{(n-k)(p-1)} \right)^{1/p} - 1 \right) + 1 \right]^p \right\} \\ &= p(x_1 - x_2)^2(x_1 + 1)^{p-1}(x_2 + 1)^{p-1}E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p) \left[ \frac{1}{(\xi + 1)^p} - 1 \right] < 0. \end{aligned}$$

It is easy to obtain that, if  $p > 1$  and  $a > 1$ , then  $E_k((\mathbf{x} + \mathbf{1})^p)$  is Schur-convex on  $(0, a^{1/p} - 1]^n$ ; if  $p > 1$  and  $a < 1$ , then  $E_k((\mathbf{x} + \mathbf{1})^p)$  is Schur-concave on  $(0, a^{-1/p} - 1]^n$ .

Using (3.3) and by Definitions 2.2 and 2.3, inequalities in (3.1) to (3.2) hold. The proof of Theorem 3.1 is complete.  $\square$

**Theorem 3.2.** *Let  $2 \leq k \leq n$  and  $p \in \mathbb{R} \setminus \{0\}$ ,  $i_1, \dots, i_k \in \mathbb{N}$ , and*

$$b = \left[ \frac{1}{2} \left( \sqrt{\frac{4(4n-3k-1)(p-1)}{k-1}} + 1 - 1 \right) \right]^{1/p}, \quad 1 < p \leq 2.$$

- (1) *If  $k = n$ ,  $0 < p \leq 2$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , or if  $k = n$ ,  $p > 2$ , and  $\mathbf{x} \in ((p-1)^{1/p} - 1, \infty)^n$ , or if  $2 \leq k \leq n-1$ ,  $0 < p \leq 1$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , then*

$$\begin{aligned} \binom{n}{k} [(G(\mathbf{x}) + 1)^p + 1]^k &\leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k [(x_{i_j} + 1)^p + 1] \\ &\leq \binom{n}{k} [(A(\mathbf{x}) + 1)^p + 1]^k. \end{aligned} \quad (3.11)$$

*If  $2 \leq k \leq n$ ,  $p < 0$ , and  $\mathbf{x} \in (0, \frac{1}{|p|}]^n$ , then the double inequality (3.11) is reversed.*

- (2) *If  $2 \leq k \leq n$ ,  $p > 2$ , and  $\mathbf{x} \in (0, (p-1)^{1/p} - 1]^n$ , or if  $2 \leq k \leq n-1$ ,  $1 < p \leq 2$ , and  $\mathbf{x} \in (0, b]^n$ , or if  $2 \leq k \leq n$ ,  $p < 0$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , then*

$$\binom{n}{k} [(A(\mathbf{x}) + 1)^p + 1]^k \leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k [(x_{i_j} + 1)^p + 1]. \quad (3.12)$$

- (3) *If  $2 \leq k \leq n-1$ ,  $p > 0$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , or if  $2 \leq k \leq n$ ,  $p < 0$ , and  $\mathbf{x} \in [\frac{2}{|p|}, \infty)^n$ , then*

$$\binom{n}{k} [(G(\mathbf{x}) + 1)^p + 1]^k \leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k [(x_{i_j} + 1)^p + 1]. \quad (3.13)$$

(4) If  $k = n$ ,  $p > 2$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , then

$$[(G(\mathbf{x}) + 1)^p + 1]^n \leq \prod_{i=1}^n [(x_i + 1)^p + 1]. \quad (3.14)$$

*Proof.* When  $k = n$ , a direct differentiation yields

$$\frac{\partial E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_1} = \frac{p(x_1 + 1)^{p-1}}{(x_1 + 1)^p + 1} E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1}). \quad (3.15)$$

From Lemma 2.4, it follows that

$$\begin{aligned} & (x_1 - x_2) \left[ \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial u_1} - \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial u_2} \right] \\ &= p(x_1 - x_2) \left[ \frac{(x_1 + 1)^{p-1}}{(x_1 + 1)^p + 1} - \frac{(x_2 + 1)^{p-1}}{(x_2 + 1)^p + 1} \right] E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1}) \\ & \begin{cases} \leq 0, & 0 < p \leq 2, & \mathbf{x} \in \mathbb{R}_+^n; \\ \geq 0, & p > 2, & \mathbf{x} \in (0, (p-1)^{1/p} - 1]^n; \\ \leq 0, & p > 2, & \mathbf{x} \in ((p-1)^{1/p} - 1, \infty)^n; \\ \geq 0, & p < 0, & \mathbf{x} \in \mathbb{R}_+^n \end{cases} \end{aligned}$$

and

$$\begin{aligned} & (\ln x_1 - \ln x_2) \left[ x_1 \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_1} - x_2 \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_2} \right] \\ &= p(\ln x_1 - \ln x_2) \left[ \frac{x_1(x_1 + 1)^{p-1}}{(x_1 + 1)^p + 1} - \frac{x_2(x_2 + 1)^{p-1}}{(x_2 + 1)^p + 1} \right] E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1}) \\ & \begin{cases} \geq 0, & p > 0, & \mathbf{x} \in \mathbb{R}_+^n, \\ \leq 0, & p < 0, & \mathbf{x} \in (0, \frac{1}{|p|}]^n, \\ \geq 0, & p < 0, & \mathbf{x} \in [\frac{2}{|p|}, \infty)^n. \end{cases} \end{aligned}$$

Therefore, from Lemmas 2.1 and 2.2, we acquire

- (1) if  $0 < p \leq 2$ , then  $E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-concave on  $\mathbb{R}_+^n$ ;
- (2) if  $p > 2$ , then  $E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-convex on  $(0, (p-1)^{1/p} - 1]^n$ ;
- (3) if  $p > 2$ , then  $E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-concave on  $((p-1)^{1/p} - 1, \infty)^n$ ;
- (4) if  $p < 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-convex on  $\mathbb{R}_+^n$ ;
- (5) if  $p > 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-geometrically convex on  $\mathbb{R}_+^n$ ;
- (6) if  $p < 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-geometrically concave on  $(0, \frac{1}{|p|}]^n$ ;
- (7) if  $p < 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-geometrically convex on  $[\frac{2}{|p|}, \infty)^n$ .

From the relations (3.3) and (3.4), employing Definitions 2.2 and 2.3, we conclude inequalities in (3.11), (3.12), (3.13), and (3.14) for  $k = n$ .

If  $2 \leq k \leq n - 1$ , since

$$\begin{aligned} \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_1} &= p(x_1 + 1)^{p-1} [(x_2 + 1)^p + 1] E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + \mathbf{1}) \\ & \quad + p(x_1 + 1)^{p-1} E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p + \mathbf{1}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \Delta_A(E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})) &\triangleq (x_1 - x_2) \left[ \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_1} - \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_2} \right] \\ &= p(x_1 - x_2) \left[ \frac{(x_1 + 1)^{p-1}}{(x_1 + 1)^p + 1} - \frac{(x_2 + 1)^{p-1}}{(x_2 + 1)^p + 1} \right] \prod_{i=1}^2 [(x_i + 1)^p + 1] E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + \mathbf{1}) \end{aligned}$$



$$\begin{aligned}
& + p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& = p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1). \quad (3.16)
\end{aligned}$$

By the equation (3.16) and Lemma 2.1, it is easy to obtain that

- (1) if  $p < 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-convex on  $\mathbb{R}_+^n$ ;
- (2) if  $0 < p \leq 1$ , then  $E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-concave on  $\mathbb{R}_+^n$ ;
- (3) if  $p > 1$  and  $\mathbf{x} \in \mathbb{R}_+^n$ ,
  - (a) when  $x_1 = x_2$ , we have  $\Delta_A(E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})) = 0$ ;
  - (b) when  $x_1 \neq x_2$ , using Cauchy's mean value theorem, we have

$$\frac{\frac{(x_1+1)^{p-1}}{(x_1+1)^{p+1}} - \frac{(x_2+1)^{p-1}}{(x_2+1)^{p+1}}}{(x_1+1)^{p-1} - (x_2+1)^{p-1}} = \frac{p-1 - (\xi+1)^p}{(p-1)[(\xi+1)^p + 1]^2}$$

for some point  $\xi \in (\min\{x_1, x_2\}, \max\{x_1, x_2\})$  such that

$$\begin{aligned}
\Delta_A(E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})) & = p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& \times \left\{ \frac{\frac{(x_1+1)^{p-1}}{(x_1+1)^{p+1}} - \frac{(x_2+1)^{p-1}}{(x_2+1)^{p+1}}}{(x_1+1)^{p-1} - (x_2+1)^{p-1}} \prod_{i=1}^2 [(x_i + 1)^p + 1] + \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1)}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1)} \right\} \\
& = p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& \times \left\{ \frac{p-1 - (\xi+1)^p}{(p-1)[(\xi+1)^p + 1]^2} \prod_{i=1}^2 [(x_i + 1)^p + 1] + \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1)}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1)} \right\}.
\end{aligned}$$

If  $p > 2$  and  $x \in (0, (p-1)^{1/p} - 1]$ , then  $p-1 - (x+1)^p \geq 0$ , so  $\Delta_A(E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})) \geq 0$  for  $\mathbf{x} \in (0, (p-1)^{1/p} - 1]^n$ .

If  $1 < p \leq 2$  and  $\mathbf{x} \in (0, b]$ , we derive  $p-1 - (x+1)^p \leq 0$ . Using

$$(\min\{x_1, x_2\} + 1)^p + 1 < (\xi + 1)^p + 1 < (\max\{x_1, x_2\} + 1)^p + 1$$

and Newton's inequality lead to

$$\begin{aligned}
\Delta_A(E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})) & = p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& \times \left\{ \frac{p-1 - (\xi+1)^p}{(p-1)[(\xi+1)^p + 1]^2} \prod_{i=1}^2 [(x_i + 1)^p + 1] + \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1)}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1)} \right\} \\
& \geq p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& \times \left\{ \frac{p-1 - (\xi+1)^p}{(p-1)[(\xi+1)^p + 1]^2} \prod_{i=1}^2 [(x_i + 1)^p + 1] + \frac{n-k}{k-1} \frac{n-2}{\sum_{i=3}^n [(x_i + 1)^p + 1]^{-1}} \right\} \\
& \geq p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& \times \left\{ \frac{p-1 - (\xi+1)^p}{(p-1)[(\xi+1)^p + 1]} [(\max\{x_1, x_2\} + 1)^p + 1] + \frac{n-k}{k-1} \frac{n-2}{\sum_{i=3}^n [(x_i + 1)^p + 1]^{-1}} \right\} \\
& \geq p(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& \quad \times \left\{ \frac{p-1 - (b+1)^p}{2(p-1)} [(b+1)^p + 1] + \frac{2(n-k)}{k-1} \right\} \\
& = \frac{p}{2(p-1)}(x_1 - x_2)[(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}]E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + 1) \\
& \quad \times \left\{ -(b+1)^p[(b+1)^p + 1] + \frac{(4n-3k-1)(p-1)}{k-1} \right\} \geq 0.
\end{aligned}$$

By Lemma 2.1, if  $1 < p \leq 2$ , then  $E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-convex with respect to  $\mathbf{x} \in (0, b]^n$ ; if  $p > 2$ , then  $E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-convex on  $(0, (p-1)^{1/p} - 1]^n$ .

When  $2 \leq k \leq n-1$ , from (3.15), we obtain

$$\begin{aligned} & (\ln x_1 - \ln x_2) \left[ x_1 \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_1} - x_2 \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})}{\partial x_2} \right] = p(\ln x_1 - \ln x_2) \\ & \times \left[ \frac{x_1(x_1 + 1)^{p-1}}{(x_1 + 1)^p + 1} - \frac{x_2(x_2 + 1)^{p-1}}{(x_2 + 1)^p + 1} \right] \prod_{i=1}^2 [(x_i + 1)^p + 1] E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p + \mathbf{1}) \\ & + p(x_1 - x_2) [x_1(x_1 + 1)^{p-1} - x_2(x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p + \mathbf{1}). \quad (3.17) \end{aligned}$$

Using the equation (3.17) and Lemma 2.2, we obtain that

- (1) if  $p > 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-geometrically convex on  $\mathbb{R}_+^n$ ;
- (2) if  $p < 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-geometrically concave on  $(0, \frac{1}{|p|}]^n$ ;
- (3) if  $p < 0$ , then  $E_k((\mathbf{x} + \mathbf{1})^p + \mathbf{1})$  is Schur-geometrically convex on  $[\frac{2}{|p|}, \infty)^n$ .

By (3.3) and Lemmas 2.1 and 2.2, we arrive at inequalities (3.11) to (3.14). The proof of Theorem 3.2 is complete.  $\square$

**Theorem 3.3.** Let  $2 \leq k \leq n$ ,  $p \in \mathbb{R} \setminus \{0\}$ ,  $i_1, \dots, i_k \in \mathbb{N}$ ,

$$c = \left[ \frac{n-1 + \sqrt{(n-1)^2 + 4(n-k)(k-1)/(p-1)}}{2(n-k)} \right]^{1/p} - 1, \quad p > 1,$$

and

$$d = \left( \frac{2n+k-3}{k-1} \right)^{1/p} - 1, \quad 0 < p < 1.$$

- (1) If  $k = n$ ,  $p \geq 1$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , or if  $2 \leq k \leq n-1$ ,  $p > 1$ , and  $\mathbf{x} \in (0, c]^n$ , or if  $2 \leq k \leq n-1$ ,  $p = 1$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , then

$$\begin{aligned} \binom{n}{k} [(G(\mathbf{x}) + 1)^p - 1]^k & \leq \sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k [(x_{i_j} + 1)^p - 1] \\ & \leq \binom{n}{k} [(A(\mathbf{x}) + 1)^p - 1]^k. \quad (3.18) \end{aligned}$$

- (2) If  $k = n$ ,  $0 < p < 1$ , and  $\mathbf{x} \in \mathbb{R}_+^n$ , or if  $2 \leq k \leq n-1$ ,  $0 < p < 1$ , and  $\mathbf{x} \in (0, d]^n$ , then

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k [(x_{i_j} - 1)^p + 1] \leq \binom{n}{k} [(G(\mathbf{x}) + 1)^p - 1]^k.$$

- (3) If  $p < 0$ ,  $k = n$  is an odd integer, and  $\mathbf{x} \in \mathbb{R}_+^n$ , or if  $2 \leq k \leq n-1$ ,  $k$  is an odd integer,  $p < 0$ , and  $\mathbf{x} \in (0, \frac{1}{|p|}]^n$ , then

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} \prod_{j=1}^k [(x_{i_j} - 1)^p + 1] \leq \binom{n}{k} [(A(\mathbf{x}) + 1)^p - 1]^k. \quad (3.19)$$

If  $p < 0$ ,  $k = n$  is an even integer, and  $\mathbf{x} \in \mathbb{R}_+^n$ , or if  $2 \leq k \leq n-1$ ,  $k$  is an even integer,  $p < 0$ , and  $\mathbf{x} \in (0, \frac{1}{|p|}]^n$ , then the inequality (3.19) is reversed.

*Proof.* The proof is divided into three cases.

Case 1. If  $k = n$ , a direct differentiation yields

$$\frac{\partial E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_1} = \frac{p(x_1 + 1)^{p-1}}{(x_1 + 1)^p - 1} E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1}).$$

From Lemma 2.4, it follows that

$$\begin{aligned} & (x_1 - x_2) \left[ \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_1} - \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_2} \right] \\ &= p(x_1 - x_2) \left[ \frac{(x_1 + 1)^{p-1}}{(x_1 + 1)^p - 1} - \frac{(x_2 + 1)^{p-1}}{(x_2 + 1)^p - 1} \right] E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1}) \\ & \begin{cases} \leq 0, & \text{if } p > 0; \\ \leq 0, & \text{if } p < 0 \text{ and } n \text{ is an odd integer;} \\ \geq 0, & \text{if } p < 0 \text{ and } n \text{ is an even integer} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & (\ln x_1 - \ln x_2) \left[ x_1 \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_1} - x_2 \frac{\partial E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_2} \right] \\ &= p(\ln x_1 - \ln x_2) \left[ \frac{x_1(x_1 + 1)^{p-1}}{(x_1 + 1)^p - 1} - \frac{x_2(x_2 + 1)^{p-1}}{(x_2 + 1)^p - 1} \right] E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1}) \\ & \begin{cases} \geq 0, & \text{if } p \geq 1; \\ \leq 0, & \text{if } 0 < p < 1; \\ \leq 0, & \text{if } p < 0 \text{ and } n \text{ is an odd integer;} \\ \geq 0, & \text{if } p < 0 \text{ and } n \text{ is an even integer.} \end{cases} \end{aligned}$$

Therefore, from Lemmas 2.1 and 2.2, we have

- (1) if  $p > 0$ , then  $E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-concave on  $\mathbb{R}_+^n$ ; if  $p < 0$  and  $n$  is an odd (or even, respectively) integer, then  $E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-concave (or Schur-convex, respectively) on  $\mathbb{R}_+^n$ ;
- (2) if  $p \geq 1$ , then  $E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-geometrically convex on  $\mathbb{R}_+^n$ ; if  $0 < p < 1$ , then  $E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-geometrically concave on  $\mathbb{R}_+^n$ ; if  $p < 0$  and  $n$  is an odd (or even, respectively) integer, then  $E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-geometrically concave (or convex, respectively) on  $\mathbb{R}_+^n$ .

For  $k = n$ , by the relations (3.3) and (3.4) and by Definitions 2.2 and 2.3, the inequalities in (3.18) to (3.19) hold.

Case 2. When  $2 \leq k \leq n - 1$ , since

$$\begin{aligned} \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_1} &= p(x_1 + 1)^{p-1} [(x_2 + 1)^p - 1] E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ & \quad + p(x_1 + 1)^{p-1} E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}), \end{aligned} \quad (3.20)$$

we have

$$\begin{aligned} & \Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) \triangleq (x_1 - x_2) \left[ \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_1} - \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_2} \right] \\ &= p(x_1 - x_2) \{ (x_1 + 1)^{p-1} [(x_2 + 1)^p - 1] - [(x_1 + 1)^p - 1] (x_2 + 1)^{p-1} \} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ & \quad + p(x_1 - x_2) [(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &= p(x_1 - x_2) \left[ \frac{(x_1 + 1)^{p-1}}{(x_1 + 1)^p - 1} - \frac{(x_2 + 1)^{p-1}}{(x_2 + 1)^p - 1} \right] \prod_{i=1}^2 [(x_i + 1)^p - 1] E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ & \quad + p(x_1 - x_2) [(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}). \end{aligned} \quad (3.21)$$

Utilizing the monotonicity of  $p(x + 1)^{p-1}$  and Lemma 2.4, we obtain that

- (1) if  $0 < p \leq 1$ , then  $\Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) \leq 0$  for  $\mathbf{x} \in \mathbb{R}_+^n$ , so the function  $E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-concave function on  $\mathbb{R}_+^n$ ;

- (2) if  $p < 0$  and  $k$  is an odd (or even, respectively) integer, then  $\Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) \stackrel{\leq}{\geq} 0$  for  $\mathbf{x} \in \mathbb{R}_+^n$ . This shows from Lemma 2.1 that, if  $p < 0$  and  $n$  is an odd (or even, respectively) integer, the function  $E_n((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-concave (or Schur-convex, respectively) on  $\mathbb{R}_+^n$ ;
- (3) if  $p > 1$  and  $\mathbf{x} \in (0, c]^n$ , from (3.21), it follows that

$$\begin{aligned} \Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) &= p(x_1 - x_2) \left\{ -(x_1 - x_2)(x_1 + 1)^{p-1}(x_2 + 1)^{p-1} \right. \\ &\quad - \left. [(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}] E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \right. \\ &\quad \left. + p(x_1 - x_2) [(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \right\} \\ &= p(x_1 - x_2) E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \left\{ -(x_1 - x_2)(x_1 + 1)^{p-1}(x_2 + 1)^{p-1} \right. \\ &\quad \left. + [(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1}] \left[ \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})} - 1 \right] \right\}. \end{aligned} \quad (3.22)$$

Suppose  $x_1 \neq x_2$  and  $\mathbf{x} \in (0, c]^n$ . By Lagrange's mean value theorem, we have

$$(x_1 + 1)^{p-1} - (x_2 + 1)^{p-1} = (p-1)(x_1 - x_2)(\xi + 1)^{p-2} \quad (3.23)$$

for some  $\xi$  in the open interval  $(\min\{x_1, x_2\}, \max\{x_1, x_2\})$  and

$$\frac{(x_1 + 1)^{p-1}(x_2 + 1)^{p-1}}{(p-1)(\xi + 1)^{p-2}} \geq \frac{1}{(p-1)(\xi + 1)^p} \geq \frac{1}{(p-1)(c+1)^p}.$$

For  $k = 2, \dots, n$  and  $\mathbf{x} \in (0, c]^n$ , using Lemma 2.5 yields

$$\frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})} \leq \frac{(n-k)\widehat{F}_1((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})}{(k-1)\widehat{F}_0((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})} \leq \frac{n-k}{k-1}(c+1)^p - \frac{n-k}{k-1}. \quad (3.24)$$

For  $\mathbf{x} \in (0, c]^n$ , by the equation (3.22) and the inequalities in (3.23) and (3.24), we obtain

$$\begin{aligned} \Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) &= p(x_1 - x_2)^2 E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &\times \left\{ -(x_1 + 1)^{p-1}(x_2 + 1)^{p-1} + (p-1)(\xi + 1)^{p-2} \left[ \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})} - 1 \right] \right\} \\ &\leq p(x_1 - x_2)^2 (p-1)(\xi + 1)^{p-2} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &\times \left[ -\frac{(x_1 + 1)^{p-1}(x_2 + 1)^{p-1}}{(p-1)(\xi + 1)^{p-2}} + \frac{n-k}{(k-1)(n-2)} \sum_{i=3}^n (x_i + 1)^p - \frac{n-1}{k-1} \right] \\ &\leq p(x_1 - x_2)^2 (p-1)(\xi + 1)^{p-2} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &\times \left[ -\frac{1}{(p-1)(c+1)^p} + \frac{n-k}{k-1}(c+1)^p - \frac{n-1}{k-1} \right] = 0. \end{aligned}$$

Therefore, if  $p > 1$ , then  $\Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) \leq 0$  for  $\mathbf{x} \in (0, c]^n$ . So, from Lemma 2.1, it follows that

- (1) if  $0 < p \leq 1$ , the function  $E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-concave on  $\mathbb{R}_+^n$ ;
- (2) if  $p < 0$  and  $k$  is an odd (or even, respectively) integer, the function  $E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-concave (or Schur-convex, respectively) on  $\mathbb{R}_+^n$ ;
- (3) if  $p > 1$ , the function  $E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-concave on  $(0, c]^n$ .

Case 3. If  $2 \leq k \leq n-1$ , then from (3.20) it follows that

$$\begin{aligned} \Delta_G(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) &\triangleq (\ln x_1 - \ln x_2) \left[ x_1 \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_1} \right. \\ &\left. - x_2 \frac{\partial E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})}{\partial x_2} \right] = p(\ln x_1 - \ln x_2) \left[ \frac{x_1(x_1 + 1)^{p-1}}{(x_1 + 1)^p - 1} - \frac{x_2(x_2 + 1)^{p-1}}{(x_2 + 1)^p - 1} \right] \end{aligned}$$

$$\begin{aligned} & \times \prod_{i=1}^2 [(x_i + 1)^p - 1] E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ & + p(\ln x_1 - \ln x_2) [x_1(x_1 + 1)^{p-1} - x_2(x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}). \end{aligned}$$

Using the monotonicity of function  $px(x+1)^{p-1}$  and Lemma 2.4 results in

- (1) if  $p \geq 1$ , then  $\Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) \geq 0$  for  $\mathbf{x} \in \mathbb{R}_+^n$ ;
- (2) if  $p < 0$  and  $k$  is an odd (or even, respectively) integer, then  $\Delta(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) \leq 0$  for  $\mathbf{x} \in (0, \frac{1}{|p|}]^n$ ;
- (3) if  $0 < p < 1$ , then

$$\begin{aligned} \Delta_G(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) &= p(\ln x_1 - \ln x_2) \{x_1(x_1 + 1)^{p-1} [(x_2 + 1)^p - 1] \\ &\quad - [(x_2 + 1)^p - 1] x_2(x_2 + 1)^{p-1}\} E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &\quad + p(\ln x_1 - \ln x_2) [x_1(x_1 + 1)^{p-1} - x_2(x_2 + 1)^{p-1}] E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &= p(\ln x_1 - \ln x_2) E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \left\{ (x_1 - x_2)(x_1 + 1)^{p-1}(x_2 + 1)^{p-1} \right. \\ &\quad \left. + [x_1(x_1 + 1)^{p-1} - x_2(x_2 + 1)^{p-1}] \left[ \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})} - 1 \right] \right\}. \quad (3.25) \end{aligned}$$

Suppose  $x_1 \neq x_2$ , by Lagrange's mean value theorem, we have

$$x_1(x_1 + 1)^{p-1} - x_2(x_2 + 1)^{p-1} = (x_1 - x_2)(\xi + 1)^{p-2}(1 + p\xi) \quad (3.26)$$

for some  $\xi \in (\min\{x_1, x_2\}, \max\{x_1, x_2\})$ . Therefore, using Lemma 2.5 leads to

$$\frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})} \leq \frac{n-k}{(k-1)(n-2)} \sum_{i=3}^n (x_i + 1)^p - \frac{n-k}{k-1}, \quad \mathbf{x} \in \mathbb{R}_+^n. \quad (3.27)$$

For  $\mathbf{x} \in (0, d]^n$ , substituting (3.26) and (3.27) into (3.25) yields

$$\begin{aligned} \Delta_G(E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})) &= p(\ln x_1 - \ln x_2)(x_1 - x_2) E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &\times \left\{ (x_1 + 1)^{p-1}(x_2 + 1)^{p-1} + (\xi + 1)^{p-2}(1 + p\xi) \left[ \frac{E_{k-1}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})}{E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1})} - 1 \right] \right\} \\ &\leq p(\ln x_1 - \ln x_2)(x_1 - x_2) E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \left\{ (x_1 + 1)^{p-1}(x_2 + 1)^{p-1} \right. \\ &\quad \left. + (\xi + 1)^{p-2}(1 + p\xi) \left[ \frac{n-k}{(k-1)(n-2)} \sum_{i=3}^n (x_i + 1)^p - \frac{n-1}{k-1} \right] \right\} \\ &= p(\ln x_1 - \ln x_2)(x_1 - x_2)(\xi + 1)^{p-2}(1 + p\xi) E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &\times \left[ \frac{(x_1 + 1)^{p-1}(x_2 + 1)^{p-1}}{(\xi + 1)^{p-2}(1 + p\xi)} + \frac{n-k}{(k-1)(n-2)} \sum_{i=3}^n (x_i + 1)^p - \frac{n-1}{k-1} \right] \\ &\leq p(\ln x_1 - \ln x_2)(x_1 - x_2)(\xi + 1)^{p-2}(1 + p\xi) E_{k-2}((\widehat{\mathbf{x}} + \mathbf{1})^p - \mathbf{1}) \\ &\quad \times \left[ (d+1)^{2p} + \frac{n-k}{k-1}(d+1)^p - \frac{n-1}{k-1} \right] = 0. \end{aligned}$$

Therefore, using Lemma 2.2, we obtain that

- (1) if  $p \geq 1$ , the function  $E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-geometrically convex on  $\mathbb{R}_+^n$ ;
- (2) if  $p < 0$  and  $k$  is an odd (or even, respectively) integer, the function  $E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-geometrically concave (or convex, respectively) function on  $(0, \frac{1}{|p|}]^n$ ;
- (3) if  $0 < p < 1$ , the function  $E_k((\mathbf{x} + \mathbf{1})^p - \mathbf{1})$  is Schur-geometrically concave on  $(0, d]^n$ .

For  $k = n$ , by the relations (3.3) and (3.4) and by Definitions 2.2 and 2.3, the inequalities in (3.18) and (3.19) hold. Theorem 3.3 is thus proved.  $\square$

**Acknowledgements.** The first three authors were partially supported by the National Natural Science Foundation of China under Grant No. 11361038, by the Foundation of the Research Program of Science and Technology at Universities of Inner Mongolia Autonomous Region under Grant No. NJZZ18154, and by the Inner Mongolia Autonomous Region Natural Science Foundation Project under Grant No. 2018LH01002, China.

#### REFERENCES

- [1] P. S. Bullen, *Handbook of Means and Their Inequalities*, Revised from the 1988 original [P. S. Bullen, D. S. Mitrinović and P. M. Vasić, Means and their inequalities, Reidel, Dordrecht], Mathematics and its Applications **560**, Kluwer Academic Publishers Group, Dordrecht, 2003; Available online at <https://doi.org/10.1007/978-94-017-0399-4>.
- [2] C.-P. Chen, F. Qi, and H. M. Srivastava, *Some properties of functions related to the gamma and psi functions*, Integral Transforms Spec. Funct. **21** (2010), no. 2, 153–164; Available online at <http://dx.doi.org/10.1080/10652460903064216>.
- [3] Y.-M. Chu, X.-M. Zhang, and G.-D. Wang, *The Schur geometrical convexity of the extended mean values*, J. Convex Anal. **15** (2008), no. 4, 707–718.
- [4] K.-Z. Guan, *Schur-convexity of the complete symmetric function*, Math. Inequal. Appl. **9** (2006), no. 4, 567–576; Available online at <https://doi.org/10.7153/mia-09-52>.
- [5] B.-N. Guo and F. Qi, *On the degree of the weighted geometric mean as a complete Bernstein function*, Afr. Mat. **26** (2015), no. 7, 1253–1262; Available online at <http://dx.doi.org/10.1007/s13370-014-0279-2>.
- [6] B.-N. Guo, F. Qi, and Q.-M. Luo, *The additivity of polygamma functions*, Filomat **29** (2015), no. 5, 1063–1066; Available online at <http://dx.doi.org/10.2298/FIL1505063G>.
- [7] S. Guo and F. Qi, *A class of completely monotonic functions related to the remainder of Binet's formula with applications*, Tamsui Oxf. J. Math. Sci. **25** (2009), no. 1, 9–14.
- [8] J.-C. Kuang, *Chángyòng Bùdēngshì (Applied Inequalities)*, 4th ed., Shandong Science and Technology Press, Ji'nan, China, 2010. (Chinese)
- [9] W.-H. Li and F. Qi, *Some Hermite-Hadamard type inequalities for functions whose  $n$ -th derivatives are  $(\alpha, m)$ -convex*, Filomat **27** (2013), no. 8, 1575–1582; Available online at <http://dx.doi.org/10.2298/FIL1308575L>.
- [10] A.-Q. Liu, G.-F. Li, B.-N. Guo, and F. Qi, *Monotonicity and logarithmic concavity of two functions involving exponential function*, Internat. J. Math. Ed. Sci. Tech. **39** (2008), no. 5, 686–691; Available online at <http://dx.doi.org/10.1080/00207390801986841>.
- [11] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and its Applications*, 2nd Ed., Springer Verlag, New York-Dordrecht-Heidelberg-London, 2011; Available online at <http://dx.doi.org/10.1007/978-0-387-68276-1>.
- [12] D. S. Mitrinović, *Analytic Inequalities*. Springer-Verlag, Berlin and New York, 1970.
- [13] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993; Available online at <http://dx.doi.org/10.1007/978-94-017-1043-5>.
- [14] F. Qi and B.-N. Guo, *Some properties of extended remainder of Binet's first formula for logarithm of gamma function*, Math. Slovaca **60** (2010), no. 4, 461–470; Available online at <http://dx.doi.org/10.2478/s12175-010-0025-7>.
- [15] F. Qi, S. Guo, and B.-N. Guo, *Complete monotonicity of some functions involving polygamma functions*, J. Comput. Appl. Math. **233** (2010), no. 9, 2149–2160; Available online at <http://dx.doi.org/10.1016/j.cam.2009.09.044>.
- [16] F. Qi and D. Lim, *Integral representations of bivariate complex geometric mean and their applications*, J. Comput. Appl. Math. **330** (2018), 41–58; Available online at <https://doi.org/10.1016/j.cam.2017.08.005>.
- [17] F. Qi, D.-W. Niu, and B.-N. Guo, *Monotonic properties of differences for remainders of psi function*, Int. J. Pure Appl. Math. Sci. **4** (2007), no. 1, 59–66.
- [18] F. Qi, X.-J. Zhang, and W.-H. Li, *An elementary proof of the weighted geometric mean being a Bernstein function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **77** (2015), no. 1, 35–38.
- [19] F. Qi, X.-J. Zhang, and W.-H. Li, *An integral representation for the weighted geometric mean and its applications*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 1, 61–68; Available online at <https://doi.org/10.1007/s10114-013-2547-8>.

- [20] F. Qi, X.-J. Zhang, and W.-H. Li, *Lévy-Khintchine representation of the geometric mean of many positive numbers and applications*, *Math. Inequal. Appl.* **17** (2014), no. 2, 719–729; Available online at <http://dx.doi.org/10.7153/mia-17-53>.
- [21] F. Qi, X.-J. Zhang, and W.-H. Li, *Lévy-Khintchine representations of the weighted geometric mean and the logarithmic mean*, *Mediterr. J. Math.* **11** (2014), no. 2, 315–327; Available online at <http://dx.doi.org/10.1007/s00009-013-0311-z>.
- [22] F. Qi, X.-J. Zhang, and W.-H. Li, *The harmonic and geometric means are Bernstein functions*, *Bol. Soc. Mat. Mex. (3)* **23** (2017), no. 2, 713–736; Available online at <https://doi.org/10.1007/s40590-016-0085-y>.
- [23] T.-C. Sun, Y.-P. Lü, and Y.-M. Chu, *Schur multiplicative and harmonic convexities of generalized Heronian mean in  $n$  variables and their applications*, *Int. J. Pure Appl. Math.* **55** (2009), no. 1, 25–33.
- [24] W.-L. Wang, *Some inequalities for symmetric functions and applications*, *J. Ningbo Univ.* **8** (1995), no. 3, 27–29. (Chinese)
- [25] B.-Y. Wang, *Foundations of Majorization Inequalities*, Beijing Normal University Press, Beijing, 1990. (Chinese)
- [26] Z. Wang and J. Chen, *Simple proof and application of a geometric inequality*, *J. Ningbo Univ.* **8** (1995), no. 1, 70–72. (Chinese)
- [27] X.-M. Zhang, *Geometrically Convex Functions*, Anhui University Press, Hefei, 2004. (Chinese)

(Xi) COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO, INNER MONGOLIA, 028043, CHINA

*Email address:* [baoyintu78@qq.com](mailto:baoyintu78@qq.com), [baoyintu68@sohu.com](mailto:baoyintu68@sohu.com), [baoyintu78@imn.edu.cn](mailto:baoyintu78@imn.edu.cn)

(Wu) COLLEGE OF MATHEMATICS, INNER MONGOLIA UNIVERSITY FOR NATIONALITIES, TONGLIAO, INNER MONGOLIA, 028043, CHINA

*Email address:* [wuying19800920@qq.com](mailto:wuying19800920@qq.com)

(Shi) DEPARTMENT OF ELECTRONIC INFORMATION, TEACHER'S COLLEGE, BEIJING UNION UNIVERSITY, 100011, CHINA

*Email address:* [shihuannan2014@qq.com](mailto:shihuannan2014@qq.com); [sfthuannan@buu.edu.cn](mailto:sfthuannan@buu.edu.cn)

(Qi) INSTITUTE OF MATHEMATICS, HENAN POLYTECHNIC UNIVERSITY, JIAOZUO, HENAN, 454010, CHINA; DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, TIANJIN POLYTECHNIC UNIVERSITY, TIANJIN, 300387, CHINA; INSTITUTE OF FUNDAMENTAL AND FRONTIER SCIENCES, UNIVERSITY OF ELECTRONIC SCIENCE AND TECHNOLOGY OF CHINA, CHENGDU, SICHUAN, 610054, CHINA

*Email address:* [qifeng618@gmail.com](mailto:qifeng618@gmail.com), [qifeng618@hotmail.com](mailto:qifeng618@hotmail.com), [qifeng618@qq.com](mailto:qifeng618@qq.com)

*URL:* <http://qifeng618.wordpress.com>