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Issues of a quasi-longitudinal study on different types of argumentation in the context of division by zero

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In this study we explore students' ways of argumentation concerning division by zero. The answers of 365 students of four different grade levels in a German secondary school were analyzed on the basis of written texts of the students explaining their results of $7:0$. Applying qualitative content analysis (Mayring, 2000), we were able to distinguish three different types of argumentation. The relative frequencies of these different types vary with the increasing age of the students: rich argumentations stagnate, apodictic references to an authority increase.

Keywords: Argumentation, reasoning, communication model, division by zero.

Motivation and interest of research

Argumentation, besides for instance modelling and problem-solving, is one of the main issues especially characterizing mathematical education. The aim to develop argumentative abilities in mathematics has been reinforced by German authorities since 2003. Consequential, there is a need to measure progress in this field on the level of learning groups and educational systems. Tests like PISA and other test series claim to perform this measuring (OECD 2015, p. 32), even though they are subject to partly strong criticism (e.g., Jahnke & Meyerhöfer, 2007). The tasks used in these tests, pretending to measure argumentative skills, necessarily are very restricted in regard to content and time on task, compared to a creative argumentation process performed in classroom. Furthermore, the dichotomous focus on right or wrong does not seem to be suited for an observation on argumentation.

The notion of “probe” in educational research

The idea of the main study is to use a so called “educational probe” (Ger. “Sonde”). This can be best described by an analogy: Car insurances ask if the car is parked in a garage. If so, the insurance fee is reduced. This is not done due to causal inference, but for statistical reasons (information from the insurance company HUK-Coburg, Coburg, by telephone, 2011): there is a robust (negative) correlation between parking in a garage and the probability of an accident. A “probe” for detecting abilities is a small bundle of easily carried out measurements, observing the patterns of reactions of learners to some standardized impulse together with an established correlation of those patterns to the intended ability of the learning group.

It is an open question if educational probes exist. As a first step, we try to find candidates for a “probe” on the ability of argumentation; in a second step, we have to validate these probes. A variety of measurements can be taken into account (e. g. videography), but here we concentrate simply on tasks given as impulses and evaluate written texts, thereby e. g. ignoring any social interaction. Several groups of tasks, all roughly of the form “Give your opinion and justify it”, were given to the same students. In this article we consider only the task concerning division by zero (in short: “ $7:0=?$ Justify your opinion.”). This special task was included in the study because of the

variety of possible justifications discussed in the literature (Knifong & Burton, 1980) and observed in classroom practice (author's experiences with eight classes of 6th graders).

Basic assumptions and research questions

Argumentative abilities are considered to be of general importance beyond mathematics. Therefore, this study does definitely not focus on proving, and argumentation is not considered as a preliminary step to establish a proof. This decision was a consequence of both, the observed argumentations of the students and a certain communication model which has shown to be compatible with the data.

Furthermore, argumentative and mathematical abilities are considered as different constructs. Therefore, the correctness of the solution of the problem cannot be a primary criterion to discern different types of argumentations. Mistakes have to be tolerated, “misconceptions” can be of a distinct rationality (Prediger, Gravemeijer, & Confrey, 2015, p. 881). This pedagogical view is supplemented by a historic mathematical fact: It is not true that division by zero is not possible or not to define. E. g. the inversion (holomorphic extension of $1/z$) on the Riemann Sphere is a continuous function which imposes $1/0=\infty$.

On first sight, one can wonder if short written answers to tasks really make a difference compared to the testing in PISA. The described preponderant disregard of “correctness” of the given result and the completely different evaluation by a qualitative content analysis (QIA, see below) are characteristics of this study, distinguishing it from PISA. The resulting category system is developed by an approach which is in a first step “grounded”, that is, constructed without reference to other theories. This methodical choice was taken because the interest of this study lies in the opportunity to compare our findings to other category systems found in the literature. However, in order to narrow the scope of this article we have to make two limitations: First, we will not report on a comparison between different category systems. Second, neither the analysis of misconceptions of division by zero nor suggestions for classroom practice are points of interest here, but presented in Fahse (2014).

Taking the hypothesis that the ability of deploying argumentation develops over time, an appropriate probe should provide different results for different ages of the learners. The research questions in this mainly descriptive study are:

- What types of mathematical argumentation can be found?
- Does the percentage of these types differ from grade level to grade level?

Theoretical background

In this section we concentrate on literature about types of argumentation and leave aside that on divisions by zero with the following exception: In congruence with our study, Tsamir and Sheffer (2000) analyze argumentation in regard to division by zero. They distinguish between concrete and formal (algebraic) arguments, and favor the formal ones (Tsamir & Sheffer, 2000, p. 94). In contrast Fahse's (2014, p. 24) empirical examples show that the use of concrete models of division can give insight into the problematics of division by zero, even if a “wrong” result is given (different from “division is impossible”, caveat see above). Therefore, in our system, the distinctions of Tsamir & Sheffer can only be considered as subcategories, not as main categories.

Different classifications of argumentation schemes without reference to any special mathematical topic can be found in the literature. Argumentation can be set in contrast to proof (Duval, 1991), or these phenomena can be treated in regard to their mutual relationship. The latter can e.g. be done by analyzing the process that leads to a proof, wherein argumentation is conceived “as a process of producing a conjecture and constructing its proof” (Boero, Douek, Morselli, & Pedemonte, 2010, p. 183). Following Pedemonte (2007), argumentations are based on a system of conceptions and related to conjectures either as “structurant” or as “constructive” argumentation. Furthermore, an argumentation can be abductive, inductive, or deductive (Pedemonte, 2007). These characteristics could be applied to our data, but since our study does not focus on proof they do not adequately describe the variety of justifications found in our study.

Harel & Sowder (1998) use the term “proof schemes” which refers to “what the person offers to convince others” (p. 275). This fits well into the model of argumentation given below. Their way of classification scheme (externally conviction, empirical, and analytical proof schemes as superordinate categories; “analytical” is changed into “deductive” in Harel, 2008, p. 491) will be compared to the findings of our study in another article.

Communication models and specification of concepts

Since essentially different (Brunner 2014, p. 231) definitions exist of the notions argumentation, reasoning (regarded here as synonymous to justification, if referring to one fixed claim), proving and explaining we have to specify these terms. They are not conceived with regard to proof, but to the argumentations notated by the tested students.

Our study uses a model of argumentation that is based on communication theories (Bühler, 1934; Kopperschmidt, 1980, following Habermas (1984)). The sender and receiver refer to a knowledge (and communication) basis assumed to be shared. The objects of justifications are statements that have different grades of plausibility for the two interlocutors. The act of justification performed by the sender is an attempt to augment this grade, conceived as an ordinal structure, on the receiver's side. Therefore, this concept of argumentation is genuinely dialogic. Nevertheless, the receiver can also be an internal entity within the sender, or a universal audience.

In the following, short definitions of the principal terms used in this article are given. The discussion and the comparison of these definitions to those found in the literature go beyond the scope of this article. But for reasons of practicability we suggest to accept these definitions in the frame of this article despite a lacking consensus in the wider scientific community (Brunner 2014, p. 231).

Argumentation is conceived as a generic term (Bezold, 2009), including the process of finding hypotheses, and checking common bases of knowledge and communication. **Reasoning** or **justification** is a communicative reaction to a questioning of a statement. The aim of reasoning is to increase the degree of the receiver’s acceptance (his attributed epistemic value) of the statement by relating the statement under discussion to the basis of knowledge and communication assumed to be shared (Kopperschmidt, 1980, p. 73). **Proof** is a sequence of argumentative steps relying on an accepted basis of statements approaching the ideal of a complete logical chain of deductive steps (Duval, 1991). A proof can be a justification, but does not necessarily be one. **Explanation** (of “why”, not “how” or “how to do”) is an addressee-oriented justification by the sender with the aim

of creating an “understanding”, which in turn is conceived as a fitting to the (possibly accommodated or enlarged) factual knowledge of the addressee (Kiel, 1999, p. 72; Hanna, 2016, “pedagogical explanations”, p. 2).

Justifications can attempt to explain, but also aim to refer to reliable sources. Furthermore, in this framework, a mathematical proof is only one method of justification and not necessarily effective, depending on the mathematical ability of the sender and receiver. E. g. an algebraic transformation $7:0 = x \mid \cdot 0 \rightarrow 7 = 0$ performed by a student does not really convince another student if their interpretation of the variable x is uncertain and the concept of equivalence transformation is known, but insufficiently familiar.

The study

The analysis of the data is not fully completed yet, but we can report first results on selected parts of the study. We asked the students to give the result of the division $7:0$ and to “justify [their] opinion in a way that someone who doesn't know the answer is able to understand [the result]” (Unabbreviated original task, translated from German). The written answers of a group of $N=365$ students in grade 7, 9, 11, 13 were analyzed. In regard to the relative abundance of argumentation types we report data from a subgroup of $N=300$ pupils which did not take part in interventional courses. These were $N=73, 86, 78, 63$ students in the four grades resp.. In this convenience sample all students were of the same secondary school (“Gymnasium”), and all students of the four chosen grades were tested (absence of students $< 5\%$, no denial).

Method of analysis

First we applied a qualitative content analysis (QCA, Mayring, 2000) with inductive category development. Therefore, we analyzed the student’s written justifications in several steps. The first step was to classify the texts only by similarity without any recourse to theory. In the next steps we aggregated items with an increasing level of abstraction (“feedback loops”) leading to the different types of argumentation described in our coding manual. To ensure reliability this manual was used to perform a separate “deductive category application” (Mayring, 2000, sec. 4.2) by a pair of university students in a second step (Interrater Reliability $\kappa=.967$, $N=365$).

Results - Three different types of argumentation

We found three types of argumentation: rich, pseudo-factual, and apodictic. A summary is given in Table 1. All examples of student justifications have been translated from German.

Rich justifications

In this category the content and the way of reasoning are essentially reasonable (see below) even if the results might be wrong, or the justification is partly false or incomplete. The statement of justification is connected to a domain which is relatively complex. Therefore, operations (e.g. changing the mode of representation, calculations) are more likely to be found.

“Essentially reasonable” means that with the same idea a correct argumentation is possible. “ $7 : 0 = 7$. So, if you have 7 apples and you divide them among 0 persons, you still have 7 apples.” The usage of a model for division in the warrant (Toulmin, 1958) can be regarded as an operation in

which the representation of division is changed from the “algebraic view” to “partitive distribution”. This latter domain is sufficiently complex: it is simply not possible to distribute 7 apples to nobody. The mistake in the original quote, disregarded for the type of argumentation, can be interpreted in the following way: the result of the division does not show what is left, but how much each person gets. Beside partitive interpretation using concrete objects, measurement interpretations of the division: “0 fits infinitely often into 7”, and algebraic calculations are typical examples of rich arguments.

	Rich	Pseudo-factual	Apodictical
Domain	Linked to a mathematically appropriate, well structured domain	Unapt or poorly structured domain, weakly linked	No mathematical domain or tautological
Warrants	Potentially useful/basic concepts	Essentially wrong	No mathematical but social warrants (authorities)
Operations	Operations e. g. change of representation	Few operations	No operations
Need for ...	Mathematical justification		Reliable authority/source
Value (see discussion)	Epistemic (Proof: Logical)	(Similarity)	Informational

Table 1: Characterization of justification types

Pseudo-factual justifications

Mathematical warrants (Toulmin, 1958) are quoted, but these are profoundly incorrect, e.g. if the link to a domain of the common knowledge basis is not reliable (one student used the analogy of $7^0=1$ - not an appropriate domain (power calculations), and weakly linked by analogy). In others, the cited domain has no sufficient structure, e.g. when including “invented” calculation rules (“all calculations with 0 produce 0 as result”), or making statements about the nature of the task or objects (“there's nothing to calculate”, “0 has no significance”). Because of the lack of rich structure in the used domains only few other acts or operations besides generalizations and analogies can be found. Other texts seem to imitate the logical and symbolic structure of a mathematical justification, or use invented terms.

Apodictic justifications

Mathematical warrants are not used, but rather references to authorities like the teacher, the calculator, the textbook, or the world-wide-web given instead. A simple tautological repetition of a statement is interpreted as a reference to one's own ultimate knowledge and thus seen as an authority in the sense of: “That's how it is, I know it.” Sometimes it is even stated that no justification is necessary. This shows a utilitarian understanding of mathematics, which can be convenient, e.g. for engineers in the course of their everyday work. Because there is no need for *mathematical* warrants, there are no domains and consequently no operations found. Typical examples: “[...] There is nothing to explain, that's the way it is”, or “The rule says you cannot divide by zero. You just have to learn and remember it.” This type seems to be very close to the authoritative type of Harel &

Sowder (1998). Precautionary it was given a different name, to be able to compare thoroughly the two types.

Quantitative results

Looking at the increasing grade level of the students, the relative abundances of the used argumentation types accordingly develop as follows: 1) Rich argumentations remain at slightly less than 40%. 2) Pseudo-factual argumentation is reduced to almost a half. 3) Apodictic argumentation almost doubles. There are very few justifications that can be seen as an algebraic proof: Only 4% use multiplication as the inverse of division. Note that the percentages in Figure 1 add up to more than 100% because each text can show more than one type of argumentation. The observed decrease, resp. increase was significant.

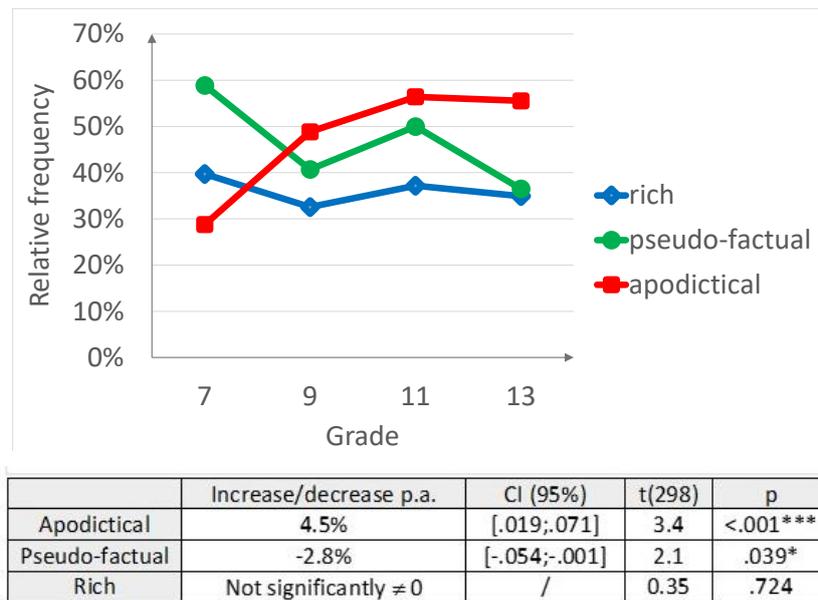


Figure 1: Relative abundance of types with increasing grade and inference-statistical characteristics of the relative abundance of the different types when applying a linear model of increase

Discussion

The classification of justifications into different types presented in this study might be useful for the teacher's practice. They offer a framework for the moderation of class discussions. At first sight, there is no valuation concerning the different types. Proof increases the logical value, non-proof justifications try to augment the epistemic value of the statement (Harel, 2007, p. 497). In our model (Table 1), we add the *informational value*, conceived as the reliability of the source, which is increased by apodictic argumentation. This can be important for non-mathematicians, e. g. in the realm of engineering. Also in school, an algebraic argumentation (close to proof) by another student can be considered hard to trust. However, the aim of school education and the standards of mathematical education favor rich argumentations. For the practice of teaching it is recommended that apodictic justifications should *not* be discredited right away, but first discussed: They may be valuable for practical needs (unexamined statements as "black boxes"), but do not initiate an understanding. In contrast, pseudo-factual justifications should be criticized, even though at this point it is an open research question if this argumentation type can be considered a preliminary stage

of rich argumentation practice. As shown in an example in Fahse and Linnemann (2015), pseudo-factual justifications can be very appreciated by fellow students, because they seem to be particularly “mathematical” at first sight. In some cases, they do not increase any value of the statement, but try to augment the acceptance of the *justification* by similarity to genuine mathematical justifications. With the help of our classification, such misleading contributions could be more clearly discerned, both in discussions and written texts. Therefore, it might even be helpful to inform the students of these types (for first experiences, see Fahse & Linnemann, 2015). For teachers, these types could be useful for diagnostic purposes: to gain insight into the individual development of argumentation skills and to foster these abilities.

Taking the special task on division by zero as an educational probe seems to be promising: In all considered grades all three types occurred and were not marginalized. The different grades (and also the 16 different learning groups) showed significantly different distributions of types (not reported here). What is more, there are clear tendencies: The stagnation of the abundance of rich argumentations and the increasing of apodictic ones. One might think that this is caused by the increasing distance (mental and in time) of the students to this topic (taught in 5th grade). The following observations, though, question this interpretation: Even in 7th grade, the majority of the students does not remember the lessons on this topic, but refer to primary school. Similar questions are topics taught in grade 9 (square-root of negative numbers) and grade 10 (7^0 , $\log(0)$). Nevertheless, the use of algebra in the answers is rare. More likely, the increase of apodictic reasoning is caused by continuous repetition of the mere algebraic rule without any explanation, and perhaps by a neglect of argumentation as an educational objective. This last hypothesis will persist only if the used probe (division by zero) can be validated as a probe for argumentation ability in general. This will be a topic of future research in our investigations.

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