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# **Fostering and investigating students' pathways to formal reasoning: A design research project on structural scaffolding for 9<sup>th</sup> graders**

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*Major obstacles for students learning formal reasoning are the lack of transparency of the logical structure of formal deductions, their theoretical status, and their verbal representation. For fostering students' learning of formal reasoning, making explicit the logical structures and unpacking their verbal representations is therefore necessary. In the design research project presented, a teaching-learning arrangement of angle theorems was designed in which given if-then-statements were to be connected with formal deductions based on the design principle of structural scaffolding. A case study of a pair of 9<sup>th</sup> graders investigated students' pathways towards becoming aware of and using the logical structures and exemplifies the functioning of structural scaffolding.*

*Keywords: Formal proof, logical structure, verbal representation, structural scaffolding.*

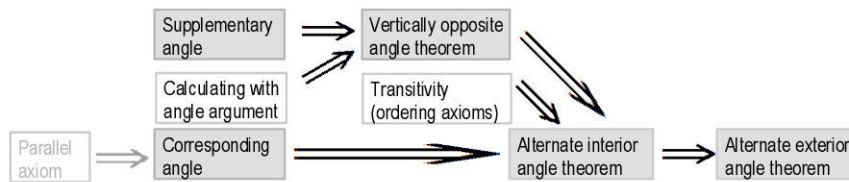
## **Introduction**

Formal reasoning, the logical deduction of new theorems from other theorems, has been shown to be a huge challenge for many students at both secondary and tertiary level. Empirical research studies have identified different reasons for these difficulties (Harel & Sowder, 1998). The design research project presented here focuses on one major obstacle, namely understanding the logical structure of deductions and deductive theory development, which is rarely explicit in mathematics classrooms (Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2011). For this obstacle to be overcome, researchers have suggested that it is important to make the logical structures of deductions and their verbal representations explicit (Durand-Guerrier et al., 2011).

This design research study follows this general suggestion. It draws upon the design principle of structural scaffolding (following general ideas of scaffolding, cf. Lajoie, 2005). It pursues two main research questions: (1) *How can a teaching and learning arrangement be developed to make logical structures of deductive reasoning explicit?* (2) *Which typical pathways towards formal reasoning can be initiated by such a teaching-learning arrangement, and which obstacles appear along the pathways?* The first two sections present the theoretical background and the methodological framework. The design outcome (a teaching-learning arrangement based on structural scaffolding) and a case study of two students in 9<sup>th</sup> grade on their pathway is presented afterwards.

## **Theoretical background: Approaching logical structures by structural scaffolding**

Formal reasoning is crucial in mathematics, not only for *convincing one self and others of the truth* of theorems or for explaining connections, but mainly for *building* (at least locally) *deductive theories* (de Villiers, 1990). Even if, for example, students immediately are convinced of the truth of angle theorems, deducing them from each other encourages students to organize them in a logical and deductive sequence and give insights in mathematical evidence instead of empirical (cf. Fig. 1).



**Figure 1: Locally deductive theory of angle theorems**

*Missing learning opportunities for formal deductions.* In school mathematics, in contrast, most reasoning activities do not refer to formal deductive reasoning but to semantical reasoning where the epistemic value is prioritized over the validity of a statement (Duval, 1991). Formal deductions are presented mostly in a ready-made form (Harel & Sowder, 1998). This does not allow students access to awareness of how to compose an argumentation using logical structure. Although composing the deductions is only the last step of proving (Boero, 1999), it is still necessary to offer learning opportunities for this last step, e.g. by linearly ordering all elements and formulating their logical relations in written forms (Russek, 1998).

*Logical structure of formal deduction and everyday argumentations.* Everyday argumentations have often been described by the argumentative structure of data, warrant, and claim (Toulmin, 1958). While in everyday argumentation the warrant may be omitted and only made explicit when an opponent raises doubts (Rapanta, Garcia-Mila, & Gilabert, 2013), it is crucial in formal reasoning and has a theoretical rather than semantic status (Duval, 1991). That means that the existence of mathematical theorems and their statements, which are not characteristics of reality, are of relevance. Thus, the range of possible warrants must be made explicit in mathematics (Douek, 1999). Also, the status of preconditions in if-then-statements differ: in most everyday argumentations, if-then-statements are only formulated when the conditions are satisfied (Nunes, Schliemann, & Carraher, 1993, p. 130ff). In mathematics, in contrast, if-then-statements are hypothetical, so the validity of the preconditions always have to be checked before applying an if-then-statement as an argument.

*Making explicit logical structures for students.* Due to these differences, many researchers have suggested explicating the logical structures of formal reasoning in the learning process (Durand-Guerrier et al., 2011). Cho and Jonassen (2002) used Toulmin's (1958) argumentation scheme for this purpose for college students in non-mathematical contexts, and we will extend this approach for 9<sup>th</sup> graders in a geometrical context by including the check of preconditions.

*Structural scaffolding as a design principle.* Explicating alone is not enough. For students to become acquainted with the logical structure, and to produce it in their own deductions, this study draws on the design principle of scaffolding. Scaffolding is characterized as enabling learners to realize supported activities before they can conduct them independently (Wood, Bruner, & Ross, 1976). Initially only applied to one-to-one interaction for language learning, the idea of scaffolding has increasingly been elaborated into a design principle for materials and computer tools, whole-class contexts, for open geometrical proofs (Miyazaki, Fujita, & Jones, 2017), and other learning contents (e.g. Lajoie, 2005).

## Methodology of the design research study

*Design research as methodological framework.* Because the study has the dual aim of designing a teaching-learning arrangement (here: on the logical structure of formal reasoning with the design principle of scaffolding) and developing an empirically grounded local theory of students' learning

pathways, we chose the methodological framework of design research with a focus on learning processes (Gravemeijer & Cobb, 2006). The concrete model of Topic-Specific Didactical Design Research (cf. Prediger & Zwetzschler, 2013) relies on the iterative and intertwined interplay of four working areas: (a) specifying and structuring learning contents; (b) developing the design of the teaching-learning arrangement, (c) conducting and analyzing design experiments; and (d) (further) developing local theories on teaching and learning processes.

*Design experiments for data collection.* Design experiments are the methodological core of design research studies (Gravemeijer & Cobb, 2006). For this project, 3 design experiment cycles were conducted with 20 ninth and tenth graders (age 14-16 years) in total. The case study reported here stems from Cycle 3 in which design experiments in laboratory settings were conducted and videotaped with 5 pairs of students, comprising two sessions of 60 minutes each (in total about 600 minutes video material). The students were familiar with the geometrical topic of angle theorems. The empirical part of this paper focuses on the case study of two female students, Katja and Emilia, from grade 9 and the first author as design experiment leader (in the following called tutor).

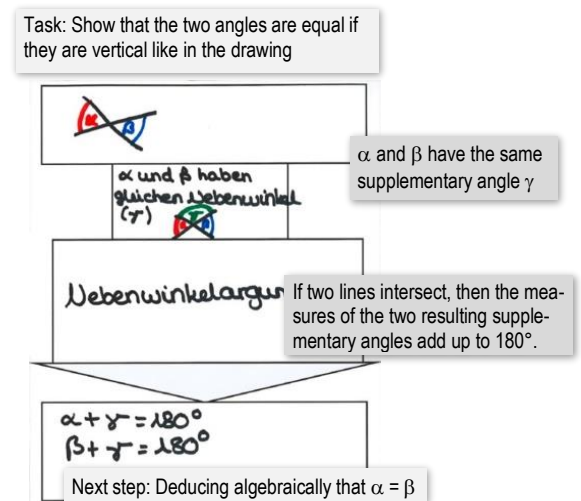
*Methods for qualitative data analysis.* The transcript of the video was analyzed with respect to students' development of explicating elements of the logical structure (using the analytic scheme of data, warrant, claim, cf. Krummheuer, 1995) and to how students articulate relations between these elements (linguistic analysis, not presented here). This makes it possible to investigate the functioning of the scaffolding tool and typical pathways and obstacles.

## Design Outcome: Teaching-learning arrangement with structural scaffolding

*Mathematical topic.* Within the iterative design experiment cycles, a teaching-learning arrangement was developed for the mathematical topic of angle sets. This topic was chosen because the if-then-statements and the set of possible warrants are well limited in this field and locally organized (cf. Fig. 1).

*Structural scaffolding.* For structural scaffolding, we use *materialized argumentation structure forms* on paper as depicted in Fig. 2. In addition to Toulmin's (1958) argumentation structure, the materialized structure also makes explicit why the preconditions of the if-then-statements (named arguments) are satisfied. Every theorem that is already proven is offered as warrant for the next step of formal reasoning. Working with this materialized structural scaffold in each step allows the students to make explicit their often implicit ideas. In the following, the boxes (from above to below) are named data box, condition check box, argument box, and conclusion box.

*Learning trajectory for introducing the structural scaffold.* The intended learning trajectory starts by activating students' previous knowledge on angle sets in cases of determining angles for *concrete* constellations ("Find  $\beta$  if  $\alpha = 120^\circ \dots$ "). When first asked to prove the *general* vertically opposite angle theorem, students' initial argumentative resources often include the critical feature, but are limited mostly by their semantic nature ("because



**Figure 2: Materialized argumentation structure forms as structural scaffolds**

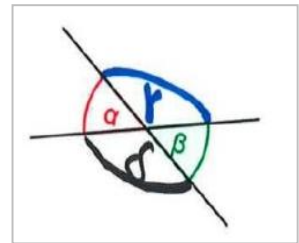
supplementary angle”). Starting from these initial reasoning resources, the tutor introduces the structural scaffold by explaining the new practice of formal reasoning as making explicit *all* aspects *implicitly* contained in the students’ brief argumentation. The condition check box for checking if the precondition of the if-then-statement is satisfied had to be introduced after the first design experiment cycle in order to clarify the theoretical and hypothetical status of if-then-statements in mathematics. The structural scaffold serves different roles along the learning trajectory, (I) as a visualizer for the extended structure; (II) as a working tool for the students to check the completeness of their explicit reasoning; and (III) as a framework for writing down the proof. In our design experiment, after 120 minutes, the students write proofs with deductive chains of reasoning, even though they do not yet find deductive chains for more complex proofs on their own.

## Empirical insights into Katja’s and Emilia’s pathways to formal reasoning

Katja and Emilia start their learning pathway in the way described above. Figure 2 shows the product of the phase of jointly introducing the structural scaffold ending with Sequence 1.

### Sequence 1: Reasoning determined by empiricism instead of validity of statements

When asked to prove that  $\alpha$  and  $\beta$  are equal, Emilia and Katja offer a typical initial, semantic three-word answer “vertically opposite angles” (unprinted Turn 339), assuming that classifying the type of relation between the two angles is enough. Becoming aware that they are supposed to prove the vertically opposite angle theorem by using arguments like the *argument of supplementary angles* (cf. Fig 2) and the *calculating argument* (“If there are angle measures, then it is possible to calculate with them like numbers.”), they start by naming the angles  $\gamma$  and  $\delta$  (Figure 3). Then they discuss the necessary conditions and conclusions.



**Figure 3: Marking angles as first step**

- 362 Emilia: [...] And now we could say actually that  $\alpha$  plus  $\gamma$  results in 180 degrees.
- 363 Tutor: Mmm.
- 364 Emilia: Also like here [*points to conclusion box of the last task with  $\alpha + 120^\circ = 180^\circ$* ]
- 365 Tutor: Yes.
- 366 Katja: Yes.
- 367 Emilia: And that, uh.
- 368 Katja:  $\gamma$  plus  $\beta$
- 369 Emilia: Yes, so actually this can be – Yes, precisely – But we have no concrete numbers [*points to the conclusion box previous task*] – and then we can go on – so, I don’t know, whether we can do this in such small steps, because we have no numbers at all, but then we could say,  $\alpha$  plus  $\gamma$  equals 120, umm, 180 degrees. And  $\beta$  plus  $\delta$  equals 120, umm, 180 degree
- 370 Tutor: Mmm.
- 371 Katja: And...
- 372 Emilia: And accordingly
- 373 Katja:  $\gamma$  plus  $\beta$  – plus  $\delta$  and then
- 374 Emilia: yes, okay, but actually, actually we need only one, don’t we? Then it is just unnecessary, this angle. [*points to the angle  $\delta$* ] – So I would say...

- 375 Katja: ... we have to – this with [6 sec break] yes,  $\alpha$  plus  $\gamma$  180 degree, then 180 degrees minus  $\beta$
- 376 Emilia: No, so I would easily write
- 377 Katja: [“unintelligible”]
- 378 Emilia:  $\alpha$  plus  $\gamma$  180 degree and  $\beta$  plus  $\gamma$  180 degree.
- 379 Tutor: Yes.
- 380 Emilia: And then, if, a system of equations could be created.

When asked to explain in more detail, the students offer details of steps of their calculation (“ $\alpha$  plus  $\gamma$  equals 120, umm, 180 degrees”, Turn 369), but do not explicate the warrants for these relations (here the argument of supplementary angles). In this way, they find out that they do not need the angle  $\delta$ . Interestingly, they formulate steps of action or calculation instead of general relations, and consequently, these steps are combined temporarily (“and then” in Turn 373) instead of logically.

## Sequence 2: Filling the argumentation structure form without verbalizing the connections

When filling in the materialized argumentation structure form (Fig. 2), the students discuss whether they need  $\delta$  and organize their process:

- 403 Emilia: Well then – eh, I would say – I know, I think, that here [points to the argument box], we first write that the supplementary-argument is our argument. Then we think which has to be there [hints to the condition-check box]
- 404 Katja: [writes “supplementary angle” in the argument box, 21 sec break] Yes, that here
- 405 Emilia: Ah, I wanted to write that
- 406 Katja: ... that we  $\gamma$  here
- 407 Emilia: Yes, that  $\alpha$  and  $\beta$  have the same supplementary angle.
- 408 Katja: [3 sec break] Where?
- 409 Emilia: Here [hints a finger at the condition check box]
- 410 Katja: [writes in the condition check box:  $\alpha$  and  $\beta$  have the same supplementary angle  $\gamma$ ] [...]
- 417 Emilia: Yes, okay. – Umm, then I would now write here, umm, -  $\alpha$  plus  $\gamma$  equals 120 degrees and  $\beta$  plus  $\gamma$  equal 120, umm, 180 degrees. Why do I always say 120? Yes,
- 418 Katja: [writes both equations in the conclusion box, cf. Fig .2]

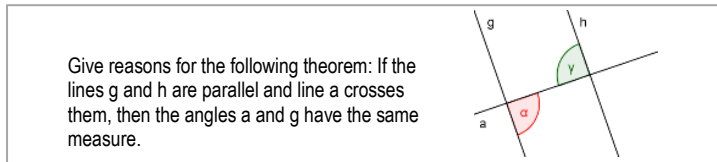


The students succeed in filling in the argumentation structure form mostly without help from the tutor. In particular, they correctly identify all elements of the logical structure, first choosing the argument and then checking whether its precondition is satisfied (Turn 403). After filling in the form, they condense the proven theorem as a new argument to be used for further proofs (in non-printed Turns 452-471): “Argument of vertically opposite angles: If two lines cross each other, then the opposite angles are equal. (They are called vertically opposite angles.)”. This illustrates how the scaffold supports them to produce a complete argumentation and to understand the logical structure. However, it is remarkable that they still do not use any logical connectors to relate the different elements to each other. The language is rather deictic (“here”, “there” in Turns 403, 404, 409, 417), but the logical relation between the elements is not verbalized by the students. To give an expert model of how the connections could be expressed, the tutor finally intervenes as follows:



474 Tutor: [...] Also this condition of point of intersection was considered, so that we have two times two supplementary angles. Here as condition, and because we have supplementary angles, we could use the supplementary-argument that says that a pair of supplementary angles add up to 180 degrees. Therefore, it can be used for our supplementary angles and umm, here two times two were regarded, this means we have two times this equation with, umm, our angles. [...]

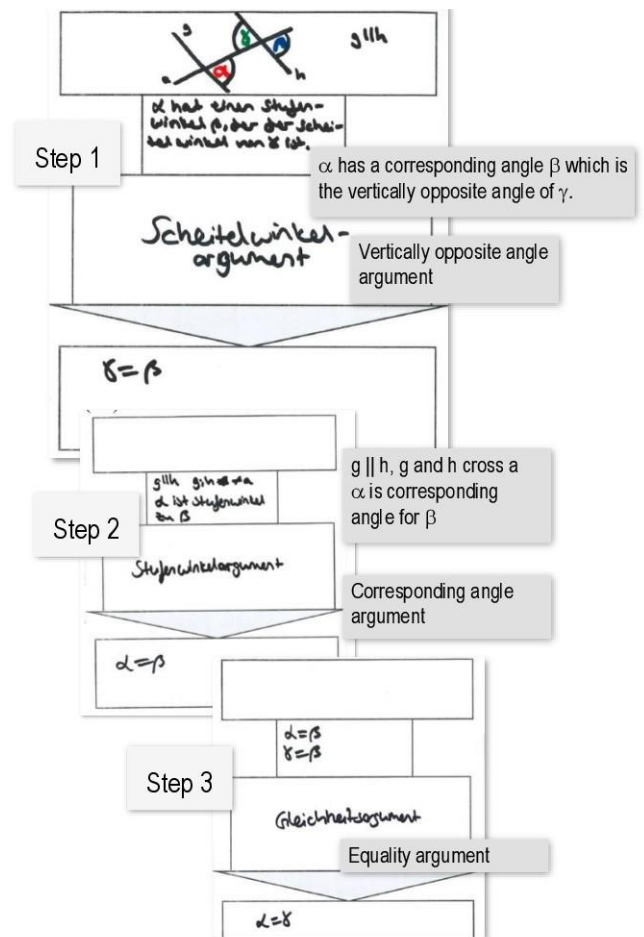
### Sequence 3: Mastering formal reasoning



**Figure 4: Alternate interior angle theorem**

After determining a specific alternate interior angle, the next task for Emilia and Katja is to prove the general alternate interior angle theorem (Fig. 4). For constructing their formal argumentation structure, the students are given the equality argument (If  $\delta = \mu$  and  $\mu = \pi$ , then  $\delta = \pi$ . (transitivity)) and the corresponding angle argument (which can only be derived from the parallel axiom and is hence left unproved for the students, cf. Fig. 1). Again, the students successfully construct a complete argumentation structure supported by the structural scaffold of the form. Based on an enriched sketch, they deduce the theorem in three steps (cf. Fig. 5): In Step 1, they use the vertically opposite angle for deriving that  $\gamma = \beta$ . In Step 2, they use the corresponding angle argument for deriving  $\alpha = \beta$ . For deriving that  $\alpha = \gamma$ , they use the equality argument and produce the last chain of reasoning in Step 3.

The written text produced by Katja for this last step shows what she has learned (cf. Fig 6). Katja's text provides at least situational evidence that she has grasped the logical structure of formal reasoning and can express some of the logical connections. In contrast to the beginning of the students' learning pathway, she makes explicit the warrant ("the equality argument says that") and the conditions of its application ("Now, we know that  $\gamma$  and  $\beta$  have the same measure and  $\alpha$  and  $\beta$ "). For expressing the logical connections, she adopts elements of a language offered by the tutor in Turn 747 ("the supplementary-argument that says"). She also expresses the deduction from the argument to the conclusion: "from this we can conclude". However, the order of aspects is still the order of discovery, not yet the strict order of formal reasoning as the conditions are again guaranteed after using the argument.



**Figure 5: Katja's and Emilia's three steps of formal proof for the alternate angle theorem**

Jetzt wissen wir, dass  $\gamma$  und  $\beta$  gleich groß sind und  $\alpha$  und  $\beta$ .  
 Das Gleichheitsargument besagt, dass wenn ein 1. Winkel genauso groß ist wie ein 2. und dieser genauso groß ist wie ein 3., dass der 1. und der 3. gleich groß sind. In dem Fall ist der 1.  $\gamma$ , der 2.  $\beta$  und der 3.  $\alpha$ .  
 Daraus können wir dann folgern, dass  $\gamma$  genauso groß ist wie  $\alpha$ .

"Now we know, that  $\gamma$  and  $\beta$  have the same measure and  $\alpha$  and  $\beta$ . The equality-argument says, that if the first angle has the same measure as a second and this has the same measure as a third, that the first and the third have the same measure. Here the first is  $\gamma$ , the second  $\beta$  and the third  $\alpha$ . From this we can conclude that  $\gamma$  has the same measure as  $\alpha$ ."

Figure 6: Katja's verbalization of the third step of reasoning

### Looking back to Sequence 1 – Sequence 3

In total, these three sequences from the students' learning process provide insights into the students' pathway from their everyday argumentative resources towards formal reasoning, their induction into mathematical proof as a cultural practice. The structural scaffold strongly supports comprehending of the logical structure, the designated Function (I). The students also capture the norm that the practice of formal reasoning is characterized by making explicit every element in the logical structure (every box must be filled, Function II). However, the scaffold alone does not sufficiently support the process of talking about the logical structures, as visible in Sequence 2. Hence, the structural scaffold had to be complemented by language scaffolds (in this case oral expert modelling offered by the tutor). The written product from Sequence 3, finally, shows that the students can adopt the language scaffolds for communicating about the formal deductions.

### Discussion and outlook

The case study of Katja and Emilia gives a first indication for the potential efficacy of the structural scaffolding. Other pairs of the 10 students in Design Experiment Cycle 3 also succeeded in mastering formal reasoning, supported by the scaffold. Filling the boxes serves as prompts for identifying every single aspect of the logical structure (data, warrant, and conclusion) and the satisfaction of preconditions of argumentations. The specific strength of the materialized structure form is that it not only makes the logical structure visible, but also permits students to complete the form in non-linear order. Based on this structural scaffold, the students' written texts are mostly produced in linear, deductive order. As with any provided format, it can be done non-generatively, passively, locally filling each box but not attending to what role the boxes play in formatting the reasoning. However, not only Katja and Emilia but also other students we observed benefited from the scaffolding as they learnt to distinguish between preconditions, if-then-statements and conclusions. The scaffold supported the students to express the relations between the elements of the logical structure verbally and to reflect amongst other things about the generality of the statements or which characteristics of the sketches are important. These features are crucial to increase awareness of formal reasoning.

Of course, the study still has methodological limitations which have to be overcome in later cycles or future research. Limitations concern the sample size (2 students presented, 10 in total) which is not yet representative. So far, the teaching-learning arrangement is focused on one specific topic, the angle theorems, which need to be extended to other topics in future research in order to gain evidence of the overall claim of efficacy. The most important limitation in view for the next cycle of the presented project concerns the language: we intend to identify the phrases and syntactic structures which appear to be necessary for students to realize the need to articulate the logical connections



between the elements in the argumentation. This will provide support for the students on the linguistic level as well as on the logical-structural level.

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