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An exploratory teaching style in promoting the learning of proof

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Our overall concern is with helping students learn to construct and re-construct proofs. Here we investigate an exploratory style which invites learners to think for themselves, with the instructor circulating among them while listening, probing, and suggesting. The objectives of this investigation are, to understand how the actions of teachers can contribute to the development of their students' thinking, and to provide explicit pedagogic strategies that teachers can use to promote their students' appreciation and understanding of mathematical proof.

Keywords: Explanation, exploration, proof, proof construction.

Theoretical underpinnings

Our paper adopts the notion of acts of teaching (Mason, 2009; 2004): something that initiates and directs; something which is acted upon; and something which mediates between these, enabling the action to take place. We proceed from this stance to analyse an exploratory style as it was observed in an advanced undergraduate geometry class, with the aim of identifying what specific contributions this style brings to the learning process.

When teachers introduce a proof task, they are likely to have a complex set of expectations of what learners will get from engaging with the task. They are (we hope) aware of, or have access to connections with pervasive mathematical themes, with other contexts in which similar ideas arise, and with the specific powers that learners have. Teachers will have views on how these powers might be developed through working on the task, and on opportunities to interact with learners during which both mathematical thinking and appreciation and comprehension of some particular mathematical topic will be deepened and enriched. Tasks can vary from following an exposition, through exploring relationships, exercising new-found procedures, and making use of newly encountered technical terms as part of their personal and collective developing narrative.

Following the *Systematics* of Bennett (1956-1966; 1993), we distinguish six modes of interaction which arise from the teacher, the learner and the content playing the roles of the three impulses comprising any action: initiating, responding and mediating, and conveniently labelled by six 'exs': *expounding, explaining, exploring, examining, expressing* and *exercising* (Mason, 2004). In *expounding*, the initiative is with the teacher who uses the presence of learners (actual or virtual) as the mediator to make contact with the mathematical content in a significant way. The teacher draws the learners into the teacher's world and ways of perceiving and acting. By contrast to expounding, and also in contrast to its every-day sense, *explaining* in this framework involves the teacher using the content as a mediator in order to make contact with the student, through listening, watching and probing. The teacher tries to enter the world of the learner. As soon as the teacher thinks they 'know' where the learner's difficulty lies, the action usually reverts to exposition. *Exploring* involves students taking initiative, mediated by the teacher who may suggest a starting point for exploration, and may

make suggestions based on what students are saying and doing. *Examining* in this framework involves the student seeking to validate their own criteria against those of the teacher'. *Expressing* is what the student does when they feel the urge to articulate insights or make conjectures. This can occur for example in response to a teacher asking probing questions. *Exercising* is what a student does when they feel the desire to practice in order to gain fluency. Our interest here is on the interactions labelled as *exploring, expressing*, and *explaining* in the technical sense used in this framework.

Our investigation builds upon previous research. Grenier (2013) has shown that an experimental teaching approach focusing on various "research situations" can succeed in helping students master mathematical reasoning and proving. Selden and Selden (2013) considered a division of proofs into a *formal-rhetorical* part and a *problem-centered* part. In their view, the formal-rhetorical part of a proof depends only on unpacking and using the logical structure of a theorem, while the problem-centered part depends on exploration and understanding, which is essential to the learning of proof.

Several researchers have investigated the use of explorations in various learning contexts. In the use of dynamic geometry, for example, Mariotti (2000), and others have shown the value of teaching proof through exploration. In the context of argumentation and proof and of the axiomatic organization of mathematics, several other researchers have examined students' use of empirical explorations (Durand-Guerrier et al., 2012; Hanna, 2010; Hemmi, 2010; Jahnke & Wambach, 2013; Reid & Knipping, 2010; Stylianides & Stylianides, 2009). Additionally, Garuti et al. (1998) and Pedemonte (2007) highlighted what they termed *cognitive unity*, the continuity between the process of conjecturing achieved through exploration and the production of an acceptable proof.

The exploratory style in this study

We investigate a particular exploratory style in which university students already familiar with linear algebra are taking an advanced geometry course in which they are exposed to proofs by means of explorations that invite them to think for themselves, with the instructor circulating among the students, listening to their discussions, asking probing questions intended to help the students think more deeply about the issue at hand, and at times offering suggestions.

Participants and classroom setting

The 24 participants were undergraduate mathematics students in a mixed-year class (2nd to 4th year) at a large urban university in central Canada. The advanced geometry course covered plane geometry, spherical geometry, and briefly, some hyperbolic geometry. It addressed the critical role of transformations (symmetries and isometries) in all of these geometries, the use of dynamic geometry software (The Geometer's Sketchpad), and proof. The instructor of the course was a geometer. The class met once a week for 3-hour sessions, 20 classroom meetings in total. The course was intended to keep equal proportion of instruction time and exploration time to facilitate the exploratory style of teaching and learning. Three classroom sequences were assigned to the investigations on conic sections. This paper focuses on one of the sessions – the exploration of ellipse.

To facilitate investigations and communication, the classroom setting of the geometry course was unconventional: 7 large round tables with chairs around each filled up the classroom, with 5 large blackboards mounted on three walls. In addition, manipulatives and visual aids associated with geometry were kept in the closet at the back of the classroom with free access for the students.

Data collection and analysis

The exploratory style of learning proofs was documented through (1) classroom observations in the form of audio recordings and field notes; (2) follow-up questions for students; (3) students' course reflections on the explorations; and (4) the researcher's research journal. In particular, the follow-up questions consist of 4 open-ended questions about students' explorations in geometry throughout the course. 17 out of 24 students completed the follow-up question sheets. All students submitted the course reflections.

The data drawn through classroom observations was organized and analyzed by the framework of 6 'exs'. The data from students' written reflections was analyzed using NVivo 10 software to explore themes and patterns of responses. The unit of analysis was a statement. Each participant's work was divided into statements and grouped in categories.

Investigations of conic sections: Findings and analysis

This paper focuses on how proving was promoted through the initiative taken by the instructor and the students, while the content connects the instructor and the students. It does not measure the students' achievement because it is concerned with perceptions of their own understanding of proof.

Initiatives of the instructor – Expounding and Explaining

The class made use of a hands-on investigation involving flashlights to explore conic sections. The instructor held a flashlight aimed at a wall at different angles. "The flashlight bulb and reflector make a "cone" of light. The wall cuts the cone with a plane, making a conic section. So moving the light changes which section we have," the instructor explained. As the beam was forming a circle, ellipse or parabola, he asked students to identify the particular shape made on the wall and to pay attention to the critical points where there was a change from one conic section to another, as the angle changed. Then the instructor raised a question about hyperbolae with a suggestion, "Now, what features will confirm the shadow is a hyperbola? You may look for asymptotes - lines which the light approaches as it goes up the wall."

Initiatives of the students - Exploring and Examining

After the demonstration, students worked in groups to create all four conic sections by using the light source of their smartphones or the flashlights provided. Students quickly discovered that when the beam was perpendicular to the wall, it gave a circle; when it was tilted a bit, it gave an ellipse; when it was tilted more, with the ellipse vanishing, a parabola emerged. Group discussions mainly focused on the creation of a hyperbola and the difference between hyperbolae and parabolae.

S8: How do you know it is a parabola or hyperbola?

S22: It depends on the angle you hold it at.

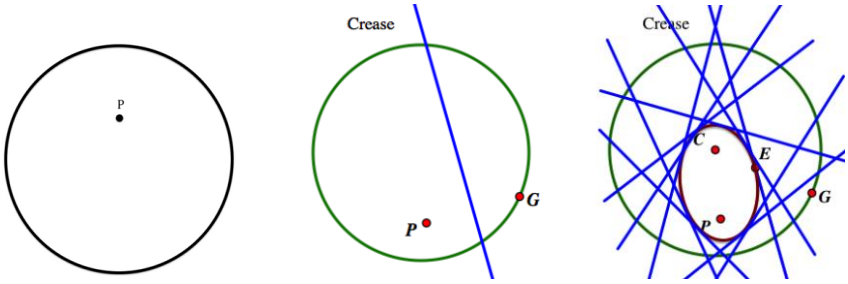
S8: Right, but how does an angle tell us whether it is a parabola or hyperbola?

S12: Well… if you look at this graph I found online, the parabola's axis is parallel to the cone's side. If it were not parallel, it would become a hyperbola.

S8: I see. So how is this related to what we are doing? The wall is the cutting plane and the light source is the cone. When the wall is not parallel to the borderline of the beam, it is a hyperbola. I cannot make them parallel precisely, but I get it.

This group discussion shows that in the course of their exploration, students did not limit their exploration to the flashlight demonstration but went on to research the problem by retrieving information online so as to better understand the features that confirm that the shadow was a hyperbola, the difference between hyperbolae and parabolae, and also to explain it to their peers.

Following the flashlight investigation, a series of paper folding activities was carried out. Taking the ellipse as an example, each student was given a clean sheet of paper with a circle and a point P inside the circle but away from the center (Figure 1a). Students were first asked to pick a point on the circle, say G, and fold the paper until P was lying directly on top of G (Figure 1b), and then to make a neat crease. Then students were asked to repeat the fold and crease action for a few dozen relatively evenly spaced points on the circle and to observe what shape emerged (Figure 1c).



a b c

Figure 1: The sheet of paper for folding to create an ellipse

Some students struggled to work out which point was being folded and to where – mistakenly folding a chosen point onto some other point on the circle, while others struggled to create a precise fold, due to the nontransparent nature of paper sheets. Although students worked on the folding at their own pace, within groups, they were talking to and helping each other as they proceeded, which allowed the ones who were struggling to listen, watch and move forward. With a number of creases created, conjectures were emerging in groups. For instance:

S6: I know that it is not going to be a circle. It is not circular. It would only be a circle if you can fold it onto the middle point. If you can fold it onto other point, it will be off-site the shape.

S17: I think it is going to be a parabola and P is going to be the focal point of the parabola. It makes sense.

Instructor: You need more folds. You can select a few more points on this side of circle (pointing at the sheet that S7 was holding).

S7: Oh, wait. It is an ellipse! I have a lot of lines. You can envision other points are going to be there. It is very clear it is an ellipse. You can really see it!

S17: (looking at S7's paper) Yeah, it is an ellipse… P looks like one of the focal point.

S6: Where is the other focal point?

Here we see that the students did not always do more than offer a conjecture about the shape of the conic created by the creases. The instructor felt it necessary to intervene and re-direct the students' attention to the core idea (the content) of the session.

Initiatives of the content - Exercising and Expressing

With more questions raised, the instructor asked each group to focus on the following questions:

(1) Now you have this ellipse, you know how to paper fold it. How do you prove it is an ellipse?
(2) If you pick one of the folds, how does this fold help us prove it is an ellipse?
(3) What can you say about this ellipse and the circle? How and why are they related?

S17: If you have this fold, and you have this distance from point P, then this distance (PE+EC) is going to equal to that distance (GC) because this is a reflection.

S6: Yes, but what does that have to do with the ellipse?

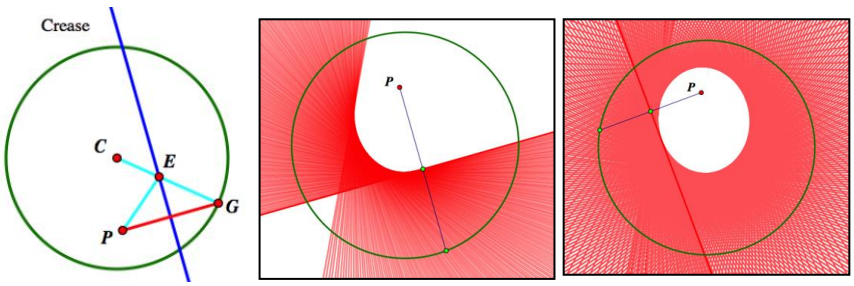
Instructor: Note: GC is the radius. These two distances (PE and EC)…

S6: No, I didn't get it.

Though the students had all the information they needed, they still had difficulty reaching the final step of the proof. By posing a prompt and question, the instructor tried to direct students' attention to the sum of the two distances (PE+EC) and the fact that it is equal to the radius. Then, a GSP graph, similar to the folding sheet in Figure 1 above, was shown to have students focus on the relationship between the two (see Figure 2a).

S17: OK. We are looking at P to E and E to C. The distance is equal to GC, which is the radius of the circle. Oh… That makes an ellipse because this distance (PE) plus this distance (EC) is fixed, the radius of the circle.

S6: And the ellipse must have something to do with the center as a focal point. If P is moving around according to the center point, the ellipse will just move as you move the P. So the center is another focal point. You can take any circle and a point off the center of the circle, and it will always be the case.



a b c

Figure 2: A GSP graph (a) and animation of the formation of an ellipse (b and c)

A Geometer's Sketchpad (GSP) animation was shown at the very end of the class. Figures 2(b) and 2(c) were two snapshots of the GSP animation indicating how an ellipse was constructed. This process taught them that, "A hypothesis is evaluated by deductively drawing consequences and by investigating whether these consequences agree with experience or should be accepted for other reasons" (Jahnke & Wambach, 2013, p. 469).

In a subsequent class, students were shown a demonstration in which sand was poured onto a circular disk with an off-center hole in it corresponding to the point in Figure 1a. A number of the students voluntarily poured the sand through a plastic strainer. As more sand poured onto the board, more a "ridge" clearly emerged. Looking down on it from above, it appeared to be an ellipse.

Students' reflections

Paper folding and sand pouring

More than half of the students admitted that they found paper folding complicated. While struggling with the first step, they missed the instructions for the second step. Following what was being said and doing folds correctly was an obvious challenge. Despite the confusion and the errors they made in folding, students appreciated the geometry embedded in the experiments. As S3 put it: "The reason for why the specific folds result in the shape never bothered me until I had this experience. I felt that I had a teacher explained the mathematical relevance." S4 observed that seeing the ellipse emerge during the sand pouring, was completely unexpected: "You can actually see an ellipse due to gravity pulling the excess parts down, forming a 'hill'." When asked about the definitions of conic sections, S14 stated that, "it is much easier, at least for me, to recall a process or property that I have physically manipulated or seen carried out visually than to recall a written definition."

The GSP demonstrations

The students were asked whether the GSP graph and demonstration directly or indirectly helped them with the proof of ellipse. The majority of the students claimed that the animation directly helped with the proof. The responses that support this claim can be categorized as follows:

- *Have attention focused.* "We had all the information but we couldn't prove it until we saw this picture (Figure 2a) which has all the lines with the colors" (S6).
- *Accuracy*. "The animation is more visual and accurate than the paper folding" (S7); "The animation showed an infinite number of straight lines without making mistakes" (S12).
- *Legitimate process*. "The animation presented a 3D visual to experiment with" (S6); and "It provided an extended version of the folds we did in class and allowed me to continue my experience in a less time consuming and more efficient way (S17).
- *Exposure of the final product*. "It showed the final product of paper folds when you fold 100 or 1000 times" (S4).

However, 2 students claimed that the GSP indirectly helped with the proof. One student explained that the animation "helped more with my understanding as opposed to helping me write down the formal proof" (S16), whereas the other student believed that the role of the GSP was to show how the ellipse was constructed and to show how the proof connects to the demonstration (S5). Compared to the hands-on investigations, S7 and S20 claimed that the GSP animation presented in class did help with their understanding, but they had difficulty interacting with it. This is so because they did not know how to use the geometry software. As S11 said, "If I knew how to use the software, I would definitely use it more".

Discussion

Adopting an exploratory style of teaching is metaphorically a bit like heading off into unknown territory; perhaps a city or forest not previously visited, and coming across blaze marks. On the surface, the 'exploratory style' of teaching involved initial stimuli provided by the instructor, then the instructor circulating listening, probing and suggesting. Questions raised by group members were fundamental and critical for the trajectory of the sessions. Beneath the surface lie the subtleties in how much time students were given to think for themselves, to discuss with each other, to try to

resolve questions that arose, and to seek assistance from the instructor. Three different contexts in which the same shapes emerge can be seen as a form of variation (Marton, 2015) with both conceptual and procedural aspects. The instructor's commitment to experiential style of engaging students with the mathematical content provided opportunity for different forms of interaction: students were stimulated to explore, to express, to seek explanation when they felt they needed it, and even to exercise their developing ideas. This prepared them to be able to make sense of what little exposition was provided during the sessions. Being fully engaged, with their hands, their own thoughts, and discussing with their peers, enabled them to produce a proof in which they had a high degree of confidence. Because the conjectures came mostly from them, they had an interest in proving, and a desire to find a proof. The physical and virtual phenomena directed student attention to the dynamic changes and alterations of the objects that they were creating, particularly when the instructor noticed students' struggles during the exploration. Student attention was directed by their peers through formulating conjectures, raising questions, and communicating their thoughts. These allowed learners to be immersed in an environment which engaged them to make conjectures, to try to express their vague thoughts, to modify their own conjectures and to challenge the conjectures of their peers, which is in line with the observations of Grenier (2013) and Hemmi (2010).

Pedagogical implications

One of the features that distinguish mathematics from other disciplines is that mathematical conjectures ultimately require proof. One importance of exposure to mathematical reasoning and proof is that it provides learners with an opportunity to "know that they know", not because someone has asserted something but because they can justify it on the basis of previously agreed properties. In the case of the paper folding and sand pouring, it is a means to provoke students' curiosity of why it works and invite them to discover a geometric proof of ellipse on their own. The exposure to the necessity of 'why' could have great impact on promoting students' learning of proof when the activity is carefully designed and chosen.

The Geometer's Sketchpad was used in the classroom throughout the course. Introducing and using geometry software in the classroom at a regular basis can gradually change the way that students approach geometry. However, as a teacher, exploiting manipulatives and geometry software effectively requires familiarity with the materials. Each context has its own trajectory, in terms of time required to make sense of the actions and to interpret the effects. Perhaps the most important pedagogic implication is the need to stimulate students to make connections, to develop their own personal narrative concerning the connections between different manifestations of the same mathematical object.

For the student teachers in the class, we believe that the exploratory teaching style allowed them to grow as a student and as a teacher. At first they looked to be told what they must do and how they will be assessed. By engaging them in mathematical activity, they had a chance to experience themselves as mathematician-learners, and to exercise and develop their own powers to imagine and express, specialize and generalize, conjecture and convince (Mason, 2004). As S3 put it, "My focus throughout the course remained on learning rather than passing, as it should be."

Thinking in terms of modes of interaction has enabled us to add a little bit of detail to the notion of an 'exploratory style' of teaching. However, in order to provide specific advice for teachers, it will be necessary to discern even finer details, which deserves further studies.

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