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# Proof-based teaching as a basis for understanding why

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*The importance of proofs as a way to gain understanding has been observed many times. In this paper we show the result of two different experiences with division of natural numbers. The first comes from children in grade 3 who have learned about division and divisibility through what we call proof-based teaching (PBT), and the second comes from students who just finished their school studies and intend to become preservice primary teachers. Our main aim is to point out how different school experiences might lead to different (divergent) ways of gaining insight into the relationship between the divisor and the remainder. We particularly focus on describing some elements we identified in the third graders' instruction that might have allowed them to articulate their own understandings.*

*Keywords: Arithmetic, elementary school mathematics, number concepts, proof.*

## Proofs as a way to gain understanding

The goal of proof-based teaching is that students gain understanding through proving. Hence, it is based on past work on the role of proof as a means to understand or explain.

In mathematics education, explanation and understanding go together. The goal is understanding, so any explanation offered is aimed at having someone understand *why* a mathematical claim is true. This implies that a proof, to be useful in the classroom, should embody explanation. It should show not only that a result is true, but also *why* it is true. It should be concerned not only with its conclusion, but also with its main ideas, its overall structure, and its relationship to other mathematical fields and concepts (De Villiers, 2004; Hanna, 1990, 2000) (Hanna, 2016, p. 2).

Hanna (2016) discusses a number of different views of what makes a proof explanatory, but for our purposes, one aspect of these views is important. Explanatory proofs make reference to what we call a key notion, but which is also called a “characterizing property” or “salient feature”. “An explanatory proof makes reference to a characterizing property of an entity or structure mentioned” (Steiner, 1978, p. 143). “A proof can be explanatory only if ‘some feature of the result is salient’ and the proof builds upon that salient feature (Lange, 2014, p. 489, cited in Hanna, 2016, p. 4).

Elsewhere (Vallejo & Ordoñez, 2015; Reid, 2011) we have suggested that proof-based teaching (PBT), in which students learn mathematics through explanatory proving that builds on a shared body of knowledge, offers an opportunity for the development of relational understanding.

In the following we first elaborate on the elements of proof-based teaching based on our experience of a 3-year design research with third graders. We then show examples of the understandings of division of the third graders after a short unit of proof-based teaching instruction. Finally, we contrast these understandings with those of students who have completed secondary school and are about to begin university studies and draw some conclusions for teacher education.

## Proof-based teaching

Reid (2011) proposed proof-based teaching as “a way to develop understanding of mathematical concepts” (p. 28), and Vallejo has elaborated this idea in a 3-year design research intervention with third graders, the main goal of which was constructing division and divisibility knowledge.

The first intervention took place in 2013 in a Peruvian public school. This intervention was framed in the context of a master thesis (Ordoñez 2014) for which Vallejo was the supervisor. The second intervention took place in 2014 with a different group of third graders in the same public school. It addressed weaknesses identified in the first intervention through observing the difficulties students encountered in the lessons. We will report on the third intervention in the next section.

In the three interventions Vallejo taught all the sessions as a guest teacher in the classroom of another teacher. Written classwork assignments and quizzes were collected which helped the researchers to assess the students’ progress in their knowledge construction. All the sessions were videotaped, and significant parts of the first and third interventions were transcribed. In all three cases, the students had no prior knowledge of these topics at the time the interventions began as the goal was to see knowledge being constructed.

## Elements of proof based teaching

Through this research several elements of proof-based teaching have been identified as important: a ‘toolbox’ of shared knowledge, an expectation for explanation, and deductive explaining.

### The toolbox

In order to prove students must share a common set of accepted principles. A ‘toolbox’ of such principles is an essential feature of PBT and this also reflects the practice of professional mathematicians. We adopt the term “toolbox” from Netz (1999) who uses the term to describe the set of theorems and assumptions that are used in classical Greek proofs without explicitly referring to them. Thurston, (1995) describes the same phenomenon in contemporary mathematical practice:

Within any field, there are certain theorems and certain techniques that are generally known and generally accepted. When you write a paper, you refer to these without proof. ... Many of the things that are generally known are things for which there may be no known written source. As long as people in the field are comfortable that the idea works, it doesn’t need to have a formal written source. (p. 33)

In the interventions, Vallejo assessed prior knowledge through an individual diagnostic test, but more importantly, she established through a class discussion three “key notions” related to division and divisibility. These provided “a framework of established knowledge from which to prove” (Vallejo & Ordoñez, 2015, p. 231). The three key notions are:

Fair distribution: Distributions must give the same number of objects to each person.

Maximum distribution: The maximum number of objects possible must be distributed.

Whole distribution: Each person must receive a whole number of objects.

These key notions were the basis for the proof-based teaching of division and divisibility employing a mixture of written (individual and groups) tasks and class discussion.

## An expectation for explanation

From the very beginning, Vallejo's students were accustomed to being asked 'why?' for every conclusion they made or in general for every answer they gave based on the "key notions". "In the course of the sessions students also gave incorrect answers. Occasions of this type were exploited to promote discussion and justification by students since they were the ones who corrected the answers" (Ordoñez 2014, p. 334). She established in this way "an expectation that answers should be justified within this framework" (Vallejo & Ordoñez 2015, p. 231). It became part of the didactic contract (Brousseau, 1997) established in the classroom. In the context of proof-based teaching this is what we call an expectation for explanation.

## Deductive explanations

As part of the common 'toolbox' the whole class also shared an understanding of conjecture and justification, explained and modelled by the teacher, which was in tune with the meaning of proof given in A. Stylianides (2007). As part of the didactic contract the students knew that they could make as many conjectures as they wanted. The teacher wrote the students' conjectures at the blackboard to be analyzed by the whole class. But they were constantly reminded that in order for their conjectures to be upgraded to 'mathematical truths', they should provide strong support in the form of deductive arguments that were evaluated by the teacher.

## Third graders' understandings of division

We report here some results from the third cycle of the research design we discussed above. This intervention took place in a public school in Peru, in 2015, with a group of 21 third graders (7-8 years old). The intervention consisted of 23 sessions, each of them made of around 90 minutes. It was session 3 when these third graders discovered the relation between remainders and divisors and explained the relation using the key notions through a whole class discussion.

At the end of the intervention (session 23), the third graders were given a final test, including two items related to remainders:

Is it true that in a division by **4** we can have a remainder of **6**?  Yes  No Justify your answer.

In a whole, fair and maximum distribution among 5 people, how many objects may be left over at most? Why can no more objects be left over? Justify your answer.

These two items were number 8 and 9 on a test with 11 items. We report here on the children's responses to these two items, which are summarized in Table 1.

	First Item	Second Item
Correct answer with explanation	9 (43%)	12 (57%)
Correct answer with unclear explanation	2 (10%)	2 (10%)
Correct answer with no explanation	1 (5%)	1 (5%)
Incorrect answer	1 (5%)	3 (14%)
No response or question misunderstood.	8 (38%)	3 (14%)

**Table 1: Summary of results from the third graders' test**

Of the 21 children, 12 (57%) answered the first question correctly and 15 (71%) answered the second question correctly. Most of those who answered the second question correctly were also able to give an explanation. Their answers are based on a relational understanding of division, bringing together knowledge of the key notions they learned and experience with explanatory proving in this context.

For example, Bruno answered the first question “No. Because if we divide by 4 the remainder is at most 3, and 6 is more than 3”. This shows that he understands why the answer is no, and can explain by making reference to specific knowledge about division by 4, and implicitly to a general rule concerning the maximum remainder possible. Some children who answered the first item correctly (2 of the 12) provided a similar explanation, although their knowledge of the possible remainders when dividing by 4 was faulty. For example, Eduardo wrote “No, because in a division by 4 the only remainders are 1, 2, 3”. Eduardo omits one possible remainder, but his explanation is still appropriate, as he points out that 6 is not among the possible remainders in a division by 4.

On the second item, Max answered “Question 1: 4 can be left; Question 2: because I can keep distributing (objects)”. His answer shows his understanding of why the remainder cannot be more than 4 when distributing objects among 5 people. Although he does not refer to the condition by name (maximum distribution), he uses a condition that makes reference to it (“because I can keep distributing objects”) as the question makes reference to cases in which the maximum number of objects has not yet been distributed. Max’s answer is an example of the kinds of arguments they were able to produce.

Similarly, Renato’s answer “There can be at most 1, 2, 3 and 4 left over. More objects can’t be left over because it wouldn’t be (a) maximum (distribution)”, shows he understands why the maximum value for the remainder in a division by 5 is 4. He is actually the only student who makes explicit reference to this condition by its name in his written work. Even though Renato’s answer is incomplete (he doesn’t consider the remainder zero) his explanation is correct. The use of this common toolbox was consistent in this intervention.

From the very beginning Vallejo invited the students to share their ideas orally, and they seemed to feel comfortable to communicate in this way. However, some students had troubles with their writing skills while communicating their ideas individually, though they could still share well-thought ideas orally. Hence, after the final test Vallejo decided to interview some of the students who had performed well in whole class discussions, but not so well on the written tasks. These semi-structured interviews revealed that some students who had not given explanations had not understood the questions being asked. For example, Piero had answered the first test item by giving an example of a division by 4 that does not result in a remainder of 6. He did not understand that the question refers to dividing by 4 in general. When Vallejo asked the same question in the interview, he answered “No, because if there would be 6 left over, it would be 6 divided by 4, and I must continue distributing (objects)”. Like Piero, most of the students who gave an answer classified as “Question misunderstood” showed in the interview that they had not understood the question in the first item. However, when the question was clarified and they were given time to reflect, most were able to provide reasons.

We feel that the explanations given by the third graders demonstrate a relational understanding of divisibility, which arose through the proof-based teaching they experienced. We have not (for practical and ethical reasons) attempted to make a comparison with a matched group of third graders taught about divisibility in another way. Instead, in the next section we compare their understandings with those of students at the end of secondary schooling, who have had many other opportunities to develop their understandings of division.

## University students' understandings of division

We analyze here the answers given on a diagnostic test given at the beginning of university studies to 148 students enrolled in primary level teacher education. These students were enrolled at a private university and had received a government scholarship to support their teacher education. Hence, they can be assumed to be among the best students enrolling in primary level teacher education. Around 65% of these students came from the capital city, Lima, where the university is located, and the other 35% came from the other parts of Peru. The test was given prior to any instruction at the university, which means that it assessed only the understanding the students retained from their school experience.

The students were asked the following question: "In a division of natural numbers with the divisor equal to 3, what are all the possible values the remainder can take? Why?" To ensure that the terminology used in the question was understood, the question was accompanied by the diagram shown (which is a translation of the real one) in Figure 1.

**Remember that in every division:**

$$\begin{array}{r|l} \text{Dividend} & \text{Divisor} \\ \hline \text{(Remainder)} & \text{Quotient} \end{array}$$

**Figure 1: Reminder included with the question**

The task presented to these students is not exactly the same as presented to the third graders. This reflects the background knowledge of these two different groups. In the case of the prospective teachers, they were not familiar with the language fair, whole and maximum distributions and the third graders were not introduced to the terms dividend, divisor, or quotient. Despite that, one can see that both tasks ask for the same knowledge about the divisor and remainder relationship.

Table 2 summarizes the results from the pre-service teachers.

	Correct answer	Partially correct	Incorrect answer	No response
With explanation	9 (6%)	1 (1%)		
Reference to a general rule	25 (17%)	11 (7%)		
Without explanation	21 (14%)	13 (9%)	38 (26%)	30 (20%)

**Table 2: Summary of results from the pre-service teachers' diagnostic test**

A correct answer was given by 55 (37%) of the pre-service teachers. But of these only 9 gave explanations that show they understood the reason why the remainder must be 0, 1 or 2. For

example, Elizabeth wrote “It can only take values less than 3, in this case they would be (0, 1, 2)  $x < 3$ , because a number multiplied by 3 cannot be less than this. (x: Remainder)]”.

Among those giving a correct answer the most common way to answer the question “Why?” was by reference to a rule such as “the maximum remainder is one less than the divisor’s value” or “the remainder is always less than the divisor”. These answers may reflect understanding, but the rule may have been memorized without understanding. The remaining 21 responses include no explanation, an unclear response, or empirical evidence as “explanation”. Figure 2 shows a response of this last kind. Note that the divisions are of small numbers, but were done using a standard algorithm. In the first two cases, the dividends 1 and 2 were treated as if they were 10 and 20 for the purpose of determining the remainder, although the first decimal place of the quotient is worked out as if 1 and 2 are being divided. The pre-service teacher writes “Por lo tanto:” [Therefore] suggesting she feels that her six examples are sufficient to explain her answer. She also wrote “¿Por qué?” [Why?] with an arrow pointed to her examples, which is consistent if she believes these examples answer the question. It seems she was not able to provide a mathematical explanation.

Figure 2 shows a handwritten response on grid paper. At the top left, it says "a) Divisor = 3". Below this, there are six division problems arranged horizontally, each with a circled remainder below it:

- $10 \overline{) 3}$  with remainder 1
- $2 \overline{) 3}$  with remainder 2
- $3 \overline{) 3}$  with remainder 0
- $4 \overline{) 3}$  with remainder 1
- $5 \overline{) 3}$  with remainder 2
- $6 \overline{) 3}$  with remainder 0

Below the first two divisions, the text "Por lo tanto:" is written. Below the last two divisions, the text "¿Por qué?" is written with an arrow pointing to the examples. At the bottom, the text "los valores del residuo serian = 0, 1, 2" is written with a checkmark.

**Figure 2: A pre-service teacher’s response, showing a correct answer without an explanation**

Another 25 (17%) pre-service teachers gave partially correct answers (listing two of the three remainders, or listing 0 and 3 as distinct possibilities resulting in four remainders), 38 (26%) gave incorrect answers and 30 (20%) gave no answer. Overall, the responses of the pre-service teachers show an instrumental understanding (in the sense of Skemp, 1987) of division and limited number sense. Only 46 could give an explanation or cite a general rule and most used procedural approaches to determine the possible remainders in spite of the small numbers involved.

## Conclusions

We do not claim that this comparison replaces an experimental design with a control group, but this was not a goal of our design based research in any case. Nevertheless, it does offer some food for thought. One might expect that adults at the conclusion of more than a decade of schooling would have had many opportunities to develop concepts related to division, a basic operation in arithmetic and one that is basic to understanding of rational numbers and algebra. Why compare them with children who have had only twenty-three lessons on the topic? What we wish to compare are the

two different school experiences with division of natural numbers these two groups have had. Most of the preservice teachers can be assumed to have had a typical school experience in mathematics. That the third graders have a better understanding of division we feel reflects the non-traditional learning context they experienced, that allowed them to make sense of division. We strongly believe that proof-based teaching was important in their achievement of this understanding, but further research is needed to confirm this.

However, this comparison also raises an important question for teacher education. If, at the conclusion of secondary school, future primary school teachers do not understand basic concepts related to division, will they be able to guide children in the development of these concepts? If they are to develop these concepts as part of their teacher education, how can this best be done? Clearly the approaches taken in their schooling were unsuccessful. Our current research focusses on such pre-service teachers, and explore whether a proof-based teaching intervention at the university level can allow adults with instrumental understandings to develop relational understandings.

## References

- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer Academic Publishers.
- De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *International Journal of Mathematical Education in Science and Technology*, 35(5), 703–724.
- Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange*, 21(1), 6–13.
- Hanna, G. (2000). Proof, explanation and exploration: An overview. *Educational Studies in Mathematics*, 44(1-2), 5–23.
- Hanna, G. (2016, July). *Reflections on proof as explanation*. Paper presented in Topic Study Group 18, 13th International Congress on Mathematical Education, Hamburg.
- Lange, M. (2014). Aspects of mathematical explanation: Symmetry, unity, and salience. *Philosophical Review*, 123(4), 485–531.
- Netz, R. (1999). *The shaping of deduction in Greek mathematics: A study in cognitive history*. Cambridge UK: Cambridge University Press.
- Ordoñez, C. (2014). La construcción de la noción de división y divisibilidad de números naturales, mediada por justificaciones, en alumnos de tercer grado de nivel primaria. (Tesis de maestría en Educación Matemática). Pontificia Universidad Católica del Perú, Lima, Perú.
- Reid, D. (2011). Understanding proof and transforming teaching. In Wiest, L., & Lamberg, T. (Eds.) *Proceedings of the 33<sup>rd</sup> Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 15–30) Reno NV: University of Nevada.
- Skemp, R. (1987). *The psychology of learning mathematics*. Expanded American edition. Hillsdale, NJ: Laurence Erlbaum.
- Steiner, M. (1978). Mathematical explanation. *Philosophical Studies*, 34, 135–151.



- Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38(3) 289–321.
- Thurston, W. (1995). On proof and progress in mathematics. *For the Learning of Mathematics*, 15(1), 29–37.
- Vallejo Vargas, E., & Ordoñez Montañez, C. (2015). An example of proof-based teaching: 3rd graders constructing knowledge by proving. In Krainer & N. Vondrová (Eds.), *Ninth Congress of the European Society for Research in Mathematics Education* (pp. 230–231). Prague: Charles University and ERME.