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Reasoning-and-proving in geometry in school mathematics textbooks in Hong Kong

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To promote learning mathematics with understanding, mathematics educators in many countries recommend that proof play a central role in school mathematics. In response to this recommendation, this study examines the opportunities for students to learn reasoning-and-proving from the geometry strand of a popular school mathematics textbook in Hong Kong. The study adopts the methodology of Stylianides (2009). Results show that Hong Kong takes the traditional but problematic approach in which proof is taught mainly in geometry and in which two-column proof is emphasized. Overall, results suggest that proof plays a marginal role in school mathematics in Hong Kong.

Keywords: Reasoning-and-proving, school mathematics textbooks, geometry, Hong Kong.

Introduction

In addition to verifying the truth of a mathematical statement, proof can have many other important functions in mathematics, including explanation, which can promote sense making and understanding in mathematics (de Villiers, 1990). As a consequence, many mathematics educators around the world, especially those in the US, recommend that proof (and proof-related reasoning) permeate school mathematics at all levels and in all content areas (e.g. Ball et al., 2002; NCTM, 2000). Furthermore, since textbooks can have an influence on what students learn, many studies have been conducted in different national curricula (e.g. US, Israel, Australia) to examine the opportunities for students to learn reasoning and proof from school mathematics textbooks. These studies were conducted at various grade levels (e.g. middle school, high school) and content areas (e.g. algebra, geometry); for example, see Stylianides (2009) and the articles devoted to this topic in Stylianides (2014). However, almost all of these studies were conducted in Western countries whereas only few studies have been conducted in East Asian countries (e.g. Singapore, South Korea, Hong Kong) where students have consistently performed very well in international studies of mathematics achievement such as TIMSS (e.g. Mullis et al., 2012). The present study is part of an on-going project aimed to complement the international research knowledge by examining the opportunities for students to learn reasoning and proof when they are using a popular secondary school mathematics textbook from Hong Kong. It is expected that the results obtained will shed light on how proof is being treated in school mathematics in one of those high-achieving countries (or regions) and provide insights into the influences that Chinese culture may have on issues concerning understanding in school mathematics. This paper reports our findings in geometry; for our findings in algebra, see Wong & Sutherland (2016).

The context: Hong Kong SAR

Being a special administrative region (SAR) of China, Hong Kong enjoys curriculum independence, in the sense that Hong Kong designs her own school curriculum, which is different from that in China. In 2009, Hong Kong launched her new academic structure, under which the number of years for senior secondary school changed from four years to just three years (Secondary 4, 5 and 6). Accordingly, at the same time Hong Kong initiated her New Senior Secondary Mathematics

Curriculum (Education Bureau HKSARG, 2007). This new curriculum consists of two parts: the Compulsory Part and the Elective Part (also called the Extended Part). It should be pointed out that teaching proof is not one of the stated goals (or processes) of the curriculum, which mentions proof only locally in the learning targets of geometry, namely, to “formulate and write geometric proofs involving 2-dimensional shapes with appropriate symbols, terminology and reasons” (ibid, p.15). The textbook series chosen for this study is the popular *New Century Mathematics* (2nd edition, Leung, Frederick K. S. et al., 2014–16). There were three reasons for choosing this textbook series: (a) this textbook series was one of the most popular ones in Hong Kong (if not the most popular one), (b) it was on the recommended booklist by the Educational Bureau, which means that it was guaranteed to be fully aligned with the new mathematics curriculum, and (c) it was coauthored by a prominent mathematics educator. Within this textbook series, there were two books (Books 4A and 4B) for Secondary 4, two books (Books 5A and 5B) for Secondary 5 and one book (Book 6) for Secondary 6. All topics in these five books were categorized into three strands: Number and Algebra, Data Handling, and Geometry (in the curriculum document (ibid.) the name “Measures, Shape and Space” was used instead).

Analytic framework and method

We followed the methodology of Stylianides (2009) in his investigation into reasoning and proof in school mathematics textbooks in the US. The framework he used was based on his conceptualization of *reasoning-and-proving* (RP), a term describing the overarching activity encompassing all of the four major proof-related mathematical activities: (a) identifying patterns, (b) making conjectures, (c) providing proofs, and (d) providing non-proof arguments. As shown in Table 1 below, the first two activities were grouped into the category of making mathematical generalizations and the latter two into the category of providing support to mathematical claims. The idea behind this conceptualization was that making mathematical generalizations (*identifying a pattern* and *conjecturing*) and providing support to mathematical claims (*proving*) are two fundamental and interrelated aspects of doing mathematics (Boero et al., 2007). Further, there were two kinds of pattern: plausible and definite; two kinds of proof: generic example and demonstration; and two kinds of non-proof argument: empirical argument and rationale. For the exact definitions of these terms, see Stylianides (2009).

| <i>Reasoning-and-Proving</i> | | | |
|---|-------------------------|--|---------------------------------------|
| I. Making Mathematical Generalizations | | II. Providing Support to Mathematical Claims | |
| (a) Identifying a Pattern | (b) Making a Conjecture | (c) Providing a Proof | (d) Providing a Non-proof Argument |
| 1. Plausible Pattern 2. Definite Pattern | 3. Conjecture | 4. Generic Example 5. Demonstration | 6. Empirical Argument 7. Rationale |

Table 1: The analytic framework (Stylianides, 2009, p. 262)

In this study, we focused on the Compulsory Part of the curriculum. We examined all of the eight chapters comprising the Geometry strand (see Table 4 below for the names of these chapters). Following Stylianides (2009), we focused on the exercises in these chapters and examined all of them. In each of these chapters, exercises were categorized under various headings: Q&A, Review Exercise, Instant Drill, Instant Drill Corner, Exercise, Supplementary Exercise, Class Activity, Inquiry & Investigation, and Unit Test. Within each category of these exercises, there were many tasks. A *task*

here means any problem in the exercises or parts thereof that have a separate marker (Stylianides, 2009, p. 270). *Task* served as unit of analysis in this study and there were totally 2,929 tasks to be analyzed and categorized into the seven subcategories of the constituent activities of reasoning-and-proving set out in Table 1 above. Additionally, we extended Stylianides' framework by further dividing the subcategory "Demonstration" into seven (sub)subcategories that correspond specifically to the different proof methods that were used in the exercises of the Geometry strand of our chosen textbook series; these included (i) Proof by Definition, (ii) Proof by Calculation, (iii) Proof by Calculation and Definition, (iv) Paragraph Proof, (v) 2-Column Proof, (vi) Proof by Contradiction, and (vii) Existence Proof (see Table 3 below). To decide if a task was an RP task, we considered how it appeared in the textbook (e.g. key phrases "Prove that", "Explain your answer"). In cases where the requirements were not clear, we consulted the Teacher's Manual (which contained suggested solutions, but only suggested solutions) in order to infer what types of response was expected for students.

Examples of analysis

Although there was a considerable amount of tasks, the forms of expression of tasks providing RP opportunities were very limited. Tasks phrased with the obvious "Prove that" or "Show that" were treated as RP activities (see Examples 2, 4 and 5 below). Those tasks ending in "Explain your answer." were also treated as RP activities, because they were asking for some kind of justification (see Examples 1 and 3 below). However, in some cases there was no explicit request to explain the answer, but judging from the solutions in the Teacher's Manuals, justifications were actually expected and hence these tasks were also treated as RP activities (see Task 3 of Example 5 below). Some tasks, usually in Class Activity or Inquiry & Investigation, were special in that they were part of a template for illustrating reasoning-and-proving. Such tasks were *dually* coded: on the one hand as a unit of analysis on its own, and on the other hand as part of the constituent activity (or activities) of reasoning-and-proving being illustrated (see Example 5 below).

Example 1

4. $Q(-1, 3)$ is rotated anticlockwise about the origin O through 90° to Q_1 .
- Write down the coordinates of Q_1 .
 - If Q_1 is reflected in the x -axis to Q_2 , are Q and Q_2 the same point? Explain your answer.
- Solution (from Teacher's Manual 4B, p.228):*
- Coordinates of $Q_1 = (-3, -1)$
 - Coordinates of $Q_2 = (-3, 1)$
Coordinates of $Q \neq$ Coordinates of Q_2
 $\therefore Q$ and Q_2 are not the same point.

Figure 1: Task 4(b) of Supplementary Exercise of Ch. 12 of Book 4B

Here Task 4(a) was not coded as any RP activity. Task 4(b) was coded as "Demonstration – Proof by Definition," by which we mean *one-step* deductive reasoning which can be derived directly from some definition (or property or theorem). This type of tasks does not involve substantive reasoning – its aim is simply to check students' understanding of the definition (or property or theorem).

Example 2

58. $A(0, \sqrt{2} + 1)$, $B(-\sqrt{2} - 1, 0)$ and $C(\sqrt{2} + 1, 0)$ are the three vertices of $\triangle ABC$.

(a) Show that $AC = 2 + \sqrt{2}$.

Solution (from Teacher's Manual 5B, p.96):

$$\begin{aligned} \text{(a)} \quad AC &= \sqrt{[0 - (\sqrt{2} + 1)]^2 + (\sqrt{2} + 1 - 0)^2} \\ &= \sqrt{2(\sqrt{2} + 1)^2} = \sqrt{2}(\sqrt{2} + 1) = 2 + \sqrt{2} \\ \therefore AC &= 2 + \sqrt{2} \end{aligned}$$

Figure 2: Task 58(a) of Supplementary Exercise of Ch. 7 of Book 5B

As shown, the solution involved substitution of values into the distance formula and algebraic manipulations to calculate AC . This task was coded as "Demonstration – Proof by Calculation". This proof method is also called "Mechanical Deduction" in the literature (e.g. Reid & Knipping, 2010, p. 124). Though involving mechanical algebraic manipulations and little reasoning, logically it should be regarded as a proof (for more on this point, see Slomson, 1996, p.11, "Proofs as Calculations").

Example 3

5. In the figure, $BM = CM = 6$ cm, $AM = 8$ cm and $AB = 10$ cm. AMD is a straight line. Is AD a diameter of the circle? Explain your answer.

Solution (from Teacher's Manual 4B, p.135):

$$AM^2 + BM^2 = (8^2 + 6^2) \text{ cm}^2 = 100 \text{ cm}^2$$

$$AB^2 = 10^2 = 100 \text{ cm}^2$$

$$\therefore AM^2 + BM^2 = AB^2$$

$$\therefore \angle AMB = 90^\circ$$

$$\therefore AM \perp CB$$

$\therefore AD$ is the perpendicular bisector of BC .

$\therefore AD$ passes through the centre of the circle.

$\therefore AD$ is a diameter of the circle.

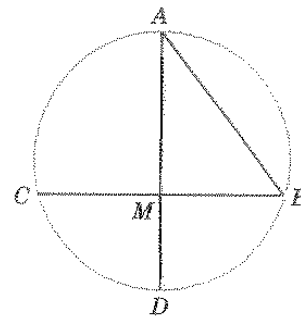


Figure 3: Task 5 of Supplementary Exercise of Ch. 10 of Book 4B

This task was coded as "Demonstration – Paragraph Proof", because it involved not just algebraic manipulations, and was written in the paragraph (or narrative) form – a less formal form in which it is not required to provide justification for every step. Paragraph proofs in geometry correspond to level 2 (informal deduction) of van Hiele Levels (Usiskin, 1982).

Example 4

11. In the figure, PCQ is a straight line. Chord AB is parallel to PQ . If $\widehat{AC} = \widehat{BC}$, prove that PQ touches the circle at C .

Solution (from Teacher's Manual 4B, p.182):

$$\therefore \widehat{AC} = \widehat{BC}$$

$$\therefore \angle ABC = \angle BAC$$

$$\angle BCQ = \angle ABC$$

$$\therefore \angle BAC = \angle BCQ$$

$\therefore PQ$ touches the circle at C .

given

equal arcs, equal angles

alt. \angle s, $AB \parallel PQ$

converse of \angle in alt. segment

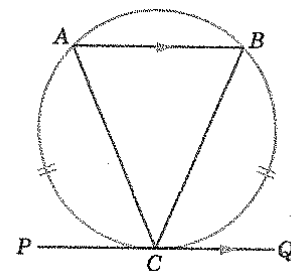


Figure 4: Task 11 of Exercise 11F of Ch. 11 of Book 4B

This task was coded as "Demonstration – 2-Column Proof", because it involved not just algebraic manipulations, and was written in the traditional two-column form – a more formal form in which every step is required to be justified with a reason and to be presented in the rigid two-column format as shown in Figure 4 above. Two-column proofs in geometry correspond to level 3 (formal deduction) of van Hiele Levels (Usiskin, 1982).

Example 5

Inquiry & Investigation 9.1: Alternative proof for the sine formula

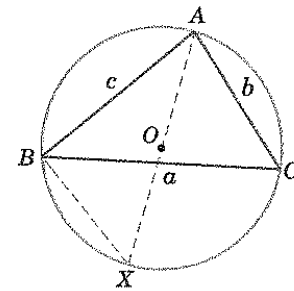
Inquiry

In $\triangle ABC$, what is the relationship among the radius r of the circumcircle, $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$?

Investigation Steps

In the figure, O is the centre of the circumcircle of $\triangle ABC$. The radius of the circumcircle is r . Produce AO to meet the circle at X . Join BX .

1. Find $\angle ABX$.
2. Consider $\triangle ABX$. Express $\sin X$ in terms of c and r .
3. What is the relationship between angles C and X ?
4. (a) Use the results of Questions 2-3 to express $\frac{c}{\sin C}$ in terms of r .
(b) Use similar method to express $\frac{a}{\sin A}$ and $\frac{b}{\sin B}$ in terms of r .



Conclusion

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \underline{\hspace{2cm}}$$

Figure 5: Task of Inquiry & Investigation 9.1 of Ch. 9 of Book 5B

This exercise consisted of six tasks (1, 2, 3, 4(a), 4(b) and Conclusion). It was a template for illustrating a direct proof. So these tasks were dually coded. First, each task was coded as a unit of analysis on its own. In this example, Task 3, and only Task 3, could be interpreted as an RP activity (Demonstration – Paragraph Proof or 2-Column Proof) in case the solution given in the Teacher's Manual would include a justification (e.g. "Angles in the same segment"). However, the solution given was just " $C = X$ ", so it was not regarded as a reasoning-and-proving task. Neither were the tasks 1, 2, 4(a), 4(b) and Conclusion. Then, each task was coded as part of the template illustrating RP activities. In this case, all of them were coded as "Demonstration – Paragraph Proof". For more examples, see the full version of this paper.

Results and Discussion

We have three major findings. Firstly, as shown in Table 2 below there were relatively limited opportunities (444 out of 2,929 tasks, i.e., 15.2%) for students to learn RP from the exercises of the Geometry strand of the chosen textbook series (Secondary 4 – 6). The majority of these exercises were to drill procedural skills. Secondly, there was a large difference between Making Mathematical Generalizations (24 tasks) and Providing Support to Mathematical Claims (420 tasks). This suggests that these two categories of activities were treated, in large part, in isolation from each other. This is problematic as they are two fundamental and interrelated aspects of doing mathematics (Boero et al., 2007; Cañadas et al., 2007; Hsieh et al., 2012). Thirdly, the majority of the RP opportunities were Demonstration (364 out of 444, i.e., 82%). However, as shown in Table 3 below, out of these 364 demonstrations, 32.1% were Proof by Definition or Proof by Calculation or Proof by Calculation and

Definition, all of which involve little reasoning. If we excluded them from Demonstration, the total RP opportunities would reduce to 11.2% (= 444 – 83 – 26 – 8 out of 2,929 tasks). A consequence that might be attributed to this lack of sufficient emphasis on proof even in geometry is that, as informed by TIMSS 2011 (Mullis et al., 2012, p.148 & p.150), “Hong Kong students in general do well in Knowing items, and relatively badly in Reasoning items” (Leung, 2015, p. 3).

| <i>Reasoning-and-proving subcategory</i> | <i>Frequency</i> | <i>(Percent)</i> |
|---|------------------|------------------|
| I. Making Mathematical Generalizations: | 24 | (5.4%) |
| (a) Identifying a Pattern: | | |
| 1. Plausible Pattern | 0 | (0.0%) |
| 2. Definite Pattern | 12 | (2.7%) |
| (b) Making a Conjecture: | | |
| 3. Conjecture | 12 | (2.7%) |
| II. Providing Support to Mathematical Claims: | 420 | (94.6%) |
| (c) Providing a Proof: | | |
| 4. Generic Example | 18 | (4.1%) |
| 5. Demonstration | 364 | (82.0%) |
| (d) Providing a Non-proof Argument: | | |
| 6. Empirical Argument | 38 | (8.6%) |
| 7. Rationale | 0 | (0.0%) |
| | Total: | 444 (100%) |

Table 2: Frequency and Distribution of RP Tasks across RP Subcategories

On the other hand, as shown in Table 4, 36% of the total RP tasks were concentrated in one chapter, namely, Book 4B Ch. 11 *More about Basic Properties of Circles* – more specifically, in Section 11.5 *Geometric Proofs on Circles*, which began with "We learnt many theorems relating to properties of circles in Book 4B Chapter 10 and this chapter. In this section we will learn how to use these theorems to prove more geometric properties." In the exercises of this section, almost every task asked for a 2-column proof, suggesting that the curriculum took a traditional approach in which proof is taught mainly in geometry and in which 2-column proof is emphasized. However, this approach to proof is problematic as it gives a misrepresentation of the nature of proof in mathematics (Wu, 1996) and its emphasis on form over meaning can lead to a shallow, syntactic kind of knowledge, rather than a connected understanding of the mathematics involved (Schoenfeld, 1988).

Given that Hong Kong teachers rely heavily on textbooks in their teaching (Tam et al., 2014), the above results not only confirm that in secondary school classrooms in Hong Kong students' activities mainly focus on practicing and memorizing mathematical concepts and procedures (Leung, 2001), but also suggest that proof plays a marginal role in school mathematics in Hong Kong. The fact that school mathematics textbooks in Hong Kong stress drilling on procedural (or calculation) skills far more than reasoning and proof may be due to influences from Chinese culture (or, more specifically, the Confucian heritage culture or CHC). According to Leung (2006, p. 43), CHC believes that “the

process of learning often starts with gaining competence in the procedure, and then through repeated practice, students gain understanding.” Additionally, CHC is an examination-oriented culture. In fact, the curriculum in Hong Kong is highly examination-driven. The fact that Hong Kong school mathematics textbooks stress practicing procedural (or calculation) skills far more than reasoning and proof may be a reflection of the strong influence of public examinations on textbook design. For more on how Chinese learn mathematics, see for example Fan et al. (2004).

| <i>Proof method</i> | <i>Frequency (Percent)</i> | |
|-------------------------------------|----------------------------|---------|
| Paragraph Proof | 174 | (47.8%) |
| Proof by Calculation and Definition | 83 | (22.8%) |
| 2-Column Proof | 71 | (19.5%) |
| Proof by Definition | 26 | (7.1%) |
| Proof by Calculation | 8 | (2.2%) |
| Proof by Contradiction | 1 | (0.3%) |
| Existence Proof | 1 | (0.3%) |

Table 3: Frequency and Distribution of Proof Methods used in Demonstration

| <i>Topic</i> | <i>Frequency (Percent)</i> | |
|---|----------------------------|---------|
| Book 4A Ch. 2 Equations of Straight Lines | 56 | (12.6%) |
| Book 4B Ch. 10 Basic Properties of Circles | 46 | (10.4%) |
| Book 4B Ch. 11 More about Basic Properties of Circles | 160 | (36.0%) |
| Book 4B Ch. 12 Basic Trigonometry | 47 | (10.6%) |
| Book 5B Ch. 7 Equations of Circles | 67 | (15.1%) |
| Book 5B Ch. 8 Locus | 26 | (5.9%) |
| Book 5B Ch. 9 Solving Triangles | 21 | (4.7%) |
| Book 5B Ch. 10 Applications in Trigonometry | 21 | (4.7%) |

Table 4: Frequency and Distribution of RP Tasks across Topics

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