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# Families of solutions to the KPI equation and the structure of their rational representations of order $N$ .

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## Abstract

We construct solutions to the Kadomtsev-Petviashvili equation (KPI) in terms of Fredholm determinants. We deduce solutions written as a quotient of wronskians of order  $2N$ . These solutions called solutions of order  $N$  depend on  $2N - 1$  parameters.

They can also be written as a quotient of two polynomials of degree  $2N(N+1)$  in  $x$ ,  $y$  and  $t$  depending on  $2N - 2$  parameters. The maximum of the modulus of these solutions at order  $N$  is equal to  $2(2N + 1)^2$ .

We explicitly construct the expressions until the order 6 and we study the patterns of their modulus in the plane  $(x, y)$  and their evolution according to time and parameters.

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## 1 Introduction

We consider the Kadomtsev-Petviashvili equation (KP) which can be written in the form

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \quad (1)$$

subscripts  $x$ ,  $y$  and  $t$  denoting partial derivatives.

Kadomtsev and Petviashvili [1] first proposed that equation in 1970. That equation is considered as a model, for example, for surface and internal water waves [2], and in nonlinear optics [3]. In 1974 Dryuma showed how the KP equation could be written in Lax form [4]. Zakharov extended the inverse scattering transform (IST) to this KPI equation, and obtained several exact solutions.

The first rational solutions were found by Manakov, Zakharov, Bordag and Matveev published in november 1977 [5] and at the same time by Krichever published in january 1978 [6].

From the end of the seventies, a lot of methods have been carried out to solve that equation. Krichever constructed for the first time in 1976 published in [7] the solutions to KPI given in terms of Riemann theta functions in the frame of algebraic geometry. Practically at the same time, a more general study in this frame was realized by Dubrovin [8].

Various researches were conducted and more general rational solutions of the KPI equation were obtained. Among many works, one can quote of the studies of Krichever in 1978 [9], Satsuma and Ablowitz in 1979 [10], Matveev in 1979 [11], Freeman and Nimmo in 1983 [12, 13], Matveev in 1987 [14], Pelinovsky and Stepanyants in 1993 [15], Pelinovsky in 1994 [16], Ablowitz, Villarroel, Chakravarty, Trubatch [17, 18, 19] in 1997-2000, Biondini and Kodama [20, 21, 22] in 2003-2007.

In the following, we recall the results of the author about the representations of solutions to the KPI equation. We have expressed the solutions in terms of Fredholm determinants of order  $2N$  depending on  $2N - 1$  parameters. We have also given another representation in terms of wronskians of order  $2N$  with  $2N - 1$  parameters. These representations allow to obtain an infinite hierarchy of solutions to the KPI equation, depending on  $2N - 1$  real parameters .

We have used these results to build rational solutions to the KPI equation, making a parameter to 0 tend to 0.

Rational solutions of order  $N$  depending on  $2N - 2$  parameters without the presence of a limit have been constructed. These families depending on  $2N - 2$  parameters for the  $N$ -th order can be written as a ratio of two polynomials of  $x, y$  and  $t$  of degree  $2N(N + 1)$ .

We prove that the maximum of the modulus of those solutions at order  $N$  is equal to  $2(2N + 1)^2$ . That provides an effective method to build an infinite hierarchy of rational solutions of order  $N$  depending on  $2N - 2$  real parameters. We present here the representations of their modulus in the plane of the coordinates  $(x, y)$  according to the  $2N - 2$  real parameters  $a_i$  and  $b_1$  and time  $t$  for  $N$  between 2 and 6.

## **2 Rational solutions to KPI equation of order $N$ depending on $2N - 2$ parameters**

### **2.1 Families of rational solutions of order $N$ depending on $2N - 2$ parameters**

We need to define some notations. First one defines real numbers  $\lambda_j$  such that  $-1 < \lambda_\nu < 1, \nu = 1, \dots, 2N$  depending on a parameter  $\epsilon$  which will be intended

to tend towards 0; they can be written as

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N, \quad (2)$$

The terms  $\kappa_\nu$ ,  $\delta_\nu$ ,  $\gamma_\nu$ ,  $\tau_\nu$  and  $x_{r,\nu}$  are functions of  $\lambda_\nu$ ,  $1 \leq \nu \leq 2N$ ; they are defined by the formulas :

$$\begin{aligned} \kappa_j &= 2\sqrt{1 - \lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1 - \lambda_j}{1 + \lambda_j}}; \\ x_{r,j} &= (r - 1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad r = 1, 3, \quad \tau_j = -12i\lambda_j^2 \sqrt{1 - \lambda_j^2} - 4i(1 - \lambda_j^2) \sqrt{1 - \lambda_j^2}, \\ \kappa_{N+j} &= \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1}, \\ x_{r,N+j} &= -x_{r,j}, \quad \tau_{N+j} = \tau_j \quad j = 1, \dots, N. \end{aligned} \quad (3)$$

$e_\nu$   $1 \leq \nu \leq 2N$  are defined in the following way :

$$\begin{aligned} e_j &= 2i \left( \sum_{k=1}^{1/2 M-1} a_k (je)^{2k+1} - i \sum_{k=1}^{1/2 M-1} b_k (je)^{2k+1} \right), \\ e_{N+j} &= 2i \left( \sum_{k=1}^{1/2 M-1} a_k (je)^{2k+1} + i \sum_{k=1}^{1/2 M-1} b_k (je)^{2k+1} \right), \quad 1 \leq j \leq N, \\ a_k, b_k &\in \mathbf{R}, \quad 1 \leq k \leq N - 1. \end{aligned} \quad (4)$$

$\epsilon_\nu$ ,  $1 \leq \nu \leq 2N$  are real numbers defined by :

$$\epsilon_j = 1, \quad \epsilon_{N+j} = 0 \quad 1 \leq j \leq N. \quad (5)$$

Let  $I$  be the unit matrix and  $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$  the matrix defined by :

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left( \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \quad (6)$$

Then we recall the following result<sup>1</sup> :

**Theorem 2.1** *The function  $v$  defined by*

$$v(x, y, t) = -2 \frac{|n(x, y, t)|^2}{d(x, y, t)^2} \quad (7)$$

where

$$n(x, y, t) = \det(I + D_3(x, y, t)), \quad (8)$$

$$d(x, y, t) = \det(I + D_1(x, y, t)), \quad (9)$$

and  $D_r = (d_{jk})_{1 \leq j, k \leq 2N}$  the matrix

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left( \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \quad (10)$$

is a solution to the KPI equation (1), dependent on  $2N - 1$  parameters  $a_k$ ,  $b_k$ ,  $1 \leq k \leq N - 1$  and  $\epsilon$ .

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<sup>1</sup>The proof of this result has been given by the author [50, 53]

We recall a second result on the solutions to KPI equation obtained recently by the author in terms of wronskians. We need to define the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3, \quad (11)$$

with the arguments

$$\Theta_{r,\nu} = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}, \quad 1 \leq \nu \leq 2N. \quad (12)$$

We denote  $W_r(w)$  the wronskian of the functions  $\phi_{r,1}, \dots, \phi_{r,2N}$  defined by

$$W_r(w) = \det[(\partial_w^{\mu-1} \phi_{r,\nu})_{\nu, \mu \in [1, \dots, 2N]}]. \quad (13)$$

We consider the matrix  $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$  defined in (10). Then we have the following statement<sup>2</sup>

**Theorem 2.2** *The function  $v$  defined by*

$$v(x, y, t) = -2 \frac{|W_3(\phi_{3,1}, \dots, \phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1}, \dots, \phi_{1,2N})(0))^2}$$

*is a solution to the KPI equation depending on  $2N - 1$  real parameters  $a_k, b_k$   $1 \leq k \leq N - 1$  and  $\epsilon$ , with  $\phi_\nu^r$  defined by*

$$\begin{aligned} \phi_{r,\nu}(w) &= \sin\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad 1 \leq \nu \leq N, \\ \phi_{r,\nu}(w) &= \cos\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad N+1 \leq \nu \leq 2N, \quad r = 1, 3, \end{aligned}$$

$\kappa_\nu, \delta_\nu, x_{r,\nu}, \gamma_\nu, e_\nu$  being defined in(3), (2) and (4).

From those two preceding results, we construct rational solutions to the KPI equation as a quotient of two determinants.

We use the following notations :

$$\begin{aligned} X_\nu &= \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{3,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2}, \\ Y_\nu &= \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{1,\nu}}{2} - i\frac{\tau_\nu}{2}t - i\frac{e_\nu}{2}, \end{aligned}$$

for  $1 \leq \nu \leq 2N$ , with  $\kappa_\nu, \delta_\nu, x_{r,\nu}$  defined in (3) and parameters  $e_\nu$  defined by (4).

We define the following functions :

$$\begin{aligned} \varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} &= \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} &= -\gamma_k^{4j+2} \cos X_k, \end{aligned} \quad (14)$$

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<sup>2</sup>The proof of this result has been given [50, 53].

for  $1 \leq k \leq N$ , and

$$\begin{aligned}\varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin X_{N+k},\end{aligned}\quad (15)$$

for  $1 \leq k \leq N$ .

We define the functions  $\psi_{j,k}$  for  $1 \leq j \leq 2N$ ,  $1 \leq k \leq 2N$  in the same way, the term  $X_k$  is only replaced by  $Y_k$ .

$$\begin{aligned}\psi_{4j+1,k} &= \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} &= \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} &= -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} &= -\gamma_k^{4j+2} \cos Y_k,\end{aligned}\quad (16)$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned}\psi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin Y_{N+k},\end{aligned}\quad (17)$$

for  $1 \leq k \leq N$ .

The following ratio

$$q(x, t) := \frac{W_3(0)}{W_1(0)}$$

can be written as

$$q(x, t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,k \in [1,2N]}}{\det(\psi_{j,k})_{j,k \in [1,2N]}}.\quad (18)$$

Then we get the following result <sup>3</sup>:

**Theorem 2.3** *The function  $v$  defined by*

$$v(x, y, t) = -2 \frac{|\det((n_{jk})_{j,k \in [1,2N]})|^2}{\det((d_{jk})_{j,k \in [1,2N]})^2}\quad (19)$$

is a rational solution to the KPI equation (1).

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0,$$

where

$$\begin{aligned}n_{j1} &= \varphi_{j,1}(x, y, t, 0), & 1 \leq j \leq 2N & & n_{jk} &= \frac{\partial^{2k-2} \varphi_{j,1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ n_{jN+1} &= \varphi_{j,N+1}(x, y, t, 0), & 1 \leq j \leq 2N & & n_{jN+k} &= \frac{\partial^{2k-2} \varphi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ d_{j1} &= \psi_{j,1}(x, y, t, 0), & 1 \leq j \leq 2N & & d_{jk} &= \frac{\partial^{2k-2} \psi_{j,1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ d_{jN+1} &= \psi_{j,N+1}(x, y, t, 0), & 1 \leq j \leq 2N & & d_{jN+k} &= \frac{\partial^{2k-2} \psi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, y, t, 0), \\ & & 2 \leq k \leq N, & & 1 \leq j \leq 2N & \end{aligned}\quad (20)$$

The functions  $\varphi$  and  $\psi$  are defined in (14), (15), (16), (17).

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<sup>3</sup>This result has been proved by the author and submitted to a review.

. The structure of the solutions to the KPI equation is given by the following result <sup>4</sup>:

**Theorem 2.4** *The function  $v$  defined by*

$$v(x, y, t) = -2 \frac{\left| \det((n_{jk})_{j,k \in [1, 2N]}) \right|^2}{\left( \det((d_{jk})_{j,k \in [1, 2N]}) \right)^2} \quad (21)$$

*is a rational solution to the KPI equation (1) quotient of two polynomials  $n(x, y, t)$  and  $d(x, y, t)$  depending on  $2N - 2$  real parameters  $a_j$  and  $b_j$ ,  $1 \leq j \leq N - 1$ .*

*$n$  and  $d$  are polynomials of degrees  $2N(N + 1)$  in  $x, y$  and  $t$ .*

*The terms  $n_{jk}$  and  $d_{jk}$  are defined by (20) and the functions  $\varphi$  and  $\psi$  are defined in (14), (15), (16), (17).*

### 3 The highest amplitude of the modulus of the N-order solution to the KPI equation equal to $2(2N + 1)^2$

#### 3.1 Another wronskian representation of solutions to KPI equation

We have recalled in the previous section a recent result obtained by the author (2.2).

We choose here to give another representation of the solutions to the KPI equation depending only on terms  $\gamma_\nu$ ,  $1 \leq \nu \leq 2N$ . For this, we express the terms  $\kappa_\nu$ ,  $\delta_\nu$ ,  $\tau_\nu$  and  $x_{r,\nu}$  in function of  $\gamma_\nu$ , for  $1 \leq \nu \leq 2N$  and we obtain :

$$\begin{aligned} \kappa_j &= \frac{4\gamma_j}{(1+\gamma_j^2)}, & \delta_j &= \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, & x_{r,j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, & \tau_j &= \frac{-8i\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3}, \\ 1 \leq j \leq N, \\ \kappa_j &= \frac{4\gamma_j}{(1+\gamma_j^2)}, & \delta_j &= -\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, & x_{r,j} &= (r-1) \ln \frac{\gamma_j + i}{\gamma_j - i}, & \tau_j &= \frac{-8i\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3}, \\ N+1 \leq j \leq 2N. \end{aligned}$$

We have the following representation :

**Theorem 3.1** *The function  $v$  defined by*

$$v(x, y, t) = -2 \frac{\left| \det[(\partial_w^{\mu-1} \tilde{\phi}_{3,\nu}(0))_{\nu, \mu \in [1, \dots, 2N]}] \right|^2}{\left( \det[(\partial_w^{\mu-1} \tilde{\phi}_{1,\nu}(0))_{\nu, \mu \in [1, \dots, 2N]}] \right)^2} \quad (22)$$

*is solution to the KPI equation (1).*

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<sup>4</sup>This result has been also proved by the author and submitted to a review.

The functions  $\tilde{\phi}_{r,\nu}$  are defined by

$$\begin{aligned}\tilde{\phi}_{r,j}(w) &= \sin\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - 4\frac{\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3}y - i\frac{(r-1)}{2}\ln\frac{\gamma_j-i}{\gamma_j+i} + \gamma_j w - i\frac{e_j}{2}\right), \\ \tilde{\phi}_{r,N+j}(w) &= \cos\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - 4\frac{\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3}y + i\frac{(r-1)}{2}\ln\frac{\gamma_j-i}{\gamma_j+i} + \frac{1}{\gamma_j}w - i\frac{e_{N+j}}{2}\right), \\ 1 \leq j, &\leq N, \\ &\text{with} \\ e_j &= 2i\left(\sum_{k=1}^{1/2M-1} a_k(je)^{2k-1} - i\sum_{k=1}^{1/2M-1} b_k(je)^{2k-1}\right), \\ e_{N+j} &= 2i\left(\sum_{k=1}^{1/2M-1} a_k(je)^{2k-1} + i\sum_{k=1}^{1/2M-1} b_k(je)^{2k-1}\right), \quad 1 \leq j \leq N, \\ a_k, b_k &\in \mathbf{R}, \quad 1 \leq k \leq N.\end{aligned}\tag{23}$$

**Remark 3.1** In the formula (22), the determinants  $\det[(\partial_w^{\mu-1} f_\nu(0))_{\nu, \mu \in [1, \dots, 2N]}]$  are the wronskians of the functions  $f_1, \dots, f_{2N}$  evaluated in  $w = 0$ . In particular  $\partial_w^0 f_\nu$  means  $f_\nu$ .

### 3.2 The highest amplitude of the modulus of the solution to the KPI equation

There is any freedom to choose  $\gamma_j$  in such a way that the conditions on  $\lambda_j$  are checked. In order to get the more simple expressions in the determinants, we choose particular solutions in the previous families.

Here we choose  $\gamma_j = j\epsilon$  for  $1 \leq j \leq N$  as simple as possible in order to have the conditions on  $\lambda_j$  checked.

**Theorem 3.2** The function  $v_0$  defined by

$$v_0(x, y, t) = -2 \left( \frac{|\det((n_{jk})_{j,k \in [1, 2N]})|^2}{(\det((d_{jk})_{j,k \in [1, 2N]}))^2} \right)_{(a_j=b_j=0, 1 \leq j \leq N-1)}\tag{24}$$

is the solution of order  $N$  solution to the KPI equation (1) whose highest amplitude in modulus is equal to  $2(2N+1)^2$ .

**Remark 3.2** In (24), the matrices  $(n_{jk})_{j,k \in [1, 2N]}$  and  $(d_{jk})_{j,k \in [1, 2N]}$  are defined in (20).

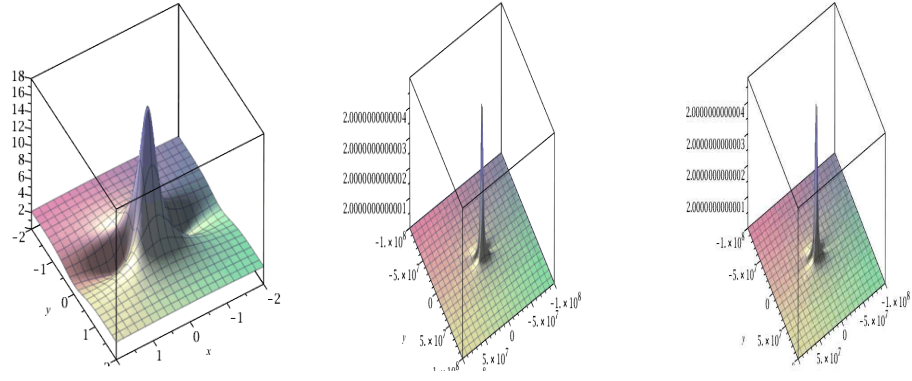
## 4 Study of the patterns of the modulus of the rational solutions to the KPI equation in function of parameters and time

We have explicitly constructed rational solutions to the KPI equation of order  $N$  depending on  $2N-2$  parameters for  $1 \leq N \leq 6$ .

In the following, we only give patterns of the modulus of the solutions in the plane  $(x, y)$  of coordinates in function of the parameters  $a_i$ , and  $b_i$ , for  $1 \leq i \leq N-1$  for  $2 \leq N \leq 6$ , and time  $t$ .

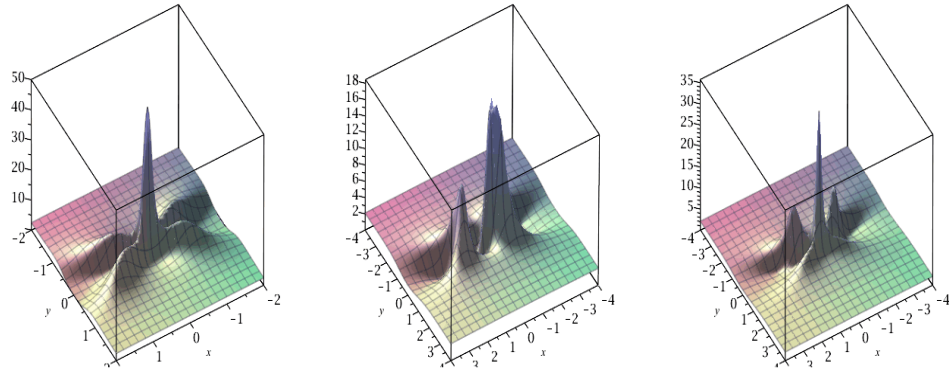


#### 4.1 Case $N = 1$

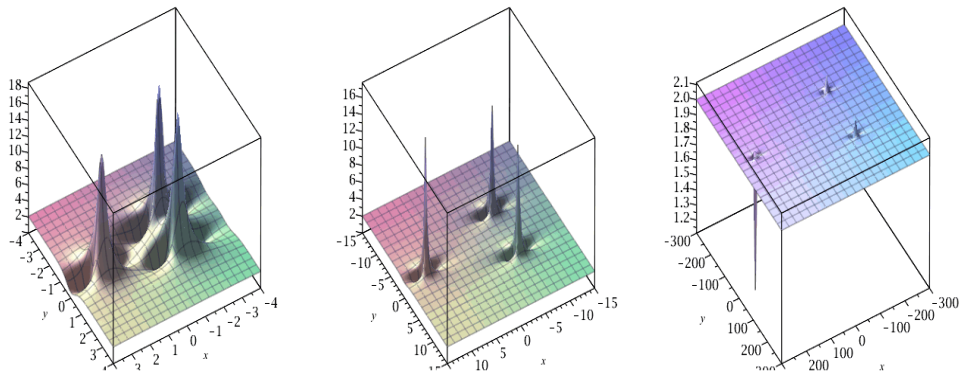


**Figure 1.** Solution of order 1 to KPI, on the left for  $t = 0$ ; in the center for  $t = 10^4$ ; on the right for  $t = 10^8$ .

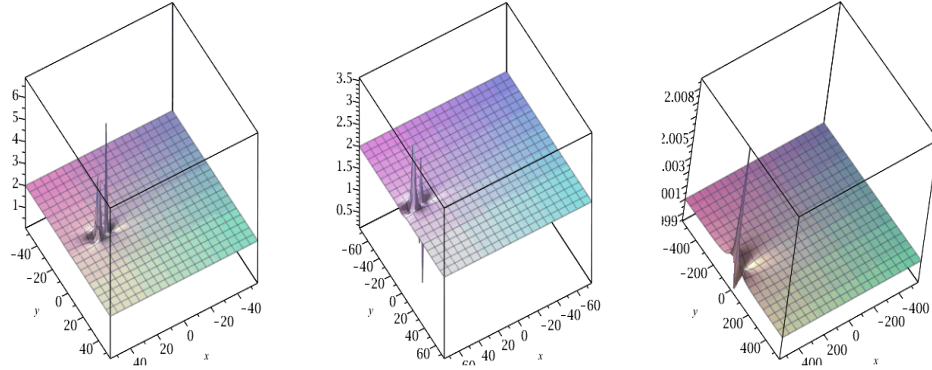
#### 4.2 Case $N = 2$



**Figure 2.** Solution of order 2 to KPI for  $t = 0$ , on the left  $a_1 = 0, b_1 = 0$ ; in the center  $a_1 = 10, b_1 = 0$ ; on the right  $a_1 = 10, b_1 = 10$ .

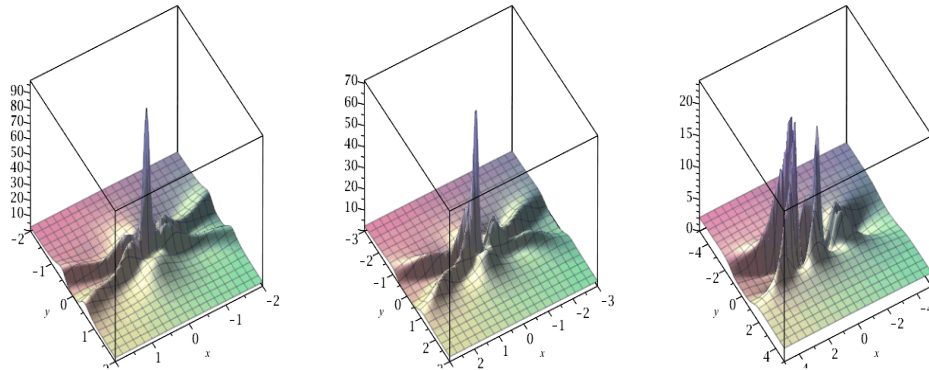


**Figure 3.** Solution of order 2 to KPI for  $t = 0$ , on the left  $a_1 = 10^2$ ,  $b_1 = 0$ ; in the center  $a_1 = 10^4$ ,  $b_1 = 0$ ; on the right for  $t = 10$ ,  $a_1 = 10^8$ ,  $b_1 = 0$ .

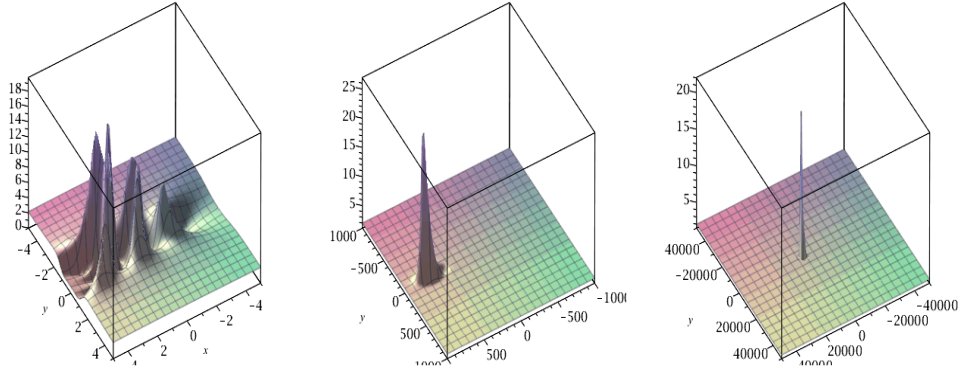


**Figure 4.** Solution of order 2 to KPI, on the left for  $t = 5$ ,  $a_1 = 0$ ,  $b_1 = 0$ ; in the center for  $t = 10$ ,  $a_1 = 0$ ,  $b_1 = 0$ ; on the right for  $t = 100$ ,  $a_1 = 0$ ,  $b_1 = 0$ .

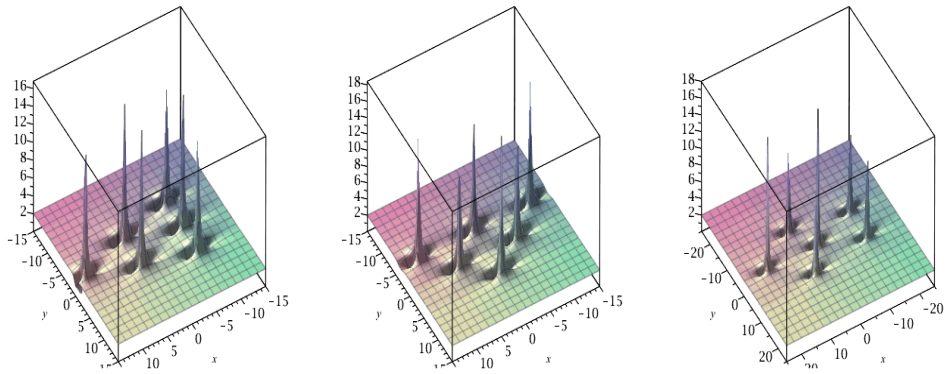
### 4.3 Case $N = 3$



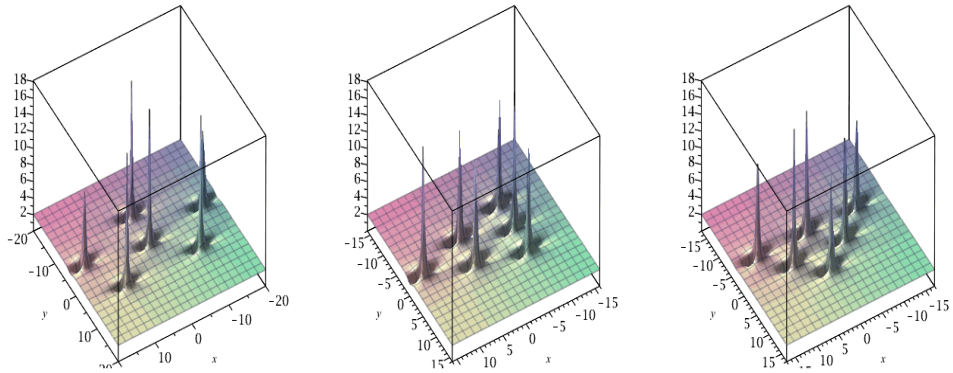
**Figure 5.** Solution of order 3 to KPI, on the left for  $t = 0$ ; in the center for  $t = 0, 01$ ; on the right for  $t = 0, 1$ ; all the parameters are equal to 0.



**Figure 6.** Solution of order 3 to KPI, on the left for  $t = 0, 2$ ; in the center for  $t = 10^2$ ; on the right for  $t = 10^3$ ; all the parameters are equal to 0.

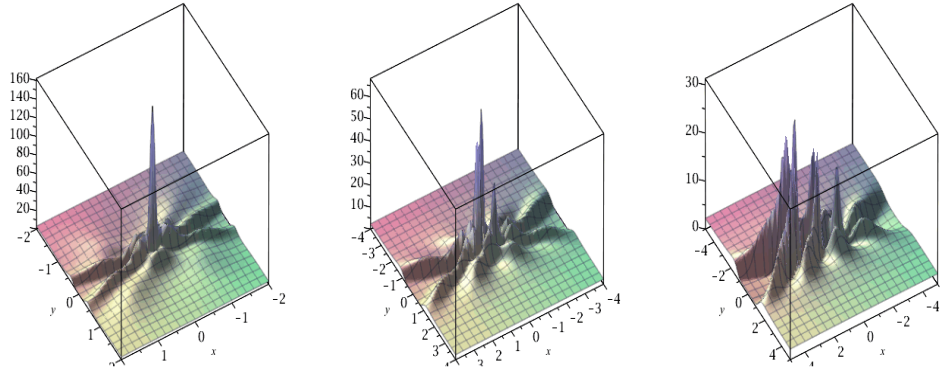


**Figure 7.** Solution of order 3 to KPI, on the left for  $a_1 = 10^3$ ; in the center for  $b_1 = 10^3$ ; on the right for  $a_2 = 10^6$ ; here  $t = 0$ .

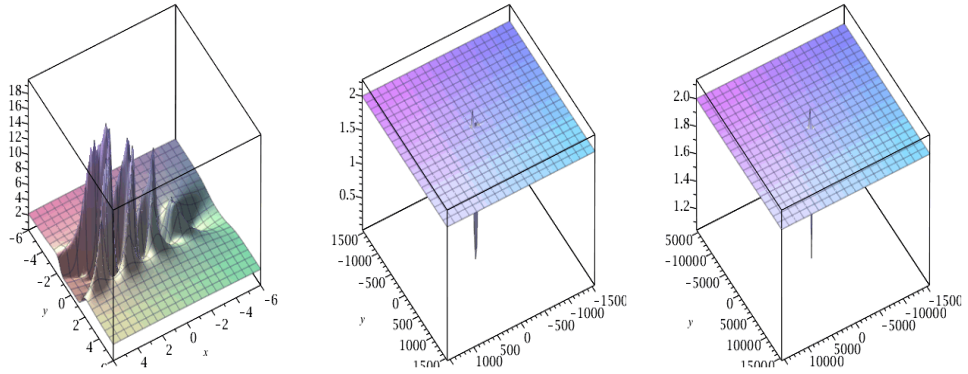


**Figure 8.** Solution of order 3 to KPI, on the left for  $t = 0, b_2 = 10^6$ ; in the center for  $t = 0, 01, a_1 = 10^3$  all the other parameters are equal to 0; on the right for  $t = 0, 1, b_1 = 10^3$  all the parameters are equal to 0.

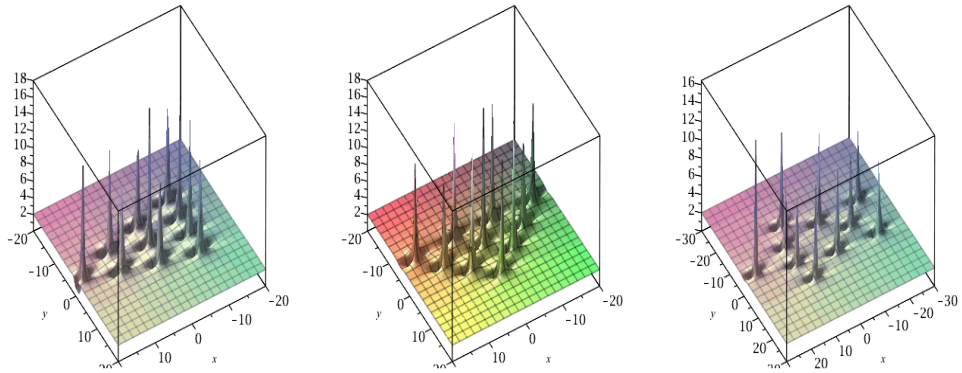
#### 4.4 Case $N = 4$



**Figure 9.** Solution of order 4 to KPI, on the left for  $t = 0$ ; in the center for  $t = 0, 01$ ; on the right for  $t = 0, 1$ ; all the parameters to equal to 0.

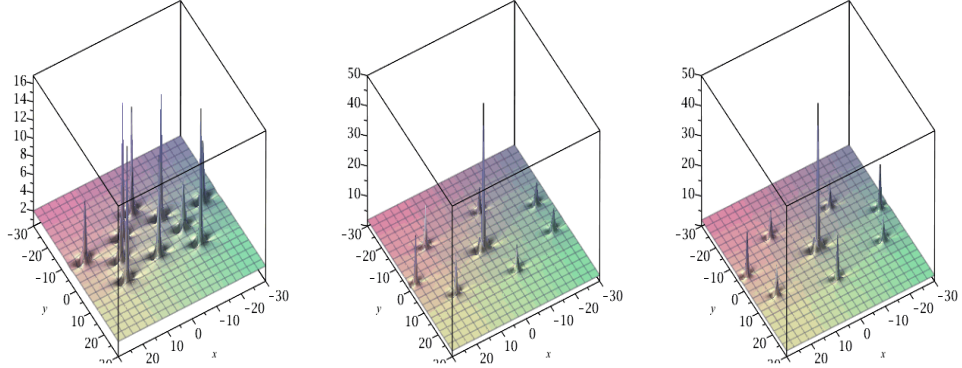


**Figure 10.** Solution of order 4 to KPI, on the left for  $t = 0, 2$ ; in the center for  $t = 10$ ; on the right for  $t = 50$ ; all the parameters to equal to 0.



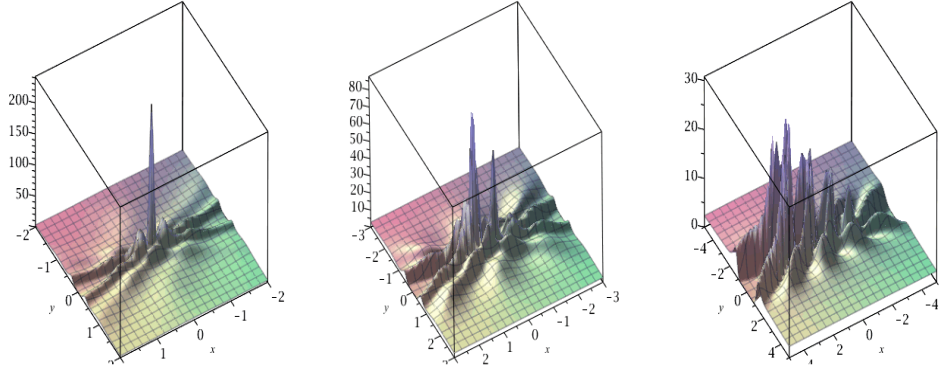
**Figure 11.** Solution of order 4 to KPI for  $t = 0$ , on the left for  $a_1 = 10^3$ ; in

the center for  $b_1 = 10^3$ ; on the right for  $a_2 = 10^6$ ; all the other parameters to equal to 0.

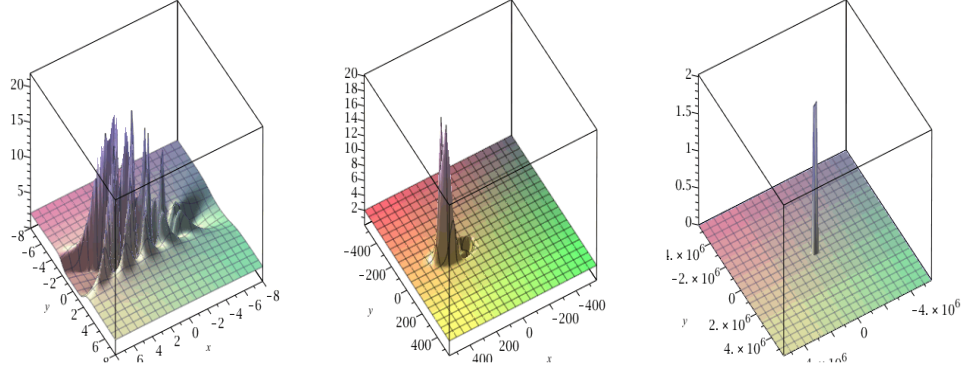


**Figure 12.** Solution of order 4 to KPI for  $t = 0$ , on the left for  $b_2 = 10^6$ ; in the center for  $a_3 = 10^9$ ; on the right for  $b_3 = 10^9$ ; all the other parameters to equal to 0.

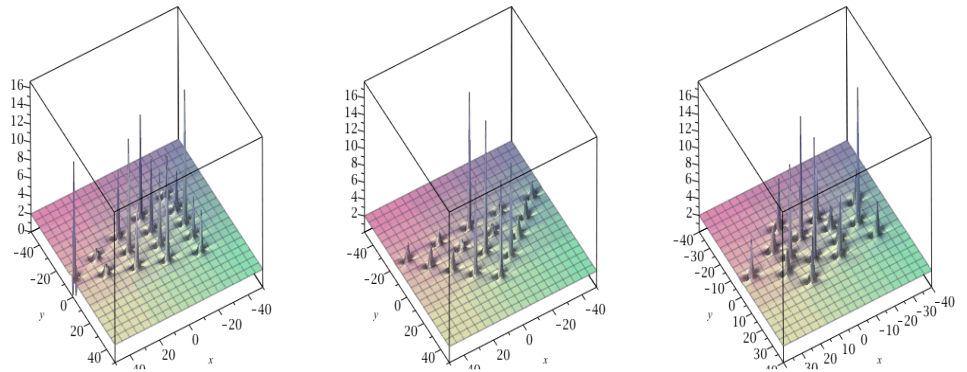
#### 4.5 Case $N = 5$



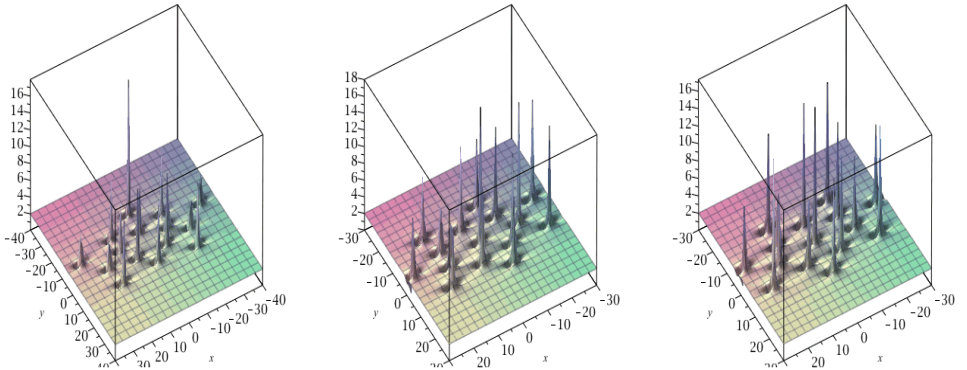
**Figure 13.** Solution of order 5 to KPI, on the left for  $t = 0$ ; in the center for  $t = 0, 01$ ; on the right for  $t = 0, 1$ ; all parameters equal to 0.



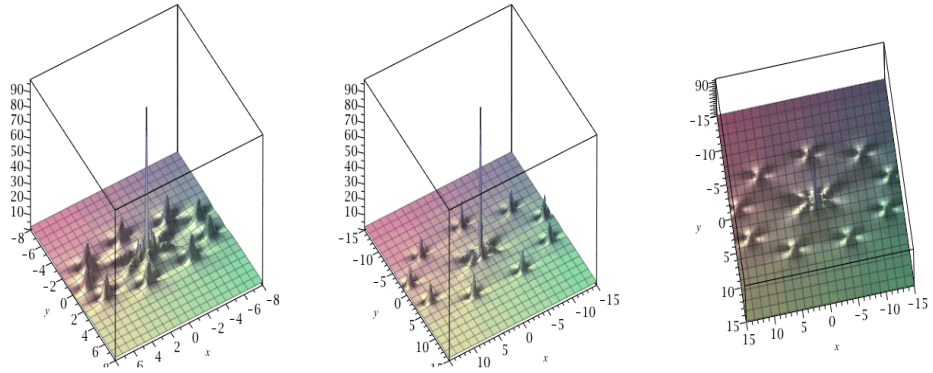
**Figure 14.** Solution of order 5 to KPI, on the left for  $t = 0, 2$ ; in the center for  $t = 20$ ; on the right for  $t = 50$ ; all parameters equal to 0.



**Figure 15.** Solution of order 5 to KPI for  $t = 0$ , on the left for  $a_1 = 10^4$ ; in the center for  $b_1 = 10^4$ ; on the right for  $a_2 = 10^6$ ; all other parameters equal to 0.

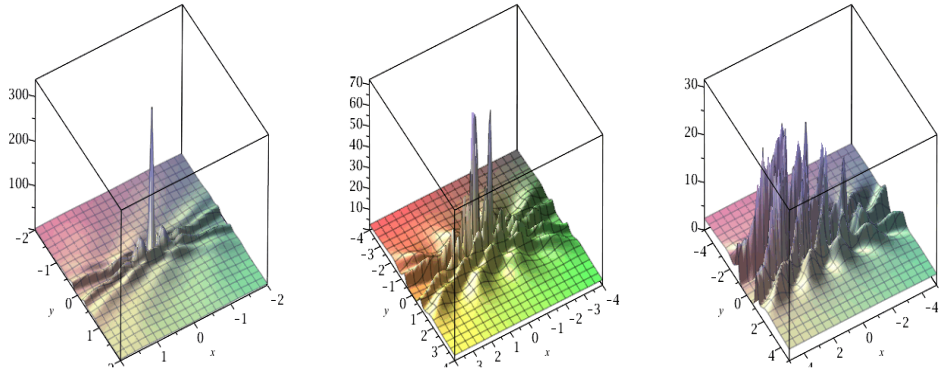


**Figure 16.** Solution of order 5 to KPI for  $t = 0$ , on the left for  $b_2 = 10^6$ ; in the center for  $a_3 = 10^8$ ; on the right for  $b_3 = 10^8$ ; all other parameters equal to 0.

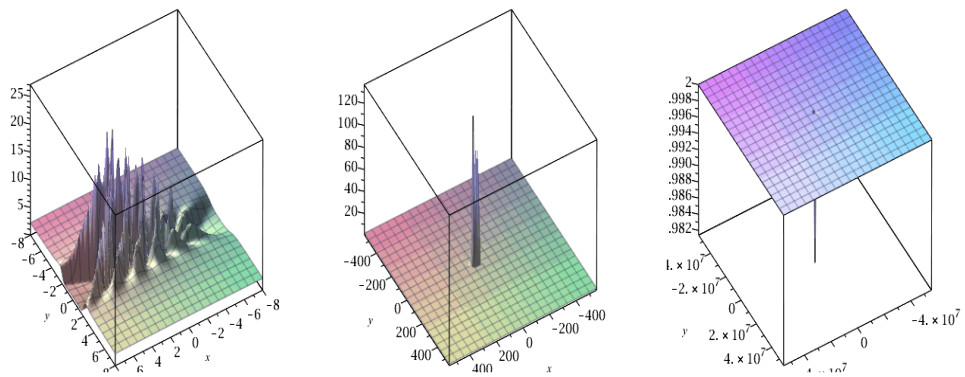


**Figure 17.** Solution of order 5 to KPI for  $t = 0$ , on the left for  $a_4 = 10^8$ ; in the center for  $b_4 = 10^8$ ; on the right for  $b_4 = 10^8$ , sight on top; all other parameters equal to 0.

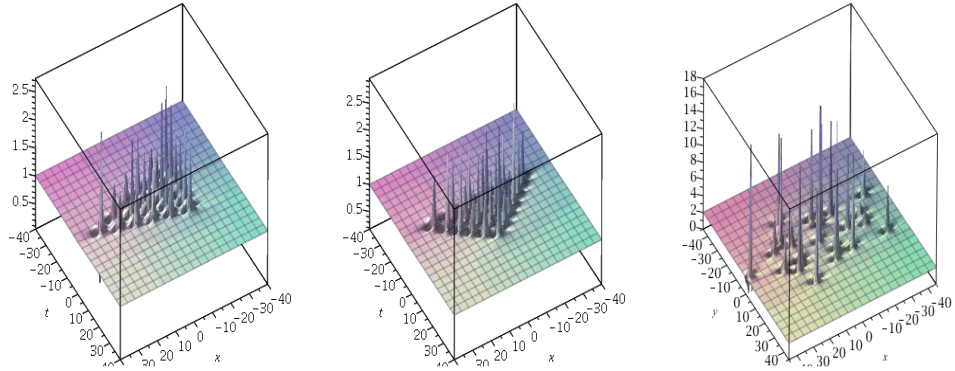
#### 4.6 Case $N = 6$



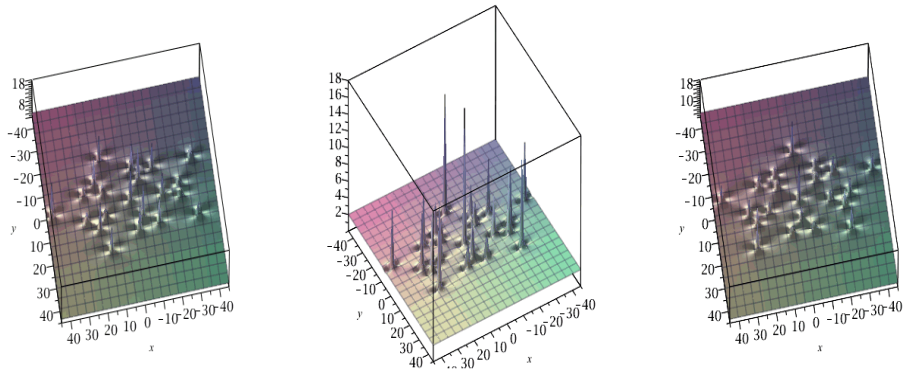
**Figure 18.** Solution of order 6 to KPI, on the left for  $t = 0$ ; in the center for  $t = 0, 01$ ; on the right for  $t = 0, 1$ ; all parameters equal to 0.



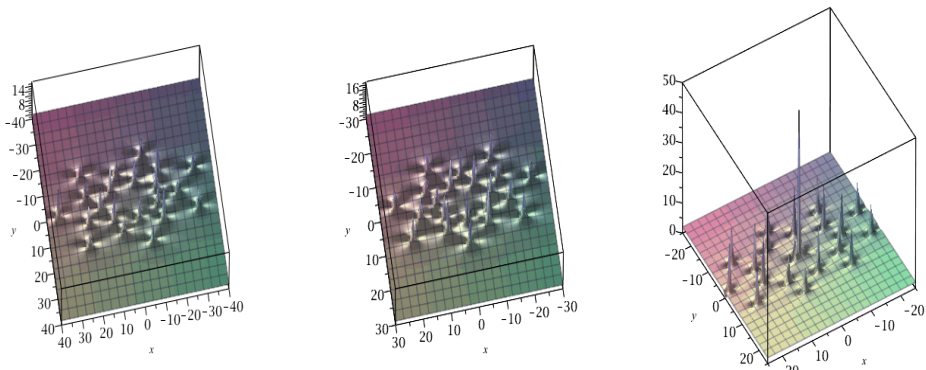
**Figure 19.** Solution of order 6 to KPI, on the left for  $t = 0, 2$ ; in the center for  $t = 3$ ; on the right for  $t = 10$ ; all parameters equal to 0.



**Figure 20.** Solution of order 6 to KPI, on the left for  $a_1 = 10^3$ ; in the center for  $b_1 = 10^3$ ; on the right for  $b_1 = 10^6$ ; all parameters equal to 0 and  $t = 0$ .

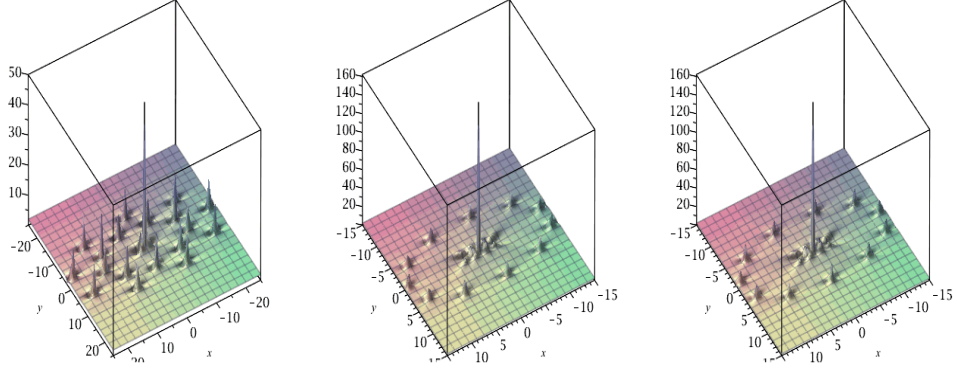


**Figure 21.** Solution of order 6 to KPI, on the left for  $a_2 = 10^6$ , sight on top; in the center for  $b_2 = 10^6$ ; on the right for  $b_2 = 10^6$ , sight on top; all parameters equal to 0 and  $t = 0$ .





**Figure 22.** Solution of order 6 to KPI, on the left for  $a_3 = 10^8$ ; in the center for  $a_3 = 10^7$ ; on the right for  $a_3 = 10^9$ ; all parameters equal to 0 and  $t = 0$ .



**Figure 23.** Solution of order 6 to KPI, on the left for  $b_4 = 10^9$ ; in the center for  $a_5 = 10^9$ ; on the right for  $b_5 = 10^9$ ; all parameters equal to 0 and  $t = 0$ .

## 5 Conclusion

From the previous representations of the solutions to the KPI equation given by the author in terms of Fredholm determinants of order  $2N - 1$  depending on  $2N - 1$  real parameters and in terms of wronskians of order  $2N$  depending on  $2N - 1$  real parameters, we succeed in obtaining rational solutions to the KPI equation depending on  $2N - 2$  real parameters. These solutions can be expressed in terms of a ratio of two polynomials of degree  $2N(N + 1)$  in  $x$ ,  $y$  and  $t$ . The maximum of the modulus of those solutions is equal to  $2(2N + 1)^2$ . That gives a new approach to find explicit solutions for higher orders and try to describe the structure of those rational solutions.

In the  $(x, y)$  plane of coordinates, different structures appear.

In the case  $N = 1$ , one obtains a peak which the height decreases very quickly as  $t$  increases.

For  $N = 2$ , the formation of three peaks is obtained when the parameters  $a_1$  or  $b_1$  are not equal to 0.

In the case  $N = 3$ , for  $a_1 \neq 0$  or  $b_1 \neq 0$  and the other parameters equal to zero, we obtain a triangle with 6 peaks; for  $a_2 \neq 0$  or  $b_2 \neq 0$ , and other parameters equal to zero, we obtain a concentric rings of 5 peaks with a peak in the center.

For  $N = 4$ , for  $a_1 \neq 0$  or  $b_1 \neq 0$  and the other parameters equal to zero, we obtain a triangle with 10 peaks; for  $a_2 \neq 0$  or  $b_2 \neq 0$ , and other parameters equal to zero, we obtain two concentric rings of 5 peaks on each of them; in the last case, when  $a_3 \neq 0$  or  $b_3 \neq 0$ , and the other parameters equal to zero, we obtain one ring with 7 peaks.

In the case  $N = 5$ , for a given  $t$ , when one parameter grows and the other ones are equal to 0 we obtain triangular or rings or concentric rings. There are four types of patterns. For  $a_1 \neq 0$  or  $b_1 \neq 0$ , and other parameters equal to zero, we

obtain a triangle with 15 peaks. For  $a_2 \neq 0$  or  $b_2 \neq 0$ , and other parameters equal to zero, we obtain three concentric rings of 5 peaks on each of them. For  $a_3 \neq 0$  or  $b_3 \neq 0$ , and other parameters equal to zero, we obtain two concentric rings of 7 peaks on each of them with a central peak; in the last case, when  $a_4 \neq 0$  or  $b_4 \neq 0$ , and other parameters equal to zero, we obtain one ring with 9 peaks with the lump  $L_3$  in the center.

In the last case studied  $N = 6$ , for a given  $t$ , when one parameter grows and the other ones are equal to 0 we obtain triangular, rings or concentric rings. There are five types of patterns. For  $a_1 \neq 0$  or  $b_1 \neq 0$ , and other parameters equal to zero, we obtain a triangle with 21 peaks. For  $a_2 \neq 0$  or  $b_2 \neq 0$ , and other parameters equal to zero, we obtain three concentric rings of 5, 10, 5 peaks respectively. For  $a_3 \neq 0$  or  $b_3 \neq 0$ , and other parameters equal to zero, we obtain three concentric rings of 7 peaks on each of them. In the case where  $a_4 \neq 0$  or  $b_4 \neq 0$ , and other parameters equal to zero, we obtain two rings with 9 peaks with the lump  $L_4$  in the center. In the last case where  $a_5 \neq 0$  or  $b_5 \neq 0$ , and other parameters equal to zero, we obtain one ring with 11 peaks with the lump  $L_4$  in the center.

It will be relevant to go on this study for higher orders to try to understand the structure of those rational solutions.

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