Misconceptions and developmental proof
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The issue of students’ misconceptions in mathematics and how to prevent and deal with them in teaching has been a major concern of mathematics educators since at least four decades. At the same time our knowledge about the processes of developing understanding and skills in proof and argumentation from early school years has increased. We argue, that there are connections between these two areas of studies important to make explicit for teachers. In this paper, we first elaborate the relation between the research on students’ misconceptions and the ideas of developmental proof. Then we present the relevant results of an empirical study about how mathematics teachers in the field interpret this relation. Our conclusion is that there are important connections between these two research fields that are not always visible for teachers.

Keywords: Misconceptions, developmental proof, teacher education, MKT.

Background

Teachers’ knowledge about students’ misconceptions in mathematics and how to deal with them has been pointed out as an important part of teachers’ mathematical knowledge for teaching (MKT) (e.g., Hill, Rowan & Ball, 2005). Knowledge about them has also been raised as an important part of educative curriculum materials and many of these provide information about common student misconceptions and suggestions on how to address them (Cengiz, Kline & Grant, 2011). At the same time a growing number of research articles have raised the importance of enhancing students’ skills and understanding of proof and proving and research has shown that it is possible and beneficial to start to develop students’ proof-related competences during early school grades (e.g. Hanna & De Villiers, 2009; Hemmi, Lepik & Viholainen, 2013). This knowledge can also be considered a part of MKT, and in line with knowledge of students’ misconceptions close to what Shulman calls pedagogical content knowledge. Research shows that teachers who have strong knowledge in different areas of MKT are more able to create opportunities for extending student thinking (e.g. Hill et al. 2008). This is especially important to consider in teacher education and teacher professional development. Yet, the relationship between specific aspects of MKT still remains unclear (cf. Cengiz et al., 2011). This study contributes to the field by investigating the relation between two particular research areas, namely the knowledge about students’ misconceptions on the one hand and the development of understanding of and skills connected to proof and proving. More specifically, we do not consider students’ misconceptions in a deficit (cf. Jaworski, 2001) manner but investigate how we could use them to enhance students’ understanding of proof and vice versa. The following questions are in the focus of this study:

1) What kind of connections can be found between the research field concerning students’ misconceptions and the development of students’ knowledge and skills with respect of proof?

2) How do teachers relate to students’ misconceptions, proof and the relation between them?
First, we offer an analysis about the connections related to the first research question. Then, we briefly describe the methodology of the empirical study focusing on upper secondary school teachers’ conceptions about these areas. Finally, we report the results of the empirical study and discuss them in the light of the connections found in the theoretical section.

**Misconceptions in mathematics related to developmental proof**

Hanna and de Villiers (2008) introduced the idea of *developmental proof* as something that should grow in sophistication in action, perception and language as the learner matures towards more coherent conceptions. We have earlier concretized the idea of developmental proof by identifying in research literature and school curricula proof-related competences that could be developed through the school grades from 1 to 12 (Hemmi et al., 2013). These competences address besides the development of argumentation and proving, the meta-level knowledge about proof, such as the character of mathematical definitions, methods of proofs, logical and formal aspects that often remain invisible for students (cf. Hemmi, 2008) as well as investigations with validation of either students’ own or others’ reasoning and results. For a comprehensive description see Hemmi et al. (2013).

Research on misconceptions and how to prevent and deal with them has roots in cognitive research and constructivism. For example, some researchers claim that students find it difficult to give up their misconceptions as they have actively constructed them (e.g. Egodawatte, 2011). Further, research on misconceptions is often concentrated within a certain topic in mathematics or science. One of the earlier ideas about reasons for misconceptions is that of Fischbein (1994). He defines three basic components in mathematical practice: the formal, the algorithmic and the intuitive. According to Fischbein, the intuitive knowledge is often experienced as self-evident and may not be problematized or deeply justified in school mathematics and therefore may conflict with the mathematical, logically proved “truths”. Scholars agree that the main function of proof in school is to offer explanations (e.g. Hanna & de Villiers, 2009). Yet, the explanations offered in early school years for operations of natural numbers do not always explain properties of operations when operating in other domains. For example, the concept of multiplication is often explained as repeated addition, in order to reveal connections between arithmetic operations. Yet, this explanation leads to an intuitive conception that multiplication always results to a bigger number than the one you multiply (cf. Fischbein, 1994). When students start to operate with rational numbers this intuitive belief could be made visible and be challenged by investigations, explanations and justifications developed by the classroom community. The transition from the domain of integers to the domain of rational numbers could offer a fruitful platform for developmental proof concerning some logical aspects of reasoning connected to universal statements’ truth-values in different domains (see e.g. Durand-Guerrier, 2003). Hence, we argue that besides prevention of future misconceptions in mathematics, this kind of testing and challenging of intuitive rules could also develop students’ understanding of proof in mathematics and the other way around.

Another example about misconceptions that can be connected to intuitive rules is “over generalizing”, often involving improper use of analogical reasoning, for example in connection to ratio between area of a figure and volume of a figure (see, for example, Tirosh & Stavy, 1999;
Chick & Baker, 2005). These kinds of misconceptions also offer excellent possibilities for students’ investigations and proofs where students could for example develop their understanding of differences between analogical and deductive reasoning. Indeed, new approaches to proof using students’ investigations have been developed and tested in order to enhance students’ skills and appreciation of proof as an important part of doing mathematics (e.g. Heinze & Reiss, 2004). These studies often advocate investigative approaches covering the whole process of proving, starting from the first experiments to generate an idea for a hypothesis up to the final step of writing down the complete proof. We think, that beside this, it is also beneficial to conduct continuously smaller investigations about truth-values of various statements, for example connected to algebraic laws. Scholars agree that several identified student misconceptions are due to students’ difficulties in algebra. The idea of developmental proof has parallels with ideas about children’s development from an understanding of arithmetic to algebra (cf. Hemmi et al., 2013). For example, the generality of reasoning is an important component in investigations and proving where the move from concrete and specific to general is needed when justifying the conjectures made on the basis of the observations of regularities.

Application of rules to situation where the rule is not valid is still another type of misconceptions found in the literature (e.g. Fischbein, 1994; Egodawatte, 2011). As an example consider the following typical misconception in simplifications of expressions (Egodawatte, 2011):

\[(1) \frac{2 + x}{x} = 2 \quad \text{(2) } \frac{12 \cdot 2x}{2} = 6x\]

The rule applied in the first example is valid for rational expressions with only multiplication in the numerator, but not with addition, while the rule applied in the second example is valid for rational expressions with addition in the numerator. Typical misconceptions also concern the use of the distributive law in situations where it is not valid (e.g. Fischbein, 1994). These kinds of misconception could be regularly used as an object for investigations in order to enhance students’ understanding of treatment of algebraic expressions and derivation of rules. Explanation in mathematics often refers to making mathematical connections explicit. Kuchemann and Hoyles (2009) emphasize the importance of the mathematics instruction to move from a computational view of mathematics to a view that conceives mathematics as a field of intricately related structures in order to develop students’ proving competences. Seeing connections and mathematical structures is also an important proof-related competence connected to developmental proof (Hemmi et al., 2013). Still another kind of misconception identified in the literature is connected to mathematical definitions. For example, several researchers present similar ideas about students who often operate as if all functions were linear (Tirosh & Stavy 1999). This is connected to development of understanding the role and character of definition in mathematics, also identified as an important aspect of developmental proof (Hemmi et al., 2013).

Scholars have also attempted to explain why some misconceptions are developed and how to deal with them to change them (e.g. Tirosh & Stavy, 1999). There are significant connections between the suggestions offered to deal with students’ misconceptions and the developmental proof, for example, the understanding of counter example, critical thinking, and argumentation with peers. Several studies show that erroneous conceptions are so stable because they might be correct in some instances. Scholars state that teachers should encourage students to critically evaluate their solutions and develop a skeptical approach to their intuitive rules. Balacheff (2010) points out that proving is
the most visible part of validation and something that cannot be separated from the ongoing controlling activity involved in solving problems or achieving tasks. Scholars also advocate the use of common misunderstandings for planning of effective sequences of instruction by both using situations where the intuitive rule is valid and where it is not valid in order to create cognitive conflict. Creating cognitive conflict by using a counter example is not always fruitful if students do not understand the role and the logic of counter example in mathematics. Here the development of students’ understanding of the role of counter example in mathematics is important and connected to developmental proof. Interestingly, the idea of creating cognitive conflict has also been used to change students’ misconception concerning the use of specific examples in validation of mathematical statements and the promotion of students’ feeling for the need of proof (Stylianides & Stylianides, 2009).

The empirical study

In Finland, proof was an important part of upper secondary school mathematics in the 1970s during a period of ‘New Math’ reforms but since then its importance has decreased significantly. Yet, the Finnish steering document for the compulsory school curriculum (2004) addresses a number of proof-related topics (Hemmi et al., 2013) and although the word proof is not mentioned in upper secondary curriculum, in textbooks for the advanced course, proof and deductive reasoning is an important part of the contents (Bergwall & Hemmi, 2017). There are two programs in upper secondary school mathematics in Finland. The basic course is for students who study humanities and social sciences while the advanced course is for those students who want to study mathematics, science and computer sciences at the university.

The empirical study was conducted with Swedish speaking upper secondary school (about the age of 16-19) mathematics teachers in Finland (Julin, 2016). The aim of the entire study was to investigate teachers’ knowledge, experiences and views of students’ misconceptions and the role of proof in mathematics and in teaching. A questionnaire comprised mostly closed statements and questions that were developed from items in literature. For example were teachers asked to judge how often they had experienced seven common misconceptions and how they usually reacted to them when encountered them in their teaching (see Figure 1). Concerning their reactions teachers could choose from five methods applied from Chick and Baker’s (2005) study: counter example, re-explain the procedure, re-explain the concept, cognitive conflict, and probe student thinking. These methods were shortly described in the questionnaire. The items in the questionnaire also addressed proof in mathematics and in teaching and finally the relation between misconceptions and proof. As a complement to the quantitative part we also posed an open question: “Explain shortly why you use/do not use proof in your teaching”, and finally there was a possibility for the teachers to freely write their own thoughts about these issues.

The electronic questionnaire was sent to all mathematics teachers working in the Swedish speaking upper secondary schools in Finland, in all 90, and of them 36 teachers responded to the questionnaire. Both the gender and age distribution were representative for the whole group and the responding teachers’ teaching experience varied from 1 to 40 years. All teachers responding to the

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1 About 5% of the Finnish population has Swedish as a mother tongue.
questionnaire were certificated mathematics teachers. The responses to the quantitative part of the study were analyzed using descriptive statistics and the open questions were analyzed inductively.

**Teachers’ relation to misconceptions and proof**

Most teachers (97%) state they recognize the common misconceptions in their own teaching and all of them consider the knowledge about misconceptions relevant for their work. Almost 70% of the teachers state that they know how to deal with these misconceptions.

Figure 1: Methods used by teachers when encountering misconceptions

Further, over one half of the teachers who state that they do not know how to deal with students’ misconceptions had less than 10 years of teaching experience and 36% of them wanted to learn more. The methods teachers would choose to deal with students’ misconceptions varied depending on the character of them. The use of a counter example and cognitive conflict are related and were dominating among the methods that the teachers preferred (Figure 1). The least popular method was the probing students thinking. Yet, it seems to be usual for the teachers to use the analysis of the steps in the reasoning when sorting out the situation and then utilize the other methods. Some teachers suggested that a teacher should focus on common misconceptions already when presenting the theory of a new topic and explain why that is not true in order to create a cognitive conflict from the beginning.

One can let students work pairwise and judge the correctness of different solutions and ask them to justify their judgments. Surprisingly, students are insecure and experience these tasks as
difficult. I have tested this with both students taking the advanced course and students studying the basic course in mathematics. This is really instructive for both a teacher and students. (All of my examples were authentic student solutions.)

Most of the teachers stated that they used their knowledge about the common misconceptions when they designed their teaching and chose tasks. Yet, peer instructions (students discuss and argue about the correctness of different solutions) was utilized (sometimes /more often) only by 25 % of the teachers (Figure 2).

![Figure 2: Teachers’ use of misconceptions](image)

1: I use my knowledge of common misconceptions when planning teaching.
2: I use my knowledge of misconceptions when selecting tasks and exercises.
3: I use Peer Instruction related elements in my teaching.

Mathematics is a cumulative subject. Elementary school has a great responsibility. Unfortunately the textbooks I have seen are quite bad. Students cannot see the structure because of all details. Mathematics is not only using of Pythagorean theorem or calculation of percent that students do without understanding. It is a logical structure.

All the teachers consider proof as more or less important for mathematics as science and they present (sometimes or more often) proofs for students studying the advanced program in mathematics. They also agree that proof somehow contributes to the teaching of mathematics. Yet, only 2 teachers present proofs for students studying the basic program, one of them states that “proof gives often a greater broadness than ‘learning by heart’” and if students learn to prove the formulas then they also can modify them so that they can solve a broader spectrum of tasks. Another view of the role of proof in school mathematics is shown by a teacher who states: “Proofs are good and beautiful but in the upper secondary school reality teaching is far away from building teaching around proving”. About 30% of the teachers seldom let their students work with proof and proving by themselves. Concerning the teachers’ views of the connections between students’
misconceptions and proof, most of them are not convinced that proof and proving would have a positive effect on students’ misconceptions. Only 9 teachers agree with the statement “Proof and proving can help to change students’ misconceptions” and 8 teachers agree with the statement “If students learned proving, their incorrect steps of reasoning would diminish.”

Concluding remarks

The paper focuses on the relation between research on students’ misconceptions and developmental proof. The elaboration of the research literature on students’ misconception from the perspective of proof reveals several important connections between the ideas and results of the two research fields. However, the empirical study shows that these connections are not clear for teachers. For example, we find it significant that the teachers in our study most often use counter examples or cognitive conflict that are closely related to developmental proof as a method for changing students’ misconceptions but at the same time only about 25 % of them consider proof and proving as beneficial to prevent and change students’ misconceptions. We also recognize different views of proof in school mathematics among the teachers that may have crucial consequences for students’ possibilities to develop their understanding and skills in proof and proving and therefore also using students’ misconceptions as a starting point for this and vice versa. The idea of developmental proof is probably not in focus in mathematics teacher education. Furthermore, it is possible that teacher educators focus on both students’ misconception and proof but because of the different research traditions the connections and the possibilities of these connections between these areas may not become clear for student teachers. More studies are needed to investigate teachers’ views of proof in relation to their views of these connections in the teaching of mathematics.

References


