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An expanded theoretical perspective for proof construction and its teaching

John Selden¹ and Annie Selden¹

¹New Mexico State University, Department of Mathematical Sciences, New Mexico, Las Cruces, U.S.A.; jselden@nmsu.edu, aselden@nmsu.edu

This paper presents a theoretical perspective for understanding and teaching university students' proof construction. It includes features of proof texts with which students may be unfamiliar. It considers psychological aspects of proving such as behavioral schemas, automaticity, working memory, consciousness, cognitive feelings, and local memory. We discuss proving actions, such as the construction of proof frameworks that could be automated, thereby reducing the burden on working memory and enabling university students to devote more resources to the truly hard parts of proofs.

Keywords: Proof construction, behavioral schemas, automaticity, consciousness, feelings.

Introduction

We report an expanded theoretical perspective to better notice, understand, and alleviate difficulties of university students' proof construction including: features of proof texts, types of proofs, situation-action links, behavioral schemas, automaticity, non-emotional cognitive feelings, and local memory. Most difficulties were observed and documented during a 10-year teaching experiment—a proof construction course. Our explanations call on the psychological as well as the education literatures. Local memory (Section 4.10) arose from observing our own proving experiences.

Features of proof texts

The proving process involves many more actions (both physical and mental) than appear in the final proof text (e.g., Mamona-Downs & Downs, 2009; Selden & Selden, 2016). Indeed, researchers have distinguished argumentation from proof, noting that an informal line of reasoning may “justify” a theorem to the prover’s satisfaction, but this often differs from the corresponding final proof text written to the standards of the mathematical community (e.g., Pedemonte, 2007).

The genre of proofs

Students sometimes find the manner in which proofs are written perplexing, as it is often at variance with other genres of writing. We have identified some significant features that generally occur in proof texts: (1) Proofs are not reports of the proving process. (2) Proofs contain little redundancy. (3) Symbols are (generally) introduced in one-to-one correspondence with mathematical objects. (4) Proofs contain only minimal explanations of inferences, that is, warrants are often left implicit. (5) Proofs contain only very short overviews or advance organizers. (6) Entire definitions, available outside the proof, are not quoted in proofs. (7) Proofs are “logically concrete” in the sense that quantifiers, especially universal quantifiers, are avoided where possible. (Selden & Selden, 2013a).

Structures of proofs

A proof can be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part that depends only on unpacking the logical structure of the statement of

the theorem, associated definitions, and earlier results. In general, this part *does not* depend on a deep understanding of the concepts or genuine problem solving in the sense of Schoenfeld (1985, p. 74). We call the remaining part of a proof the *problem-centered* part. It *does* depend on problem solving, intuition, heuristics, and understanding the concepts involved (Selden & Selden, 2011).

Proof frameworks

A feature that can help write the formal-rhetorical part of a proof is what we call a *proof framework*, of which there are several kinds, and in most cases, both a *first-* and a *second-level* framework. For example, given a theorem of the form “For all real numbers x , if $P(x)$ then $Q(x)$ ”, a first-level proof framework would be “Let x be a real number. Suppose $P(x)$ Therefore, $Q(x)$,” with the remainder of the proof ultimately replacing the ellipsis. A second-level framework can often be obtained by “unpacking” the meaning of $Q(x)$ and putting the second-level framework between the lines already written for the first-level framework. Thus, the proof would “grow” from both ends toward the middle, instead of being written from the top down.

Operable interpretations

Another feature that can help write the formal-rhetorical part of a proof is converting definitions and previously proved results into *operable interpretations*. These interpretations are similar to Bills and Tall’s (1998) idea of operable definitions. For example, given a function $f: X \rightarrow Y$ and $A \subseteq Y$, one defines $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. An operable interpretation would say, “If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa.” One might think translation into an operable form would be unnecessary or easy especially because the symbols in $\{x \in X \mid f(x) \in A\}$ can be translated into words in a one-to-one way. But for some students this requires practice.

Dimensions of potential proof construction difficulty

The need for previous results—proof types: 0, 1, 2, 3

We have classified theorems of increasing difficulty to refine our inquiry-based “proof course” notes (Selden & Selden, 2013b). *Type 0* often follows immediately from definitions. *Type 1* may need a result in the notes. *Type 2* needs a lemma, not in the notes, but relatively easily to discern, formulate, and prove. For *Type 3*, at least one of discern, formulate and prove should be difficult. A sample *Type 3* theorem is: *A commutative semigroup S with no proper ideals is a group*, when provided only the definitions of semigroup and ideal. One needs to observe that, for $a \in S$, aS is an ideal, so $aS=S$. This implies equations of the form $ax=b$ are solvable for any $b \in S$. Using some clever instantiations of this equation, one can obtain an identity and inverses, and conclude S is a group.

The need for unguided exploration

In constructing some proofs, one may reach a point where there is no “natural” way forward. In what we call *unguided exploration*, one may need to find, or define, an object and prove something about it, with no idea of its usefulness, that is, one may need to “explore” the situation. For example, in proving the above semigroup theorem, this can happen three times. First, one notes aS is an ideal and thus $aS=S$. Then one sees equations of the form $ax=b$ are solvable for any $b \in S$. Such exploration may require self-efficacy (Bandura, 1994), which can be encouraged by arranging early student successes.

The need to unpack the logical structure of a theorem statement

An *informal statement* is one that departs from the usual use of predicate and propositional calculus or fails to specify variables. For example, *Differentiable functions are continuous*, is informal because a universal quantifier and a variable are omitted, and because it departs from the usual “if-then” form of the conditional. Such statements are commonplace in everyday mathematics. They are not ambiguous or ill-formed because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians and less reliably by students. An informally stated theorem can be memorable and easily brought to mind, but it may be difficult to unpack and prove (Selden & Selden, 1995).

Psychological aspects of the proving process

We view proof construction as a sequence of actions that can be physical (e.g., writing a line of the proof or drawing a sketch) or mental (e.g., changing one’s focus from the hypothesis to the conclusion or trying to recall a theorem). The sequence of actions that eventually leads to a proof is usually considerably longer than the final proof text and is often not constructed from the top down.

Situations and actions

When considering proving, we use the term, *action*, broadly as a response to a situation in a partly completed proof. We include not only physical actions, but also mental actions. The latter can include trying to recall something or bringing up a feeling, such as a feeling of caution or of self-efficacy (Selden & Selden, 2014). In addition, we include “meta-actions” meant to alter one’s own thinking, such as changing focus to another part of a developing proof construction.

Situation-action links, automaticity, and behavioral schemas

If, during several proof constructions in the past, similar situations have corresponded to similar reasoning leading to similar actions, then, just as in classical associative learning (Machamer, 2009), a *link* may be learned between them, so that another similar situation evokes the corresponding action in future proof constructions without the need for the earlier intermediate reasoning. Using such *situation-action links* strengthens them, and after sufficient experience/practice, they can become overlearned and automated. We call automated situation-action links *behavioral schemas*.

Features of automaticity

In general, it is known that a person executing an automated action tends to: (1) be unaware of any needed mental process; (2) be unaware of intentionally initiating the action; (3) execute the action while putting little load on working memory; and (4) find it difficult to stop or alter the action (Bargh, 1994). However, not necessarily all four of these tendencies occur in every situation.

Forming behavioral schemas converts S2 cognition, which is slow, conscious, effortful, evolutionarily recent, and calls on considerable working memory, into S1 cognition, which is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory (Stanovich & West, 2000). This conversion into S1 cognition conserves working memory resources.

Behavioral schemas as a kind of knowledge

We view behavioral schemas as belonging to a person's knowledge base. They can be considered as partly conceptual knowledge (recognizing and interpreting the situation) and partly procedural knowledge (doing the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment". We suggest that, in using a situation-action link, or a behavioral schema, almost always both the situation and the action (or its result) will be at least partly conscious.

Here is an example of a behavioral schema that can conserve resources. One might be starting to prove a statement having a conclusion of the form p or q . This would be the situation. If one had encountered this situation a number of times before, one might readily write into the proof "Assume *not p*" and prove q or vice versa. While this action can be warranted by logic (if *not p* then q , is logically to, p or q), there would no longer be a need to bring the warrant to mind.

The genesis and enactment of behavioral schemas

The action produced by the enactment of a behavioral schema might be simple. It might also be compound, such as a procedure consisting of several smaller actions, each produced by the enactment of its own behavioral schema that was "triggered" by the action of the preceding schema in the procedure. We have developed a six-point theoretical sketch of the genesis and enactment of behavioral schemas (Selden, McKee, & Selden, 2010, pp. 205-206). Very briefly, here are the six points: 1) Within very broad contextual considerations, behavioral schemas are immediately available. 2) Simple behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action – one just does it. 3) Behavioral schemas tend to produce immediate action, which may lead to subsequent action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs. 4) Behavioral schemas were once actions arising from situations through warrants that no longer need to be brought to mind. Behavioral schemas cannot be "chained together" and act outside of consciousness, as if they were one schema. 5) An action due to a behavioral schema depends on conscious input, at least in large part. In general, a stimulus need not become conscious to influence a person's actions, but such influence is normally not precise enough for doing mathematics. 6) Behavioral schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing a detrimental behavioral schema requires similar, perhaps longer, practice.

Implicit learning of behavioral schemas

It appears that the process of learning a behavioral schema can be implicit, although the situation and the action are in part conscious. That is, a person can acquire a behavioral schema without being aware that it is happening. Indeed, such unintentional, or implicit, learning happens frequently and has been studied by psychologists and neuroscientists (e.g., Cleeremans, 1993). In the case of proof construction, we suggest that with the experience of proving a considerable number of theorems in which similar situations occur, an individual might implicitly acquire a number of relevant beneficial behavioral schemas. As a result, he or she might simply not have to think quite so deeply as before about certain portions of the proving process, and might, as a consequence of having more working memory available, take fewer "wrong turns".

Detrimental behavioral schema

Many teachers can recall having a student write $\sqrt{a^2 + b^2} = a + b$, giving a counterexample, and then having the student make the same error somewhat later, perhaps in a different context. Rather than being a misconception (i.e., believing something that is false), this may well be the result of an implicitly learned detrimental behavioral schema. If so, the student would not have been thinking very deeply about this calculation when writing it. Furthermore, having previously understood the counterexample would also have little effect in the moment. It seems that to weaken/remove this particular detrimental schema, the triggering situation of the form $\sqrt{a^2 + b^2}$ should occur a number of times when the student can be prevented from automatically writing “ $= a + b$ ” in response.

Feelings and proof construction

The word “feeling” is used in a variety of ways in the literature so we first indicate how we use it. Often feelings and emotions are used more or less interchangeably, perhaps because both appear to be conscious reports of unconscious mental states, and each can, but need not, engender the other. We will follow Damasio (2003) in separating feelings from emotions because emotions are expressed by observable physical characteristics, such as temperature, facial expression, blood pressure, pulse rate, perspiration, and so forth, while feelings are not.

Feelings, such as a feeling of knowing, can play a considerable role in proof construction (Selden, McKee, & Selden, 2010). For example, one might experience a feeling of knowing that one has seen a theorem useful for constructing a proof, but not be able to bring it to mind at the moment. Such feelings of knowing can guide cognitive actions because they can influence whether one continues a search or aborts it (Clore, 1992, p. 151). We call such feelings *non-emotional cognitive feelings*.

For the nature of feelings, we will follow Mangan (2001), who has drawn somewhat on William James (1890). Feelings seem to be summative in nature and to pervade one’s whole field of consciousness at any particular moment. Non-emotional cognitive feelings, different from a feeling of knowing, are: a feeling of familiarity and a feeling of rightness. Rightness is “the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge”. (Mangan, 2001). About such feelings, Mangan (2001) has written that “people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction.” Finally, we conjecture that feelings may eventually be found to play a larger role in proof construction than indicated above, because they provide a direct link between the conscious mind and the structures and possible actions of the unconscious mind.

The roles of affect and self-efficacy

In order to prove harder theorems--ones with a substantial problem-centered part--students need to persist in their efforts, and such persistence is facilitated by a sense of self-efficacy. According to Bandura (1995), self-efficacy is “a person’s belief in his or her ability to succeed in a particular situation”. Of developing a sense of self-efficacy, Bandura (1994) stated that “The most effective way of developing a strong sense of self-efficacy is through mastery experiences,” that performing a task successfully strengthens one’s sense of self-efficacy. Also, according to Bandura, “Seeing people similar to oneself succeed by sustained effort raises observers’ beliefs that they too possess the capabilities to master comparable activities to succeed.”

Bandura's ideas "ring true" with our past experiences as mathematicians teaching courses by the Moore Method (Mahavier, 1999). Such courses are taught from a brief set of notes consisting of definitions, requests for examples, and statements of major results, together with lesser results needed to prove them, but no proofs. The students provide the proofs and present them in class.

The development and uses of local memory

Some may think that proof construction consists mainly of conscious thought (i.e., as communication with oneself or others using speech, vision, etc., or their inner versions, as suggested by Sfard, 2010). However, we take a somewhat different view. In constructing a proof of some complexity, often much more relevant information can be activated than can be held in one's short-term working memory (ST-WM). When such information is lost from consciousness, it may not return to its original state, but rather to a state of partial activation. Nonetheless, conscious thought can sometimes influence the activation of related information in long-term memory (LTM), that is, help bring something to mind. Ericsson and Kintsch (1995) stated that "reliance on acquired memory skills will enable individuals [experts] to use LTM as an efficient extension of ST-WM in particular domains and activities after sufficient practice and training." We speculate that mathematicians can do this when conducting their own research. We have observed of ourselves, when attempting an intricate complex proof, that a considerable amount of information is generated, but cannot all be kept in mind; however, it is easily recalled. We refer to such partially activated information as *local memory* -- it is available as long as we are seriously engaged with the proof. It seems analogous to Ericsson and Kintsch's (1995) idea of long term working memory (LT-WM).

Teaching and future research considerations

We believe this perspective on proving, using situation-action links and behavioral schemas, together with information from psychology, is mostly new to the field. Thus, it is likely to lead to additional insights and teaching interventions, which brings up the question of priorities. Which proving actions of the kinds discussed above are most useful for mid-level university mathematics students to automate when they are learning how to construct proofs? Since such students are often asked to prove relatively easy theorems—ones that follow directly from definitions and recently proved theorems—it would seem that noting the kinds of structures that occur most often might be a place to start. Indeed, since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start. Furthermore, we have observed that some students do not write a second-level proof framework, perhaps because they have difficulty unpacking the meaning of the conclusion. This may be because a relevant definition needs to be converted into an operable interpretation in order to construct the second-level proof framework. Thus, helping students interpret formal mathematical definitions so that these become operable might be another place to start, even though students should eventually learn to make such operable interpretations themselves.

Finally, this theoretical perspective is likely to allow one to see parts of the teaching of proof construction in unusual ways and lead to new questions. For example, unguided exploration can be helpful for some proofs, but a student could easily feel the time required for exploration might reduce (timed) test grades. A feeling of self-efficacy might overcome that, but how are feelings "taught"? Early successes with proofs can help, but arranging for these might require detailed

planning of the course before it starts. Such planning could perhaps be aided by following a textbook, but most advanced mathematics textbooks prove the most important and useful theorems themselves, thereby taking away from students the opportunity to experience the proving of even parts of such theorems.

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