A reference for studying the teaching of logic

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This paper presents a work developed in my thesis on the teaching of logic in high school in France. The current official instructions specify that teachers don't have *to make a mathematical logic course, but have to help their students develop a relevant use of some notions of logic as tools. Therefore, I point out in this paper that in order to take this constraint into account in the study of didactic transposition process, it is relevant to describe it from a reference knowledge for logic, but that such knowledge has never been established by the mathematics community. In order to offer such a reference, I base myself on a double epistemological and didactical study in which I favor the links between logic and language. I will also explain the choices made for this reference and its use as a methodological tool through the example of quantifiers.*

Keywords: Logic, teaching, didactic transposition, quantifiers.

Introduction

In France, mathematics syllabuses for high school (students between 15 and 18) mention explicit goals concerning certain notions of logic. For example, they recommend that "students [be] trained on examples to wisely use the universal and existential quantifiers (the symbols $\forall$, $\exists$ are not required) and to identify the implicit quantifications in some propositions, particularly in conditional propositions". These current recommendations come after twenty years during which the logic was excluded from syllabuses. This exclusion itself is a reaction to the too abstract and formal aspect of modern mathematics (taught in France between 1969 and 1981) for which mathematical logic and set theory were basic elements. They strongly highlight the logical notions as tools and even show distrust against their feature as objects. This distrust can be interpreted as a resistant mark of turning down modern mathematics. The syllabuses specify that "the concepts and methods of the mathematical logic should not be subject of specific courses but must naturally take place in all chapters of the syllabus". I described more precisely the characteristics of the conditions and constraints due to these official instructions in a contribution to CERME 8 (Mesnil, 2013): teaching goals are ill-defined and subjected to strong constraints. Moreover, mathematics teachers do not share a common reference for interpreting these syllabuses. Indeed, mathematical logic is not part of the domains necessarily studied by a mathematics student, bringing a diversity of teachers' knowledge, that is usually not filled during their training in which teaching of logic is often only superficially and quickly addressed.

In this contribution, I would like to pursue this issue of a reference for the teaching of logic. In my thesis, I studied the teaching of logic in high school as the result of a didactic transposition process, and I'll explain first why the nature of mathematicians' logical knowledge requires thinking this transposition not from a *mathematical knowledge* like mathematical logic, but from a *reference knowledge*, nonexistent for the moment in the mathematical community (Mesnil, 2014). Secondly, I will justify the choice I made then to construct a methodological tool to conduct the analysis of syllabuses and textbooks. This tool is a *reference* in which the notions of logic are presented from three points of view that root them deeply in mathematics, in mathematical activity, and in the classroom. I will conclude by illustrating the choices I made for this *reference,* and its use through the example of quantifiers.

The need for a *reference*

During the didactic transposition process (Chevallard, 1985), a mathematical object is identified in a body of knowledge (in French, *savoir savant*), and a first succession of adaptations will make it able to become an object of teaching in a particular institution. It is then identified by mathematics teachers in the knowledge to be taught (in French, *savoir à enseigner*) and is subjected to a second succession of adaptations to become a taught object.

But except in some university courses, teaching logic does not mean teaching mathematical logic, but the logic at work in mathematical activity, which supports mathematicians' expression and reasoning. We can consider that mathematicians have a logical knowledge, which may be the subject of a didactic transposition process, but this knowledge is more visible in practices than in treaties. Mathematical logic can be perceived as a description of the principles of this logic. It was then explicitly taught during the time of modern mathematics, in France and elsewhere, but these experiments showed that this teaching, conducted formally and isolated, did not help pupils to express themselves and to reason (Adda, 1988). Several researchers in mathematics education now agree that such teaching should be explicitly linked together with mathematical activity in which the logic is omnipresent (see for example Epp, 2003; Durand-Guerrier, 2005). However, there are few research studies on the effect of teaching logic on reasoning and expression capacities, which is still an open issue.

Thus, to take into account this particular nature of the logical knowledge and its connection to mathematics, and to appropriately study the didactic transposition process of this knowledge, I propose to describe this transposition not from a mathematical knowledge, but from a reference knowledge. I take this notion from Rogalski and Samurçay (1994) who thus characterize a knowledge produced by practices, but decontextualized from situations where knowledge is apparent into action. These authors state that it is necessary that this reference knowledge can "be expressed with its concepts, its methods, its systems of representation and its language" (*ibid*, p. 46). However, concerning notions of logic, such a reference knowledge does not appear in that there is no corpus collecting the logical knowledge necessary to mathematical activity and about which everyone agree the choices of concepts which are featured and of their representation.

I then conducted a study to answer the following research question: what kind of reference knowledge would be epistemologically and didactically relevant to the logic teaching? This study allowed me to construct a *reference*, which I then used to analyze the knowledge to be taught in high school in France. I call this analysis tool *the reference*, and not reference knowledge, because the production of a knowledge falls under a long and collective process.

The importance of language for reasoning, and its links with logic

The study of various logical systems across the ages allowed me to identify invariants and differences in the role assigned to logic and in the ways that are given to it to fulfil it. The study of didactic studies shows how issues relating to the teaching of logic meet the concerns and choices of these logicians.

All these logical systems are built from a work on language. The concept of proposition is primordial in them all. Aristotle describes it in terms of subject-copula-predicate, and it was not until Frege at the end of the XIX^{e} century that this analysis was to be replaced by an analysis in terms of function and argument allowing two things essential for mathematical language: on one hand to consider predicates with several arguments, on the other hand to pull out the act of quantification from the proposition by making it expressed by quantifiers which act on variables. Predicate logic that is then born is able to model mathematical propositions. From a didactic perspective, several research studies have shown many examples of situations in which the predicate logic is a relevant reference for didactic analysis which allows to highlight the importance of issues on quantification (Durand-Guerrier, 2005).

The current language of mathematicians is inspired by Frege's formalism, but it isn't a strict use of a formal language. Focusing particularly on problems of language in mathematics teaching, Laborde (1982) showed that there is a particular use of language in mathematics, due to the interaction of the two codes of symbolic writing and natural language. This interaction allows mathematicians to use reformulations useful for conceptualization. Teachers are familiar with the particular features of mathematicians' language, but they can cause difficulties for students who "discover together the concepts and the way we talk about them" (Hache, 2015, p.28).

The common goal of the studied logical systems is to ensure the validity of reasoning, with a preliminary work on language. But for the authors of *The Logic of Port Royal*[1], logic above all needs to be trained and the formalization of reasoning is seen as an obstacle to the use of intuition, whereas for Leibniz and Frege on the contrary, logic must provide a system of signs in which reasoning can be expressed, and this formal expression guaranteeing its infallibility. Gandit (2004) denounces the excessive place taken by the formal aspect in the beginning of proof learning. But being careful about formalization at the time of the discovery of deductive reasoning does not mean that it cannot subsequently help those who begins to have a good practice of it. Thus, in higher education, Selden and Selden (1995) suggest presenting theorems and definitions in an informal formulation, which allows intuitive understanding, and in a formal language, which allows linking structure of the statements and structure of its proof.

Organization of *the reference*

In the *reference* I proposed, I decided to give a broad place to language. Entering into logic by the language is consistent with the epistemological study, granting it an important place is consistent with didactic studies. Of course, in the same way as it is done in the studied logic systems, especially in a didactic perspective, the study of notions of logic as components of mathematical language has to be articulated with their use in reasoning.

Finally, these epistemological and didactic studies led me to propose a *reference* in which the presentation of logical concepts combines three approaches:

[1] Name of a famous french treatise, which original title is « La logique ou l'art de penser », written in 1662 by A. Arnauld and P. Nicole, who were very influenced by Descarte's method.

- The mathematical logic. It is a recent branch of mathematics that can be considered as a result of what has been sought by different logical systems creators since Greek antiquity. Mathematical logic seems so particularly suitable as a formal reference to describe the logic at work in mathematics.
- The study of the language practices of mathematicians. In this way, the presentation of the logical concepts is rooted in mathematical activity taking into account how they are expressed in mathematical discourse, using predicate logic to uncover some complex and sometimes ambiguous formulations that are yet a part of the language practices of mathematicians, widely imported in the mathematics classroom.
- The research in mathematics education. In this way, the presentation of the logical concepts is rooted in mathematics classroom, taking into account the difficulties that the complexity of these notions can bring for students.

In this *reference,* the components of mathematical language are shown, beginning with the primordial notions of proposition and variable. Then for the connectives AND and OR, implication, negation, quantifiers, I have consistently adopted the three approaches mentioned[2]. Although the focus is on the language, reasoning is of course not absent from the *reference*. A difficulty for pupils and students is to distinguish, in a text of a proof, mathematical propositions concerning mathematical objects, and parts of the text which allow to follow the progression of reasoning, such as variable introductions, or justification of an inference. The confusion between implication and deduction falls under this type of difficulty.

The example of quantifiers in the *reference*

As announced, the *reference* contains first a presentation of logical concepts from mathematical logic. Predicate logic uses two quantifiers: applied to a variable x, and from a proposition P, the universal quantifier allows to obtain the proposition $\forall x$ P, and the existential quantifier allows to obtain the proposition $\exists x$ P (description of the syntactic aspect of the quantifiers: they operate on a variable and a proposition to build a new proposition).

Let E be a set in which the variable x can take values. The proposition $\forall x$ P[x] is true[3] when for any element a of E, the proposition P[a] is true. The proposition $\exists x$ P[x] is true when there is at least one element a of E such that P[a] is true (description of the semantic aspect of quantifiers: truth conditions of a quantified proposition). Quantifiers have an important effect on the variables: a variable that is in the scope of a quantifier is a dummy variable in the quantified proposition, and this proposition does not give information on the object designated by the variable, but on the set in which it can take its values.

Some important results on quantified propositions may be established by a semantic way, using the sense, as well known equivalence between NOT($\forall x$ P[x]) and $\exists x$ NOT(P[x]), or the fact that if $\exists y$

[2] For each of these themes I have summarized the mathematical content, the language related issues, and research findings on student difficulties.

[3] This semantic characterization can be described as "naive" because I do not strictly define what "being true" means. But of course, this characterization may be more rigorous with the notion of satisfaction of a formula in a model introduced by A. Tarski.

$\forall x$ P[x,y] is true, then $\forall x\ \exists y$ P[x,y] is true. These results are then used in syntactic manipulations, independent of the sense, in the same way we manipulate algebraic equalities.

In mathematical language, the quantifiers are a way to express the quantification, but there are many others. We can see this through some examples of mathematical propositions[4]:

1) Le carré d'un nombre réel est positif (The square of a real number is positive)
2) Le carré d'un nombre réel est toujours positif (The square of a real number is always positive)
3) Tous les réels ont un carré positif (The square of any real number is positive
4) Tout réel x est tel que x^2 est positif (Any real number x is so that x^2 is positive)
5) Pour tout réel x, x^2 est un réel positif (For all real number x, x^2 is a positive real number)
6) $\forall x \in \mathbb{R}, x^2 \geq 0$

They are several formulations of the same property, but universal quantification is expressed very differently. In proposition (1), quantification is implicit, implied by the word *un* (translated with *a*). We frequently use the indefinite article *un* to mark a universal quantification, in everyday language as in mathematics. But *un* is also sometimes used to mark an existential quantification, which is obviously confusing! Sometimes, the two usages coexist in the same proposition, such as in "un réel positif possède une racine carrée" ("any positive real number has a square root", in English, the first *un* is rather translated with *any* and the second *une* is rather translated with *a*, and there is no confusion). In proposition (2), the adverb *toujours* (always) is used to explicitly mark this universal quantification, as the word *tous* (any) in proposition (3). Propositions (4) to (6) are distinct from the first by the use of a variable. Furthermore, one can identify in each of these propositions an expression which express the quantification (here universal quantification) and which has the property that it can be separated from the proposition "x^2 is positive" (or equivalent formulation). Such expression works as quantifiers of mathematical logic, and I therefore also calls them *quantifier*.

Finally, we saw that in the language practices of mathematicians, quantification can be implicit or explicit, and in the second case, possibly marked by a quantifier which is an expression observing syntactic rules of use. Propositions (4) and (5) may seem closer to propositions (1) to (3) as they are formulated "with words" contrary to the proposition (6) which only uses mathematical symbols, and that may seems much more formal. I would like to stress that such a vision hide formalization still existing in these propositions, in the sense of a shaping according to certain rules, even if that formalization is not accompanied by a symbolization.

I will conclude by mentioning some of the difficulties of high school students or senior students in related to the use of quantifiers. First, the implicit quantifications are not always perceived by students. The case of the universal quantification associated with the implications and the formulation *if ... then ...* is highlighted for a long time (Durand-Guerrier, 1999). Quantification is often encapsulated in stiffened structures (for example, "u_n is as big as we want by taking n big enough") that the expert mathematician knows how to reformulate by explaining the quantifications, but these reformulations in more formal language tend to disappear from the language used in high school, and

[4] I give the examples in French first because I will explain some difficulties linked to the word un which is used in this language in different meanings, and which is translated in English with *one*, with *a*, with *the*… Commentaries following the examples refer to the French expression.

are a source of difficulty when students meet them in higher education. Another difficulty concerns the failure to take the order of quantifiers into account when there is an alternation. We know that students have rather an interpretation *for all... there exists...* even though they are facing a proposition *there exists... for all...* (Dubinsky & Yiparaki, 2000). Furthermore, Chellougui (2004) showed student difficulties with the use of an existential proposition. In a proof text, there is generally a confusion between the affirmation of the existence of an element checking a property, and the act which consist to consider it and give it a name. Likewise, mathematicians do not get trapped by the "dependence rule" in the statements *for all... there exists...* and identify easily this error in a student production. However, they do not necessarily explain this error to the students by linking it to formal rules of manipulation of variables and quantifiers (Durand-Guerrier & Arsac, 2005).

Examples of the treatment of quantifiers in school textbooks

The *reference* I have developed allows an analysis of resources available to teachers highlighting sensitive issues that need to be paid attention to. I will then conclude with an analysis of two extracts from school textbooks.

The reintroduction of logical concepts in the syllabuses had an effect on the textbooks: those published in 2010 for the first class of high-school (15 years old students) all contained passages identified as speaking of these concepts. Nine textbooks out of ten have chosen to dedicate a few pages (between one page and nine pages) to notions of logic, usually located at the beginning or at the end of the textbook (only one textbook does it in a disseminated way). Moreover, they all contain exercises with a stamp "logic" (from ten to fifty-four exercises in the studied textbooks).

In the eight textbooks that deal with quantifiers, the letters are introduced by examples. Seven of the eight textbooks give only examples of true quantified proposition, and this choice eliminate the syntactic aspect of quantifiers: quantifiers are used only to affirm something, there isn't the idea of a proposition built with a quantifier that one could wonder whether it is true or false.

The textbook *Indice* gives the example of the proposition "le carré d'un réel est positif" ("the square of a real number is positive") and states that "cette proposition est vraie quel que soit le nombre réel" ("this statement is true for all real numbers"). The authors of this textbook probably want to emphasize on the various possible meanings of the word *un*, but do not offer at the same time an example of the meaning as existential quantification. Moreover, to know that the meaning of *un* in this proposition is a universal quantification, it is necessary… to know that the universal proposition is true! Mathematical knowledge is therefore needed to decide between the two possible meanings of the word *un*, which calls for caution when using this word in a context where student knowledge is potentially fragile. Let's go back now on the comment "this statement is true for all real numbers". It makes no sense to say that the proposition "the square of a real number is positive", which is equivalent to "for all real x, the square of x is positive" is true for all real numbers, since the variable x is dummy in this proposition. The proposition referred to in this commentary is not the quantified one, but the not quantified proposition "the square of x is positive". Finally, there is a confusion between the use of "quel que soit" ("for all") to simply mark the universal quantification, and its use to assure that this universal proposition is true.

Now let's look at an example of exercise, taken from the textbook *Repères*, but it is an exercise that is found in many textbooks. Students must "complete the sentences (for example "… real number

x… $f(x)>0$") using either *for all… we have…* or *there exists… such that…*" from the graphical representation of the function *f*. Note first that the application is not explicitly to complete so that sentences are true! Furthermore, the instruction "complete using either… or…" suggests that each time only one of the both quantifiers is correct. Yet, when the proposition "for all *x* P[*x*]" is true, the proposition "there exists *x* such that P[*x*]" is also true, so when it is possible to complete with the universal quantifier, it is also possible to complete with the existential quantifier. In everyday language, we respect the principle of maximum information, according to which we give to our interlocutor all information in our possession. So, if I say "on my holidays, it rained some days", I say in the same time that it did not rain every day. The practice of this principle leads us in this exercise to complete naturally with the universal quantifier when possible. However, the notion of truth of a proposition will be contradicted by saying that using the existential quantifier is a mistake, because in mathematics, when the proposition "for all *x* P[*x*]" is true, it is not "more true" than the proposition "there exists *x* such that P[*x*]". Some students, however, adopt this position, and we can doubt position of the authors of the teacher's textbook who offers as a correction only the universal quantifier when it is possible.

Conclusion

I presented in this paper a methodological tool, a *reference* to study the teaching of logic. It is of course to be completed, to be improved, both from an epistemological and from a didactic point of view. Important work remains in particular on the concept of proposition, generally not made explicit in teaching, and on the notion of variable (didactic of algebra is very concerned about the status of the letters, but it seems to me that a logical point of view on the concept of variable, such as taking in Epp, 2011, or as I suggested in Mesnil, 2014, is more unusual), especially to identify students difficulties with these concepts that can be related to their epistemological complexity, or their use in the classroom.

Moreover, I have used for the moment this *reference* to analyze the syllabuses and textbooks, and a training for teachers who offers a similar approach of notions of logic. But it could also be used to study the practices of teachers, students' activity and conceptions. It would be particularly interesting to compare the effect of knowledge in mathematical logic that teachers have or have not.

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