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# ON A SENARY QUARTIC FORM

JIANYA LIU, JIE WU & YONGQIANG ZHAO

ABSTRACT. We count rational points of bounded height on the non-normal senary quartic hypersurface  $x^4 = (y_1^2 + \cdots + y_n^2)z^2$  in the spirit of Manin's conjecture.

## 1. INTRODUCTION

Recently, we [7] proved Manin's conjecture for singular cubic hypersurfaces

$$(1.1) \quad x^3 = (y_1^2 + \cdots + y_n^2)z,$$

where  $n$  is a positive multiple of 4. In this short note, we show that our method used in [7] also works for higher degree forms like

$$(1.2) \quad x^m = (y_1^2 + \cdots + y_n^2)z^{m-2},$$

where  $n \geq 4$  and  $m \geq 4$ . To illustrate, we establish an asymptotic formula for the number of rational points of bounded height on the quartic hypersurface

$$(1.3) \quad Q : x^4 = (y_1^2 + y_2^2 + y_3^2 + y_4^2)z^2,$$

in the spirit of Manin's conjecture.

It is easy to see that the subvariety  $x = z = 0$  of  $Q$  already contains  $\gg B^4$  rational points with  $|x| \leq B$ ,  $|z| \leq B$ , and  $|y_j| \leq B$  with  $1 \leq j \leq 4$ , which is predominant and is much larger than the heuristic prediction that is of order  $B^2$ . One therefore counts rational points on the complement subset  $U = Q \setminus \{x = z = 0\}$ . Let  $H$  be the height function

$$H(x : y_1 : \cdots : y_4 : z) = \max \left\{ |x|, \sqrt{y_1^2 + \cdots + y_4^2}, |z| \right\}$$

for  $(x, y_1, \dots, y_4, z) = 1$ . Let  $B$  be a large integer, and define

$$N_U(B) := \left| \left\{ (x : y_1 : \cdots : y_4 : z) \in U : H(x : y_1 : \cdots : y_4 : z) \leq B \right\} \right|.$$

This counts rational points in  $U$  whose height is bounded by  $B$ , and the aim of this note is obtain an asymptotic formula for it. To this end, we need to understand in advance a similar quantity

$$N_U^*(B) := \sum_{\substack{1 \leq |x| \leq B, 1 \leq y_1^2 + \cdots + y_4^2 \leq B^2, |z| \leq B \\ x^4 = (y_1^2 + \cdots + y_4^2)z^2}} 1.$$

One sees, in  $N_U^*(B)$ , that the co-prime condition  $(x, y_1, \dots, y_4, z) = 1$  in  $N_U(B)$  is relaxed. Our main result is as follows.

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**Theorem 1.1.** *As  $B \rightarrow \infty$ , we have*

$$(1.4) \quad N_U(B) = \mathcal{C}_4 B^3 \log B \left\{ 1 + O\left(\frac{1}{\sqrt[4]{\log B}}\right) \right\},$$

$$(1.5) \quad N_U^*(B) = \mathcal{C}_4^* B^3 \log B \left\{ 1 + O\left(\frac{1}{\sqrt[4]{\log B}}\right) \right\}$$

with  $\mathcal{C}_4 := \frac{192}{5\zeta(3)} \mathcal{C}_4$  and  $\mathcal{C}_4^* := \frac{192}{5} \mathcal{C}_4$ , where  $\mathcal{C}_4$  is defined as in (2.6) below, and  $\zeta$  is the Riemann zeta-function.

We note that the exponent of  $B$  in the main terms of the above theorem is 3 instead of 2 as predicted by the usual heuristic. This phenomenon may be explained by the fact that the hypersurface  $Q$  is not normal.

It is easy to check that  $Q$  has an obvious quadric bundle structure given by

$$(1.6) \quad Q_{[a:b]} : \begin{cases} b^2 x^2 = a^2 (y_1^2 + y_2^2 + y_3^2 + y_4^2), \\ ax - by = 0, \end{cases}$$

and  $\{Q_{[a:b]}\}$  covers  $Q$  as long as  $[a : b]$  goes thorough  $\mathbb{P}^1(\mathbb{Q})$ . From this, it is possible to interpret Theorem 1.1 in the framework of the generalized Manin's conjecture by Batyrev and Tschinkel [1], as was done in the work of de la Bretèche, Browning, and Salberger [3]. However, we will not pursue such an explanation here. The only sole purpose of this short note is to show that our method used in [7] also works for higher degree forms  $Q$ .

Finally, we remark that using the method in our joint paper [4] with de la Bretèche, one can get power-saving error terms in Theorem 1.1, which we will not pursue here.

## 2. OUTLINE OF THE PROOF OF THEOREM 1.1

Denote by  $r_4(d)$  the number representations of a positive integer  $d$  as the sum of four squares :  $d = y_1^2 + \cdots + y_4^2$  with  $(y_1, \dots, y_4) \in \mathbb{Z}^4$ . It is well-known (cf. [5, (3.9)]) that

$$(2.1) \quad r_4(d) = 8r_4^*(d) \quad \text{with} \quad r_4^*(d) := \sum_{\ell|d, \ell \not\equiv 0 \pmod{4}} \ell.$$

Let  $\mathbb{1}_{\square}(n)$  be the characteristic function of squares. In view of the above, we can write

$$(2.2) \quad N_U^*(B) = 32 \left\{ \sum_{n \leq B} \sum_{\substack{d|n^4 \\ d \leq B^2}} r_4^*(d) \mathbb{1}_{\square}\left(\frac{n^4}{d}\right) - \sum_{n \leq B} \sum_{\substack{d|n^4 \\ d < n^4/B^2}} r_4^*(d) \mathbb{1}_{\square}\left(\frac{n^4}{d}\right) \right\}.$$

Hence to prove (1.5) in Theorem 1.1, it is sufficient to establish asymptotic formulae for the following two quantities

$$(2.3) \quad S(x, y) := \sum_{n \leq x} \sum_{\substack{d|n^4 \\ d \leq y}} r_4^*(d) \mathbb{1}_{\square}\left(\frac{n^4}{d}\right), \quad T(B) := \sum_{n \leq B} \sum_{\substack{d|n^4 \\ d < n^4/B^2}} r_4^*(d) \mathbb{1}_{\square}\left(\frac{n^4}{d}\right).$$

For  $S(x, y)$ , our result is as follows.

**Theorem 2.1.** *Let  $\varepsilon > 0$  be arbitrary. We have*

$$(2.4) \quad S(x, y) = xy(4P(\psi) + \frac{3}{2}P'(\psi)) + O_\varepsilon(x^{\frac{5}{4}}y^{\frac{7}{8}} + x^{\frac{1}{2}+\varepsilon}y^{\frac{9}{8}})$$

*uniformly for  $x^3 \geq y \geq x \geq 10$ , where  $\psi := \log x - \frac{1}{4}\log y$  and  $P(t)$  is a quadratic polynomial, defined as in (4.18) below. In particular, for any fixed  $\eta \in (0, 1]$  we have*

$$(2.5) \quad S(x, y) = 4\mathcal{C}_4xy \left( \log x - \frac{1}{4}\log y \right) \left\{ 1 + O\left( \frac{1}{(\log x)^\eta} \right) \right\}$$

*uniformly for  $x \geq 10$  and  $x^2(\log x)^{-8(1-\eta)} \leq y \leq x^3$ , where*

$$(2.6) \quad \mathcal{C}_4 := \frac{23}{150}\zeta(5) \prod_p \left( 1 + \frac{1}{p} + \frac{2}{p^2} + \frac{2}{p^3} + \frac{1}{p^4} + \frac{1}{p^5} \right) \left( 1 - \frac{1}{p} \right)$$

*is the leading coefficient of  $P(t)$ .*

Now we turn to analyze  $T(B)$  which is more difficult, since the range of its second summation depends on the variable  $n$  of the first summation. Thus Theorem 2.1 does not apply to  $T(B)$  directly. In §5 we show that Theorem 2.1 together with delicate analysis is sufficient to establish the following result.

**Theorem 2.2.** *As  $B \rightarrow \infty$ , we have*

$$(2.7) \quad T(B) = \frac{2}{5}\mathcal{C}_4B^3 \log B \left\{ 1 + O\left( \frac{1}{\sqrt[4]{\log B}} \right) \right\},$$

*where  $\mathcal{C}_4$  is as in (2.6) above.*

As in [2, 7], we shall firstly establish an asymptotic formula for the quantity

$$(2.8) \quad M(X, Y) := \int_1^Y \int_1^X S(x, y) dx dy.$$

by applying the method of complex integration. Then we derive the asymptotic formula (2.4) for  $S(x, y)$  in Theorem 2.1 by the operator  $\mathcal{D}$  defined below. Let  $\mathcal{E}_k$  be the set of all functions of  $k$  variables. Define the operator  $\mathcal{D} : \mathcal{E}_2 \rightarrow \mathcal{E}_4$  by

$$(2.9) \quad (\mathcal{D}f)(X, H; Y, J) := f(H, J) - f(H, Y) - f(X, J) + f(X, Y).$$

The next lemma summarises all properties of  $\mathcal{D}$  needed later.

**Lemma 2.1.** (i) *Let  $f \in \mathcal{E}_2$  be a function of class  $C^3$ . Then we have*

$$(\mathcal{D}f)(X, H; Y, J) = (J - Y)(H - X) \left\{ \frac{\partial^2 f}{\partial x \partial y}(X, Y) + O(R(X, H; Y, J)) \right\}$$

*for  $X \leq H$  and  $Y \leq J$ , where*

$$R(X, H; Y, J) := (H - X) \max_{\substack{X \leq x \leq H \\ Y \leq y \leq J}} \left| \frac{\partial^3 f}{\partial x^2 \partial y}(x, y) \right| + (J - Y) \max_{\substack{X \leq x \leq H \\ Y \leq y \leq J}} \left| \frac{\partial^3 f}{\partial x \partial y^2}(x, y) \right|.$$

(ii) *Let  $S(x, y)$  and  $M(X, Y)$  be defined as in (2.3) and (2.8). Then*

$$(\mathcal{D}M)(X - H, X; Y - J, Y) \leq HJS(X, Y) \leq (\mathcal{D}M)(X, X + H; Y, Y + J)$$

*for  $H \leq X$  and  $J \leq Y$ .*

The next elementary estimate ([2, Lemma 6(i)] or [7, Lemma 4.3]) will also be used several times in the paper.

**Lemma 2.2.** *Let  $1 \leq H \leq X$  and  $|\sigma| \leq 10$ . Then for any  $\beta \in [0, 1]$ , we have*

$$(2.10) \quad |(X + H)^s - X^s| \ll X^\sigma (|\tau| + 1)H/X^\beta,$$

where the implied constant is absolute.

### 3. DIRICHLET SERIES ASSOCIATED WITH $S(x, y)$

In view of the definition of  $S(x, y)$  in (2.3), we define the double Dirichlet series

$$(3.1) \quad \mathcal{F}(s, w) := \sum_{n \geq 1} n^{-s} \sum_{d|n^4} d^{-w} r_4^*(d) \mathbb{1}_\square \left( \frac{n^4}{d} \right)$$

for  $\Re s > 5$  and  $\Re w > 0$ . The next lemma states that the function  $\mathcal{F}(s, w)$  enjoys a nice factorization formula.

**Lemma 3.1.** *For  $\min_{0 \leq j \leq 2} \Re(s + 2jw - 2j) > 1$ , we have*

$$(3.2) \quad \mathcal{F}(s, w) = \prod_{0 \leq j \leq 2} \zeta(s + 2jw - 2j) \mathcal{G}(s, w),$$

where  $\mathcal{G}(s, w)$  is an Euler product, given by (3.8), (3.10) and (3.11) below. Further, for any  $\varepsilon > 0$  and for  $\min_{0 \leq j \leq 2} \Re(s + 2jw - 2j) \geq \frac{1}{2} + \varepsilon$ ,  $\mathcal{G}(s, w)$  converges absolutely and

$$(3.3) \quad \mathcal{G}(s, w) \ll_\varepsilon 1.$$

*Proof.* Since the functions  $r_4^*(d)$  and  $n^{-s} \sum_{d|n^4} d^{-w} r_4^*(d) \mathbb{1}_\square(n^4/d)$  are multiplicative, for  $\Re s > 5$  and  $\Re w > 0$  we can write the Euler product

$$\mathcal{F}(s, w) = \prod_p \sum_{\nu \geq 0} p^{-\nu s} \sum_{0 \leq \mu \leq 2\nu} p^{-2\mu w} r_4^*(p^{2\mu}) = \prod_p \mathcal{F}_p(s, w).$$

In the above computations, special attention should be paid to the effect of the function  $\mathbb{1}_\square$ . The next is to simplify each  $\mathcal{F}_p(s, w)$ . To this end, we recall (2.1) so that

$$(3.4) \quad r_4^*(p^\mu) = \frac{1 - p^{\mu+1}}{1 - p} \quad (p > 2), \quad r_4^*(2^\mu) = 3$$

for all integers  $\mu \geq 1$ . On the other hand, a simple formal calculation shows

$$(3.5) \quad \begin{aligned} \sum_{\nu \geq 0} x^\nu \sum_{0 \leq \mu \leq 2\nu} y^{2\mu} \frac{1 - z^{2\mu+1}}{1 - z} &= \frac{1}{1 - z} \sum_{\nu \geq 0} x^\nu \left( \frac{1 - y^{4\nu+2}}{1 - y^2} - z \frac{1 - (yz)^{4\nu+2}}{1 - y^2 z^2} \right) \\ &= \frac{1 + xy^2(1 + z + z^2) + xy^4(z + z^2 + z^3) + x^2 y^6 z^3}{(1 - x)(1 - xy^4)(1 - xy^4 z^4)} \end{aligned}$$

and

$$(3.6) \quad \begin{aligned} 1 + \sum_{\nu \geq 1} x^\nu \left( 1 + a \sum_{1 \leq \mu \leq 2\nu} y^{2\mu} \right) &= 1 + \sum_{\nu \geq 1} x^\nu \left( 1 + a \frac{y^2 - y^{4\nu+2}}{1 - y^2} \right) \\ &= \frac{1 + axy^2 + (a - 1)xy^4}{(1 - x)(1 - xy^4)}. \end{aligned}$$

When  $p > 2$ , in view of (3.4), we can apply (3.5) with  $(x, y, z) = (p^{-s}, p^{-w}, p)$  to write

$$(3.7) \quad \mathcal{F}_p(s, w) = \prod_{0 \leq j \leq 2} (1 - p^{-(s+2jw-2j)})^{-1} \mathcal{G}_p(s, w),$$

where

$$(3.8) \quad \mathcal{G}_p(s, w) := \left(1 + \frac{p^2 + p + 1}{p^{s+2w}} + \frac{p^3 + p^2 + p}{p^{s+4w}} + \frac{p^3}{p^{2s+6w}}\right) \left(1 - \frac{p^2}{p^{s+2w}}\right) \left(1 - \frac{1}{p^{s+4w}}\right)^{-1}.$$

While for  $p = 2$ , the formula (3.6) with  $(x, y, z, a) = (2^{-s}, 2^{-w}, 2, 3)$  gives

$$(3.9) \quad \mathcal{F}_2(s, w) = \prod_{0 \leq j \leq 2} (1 - 2^{-(s+2jw-2j)})^{-1} \mathcal{G}_2(s, w),$$

where

$$(3.10) \quad \mathcal{G}_2(s, w) := \frac{1 + 3 \cdot 2^{-s-2w} + 2^{-s-4w+1}}{1 - 2^{-s-4w}} \prod_{1 \leq j \leq 2} (1 - 2^{-(s+2jw-2j)}).$$

Combining (3.7)–(3.10), we get (3.2) with

$$(3.11) \quad \mathcal{G}(s, w) := \prod_p \mathcal{G}_p(s, w) \quad (\Re s > 5, \Re w > 0).$$

It is easy to verify that for  $\min_{0 \leq j \leq 2} (\sigma + 2ju - 2j) \geq \frac{1}{2} + \varepsilon$ , we have  $|\mathcal{G}_p(s, w)| = 1 + O(p^{-1-\varepsilon})$ . This shows that under the same condition, the Euler product  $\mathcal{G}(s, w)$  converges absolutely and (3.3) holds. By analytic continuation, (3.2) is also true in the same domain. This completes the proof.  $\square$

#### 4. PROOF OF THEOREM 2.1

In the sequel, we suppose

$$(4.1) \quad 10 \leq X \leq Y \leq X^3, \quad (XY)^3 \leq 4T \leq U \leq X^{12}, \quad H \leq X, \quad J \leq Y,$$

and for brevity we fix the following notation:

$$(4.2) \quad s := \sigma + i\tau, \quad w := u + iv, \quad \mathcal{L} := \log X, \quad \kappa := 1 + \mathcal{L}^{-1}, \quad \lambda := 1 + 4\mathcal{L}^{-1}.$$

The following proposition is an immediate consequence of Lemmas 4.2-4.5 below.

**Proposition 4.1.** *Under the previous notation, we have*

$$M(X, Y) = X^2 Y^2 P(\log X - \frac{1}{4} \log Y) + R_0(X, Y) + R_1(X, Y) + R_2(X, Y) + O(1)$$

uniformly for  $(X, Y, T, U, H, J)$  satisfying (4.1), where  $R_0, R_1, R_2$  and  $P(t)$  are defined as in (4.8), (4.13), (4.16) and (4.18) below, respectively.

The proof is divided into several subsections.

**4.1. Application of Perron's formula.** The first step is to apply Perron's formula twice to transform  $M(X, Y)$  into a form that is ready for future treatment.

**Lemma 4.2.** *Under the previous notation, we have*

$$(4.3) \quad M(X, Y) = M(X, Y; T, U) + O(1)$$

uniformly for  $(X, Y, T, U)$  satisfying (4.1), where the implied constant is absolute and

$$(4.4) \quad M(X, Y; T, U) := \frac{1}{(2\pi i)^2} \int_{\kappa-iT}^{\kappa+iT} \left( \int_{\lambda-iU}^{\lambda+iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw \right) \frac{X^{s+1}}{s(s+1)} ds.$$

The proof is the same as that of [7, Lemma 6.2].

**4.2. Application of Cauchy's theorem.** In this subsection, we shall apply Cauchy's theorem to evaluate the integral over  $w$  in  $M(X, Y; T, U)$ . We write

$$(4.5) \quad w_j = w_j(s) := (2j + 1 - s)/(2j) \quad (1 \leq j \leq 2)$$

and

$$(4.6) \quad \mathcal{F}_1^*(s) := \zeta(s)\zeta(2-s)\mathcal{G}(s, w_1(s)), \quad \mathcal{F}_2^*(s) := \zeta(s)\zeta\left(\frac{s+1}{2}\right)\mathcal{G}(s, w_2(s)).$$

**Lemma 4.3.** *Under the previous notation, for any  $\varepsilon > 0$  we have*

$$(4.7) \quad M(X, Y; T, U) = I_1 + I_2 + R_0(X, Y) + O_\varepsilon(1)$$

uniformly for  $(X, Y, T, U)$  satisfying (4.1), where

$$I_1 := \frac{4}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{\mathcal{F}_1^*(s) X^{s+1} Y^{(5-s)/2}}{(3-s)(5-s)s(s+1)} ds,$$

$$I_2 := \frac{16}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} \frac{\mathcal{F}_2^*(s) X^{s+1} Y^{(9-s)/4}}{(5-s)(9-s)s(s+1)} ds,$$

and

$$(4.8) \quad R_0(X, Y) := \frac{1}{(2\pi i)^2} \int_{\kappa-iT}^{\kappa+iT} \left( \int_{\frac{11}{12}+\varepsilon-iU}^{\frac{11}{12}+\varepsilon+iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw \right) \frac{X^{s+1}}{s(s+1)} ds.$$

Furthermore we have

$$(4.9) \quad \left. \begin{aligned} &(\mathcal{D}R_0)(X, X+H; Y, Y+J) \\ &(\mathcal{D}R_0)(X-H, X; Y-J, Y) \end{aligned} \right\} \ll_\varepsilon X^{\frac{7}{6}+\varepsilon} Y^{\frac{11}{12}+\varepsilon} H^{\frac{5}{6}} J + X^{1+\varepsilon} Y^{\frac{13}{12}+\varepsilon} H J^{\frac{5}{6}}$$

uniformly for  $(X, Y, T, U, H, J)$  satisfying (4.1).

*Proof.* We want to calculate the integral

$$\frac{1}{2\pi i} \int_{\lambda-iU}^{\lambda+iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw$$

for any individual  $s = \sigma + i\tau$  with  $\sigma = \kappa$  and  $|\tau| \leq T$ . We move the line of integration  $\Re w = \lambda$  to  $\Re w = \frac{3}{4} + \varepsilon$ . By Lemma 3.1, for  $\sigma = \kappa$  and  $|\tau| \leq T$ , the points  $w_j(s)$  ( $j = 1, 2$ ), given by (4.5), are the simple poles of the integrand in the rectangle  $\frac{3}{4} + \varepsilon \leq u \leq \lambda$  and  $|v| \leq U$ . The residues of  $\frac{\mathcal{F}(s, w)}{w(w+1)} Y^{w+1}$  at the poles  $w_j(s)$  are

$$(4.10) \quad \frac{4\mathcal{F}_1^*(s) Y^{(5-s)/2}}{(3-s)(5-s)}, \quad \frac{16\mathcal{F}_2^*(s) Y^{(9-s)/4}}{(5-s)(9-s)},$$

respectively, where  $\mathcal{F}_j^*(s)$  ( $j = 1, 2$ ) are defined as in (4.6).

It is well-known that (cf. e.g. [8, page 146, Theorem II.3.7])

$$(4.11) \quad \zeta(s) \ll |\tau|^{\max\{(1-\sigma)/3, 0\}} \log |\tau| \quad (\sigma \geq \frac{1}{2}, |\tau| \geq 2)$$

where  $c > 0$  is a constant. When  $\sigma = \kappa$  and  $\frac{11}{12} + \varepsilon \leq u \leq \lambda$ , it is easily checked that

$$\min_{0 \leq j \leq 2} (\sigma + 2ju - 2j) \geq 1 + 4(\frac{11}{12} + \varepsilon - 1) = \frac{3}{4} + 3\varepsilon > \frac{1}{2} + \varepsilon.$$

It follows from (4.11) and (3.3) that  $\mathcal{F}(s, w) \ll_{\varepsilon} U^{2(1-u)} \mathcal{L}^4$  for  $\sigma = \kappa, |\tau| \leq T, \frac{11}{12} + \varepsilon \leq u \leq \lambda$  and  $v = \pm U$ . This implies that

$$\int_{\frac{11}{12} + \varepsilon \pm iU}^{\lambda \pm iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw \ll_{\varepsilon} Y \mathcal{L}^4 \int_{\frac{11}{12}}^{\lambda} \left( \frac{Y}{U^2} \right)^u du \ll_{\varepsilon} \frac{Y^{\frac{23}{12}} \mathcal{L}^4}{U^{\frac{11}{6}}} \ll_{\varepsilon} 1.$$

Cauchy's theorem then gives

$$\begin{aligned} \frac{1}{2\pi i} \int_{\lambda - iU}^{\lambda + iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw &= \frac{4\mathcal{F}_1^*(s) Y^{(5-s)/2}}{(3-s)(5-s)} + \frac{16\mathcal{F}_2^*(s) Y^{(9-s)/4}}{(5-s)(9-s)} \\ &\quad + \frac{1}{2\pi i} \int_{\frac{11}{12} + \varepsilon - iU}^{\frac{11}{12} + \varepsilon + iU} \frac{\mathcal{F}(s, w) Y^{w+1}}{w(w+1)} dw + O_{\varepsilon}(1). \end{aligned}$$

Inserting the last formula into (4.4), we obtain (4.7).

Finally we prove (4.9). For  $\sigma = \kappa, |\tau| \leq T, u = \frac{11}{12} + \varepsilon$  and  $|v| \leq U$ , we apply (4.11) and (3.3) as before, to get

$$\mathcal{F}(s, w) \ll (|\tau| + |v| + 1)^{\frac{1}{6}} \mathcal{L}^4 \ll \{(|\tau| + 1)^{\frac{1}{6}} + (|v| + 1)^{\frac{1}{6}}\} \mathcal{L}^4.$$

Also, for  $\sigma, \tau, u, v$  as above, we have

$$\begin{aligned} r_{s,w}(X, H; Y, J) &:= ((X + H)^{s+1} - X^{s+1})((Y + J)^{w+1} - Y^{w+1}) \\ &\ll X^2 ( (|\tau| + 1) H/X )^{\frac{5}{6} - \varepsilon} Y^{\frac{23}{12} + \varepsilon} ( (|v| + 1) J/Y )^{1 - \varepsilon} \\ &\ll X^{\frac{7}{6} + \varepsilon} Y^{\frac{11}{12} + \varepsilon} H^{\frac{5}{6}} J (|\tau| + 1)^{\frac{5}{6} - \varepsilon} (|v| + 1)^{1 - \varepsilon} \end{aligned}$$

by (2.10) of Lemma 2.2 with  $\beta = \frac{5}{6} - \varepsilon$  and with  $\beta = 1 - \varepsilon$ . Similarly,

$$\begin{aligned} r_{s,w}(X, H; Y, J) &= ((X + H)^{s+1} - X^{s+1})((Y + J)^{w+1} - Y^{w+1}) \\ &\ll X^2 ( (|\tau| + 1) H/X )^{1 - \varepsilon} Y^{\frac{23}{12} + \varepsilon} ( (|v| + 1) J/Y )^{\frac{5}{6} - \varepsilon} \\ &\ll X^{1 + \varepsilon} Y^{\frac{13}{12} + \varepsilon} H J^{\frac{5}{6}} (|\tau| + 1)^{1 - \varepsilon} (|v| + 1)^{\frac{5}{6} - \varepsilon} \end{aligned}$$

by Lemma 2.2 with  $\beta = 1 - \varepsilon$  and with  $\beta = \frac{5}{6} - \varepsilon$ . These and Lemma 2.1(i) imply

$$\begin{aligned} (\mathcal{D}R_0)(X, X + H; Y, Y + J) &= \int_{\kappa - iT}^{\kappa + iT} \int_{\frac{11}{12} + \varepsilon - iU}^{\frac{11}{12} + \varepsilon + iU} \frac{\mathcal{F}(s, w)}{(2\pi i)^2} \frac{r_{s,w}(X, H; Y, J)}{s(s+1)w(w+1)} dw ds \\ &\ll_{\varepsilon} X^{\frac{7}{6} + \varepsilon} Y^{\frac{11}{12} + \varepsilon} H^{\frac{5}{6}} J + X^{1 + \varepsilon} Y^{\frac{13}{12} + \varepsilon} H J^{\frac{5}{6}}. \end{aligned}$$

This completes the proof.  $\square$



4.3. Evaluation of  $I_1$ .

**Lemma 4.4.** *Under the previous notation, we have*

$$(4.12) \quad I_1 = R_1(X, Y) + O(1)$$

uniformly for  $(X, Y, T)$  satisfying (4.1), where

$$(4.13) \quad R_1(X, Y) := \frac{4}{2\pi i} \int_{\frac{5}{4}-iT}^{\frac{5}{4}+iT} \frac{\mathcal{F}_1^*(s) X^{s+1} Y^{(5-s)/2}}{(3-s)(5-s)s(s+1)} ds.$$

Further we have

$$(4.14) \quad \left. \begin{aligned} &(\mathcal{D}R_1)(X, X+H; Y, Y+J) \\ &(\mathcal{D}R_1)(X-H, X; Y-J, Y) \end{aligned} \right\} \ll X^{\frac{5}{4}} Y^{\frac{7}{8}} HJ$$

uniformly for  $(X, Y, T, H, J)$  satisfying (4.1).

*Proof.* We shall prove (4.12) by moving the contour  $\Re s = \kappa$  to  $\Re s = \frac{5}{4}$ . When  $\kappa \leq \sigma \leq \frac{5}{4}$ , it is easy to check that

$$\min_{0 \leq j \leq 2} (\sigma + 2jw_1(\sigma) - 2j) = \min_{0 \leq j \leq 2} (j + (1-j)\sigma) \geq \frac{3}{4}.$$

By Lemma 3.1 the integrand is holomorphic in the rectangle  $\kappa \leq \sigma \leq \frac{5}{4}$  and  $|\tau| \leq T$ ; and we can apply (4.11) and (3.3) to get  $\mathcal{F}_1^*(s) \ll T^{(\sigma-1)/3} \mathcal{L}^2$  in this rectangle, which implies that

$$\begin{aligned} \int_{\kappa \pm iT}^{\frac{5}{4} \pm iT} \frac{\mathcal{F}_1^*(s) X^{s+1} Y^{(5-s)/2}}{(3-s)(5-s)s(s+1)} ds &\ll \frac{X^2 Y^2 \mathcal{L}^2}{T^4} \int_{\kappa}^{\frac{5}{4}} \left( \frac{X T^{1/3}}{Y^{1/2}} \right)^{\sigma-1} d\sigma \\ &\ll \frac{X^2 Y^2 \mathcal{L}^2}{T^4} + \frac{X^{\frac{9}{4}} Y^{\frac{15}{8}} \mathcal{L}^2}{T^{\frac{47}{12}}} \ll 1. \end{aligned}$$

This proves (4.12).

To establish (4.14), we note that  $\mathcal{F}_1^*(s) \ll (|\tau| + 1)^{\frac{1}{4}}$  for  $\sigma = \frac{5}{4}$  and  $|\tau| \leq T$ . By (2.10) of Lemma 2.2 with  $\beta = 1$ ,

$$\begin{aligned} r_{s, w_1(s)}(X, H; Y, J) &:= ((X+H)^{s+1} - X^{s+1})((Y+J)^{(5-s)/2} - Y^{(5-s)/2}) \\ &\ll X^{\frac{5}{4}} Y^{\frac{7}{8}} HJ (|\tau| + 1)^2. \end{aligned}$$

Combining these with Lemma 2.1(ii), we deduce that

$$\begin{aligned} (\mathcal{D}R_1)(X, X+H; Y, Y+J) &= \frac{4}{2\pi i} \int_{\frac{5}{4}-iT}^{\frac{5}{4}+iT} \frac{\mathcal{F}_1^*(s) r_{s, w_1(s)}(X, H; Y, J)}{(3-s)(5-s)s(s+1)} ds \\ &\ll X^{\frac{5}{4}} Y^{\frac{7}{8}} HJ, \end{aligned}$$

from which the desired result follows.  $\square$

4.4. Evaluation of  $I_2$ .

**Lemma 4.5.** *Under the previous notation, for any  $\varepsilon > 0$  we have*

$$(4.15) \quad I_2 = X^2 Y^2 P(\log X - \frac{1}{4} \log Y) + R_2(X, Y) + O_\varepsilon(1)$$

uniformly for  $(X, Y, T)$  satisfying (4.1), where  $P(t)$  is defined as in (4.18) below and

$$(4.16) \quad R_2(X, Y) := \frac{16}{2\pi i} \int_{\frac{1}{2} + \varepsilon - iT}^{\frac{1}{2} + \varepsilon + iT} \frac{\mathcal{F}_2^*(s) X^{s+1} Y^{(9-s)/4}}{(5-s)(9-s)s(s+1)} ds.$$

Further we have

$$(4.17) \quad \left. \begin{aligned} (\mathcal{D}R_2)(X, X+H; Y, Y+J) \\ (\mathcal{D}R_2)(X-H, X; Y-J, Y) \end{aligned} \right\} \ll_\varepsilon X^{\frac{1}{2} + \varepsilon} Y^{\frac{9}{8}} HJ$$

uniformly for  $(X, Y, T, H, J)$  satisfying (4.1).

*Proof.* We move the line of integration  $\Re s = \kappa$  to  $\Re s = \frac{1}{2} + \varepsilon$ . Obviously  $s = 1$  is the unique pole of order 2 of the integrand in the rectangle  $\frac{1}{2} + \varepsilon \leq \sigma \leq \kappa$  and  $|\tau| \leq T$ , and the residue is  $X^2 Y^2 P(\log X - \frac{1}{4} \log Y)$  with

$$(4.18) \quad P(t) := \left( \frac{16(s-1)^2 \mathcal{F}_2^*(s) e^{t(s-1)}}{(5-s)(9-s)s(s+1)} \right)' \Big|_{s=1}.$$

Here  $P(t)$  is a linear polynomial with the leading coefficient  $\mathcal{C}_4$  given by (2.6) above.

When  $\frac{1}{2} + \varepsilon \leq \sigma \leq \kappa$ , we check that

$$\min_{0 \leq j \leq 2} (\sigma + 2jw_2(\sigma) - 2j) = \frac{1}{2} \min_{0 \leq j \leq 2} (j + (2-j)\sigma) \geq \frac{1}{2} + \varepsilon.$$

Hence when  $\frac{1}{2} + \varepsilon \leq \sigma \leq \kappa$  and  $|\tau| \leq T$ , (4.11) and (3.3) yields  $\mathcal{F}_2^*(s) \ll T^{(1-\sigma)/2} \mathcal{L}^3$ . It follows that

$$\begin{aligned} \int_{\frac{1}{2} + \varepsilon - iT}^{\kappa + iT} \frac{\mathcal{F}_2^*(s) X^{s+1} Y^{(9-s)/4}}{(5-s)(9-s)s(s+1)} ds &\ll \frac{X^2 Y^2 \mathcal{L}^3}{T^4} \int_{\frac{1}{2}}^{\kappa} \left( \frac{YT^2}{X^4} \right)^{(1-\sigma)/4} ds \\ &\ll \frac{X^2 Y^2 \mathcal{L}^3}{T^4} + \frac{X^{\frac{3}{2}} Y^{\frac{9}{4}} \mathcal{L}^3}{T^{\frac{7}{2}}} \ll 1. \end{aligned}$$

These establish (4.15). To prove (4.17), we note that for  $\sigma = \frac{1}{2} + \varepsilon$  and  $|\tau| \leq T$ , we have  $\mathcal{F}_2^*(s) \ll_\varepsilon (|\tau| + 1)^{1/3}$  thanks to (4.11) and (3.3), and

$$\begin{aligned} r_{s, w_2(s)}(X, H; Y, J) &:= ((X+H)^{s+1} - X^{s+1})((Y+J)^{(9-s)/4} - Y^{(9-s)/4}) \\ &\ll_\varepsilon X^{\frac{1}{2} + \varepsilon} Y^{\frac{9}{8}} HJ (|\tau| + 1)^2 \end{aligned}$$

by Lemma 2.2 with  $\beta = 1$ . Combining these with Lemma 2.1(i), we deduce that

$$\begin{aligned} (\mathcal{D}R_2)(X, X+H; Y, Y+J) &= \frac{16}{2\pi i} \int_{\frac{1}{2} + \varepsilon - iT}^{\frac{1}{2} + \varepsilon + iT} \frac{\mathcal{F}_2^*(s) r_{s, w_2(s)}(X, H; Y, J)}{(5-s)(9-s)s(s+1)} ds \\ &\ll_\varepsilon X^{\frac{1}{2} + \varepsilon} Y^{\frac{9}{8}} HJ. \end{aligned}$$

This proves the lemma.  $\square$

**4.5. Completion of proof of Theorem 2.1.** Denote by  $\mathcal{M}(X, Y)$  the main term in the asymptotic formula of  $M(x, y)$  in Proposition 4.1, that is  $\mathcal{M}(X, Y) := X^2 Y^2 P(\psi)$  and  $\psi := \log(X/Y^{1/4})$ . Then Lemma 2.1(i) gives

$$(\mathcal{D}\mathcal{M})(X, X+H; Y, Y+J) = \{XY(4P(\psi) + \frac{3}{2}P'(\psi)) + O(XJ\mathcal{L}^2 + YH\mathcal{L}^2)\}HJ.$$

Since  $\mathcal{D}$  is a linear operator, this together with Proposition 4.1 implies that

$$(\mathcal{D}M)(X, X+H; Y, Y+J) = \{XY(4P(\psi) + \frac{3}{2}P'(\psi)) + O_\varepsilon(\mathcal{R})\}HJ$$

with

$$\mathcal{R} := X^{\frac{7}{6}+\varepsilon}Y^{\frac{11}{12}}H^{-\frac{1}{6}} + X^{1+\varepsilon}Y^{\frac{13}{12}}J^{-\frac{1}{6}} + X^{\frac{5}{4}}Y^{\frac{7}{8}} + X^{\frac{1}{2}+\varepsilon}Y^{\frac{9}{8}} + XJ\mathcal{L}^2 + YH\mathcal{L}^2.$$

The same formula also holds for  $(\mathcal{D}M)(X-H, X; Y-J, Y)$ . Now Lemma 2.1(ii) with  $H = XY^{-\frac{1}{14}}$  and  $J = Y^{\frac{13}{14}}$  allows us to deduce

$$S(X, Y) = XY(4P(\psi) + \frac{3}{2}P'(\psi)) + O_\varepsilon(X^{\frac{5}{4}}Y^{\frac{7}{8}} + X^{\frac{1}{2}+\varepsilon}Y^{\frac{9}{8}}),$$

where we have used the following facts

$$\begin{aligned} (X^{\frac{5}{4}}Y^{\frac{7}{8}})^{\frac{5-24\varepsilon}{6}}(X^{\frac{1}{2}+\varepsilon}Y^{\frac{9}{8}})^{\frac{1+24\varepsilon}{6}} &= X^{\frac{27}{24}-\frac{4(17-24\varepsilon)\varepsilon}{24}}Y^{\frac{11}{12}+\varepsilon} \geq X^{1+\varepsilon}Y^{\frac{11}{12}+\varepsilon}, \\ (X^{\frac{5}{4}}Y^{\frac{7}{8}})^{\frac{11}{14}}(X^{\frac{1}{2}+\varepsilon}Y^{\frac{9}{8}})^{\frac{3}{14}} &= X^{\frac{61+12\varepsilon}{56}}Y^{\frac{13}{14}} \geq X^{1+\varepsilon}Y^{\frac{13}{14}}. \end{aligned}$$

This finally completes the proof of Theorem 2.1.

## 5. PROOF OF THEOREMS 2.2 AND 1.1

*Proof of Theorems 2.2.* The idea is to apply Theorems 2.1 in a delicate way. Trivially we have  $r_4^*(d) \leq d\tau(d)$  (here  $\tau(n)$  is the divisor function), and therefore

$$(5.1) \quad S(x, y) \leq y \sum_{n \leq x} \sum_{d|n^4} \tau(d) \ll xy(\log x)^{14}$$

for all  $x \geq 2$  and  $y \geq 2$ , where the implied constant is absolute.

Let  $\delta := 1 - (\log B)^{-1}$  and let  $k_0$  be a positive integer such that  $\delta^{k_0} < (\log B)^{-3} \leq \delta^{k_0-1}$ . Note that  $k_0 \asymp (\log B) \log \log B$ . In view of (5.1), we can write

$$\begin{aligned} (5.2) \quad T(B) &= \sum_{1 \leq k \leq k_0} \sum_{\delta^k B < n \leq \delta^{k-1} B} \sum_{\substack{d|n^4 \\ d < n^4/B^2}} r_4^*(d) \mathbb{1}_\square\left(\frac{n^4}{d}\right) + O(B^3) \\ &\leq \sum_{1 \leq k \leq k_0} (S(\delta^{k-1}B, \delta^{4(k-1)}B^2) - S(\delta^k B, \delta^{4(k-1)}B^2)) + O(B^3). \end{aligned}$$

Similarly (even easily),

$$(5.3) \quad T(B) \geq \sum_{1 \leq k \leq k_0} (S(\delta^{k-1}B, \delta^{4k}B^2) - S(\delta^k B, \delta^{4k}B^2)).$$

By (2.5) of Theorem 2.1 with  $\eta = \frac{1}{4}$ , a simple computation shows that

$$(5.4) \quad T(B) \leq 2(1-\delta) \frac{1-\delta^{5k_0}}{1-\delta^5} \cdot \mathcal{C}_4 B^3 \log B \left\{ 1 + O\left(\frac{1}{\sqrt[4]{\log B}}\right) \right\} + O(B^3),$$

$$(5.5) \quad T(B) \geq 2(\delta^{-1}-1) \frac{\delta^5 - \delta^{5(k_0+1)}}{1-\delta^5} \cdot \mathcal{C}_4 B^3 \log B \left\{ 1 + O\left(\frac{1}{\sqrt[4]{\log B}}\right) \right\}.$$

By noticing that

$$(1 - \delta) \frac{1 - \delta^{5k_0}}{1 - \delta^5} = \frac{1 - \delta^{5k_0}}{1 + \delta + \delta^2 + \delta^3} = \frac{1}{5} + O\left(\frac{1}{\sqrt[4]{\log B}}\right),$$

$$(\delta^{-1} - 1) \frac{\delta^5 - \delta^{5(k_0+1)}}{1 - \delta^5} = \frac{\delta^4 - \delta^{5k_0+4}}{1 + \delta + \delta^2 + \delta^3 + \delta^4} = \frac{1}{5} + O\left(\frac{1}{\sqrt[4]{\log B}}\right).$$

The desired asymptotic formula (2.7) follows from (5.4) and (5.5).  $\square$

*Proof of Theorem 1.1.* Applying (2.4) of Theorem 2.1 with  $(x, y) = (B, B^2)$ , we have

$$(5.6) \quad \sum_{n \leq B} \sum_{\substack{d|n^4 \\ d \leq B^2}} r_4^*(d) \mathbb{1}_{\square}\left(\frac{n^4}{d}\right) = 2\mathcal{C}_4 B^3 \log B \left\{ 1 + O\left(\frac{1}{\sqrt[4]{\log B}}\right) \right\}.$$

Inserting this and (2.7) into (2.2), we obtain (1.5) with  $\mathcal{C}_4^* = \frac{192}{5}\mathcal{C}_4$ .

Finally (1.4) follows from (1.5) via the inversion formula of Möbius.  $\square$

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