



HAL
open science

Real structures on horospherical varieties

Lucy Moser-Jauslin, Ronan Terpereau, Mikhail Borovoi

► **To cite this version:**

Lucy Moser-Jauslin, Ronan Terpereau, Mikhail Borovoi. Real structures on horospherical varieties. 2018. hal-01865442v1

HAL Id: hal-01865442

<https://hal.science/hal-01865442v1>

Preprint submitted on 31 Aug 2018 (v1), last revised 27 Jul 2021 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

REAL STRUCTURES ON HOROSPHERICAL VARIETIES

LUCY MOSER-JAUSLIN AND RONAN TERPEREAU
WITH AN APPENDIX BY MIKHAIL BOROVOI

ABSTRACT. We study the equivariant real structures on complex horospherical varieties, generalizing classical results known for toric varieties and flag varieties. In particular, we obtain a necessary and sufficient condition for the existence of such real structures and determine the number of equivalence classes. We then apply our results to classify the equivariant real structures on smooth projective horospherical varieties of Picard rank 1.

CONTENTS

Introduction	1
1. Real group structures on complex algebraic groups	5
2. Equivariant real structures on quasi-homogeneous spaces	13
3. Equivariant real structures on horospherical varieties	16
Appendix A. Tits classes for the simply-connected simple algebraic groups (by Mikhail Borovoi)	26
References	29

INTRODUCTION

A *real structure* on a complex algebraic variety X is an antiregular involution μ on X , where the word *antiregular* means that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\mathrm{Spec}(z \mapsto \bar{z})} & \mathrm{Spec}(\mathbb{C}) \end{array}$$

Two real structures μ and μ' are called *equivalent* if there exists $\varphi \in \mathrm{Aut}(X)$ such that $\mu' = \varphi \circ \mu \circ \varphi^{-1}$. To any real structure μ on X one can associate the quotient $\mathcal{X} = X/\mu$, which is a real algebraic space satisfying $\mathcal{X} \times_{\mathrm{Spec}(\mathbb{R})} \mathrm{Spec}(\mathbb{C}) \simeq X$ as complex varieties. Moreover, if X is quasi-projective, then \mathcal{X} is actually a real variety. The quotient \mathcal{X} is called a *real form* of X . Two real forms \mathcal{X} and \mathcal{X}' are \mathbb{R} -isomorphic if and only if the corresponding real structures are equivalent. Describing all the equivalence classes of real structures on a given complex variety is a classical problem in algebraic geometry. We refer to [Man17, Chp. 2] and [Ben16, Chp. 3] for an exposition (in French) of the foundations of this theory.

When X carries some extra structure, it is natural to ask for \mathcal{X} to also carry this extra structure. For instance, if $X = G$ is a complex algebraic group, then it is particularly interesting to describe the real structures σ on G which are group

2010 *Mathematics Subject Classification.* 14M27, 14M17, 20G20, 11E72, 14P99.

Key words and phrases. Horospherical variety, homogeneous space, real structure, real form.

This research was partially supported by the ISITE-BFC Project ANR-IS-IDEX-OOOB .

involutions, so that the real form $\mathcal{G} = G/\sigma$ is a real algebraic group (and not just a real variety). Such real structures are called *real group structures*. Two real group structures σ and σ' are called *equivalent* if there exists $\varphi \in \text{Aut}_{gr}(G)$ such that $\sigma' = \varphi \circ \sigma \circ \varphi^{-1}$. Then two real forms \mathcal{G} and \mathcal{G}' are \mathbb{R} -isomorphic as real algebraic groups if and only if the corresponding real group structures are equivalent. Let us note that the real group structures on complex reductive algebraic groups are well-known (see §§1.1-1.2 for a recap); their classification reduces to the cases of complex tori and complex simply-connected simple algebraic groups.

Another class of complex varieties carrying extra structure are the complex varieties endowed with an algebraic group action, which yields the key notion of equivariant real structure. Let G be a complex algebraic group, let σ be a real group structure on G , and let X be a complex G -variety. A real structure μ on a X is called a (G, σ) -equivariant real structure if $\mu(g \cdot x) = \sigma(g) \cdot \mu(x)$ for all $g \in G$ and all $x \in X$. Two (G, σ) -equivariant real structures μ and μ' are called *equivalent* if there exists $\varphi \in \text{Aut}^G(X)$ such that $\mu' = \varphi \circ \mu \circ \varphi^{-1}$. Then the real form $\mathcal{X} = X/\mu$ is a real \mathcal{G} -variety, and two real forms \mathcal{X} and \mathcal{X}' are isomorphic as \mathcal{G} -varieties if and only if the corresponding equivariant real structures are equivalent.

Let us illustrate the notion of equivariant real structure with a few classical examples. Let $X = G/P$ be a flag variety. Then X has a (G, σ) -equivariant real structure if and only if $\sigma(P) = cPc^{-1}$ for some $c \in G$, in which case any such real structure is equivalent to $\mu : gP \mapsto \sigma(g)cP$. Indeed, if $\sigma(P) = cPc^{-1}$, then $\sigma(c)c \in P$, and therefore μ is an involution. Let now $G = T$ be a complex torus, and let X be a toric variety with open orbit $X_0 \simeq T$. Any real group structure σ on $T \simeq \mathbb{G}_m^r$ is equivalent to a product $\sigma_0^{\times p} \times \sigma_1^{\times n} \times \sigma_2^{\times q}$, where $\sigma_0(t) = \bar{t}$, $\sigma_1(t) = \bar{t}^{-1}$, and $\sigma_2(t_1, t_2) = (\bar{t}_2, \bar{t}_1)$. Then X_0 has 2^n equivalence classes of (T, σ) -equivariant real structures, and a given (T, σ) -equivariant real structure extends to X if and only if the μ_2 -action on $\mathbb{X}^\vee(T) = \text{Hom}_{gr}(\mathbb{G}_m, T)$ induced by σ stabilizes the fan of X . (Here, μ_2 is the cyclic group of order 2.) This case will be treated in detail in this article (see Examples 2.6 and 3.25).

Flag varieties and toric varieties belong to the larger class of horospherical varieties. From now on we suppose that G is reductive. A *horospherical G -variety* is a normal G -variety with a finite number of G -orbits and whose general stabilizer contains a maximal unipotent subgroup of G . Horospherical varieties form a subclass of spherical varieties (see [Pau81, Kno91, Tim11, Per14]) whose combinatorial description is easier. A presentation of the theory of horospherical varieties, and their relation to Fano varieties, can be found in [Pas06, Pas08]. They form a fertile ground to tackle many problems in algebraic geometry: the Mukai conjecture [Pas10], the (log) minimal model program [Pas15a, Pas18], the stringy invariants [BM13, LPR], the quantum cohomology [GPPS]... However, as far as we know, the theory of equivariant real structures on horospherical varieties has never been systematically studied. The present article aims at filling this gap.

Before stating our main results we briefly recall the combinatorial description of horospherical subgroups from [Pas08] (more details will be given in §3.1). A *horospherical subgroup* $H \subset G$ is a subgroup containing a maximal unipotent subgroup of G , or equivalently, a subgroup such that G/H is a horospherical G -variety. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let us note that if σ is a real group structure on G , then it induces a μ_2 -action on the character group $\mathbb{X} = \mathbb{X}(T)$; see §1.2 for details. Let \mathcal{S} be the set of simple roots associated with the

triple (G, B, T) , let $I \subset \mathcal{S}$, and let P_I be the *standard parabolic subgroup* containing B . Let M be a sublattice of \mathbb{X} such that $\langle \chi, \alpha^\vee \rangle = 0$ for all $\alpha \in I$ and all $\chi \in M$, where $\langle \cdot, \cdot \rangle$ is the duality bracket and $\alpha^\vee \in \mathbb{X}^\vee$ is the coroot associated to the root α . Then $H_{(I, M)} = \bigcap_{\chi \in M} \text{Ker}(\chi)$ is a *horospherical subgroup* of G whose normalizer is P_I . For any horospherical subgroup $H \subset G$, there exists a unique pair (I, M) as above such that H is conjugate to $H_{(I, M)}$; we say that (I, M) is the datum of H . For instance the datum of a parabolic subgroup conjugate to P_I is $(I, \{0\})$ and the datum of a maximal unipotent subgroup of G is (\emptyset, \mathbb{X}) .

Theorem 0.1. *Let G be a complex reductive algebraic group with a real group structure σ . Let H be a horospherical subgroup of G with datum (I, M) as above. Then a (G, σ) -equivariant real structure exists on G/H if and only if (I, M) is stable for the μ_2 -action on \mathbb{X} induced by σ and $\Delta_H(\sigma) = 1$, where Δ_H is the map defined in §1.3. Moreover, if such a structure exists, then there are exactly 2^n equivalence classes of (G, σ) -equivariant real structures on G/H , where n is a non-negative integer that can be calculated explicitly (see the beginning of §3.4 for details).*

We refer to [Kno91, Tim11, Gan18] and references therein for the classification of equivariant embeddings of spherical homogeneous spaces in terms of combinatorial gadgets called *colored fans*. As explained in [Hur11b], if σ is a real group structure on G , and if G/H is a spherical homogeneous space endowed with a (G, σ) -equivariant real structure, then σ induces a μ_2 -action on the set of colored fans defining a G -equivariant embedding $G/H \hookrightarrow X$.

The next result is an immediate consequence of [Wed, Theorem 9.1] (see also [Hur11b, Theorem 2.23]) and the characterization of quasi-projective spherical varieties via the existence of a strictly convex \mathbb{Q} -valued piecewise linear function.

Theorem 0.2. *Let μ be a (G, σ) -equivariant real structure on a horospherical homogeneous space G/H , and let X be a horospherical G -variety with open orbit G/H . Then the real structure μ extends on X if and only if the colored fan of the embedding $G/H \hookrightarrow X$ is invariant for the μ_2 -action induced by σ , in which case the corresponding real form X/μ is a real variety.*

To illustrate the effectiveness of our results, we then consider the equivariant real structures on smooth projective horospherical G -variety of Picard rank 1 (the odd symplectic grassmannians are examples of such varieties). They were classified by Pasquier in [Pas09] and since then their geometry has been extensively studied (see e.g. [PP10, Hon16, GPPS]).

Theorem 0.3. *Let G be a complex simply-connected semisimple algebraic group with a real group structure σ . Let X be a smooth projective horospherical G -variety of Picard rank 1. If a (G, σ) -equivariant real structure exists on X , then it is unique up to equivalence. The cases where such a real structure exists are classified in Example 3.20 (when $X = G/P$ with P a maximal parabolic subgroup) and in Theorem 3.35 (when X is non-homogeneous).*

As mentioned before, horospherical varieties are a subclass of spherical varieties. Equivariant real structures on spherical varieties already appeared in the literature, but the scope was not the same as in this article. More precisely:

- In [Hur11a, Hur11b, Wed] the authors consider the situation where a real structure on the open orbit is given and they want to determine in which cases this

real structure extends to the whole spherical variety. They do not treat the case of equivariant real structures on homogeneous spaces. Our results give a new approach which allows us to classify the equivariant real structures on horospherical homogeneous spaces and then, for the quasi-homogeneous case, we apply their results. (Note also that they work over an arbitrary field and not just over \mathbb{R} .)

- In [ACF14, Akh15, CF15] the authors consider equivariant real structures on spherical homogeneous spaces G/H and their equivariant embeddings, assuming that $N_G(H)/H$ is finite. Such varieties are never horospherical, except the flag varieties.
- In [BG] the author extends part of the results in [ACF14, Akh15, CF15] and works over an arbitrary base field of characteristic zero.

Besides their easy combinatorial description and their ubiquity in the world of algebraic group actions, horospherical varieties lend themselves very nicely to the study of equivariant real structures for several reasons: First, the group of G -equivariant automorphism of G/H is a torus (see §3.1). Second, if σ is a quasi-split real group structure on G such that $\sigma(H)$ is conjugate to H , then there exists a conjugate H' of H such that $\sigma(H') = H'$ (Proposition 3.9). These two facts are essential in the proof of Theorem 0.1. Third, if we assume that the homogeneous space G/H is spherical instead of horospherical in Theorem 0.2, then in general the real form X/μ is a real algebraic space but not a real variety.

In §§1.1-1.2 we recall definitions and well-known facts about real group structures on complex algebraic groups. In §1.3 we define the cohomological invariant that appears in Theorem 0.1, namely the $\Delta_H(\sigma)$.

Then in §2 we introduce the notion of equivariant real structures on complex varieties endowed with a complex algebraic group action. In particular, we show how to determine if such a structure exists on homogeneous spaces, and if so, then how to use Galois cohomology to determine the set of equivalence classes of these equivariant real structures.

The main part of this article is §3 in which we prove the results above. In §3.1 we recall the basic notions regarding the horospherical homogeneous spaces and their combinatorial classification. In §§3.2-3.3 we prove the part existence in Theorem 0.1 (this is Theorem 3.18), and in §3.4 we prove the quantitative part in Theorem 0.1 (this is Proposition 3.23). In §3.5 we recall the main result of [Hur11b, Wed] regarding the extension of equivariant real structures from a spherical homogeneous space to the whole spherical variety and we apply it to prove Theorem 0.2 (which is Corollary 3.31). Then, we apply our results to classify the equivariant real structures on smooth projective horospherical varieties of Picard rank 1 and prove Theorem 0.3 (see §3.6).

Finally the list of real group structures on complex simply-connected simple algebraic groups together with the list of the corresponding Tits classes is recalled in Appendix A. These cohomology classes are useful to compute the cohomological invariant $\Delta_H(\sigma)$ in examples.

Acknowledgments. The authors are very grateful to Mikhail Borovoi for stimulating discussions and e-mail exchanges about this project; in particular, the cohomological interpretation of our existence criterion for an equivariant real structure on a homogeneous space in §2 was inspired by [Bor].

Notation. In this article we work over the field of real numbers \mathbb{R} and the field of complex numbers \mathbb{C} . We will denote by μ_n the group of n -th roots of unity and by Γ the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\} \simeq \mu_2$. A *variety* over a field k is a geometrically reduced separated scheme of finite type over k ; in particular, varieties can be reducible.

An *algebraic group* over k is a group scheme over k . By an *algebraic subgroup* we mean a closed subgroup scheme. Reductive algebraic groups are always assumed to be connected for the Zariski topology. When we talk about a *group involution* σ we mean that σ is an automorphism of algebraic groups (not necessarily over k) such that $\sigma \circ \sigma = \text{Id}$. We refer the reader to [Hum75] for the standard background on algebraic groups.

We always denote by G a complex algebraic group, by B a Borel subgroup of G , by T a maximal torus of B , by $\mathbb{X} = \mathbb{X}(T) = \text{Hom}_{gr}(T, \mathbb{G}_m)$ the character group of T , by $\mathbb{X}^\vee = \mathbb{X}^\vee(T) = \text{Hom}_{gr}(\mathbb{G}_m, T)$ the cocharacter group of T , and by $\text{Dyn}(G)$ the Dynkin diagram of G (when G is semisimple).

1. REAL GROUP STRUCTURES ON COMPLEX ALGEBRAIC GROUPS

In this section, we start by recalling definitions and well-known facts about real group structures on complex algebraic groups. We are mostly interested in the case of reductive groups. In particular, we will show how to obtain all real group structures on complex reductive algebraic groups, by piecing together the structures on complex tori and on complex simply-connected simple algebraic groups.

The main references we use are [Con14] for results concerning the structure of reductive algebraic groups, [Ser02] for general results concerning Galois cohomology and [Man17, Ben16] for generalities on real structures. In *loc.cit.* the author treats the case of real structures on complex quasi-projective varieties; the corresponding results referred to here concern complex algebraic groups and can be treated in exactly the same way.

1.1. Generalities and first classification results.

Definition 1.1. (Real group structures on complex algebraic groups.)

- (i) Let G be a complex algebraic group. A *real group structure on G* is an antiregular group involution $\sigma : G \rightarrow G$, i.e., a group involution over $\text{Spec}(\mathbb{R})$ which makes the following diagram commute:

$$\begin{array}{ccc}
 G & \xrightarrow{\sigma} & G \\
 \downarrow & & \downarrow \\
 \text{Spec}(\mathbb{C}) & \xrightarrow{\text{Spec}(z \mapsto \bar{z})} & \text{Spec}(\mathbb{C}) \\
 & \searrow & \swarrow \\
 & \text{Spec}(\mathbb{R}) &
 \end{array}$$

- (ii) Two real group structures σ and σ' on G are *equivalent* if there exists a (regular) group automorphism $\varphi \in \text{Aut}_{gr}(G)$ such that $\sigma' = \varphi \circ \sigma \circ \varphi^{-1}$.

Remark 1.2. If σ is an antiregular group involution on G , then any antiregular group automorphism is of the form $\varphi \circ \sigma$ for some group automorphism φ . If φ and σ commute, then $\varphi \circ \sigma$ is an involution if and only if φ is an involution.

If (G, σ) is a complex algebraic group with a real group structure, then the quotient scheme $\mathcal{G} = G/\sigma$ is a real algebraic group which satisfies $\mathcal{G} \times_{\text{Spec}(\mathbb{R})} \text{Spec}(\mathbb{C}) \simeq$

G as complex algebraic groups. The real group \mathcal{G} is called a *real form of G* . Two real forms are \mathbb{R} -isomorphic if and only if the corresponding real group structures are equivalent (see [Ben16, Corollary 3.13]).

Definition 1.3. If (G, σ) is a complex algebraic group with a real group structure, then $G_0 = G(\mathbb{C})^\sigma$ is called the *real part* (or *real locus*) of (G, σ) ; it coincides with the set of \mathbb{R} -points of the real algebraic group G/σ (see [Ben16, Proposition 3.14] for details).

With the notation of Definition 1.3 the group of \mathbb{C} -points $G(\mathbb{C})$ is a complex Lie group and $G_0 = G(\mathbb{C})^\sigma$ is a real Lie group. In fact, the real part G_0 determines the real group structure for a connected complex algebraic group:

Theorem 1.4. ([Ben16, Theorem 3.41]) *Let G be a connected complex algebraic group with two real group structures σ and σ' . Then σ and σ' are equivalent if and only if there is a (scheme) automorphism $\varphi \in \text{Aut}(G)$ such that $\varphi(G(\mathbb{C})^\sigma) = G(\mathbb{C})^{\sigma'}$.*

Because of Theorem 1.4 we will sometimes consider the real part $G_0 = G(\mathbb{C})^\sigma$ instead of the real group structure σ when describing all the possible equivalence classes of real group structures on connected complex algebraic groups.

Let G be a complex reductive algebraic group, let $T = Z(G)^0$ be the neutral component of the center of G , and let G' be the derived subgroup of G . Then the homomorphism $T \times G' \rightarrow G$, $(t, g') \mapsto t^{-1}g'$ is a central isogeny with kernel $T \cap G'$. Also, there is a 1-to-1 correspondence

$$\{\text{real group structures } \sigma \text{ on } G\} \leftrightarrow \left\{ \begin{array}{l} \text{real group structures } (\sigma_1, \sigma_2) \text{ on } T \times G' \\ \text{such that } \sigma_1|_{T \cap G'} = \sigma_2|_{T \cap G'} \end{array} \right\}$$

given by $\sigma \mapsto (\sigma|_T, \sigma|_{G'})$. Therefore, to determine real group structures on complex reductive algebraic groups, it suffices to determine real group structures on complex tori and on complex semisimple algebraic groups.

Proposition 1.5. (Classification of real group structures on complex tori.)

Let $T \simeq \mathbb{G}_m^n$ be an n -dimensional complex torus.

- (i) *If $n = 1$, then T has exactly two inequivalent real group structures, defined by $\sigma_0 : t \mapsto \bar{t}$ and $\sigma_1 : t \mapsto \bar{t}^{-1}$.*
- (ii) *If $n = 2$, then $\sigma_2 : (t_1, t_2) \mapsto (\bar{t}_2, \bar{t}_1)$ defines a real group structure on T .*
- (iii) *If $n \geq 2$, then every real group structure on T is equivalent to exactly one real group structure of the form $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$, where $n = n_0 + n_2 + 2n_2$.*

Proof. This result is well-known to specialists (see for instance [Vos98, Chp.4, §10.1]) but we give a sketch of the proof for the sake of completeness.

Clearly, each σ_i for $i = 1, 2$ or 3 defines a real group structure on T . For $n = 1$, the structures σ_0 and σ_1 are not equivalent, since the real parts are not diffeomorphic. Also since $\text{Aut}_{gr}(\mathbb{G}_m) \simeq \mu_2$, these are the only two real group structures on a one-dimensional torus. For dimension two, σ_2 defines a new real group structure since it is an antiregular group involution. Also, for $n \geq 2$ all the real group structures on T given in (iii) exist and are inequivalent since the corresponding real parts are $(\mathbb{R}^*)^{n_0} \times (S^1)^{n_1} \times (\mathbb{C}^*)^{n_2}$ which are pairwise non-diffeomorphic.

It remains to show that all real group structures are equivalent to one of the structures given in (iii). This is done by determining all the conjugacy classes of

$\text{Aut}_{gr}(\mathbb{G}_m^n) \simeq \text{GL}_n(\mathbb{Z})$. More precisely, any real group structure in dimension n is equivalent to $\sigma = \varphi \circ (\sigma_0^{\times n})$ for some $\varphi \in \text{Aut}_{gr}(\mathbb{G}_m^n)$ (see Remark 1.2). Since all group automorphisms commute with $\sigma_0^{\times n}$, we see that σ is an involution if and only if φ is an involution. Also, two involutions define equivalent real group structures if and only if they are in the same conjugacy class.

Finally, note that the conjugacy classes of elements of order 2 in $\text{GL}_n(\mathbb{Z})$ are represented exactly by block diagonal matrices of the form

$$\text{diag} \left(1, \dots, 1, -1, \dots, -1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right).$$

Each block corresponds to σ_0 , σ_1 and σ_2 respectively, which proves the result. \square

Remark 1.6. If one forgets the group structure and considers $T \simeq \mathbb{G}_m^n$ as a complex variety, then there are other real structures on T . For instance $\tau_1 : t \mapsto -\bar{t}^{-1}$ is a real structure on \mathbb{G}_m but not a real group structure. In fact, one can show that any real structure on T is equivalent to a product $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2} \times \tau_1^{\times m}$ for some $n_0, n_1, n_2, m \in \mathbb{N}$.

It remains to determine the real group structures on complex semisimple algebraic groups. For any complex semisimple algebraic group G' , there exists a central isogeny $\varphi : \widetilde{G}' \rightarrow G'$, where \widetilde{G}' is a simply-connected semisimple algebraic group (uniquely defined by G' up to isomorphism); see [Con14, Exercise 1.6.13]. Then \widetilde{G}' is isomorphic to a product of simply-connected simple algebraic groups [Con14, §6.4]; the later is uniquely defined up to permutation of the factors.

The next lemma reduces the classification of real group structures on simply-connected semisimple groups to the classification of real group structures on simply-connected simple groups.

Lemma 1.7. *Let σ be a real group structure on a complex simply-connected semisimple algebraic group $G' \simeq \prod_{i \in I} G_i$, where the G_i are the simple factors of G' . Then, for a given $i \in I$, we have the following possibilities:*

- (i) $\sigma(G_i) = G_i$ and $\sigma|_{G_i}$ is a real group structure on G_i ; or
- (ii) there exists $j \neq i$ such that $\sigma(G_i) = G_j$, then $G_i \simeq G_j$ and $\sigma|_{G_i \times G_j}$ is equivalent to $(g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$, where σ_0 is an arbitrary real group structure on $G_i \simeq G_j$.

Proof. We use the fact that the factors G_i are the unique simple normal subgroups of G (see [Con14, Theorem 5.1.19]). In particular any group automorphism of G permutes the factors. Since σ is a group involution, either $\sigma(G_i) = G_i$ and we get (i), or $\sigma(G_i) = G_j$ for some $j \neq i$. In the second case, G_i and G_j are \mathbb{R} -isomorphic, and since they are both simply-connected simple groups they must be \mathbb{C} -isomorphic (this follows for instance from the classification of simply-connected simple algebraic groups in terms of Dynkin diagrams). Therefore $G_i \times G_j \simeq H \times H$, for some simply-connected simple group H , and $\sigma|_{G_i \times G_j}$ identifies with $\sigma_{H \times H} : (h_1, h_2) \mapsto (\sigma_1(h_2), \sigma_1^{-1}(h_1))$ for some antiregular automorphism σ_1 on H . But then it suffices to conjugate $\sigma_{H \times H}$ with the group automorphism φ defined by $(h_1, h_2) \mapsto (\sigma_1 \circ \sigma_0(h_2), h_1)$ to get the real group structure $(g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$, where σ_0 is an arbitrary real group structure on H . \square

The real group structures on complex simply-connected simple algebraic groups are well-known (a recap can be found in Appendix A); they correspond to real

Lie algebra structures on complex simple Lie algebras (see [Kna02, §VI.10] for the classification of those). Therefore we can determine all the real group structures on complex simply-connected semisimple algebraic groups from Lemma 1.7. In the next subsection, we will give a brief outline of a way to classify them, using quasi-split structures and inner twists.

Example 1.8. Up to equivalence, there are two real group structures on SL_2 given by $\sigma_0(g) = \bar{g}$ and $\sigma_1(g) = {}^t\bar{g}^{-1}$. Up to equivalence, there are four real group structures on $\mathrm{SL}_2 \times \mathrm{SL}_2$ given by $\sigma_i \times \sigma_j$ with $(i, j) \in \{(0, 0), (0, 1), (1, 1)\}$ and $\sigma_2 : (g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$. Similarly, we let the reader check that, up to equivalence, there are six real group structures on $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and nine real group structures on $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$.

1.2. Quasi-split real group structures and inner twists. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\} \simeq \mu_2$. Let (G, σ) be a complex reductive algebraic group with a real group structure. Note that the sets of Borel subgroups of G and, for any $n \in \mathbb{N}$, the sets of n -dimensional tori of G are each preserved by σ (this is a direct consequence of the definition of Borel subgroups and tori).

Definition 1.9.

- (i) If there exists a Borel subgroup $B \subset G$ such that $\sigma(B) = B$, then σ is called *quasi-split*. We will see in Theorem 1.20 (iii) that a quasi-split real group structure preserves some torus $T \subset B$. Moreover, with the notation of Proposition 1.5, if the restriction $\sigma|_T$ is equivalent to a product $\sigma_0^{\times \dim(T)}$, then σ is called *split*.
- (ii) For $c \in G$ we denote by inn_c the inner automorphism of G defined by

$$\mathrm{inn}_c : G \rightarrow G, g \mapsto cgc^{-1}.$$

If σ_1 and σ_2 are two real group structures on G such that $\sigma_2 = \mathrm{inn}_c \circ \sigma_1$, for some $c \in G$, then σ_2 is called an *inner twist* of σ_1 .

Remark 1.10. If σ is a quasi-split real group structure on G , then for any pair (B', T') with $B' \subset G$ a Borel subgroup and $T' \subset B'$ a maximal torus, there exists an equivalent real group structure $\sigma' = \mathrm{inn}_c \circ \sigma \circ \mathrm{inn}_c^{-1}$, for some $c \in G$, stabilizing both B' and T' . Indeed all Borel subgroups of G are G -conjugate and all maximal tori contained in a Borel subgroup B are B -conjugate (see [Hum75, §21.3]).

There exists a unique split real group structure on G up to equivalence (see [Con14, Theorem 6.1.17] or [OV90, Chp.5, §4.4] when G is semisimple) that we will always denote by σ_0 . There is also a unique compact real group structure σ_c on G up to equivalence (see [OV90, Chp. 5, §§1.3-1.4] when G is semisimple and Proposition 1.5 when G is a torus, the general case follows from these two cases), i.e. a real group structure such that $G(\mathbb{C})^{\sigma_c}$ is a maximal compact subgroup of $\mathbb{C}(G)$, but we will not use it in a crucial way.

In general, G may have several inequivalent quasi-split real group structures. If G is a simple algebraic group, then G has at most two inequivalent quasi-split real group structures (this will follow from Theorem 1.20 (iv)). On the other hand, for tori all real group structures are quasi-split.

The next lemma yields a description of the set of real group structures obtained as inner twists of a given real group structure.

Lemma 1.11. *For a given $c \in G$, the antiregular group automorphism $\text{inn}_c \circ \sigma$ is a real group structure on G if and only if $c\sigma(c) \in Z(G)$ (and then $c\sigma(c) = \sigma(c)c$). Also, $\text{inn}_c \circ \sigma = \text{inn}_{c'} \circ \sigma$ if and only if $c^{-1}c' \in Z(G)$.*

Proof. Since $\text{inn}_c \circ \sigma$ is an antiregular group automorphism, it is a real group structure if and only if it is an involution, i.e.

$$\begin{aligned} (\text{inn}_c \circ \sigma) \circ (\text{inn}_c \circ \sigma) &= \text{Id} \Leftrightarrow \forall g \in G, c\sigma(c\sigma(g)c^{-1})c^{-1} = g \\ &\Leftrightarrow \forall g \in G, c\sigma(c)g = gc\sigma(c) \\ &\Leftrightarrow c\sigma(c) \in Z(G). \end{aligned}$$

A similar computation yields the second equivalence stated in the lemma. \square

Example 1.12. Let $c = c^{-1} = \begin{bmatrix} 0 & 0 & -i \\ 0 & -1 & 0 \\ i & 0 & 0 \end{bmatrix} \in \text{SL}_3$. Up to equivalence, the group

SL_3 has three real group structures given by $\sigma_0(g) = \bar{g}$, which is split and whose real part is $\text{SL}_3(\mathbb{R})$, $\sigma_1(g) = c({}^t\bar{g}^{-1})c^{-1}$, which is quasi-split and whose real part is $\text{SU}(1, 2)$, and $\sigma_2(g) = {}^t\bar{g}^{-1} = \text{inn}_c \circ \sigma_1$, which is an inner twist of σ_1 and whose real part is $\text{SU}(3)$.

We now introduce a Γ -action that will play an important role in §3 when studying the equivariant real structures on horospherical varieties.

Definition 1.13. Let (G, σ_{qs}) be a complex reductive algebraic group with a quasi-split real group structure that preserves a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Then σ_{qs} induces a Γ -action on the character group $\mathbb{X} = \text{Hom}_{gr}(T, \mathbb{G}_m)$ and on the cocharacter group $\mathbb{X}^\vee = \text{Hom}_{gr}(\mathbb{G}_m, T)$ as follows:

$$\forall \chi \in \mathbb{X}, \quad \gamma\chi = \tau \circ \chi \circ \sigma_{qs} \quad \text{and} \quad \forall \lambda \in \mathbb{X}^\vee, \quad \gamma\lambda = \sigma_{qs} \circ \lambda \circ \tau,$$

where $\tau(t) = \bar{t}$ is the complex conjugation. Also, the sets of roots, coroots, simple roots, and simple coroots associated with the triple (G, B, T) are preserved by this Γ -action [Con14, Remark 7.1.2], and so Γ acts on the based root datum of G .

Remark 1.14. If $\sigma_{qs} = \sigma_0$ is a split real group structure, then the Γ -action on \mathbb{X} and \mathbb{X}^\vee is trivial.

We now recall the definition of the first Galois cohomology pointed set as it will appear several times in the rest of this article. Since we are concerned with real structures, we will restrict the presentation to Galois cohomology for $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\} \simeq \mu_2$ and Γ -groups. More details on Galois cohomology in a more general setting can be found in [Ser02].

Definition 1.15. If A is a Γ -group, then the first Galois cohomology pointed set is $H^1(\Gamma, A) = Z^1(\Gamma, A) / \sim$, where $Z^1(\Gamma, A) = \{a \in A \mid a^{-1} = \gamma a\}$ and two elements $a_1, a_2 \in Z^1(\Gamma, A)$ satisfy $a_1 \sim a_2$ if $a_2 = b^{-1}a_1\gamma b$ for some $b \in A$.

Remark 1.16. If A is an abelian group, then $H^1(\Gamma, A)$ is an abelian group. In this case, we can also define $H^2(\Gamma, A)$ which identifies with the group $A^\Gamma / \{a\gamma a \mid a \in A\}$; see [Ser02, §I.2] for details.

Remark 1.17. We have $a^2 = a(a^{-1})^{-1} = a(\gamma a)^{-1} \sim 1$ for all $a \in Z^1(\Gamma, A)$. In the case where $H^1(\Gamma, A)$ is finite, this implies that its cardinal is a power of 2.

Notation 1.18. Of course, the Galois cohomology depends on the Γ -action on the group A . In this article, we will sometimes consider different Γ -actions for the same group A . If the action is not clear from the context, then we will say $H^i(\Gamma, A)$ for the Γ -action on A induced by σ ; this means that σ is an involution on A , and that the non-trivial element $\gamma \in \Gamma$ acts on A by applying σ .

For example, if A an automorphism group of a variety X , and μ is an involution on X , then when we say $H^i(\Gamma, A)$ for the Γ -action on A induced by μ -conjugation, we mean that the non-trivial element $\gamma \in \Gamma$ acts on A by conjugating automorphisms by μ .

We now calculate the first and second cohomology groups for the case where $A = \mathbb{T}$ is a torus, and the Γ -action on \mathbb{T} is induced by a real group structure. The result for the first cohomology group will be used later in the article, in Lemma 1.24 and in Proposition 3.23. We will use the result for the second cohomology group in Remark 1.23.

Proposition 1.19. *Let \mathbb{T} be a torus endowed with a real group structure equivalent to a product $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$ (with the notation of Proposition 1.5), then*

- (i) $H^1(\Gamma, \mathbb{T}) \simeq (\mu_2)^{n_1}$; and
- (ii) $H^2(\Gamma, \mathbb{T}) \simeq (\mu_2)^{n_0}$.

Proof. This result comes from a direct computation. More precisely, if $\mathbb{T} \simeq \mathbb{G}_m$ with the real structure σ_0 , given $a \in \mathbb{T}^\Gamma$, there exists $b \in \mathbb{T}$ such that $a = \pm b \gamma b$. If $\mathbb{T} \simeq \mathbb{G}_m$ with the real structure σ_1 , any $a \in \mathbb{T}^\Gamma$ is of the form $b \gamma b$ for a choice of $b \in T$, and the same holds for $\mathbb{T} \simeq \mathbb{G}_m^2$ endowed with the real structure σ_2 . Now by using the definitions of the two cohomology groups, one finds that each σ_0 -component of the real structure induces a non-trivial component of $H^2(\Gamma, \mathbb{T})$, isomorphic to μ_2 , and each σ_1 -component of the real structure induces a non-trivial component of $H^1(\Gamma, \mathbb{T})$, isomorphic to μ_2 . The σ_0 - and σ_2 -components have no effect on the first cohomology group, and the σ_1 - and σ_2 -components have no effect on the second cohomology group. \square

Let σ be a real group structure on the complex reductive algebraic group G . As σ is a group involution, it preserves $Z(G)$. Let $\text{Inn}(G) \simeq G/Z(G)$ be the group of inner automorphisms of G and let $\text{Out}(G)$ be the quotient group $\text{Aut}_{gr}(G)/\text{Inn}(G)$. The Galois group Γ acts on $\text{Aut}_{gr}(G)$ by σ -conjugation, i.e. $\gamma\varphi = \sigma \circ \varphi \circ \sigma$ for all $\varphi \in \text{Aut}_{gr}(G)$; this Γ -action stabilizes $\text{Inn}(G)$ on which it coincides with the Γ -action induced by σ on $G/Z(G)$. The short exact sequence

$$(A) \quad 1 \rightarrow \text{Inn}(G) \rightarrow \text{Aut}_{gr}(G) \rightarrow \text{Out}(G) \rightarrow 1$$

induces a long exact sequence in Galois cohomology (see [Ser02, §5.5]):

$$1 \rightarrow \text{Inn}(G)^\Gamma \rightarrow \text{Aut}_{gr}(G)^\Gamma \rightarrow \text{Out}(G)^\Gamma \rightarrow H^1(\Gamma, \text{Inn}(G)) \rightarrow H^1(\Gamma, \text{Aut}_{gr}(G)) \xrightarrow{\theta} H^1(\Gamma, \text{Out}(G)).$$

The next theorem gathers the main results we will need regarding the classification of the real group structures on G via Galois cohomology.

Theorem 1.20. *We keep the previous notation, and we fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. The following statements hold:*

- (i) *Let Γ act on $\text{Aut}_{gr}(G)$ by σ -conjugation as above. Then the map*

$$H^1(\Gamma, \text{Aut}_{gr}(G)) \rightarrow \{\text{real group structures on } G\}/\text{equiv} \text{ induced by } \varphi \mapsto \varphi \circ \sigma$$

is a bijection that sends the identity element to the equivalence class of σ .

- (ii) We have $\theta(\sigma_1) = \theta(\sigma_2)$ if and only σ_2 is equivalent to an inner twist of σ_1 .
- (iii) There is exactly one equivalence class of quasi-split real group structures in each non-empty fiber of the map θ . Moreover, in each of them there is a quasi-split real group structure that stabilizes B and T .
- (iv) If moreover G is simply-connected semisimple, then the map

$$\{q\text{-s. real gr. structures on } G \text{ preserving } B \text{ and } T\} \rightarrow \text{Aut}(\text{Dyn}(G))$$

of Definition 1.13 induces a bijection between the set of equivalence classes of quasi-split real group structures and the set of conjugacy classes of elements of order ≤ 2 in $\text{Aut}(\text{Dyn}(G))$.

Proof. All the proofs and details of the statements can be found in [Con14, §7]. More precisely:

- (i): [Con14, Lemma 7.1.1 and the beginning of §7.2].
- (ii): [Con14, Remark 7.1.2, Propositions 7.2.2 and 7.2.12, and comments on p. 323].
- (iii): [Con14, Proposition 7.2.12].
- (iv): This follows from [Con14, Remarks 7.1.2 and 7.1.7]. \square

Example 1.21.

- (1) We keep the notation of Example 1.12 and we write $\sigma_{ij} = \sigma_i \times \sigma_j$. The group $G = \text{SL}_3 \times \text{SL}_3$ has seven inequivalent real group structures: σ_{00} (split), σ_{01} (quasi-split) and its inner twist $\sigma_{02} = \text{inn}_{(1,c)} \circ \sigma_{01}$, σ_{11} (quasi-split) and its two inner twists $\sigma_{12} = \text{inn}_{(1,c)} \circ \sigma_{11}$ and $\sigma_{22} = \text{inn}_{(c,c)} \circ \sigma_{11}$, and the quasi-split real group structure $\sigma : (g_1, g_2) \mapsto (\sigma_0(g_2), \sigma_0(g_1))$. Also, $\text{Aut}(\text{Dyn}(G))$ is the dihedral group $\langle r, s \mid r^4 = s^2 = (sr)^2 = 1 \rangle$ which is the union of five conjugacy classes $\{1\}$, $\{r^2\}$, $\{s, rsr^{-1}\}$, $\{sr, rs\}$, and $\{r, r^{-1}\}$, four of which consist of elements of order ≤ 2 . We may check that the bijection in Theorem 1.20 (iv) is the following one: $\sigma_{00} \leftrightarrow \{1\}$, $\sigma_{01} \leftrightarrow \{sr, rs\}$, $\sigma_{11} \leftrightarrow \{r^2\}$, and $\sigma \leftrightarrow \{s, rsr^{-1}\}$.
- (2) Let $G = \text{Spin}_8$. Then G has a unique (up to equivalence) split real group structure denoted, as usual, by σ_0 . Note that $\text{Dyn}(G) = D_4$ and that $\text{Aut}(\text{Dyn}(G)) \simeq \mathfrak{S}_3$. This group has exactly one conjugacy class of non-trivial elements of order 2, which corresponds via the bijection in Theorem 1.20 (iv) to the unique (up to equivalence) quasi-split not split real group structure σ_1 . Moreover, there are two other inequivalent real group structures on G , which are both inner twists of σ_0 , and one other inequivalent real group structure on G which is an inner twist of σ_1 . The real parts of these real groups structures are the real Lie groups $\text{Spin}(8 - m, m)$ with $0 \leq m \leq 4$. The case $m = 4$ corresponds to σ_0 , and its inner twists correspond to $m = 2$ and $m = 0$ (the later is the real part of the compact real group structure on G). The case $m = 3$ corresponds to σ_1 , and the case $m = 1$ corresponds to the inner twist of σ_0 .

1.3. A cohomological invariant. Let (G, σ_{qs}) be a complex reductive algebraic groups with a quasi-split real group structure. We consider the short exact sequence of Γ -groups

$$1 \rightarrow Z(G) \rightarrow G \rightarrow G/Z(G) \rightarrow 1.$$

where the Γ -action is induced by σ_{qs} . In other words, the element $\gamma \in \Gamma$ acts on G and $Z(G)$ by σ_{qs} , and on $G/Z(G)$ by the induced real group structure $\overline{\sigma_{qs}}$. Since $Z(G)$ is an abelian group, there is a connecting map (see [Ser02, §I.5.7])

$$(B) \quad \delta : H^1(\Gamma, G/Z(G)) \rightarrow H^2(\Gamma, Z(G)).$$

It follows from Lemma 1.11 and Definition 1.15 that $Z^1(\Gamma, G/Z(G))$ identifies with the set of inner twists of σ_{qs} and

$$H^1(\Gamma, G/Z(G)) \simeq \{c \in G \mid c\sigma_{qs}(c) \in Z(G)\} / \sim$$

where $c \sim c'$ if $c^{-1}b^{-1}c'\sigma_{qs}(b) \in Z(G)$ for some $b \in G$. Also, there is an isomorphism of abelian groups (see Remark 1.16):

$$H^2(\Gamma, Z(G)) \simeq Z(G)^\Gamma / \{a\sigma_{qs}(a) \mid a \in Z(G)\}.$$

With these identifications, the connecting map δ is the map induced by

$$\{c \in G \mid c\sigma_{qs}(c) \in Z(G)\} \rightarrow Z(G)^\Gamma, \quad c \mapsto c\sigma_{qs}(c).$$

If σ is a real group structure on G equivalent to $\text{inn}_c \circ \sigma_{qs}$, then we will also write $\delta(\sigma)$ instead of $\delta(\bar{c})$.

Definition 1.22. When G is a simply-connected semisimple algebraic group, the element $\delta(\sigma)$ is called the *Tits class* of the real group structure (G, σ) .

Tables where the Tits classes are determined for any (G, σ) , with G a simply-connected simple algebraic group and σ a real group structure on G , can be found in Appendix A.

Consider now the case where H is a subgroup of G such that:

- the algebraic group $N_G(H)/H$ is abelian (this is the case, for example, if H is a spherical subgroup of G by [Per14, Proposition 3.4.1]); and
- H is conjugate to a subgroup H' stable by σ_{qs} .

Replacing H by H' , we may assume that $\sigma_{qs}(H) = H$ to simplify the situation. Then σ_{qs} induces a real group structure on $N_G(H)/H$, namely $\overline{\sigma_{qs}}(nH) = \sigma_{qs}(n)H$, and we can consider the second cohomology group $H^2(\Gamma, N_G(H)/H)$. The natural homomorphism $\chi_H : Z(G) \rightarrow N_G(H)/H$, induced by the inclusion $Z(G) \rightarrow N_G(H)$, yields an homomorphism between the second cohomology groups

$$\chi_H^* : H^2(\Gamma, Z(G)) \rightarrow H^2(\Gamma, N_G(H)/H).$$

In the rest of this article we will denote the composed map $\chi_H^* \circ \delta$ by

$$\Delta_H : H^1(\Gamma, G/Z(G)) \rightarrow H^2(\Gamma, N_G(H)/H).$$

The element $\Delta_H(\sigma) \in H^2(\Gamma, N_G(H)/H)$ is the *cohomological invariant* that the title of this subsection is referring to.

Remark 1.23. A consequence of Proposition 1.19 is that if $N_G(H)/H$ is a torus, and if the Γ -action is induced by a real group structure on this torus with $n_0 = 0$, then $H^2(\Gamma, N_G(H)/H) = \{0\}$ and so Δ_H is the trivial map.

Lemma 1.24. *With the previous notation, let H be a maximal unipotent subgroup of G . Then the two conditions above are satisfied and $\Delta_H(\sigma) = 1$ if and only if $\delta(\sigma) = 1$.*

Proof. First, note that σ_{qs} stabilizes a Borel subgroup B , and therefore also its maximal unipotent subgroup U . Also, $N_G(H) = B$, and therefore $N_G(H)/H = \mathbb{T}$ is a torus (isomorphic to a maximal torus of G). Thus the two conditions above hold. By applying an appropriate conjugation, we can assume that $H = U$ and then $\sigma_{qs}(H) = H$.

Now we will show that χ_U^* is injective, which will imply the result. Consider the short exact sequence

$$0 \mapsto Z(G) \rightarrow \mathbb{T} \rightarrow \overline{\mathbb{T}} = \mathbb{T}/Z(G) \rightarrow 0.$$

This exact sequence induces an exact sequence of cohomology groups:

$$H^1(\Gamma, \overline{\mathbb{T}}) \rightarrow H^2(\Gamma, Z(G)) \rightarrow H^2(\Gamma, \mathbb{T}),$$

where the second map is simply χ_U^* . The torus $\overline{\mathbb{T}}$ is isomorphic to a maximal torus of the adjoint semi-simple group $G/Z(G)$ and the quasi-split real group structure $\overline{\sigma}_{qs}$ on $G/Z(G)$, induced by σ_{qs} on G , acts on the character group of $\overline{\mathbb{T}}$ by permutations. Indeed, since $\overline{\sigma}_{qs}$ stabilizes the Borel subgroup \overline{B} of $G/Z(G)$, it preserves the positive roots of $(G/Z(G), \overline{B}, \overline{\mathbb{T}})$. This means in particular that the restriction of $\overline{\sigma}_{qs}$ on $\overline{\mathbb{T}}$ is equivalent to $\sigma_0^{\times n_0} \times \sigma_2^{\times n_2}$ (that is, there are no factors of type σ_1).

In the cohomology group $H^1(\Gamma, \overline{\mathbb{T}})$, the Γ -action on $\overline{\mathbb{T}}$ is induced by σ_{qs} . Thus, by Proposition 1.19, the group $H^1(\Gamma, \overline{\mathbb{T}})$ is trivial. This implies that χ_U^* is injective. \square

The map Δ_H will be a key-ingredient in §3.3 to determine the existence of a (G, σ) -equivariant real structure on a horospherical homogeneous space G/H when σ is a non quasi-split real group structure on G .

2. EQUIVARIANT REAL STRUCTURES ON QUASI-HOMOGENEOUS SPACES

In this section, we introduce the notion of equivariant real structures on G -varieties which are real structures compatible with a given real group structure on the complex algebraic group G . We show first how to determine if such a structure exists on homogeneous spaces, and, if so, then how to use Galois cohomology to determine the set of equivalence classes of these equivariant real structures. We will also make some observations on the existence of extensions of these real structures on quasi-homogeneous spaces. These results will be used in the next section.

Definition 2.1. Let (G, σ) be a complex algebraic group with a real group structure, and let X be a G -variety.

- (i) A (G, σ) -equivariant real structure on X is an antiregular involution μ on X such that

$$\forall g \in G, \forall x \in X, \mu(g \cdot x) = \sigma(g) \cdot \mu(x).$$

- (ii) Two equivariant real structures μ and μ' on a (G, σ) -variety X are *equivalent* if there exists a G -automorphism $\varphi \in \text{Aut}^G(X)$ such that $\mu' = \varphi \circ \mu \circ \varphi^{-1}$.

Remark 2.2. Let (X, μ) be a (G, σ) -equivariant real structure. Whenever the real part $X_0 = X(\mathbb{C})^\mu$ of X is non-empty, it defines a real manifold endowed with an action of the real Lie group $G_0 = G(\mathbb{C})^\sigma$. In particular, if X is a spherical G -variety, then this construction provides examples of real spherical spaces.

Remark 2.3. If X is a *quasi-homogeneous* G -variety, i.e. a G -variety with an open orbit G/H , then a given equivariant real structure μ on G/H does not always extend to X . If μ extends to X , then this extension is unique.

Lemma 2.4. *Let (G, σ) be a complex algebraic group with a real group structure, and let $X = G/H$ be a homogeneous space. Then X has (G, σ) -equivariant real structure if and only if there exists $g \in G$ such that these two conditions hold:*

- (1) (G, σ) -compatibility condition: $\sigma(H) = gHg^{-1}$

(2) involution condition: $\sigma(g)g \in H$

in which case μ is defined by $\mu(kH) = \sigma(k)gH$ for all $k \in G$.

Proof. If μ defines a (G, σ) -equivariant real structure on X , then μ is determined by $\mu(eH) = gH$ for some $g \in G$. Then μ must be compatible with σ , i.e. $\mu(eH) = \mu(hH) = \sigma(h)\mu(eH)$ for all $h \in H$, this yields condition (1). Also, μ must be an involution, this yields condition (2). For the converse, if $g \in G$ satisfies the conditions (1) and (2), then the map $\mu : G \rightarrow G$ defined by $\mu(kH) = \sigma(k)gH$ is clearly a (G, σ) -equivariant real structure on X . \square

Remark 2.5. If H' is conjugate to H , then G/H has a (G, σ) -equivariant real structure μ if and only if G/H' has a (G, σ) -equivariant real structure μ' . Indeed, if $H' = kHk^{-1}$ and $\mu(eH) = gH$, then we can define μ' by $\mu'(eH') = g'H$ with $g' = \sigma(k)gk^{-1}$. We check that $\sigma(H') = g'H'g'^{-1}$ if and only if $\sigma(H) = gHg^{-1}$, and that $\sigma(g')g' \in H'$ if and only if $\sigma(g)g \in H$.

Example 2.6. (Equivariant real structures on toric varieties.)

Let (T, σ) be a complex torus with a real group structure (those were described in Proposition 1.5). Let X be a complex toric variety with open orbit $X_0 \simeq T$.

We start with the case of a homogeneous toric variety, that is, we consider the case $X = X_0$. It is easy to check, using Lemma 2.4 that the homogeneous space X_0 always has a (T, σ) -equivariant real structure; we can for instance choose $t_0 = 1$ and consider $\mu = \sigma$. Now we can find all the equivalence classes of (T, σ) -equivariant structures on X_0 . We use the notation of Remark 1.6. Suppose that T is endowed with the real group structure $\sigma = \sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$. Then each anti-regular involution of the form $\sigma_0^{\times n_0} \times \mu'_1 \times \cdots \times \mu'_{n_1} \times \sigma_2^{\times n_2}$, where $\mu'_i = \sigma_1$ or τ_1 for each $i = 1, \dots, n_1$, defines a (T, σ) -equivariant real structure on T , and no two of these involutions are (equivariantly) equivalent. Moreover, any (T, σ) -equivariant real structure on T is of this form.

As for the quasi-homogeneous case, by [Hur11b, Theorem 1.25] (see also [Hur11a, Chp. 1]) the equivariant real structure μ on X_0 extends on X if and only if the Γ -action on $\mathbb{X}^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$, introduced in Definition 1.13, stabilizes the fan of X . (For a complete account on toric varieties, we refer the interested reader to [Ful93].)

Lemma 2.7. *Let (G, σ_1) be a complex algebraic group with a real group structure, and let $X = G/H$ be a homogeneous space. Let $\sigma_2 = \text{inn}(c) \circ \sigma_1$ be an inner twist of σ_1 , and suppose that $\sigma_1(H) = H$. Then*

- (i) X has (G, σ_1) -equivariant real structure; and
- (ii) X has a (G, σ_2) -equivariant real structure if and only if there exists $n \in N_G(H)$ such that $c\sigma_1(c)\sigma_1(n)n \in H$.

Proof. The first statement follows from Lemma 2.4 (take $g = 1$). For the second statement, note that $\sigma_2(H) = cHc^{-1}$. According to Lemma 2.4, the variety X has a (G, σ_2) -equivariant real structure if and only if there exists $g \in G$ such that $gHg^{-1} = \sigma_2(H) = cHc^{-1}$, and $g\sigma_2(g) \in H$. But $gHg^{-1} = cHc^{-1}$ if and only if there exists $n \in N_G(H)$ such that $g = cn$. Then the second condition in Lemma 2.4 yields $\sigma_2(g)g = c\sigma_1(g)c^{-1}g = c\sigma_1(c)\sigma_1(n)n \in H$. The converse is straightforward, it suffices to take $g = cn$. \square

In the next proposition, we use Lemma 2.7 to give a cohomological condition to determine the existence of an equivariant real structure on G/H . Because of the

well-known tables of structures on semisimple algebraic groups, this cohomological interpretation is particularly well-adapted to calculate examples.

Proposition 2.8. *Let (G, σ) be a complex reductive algebraic group with a quasi-split real group structure, and let $\sigma_{qs} = \text{inn}_c \circ \sigma_{qs}$ be an inner twist of σ_{qs} . Let $X = G/H$ be a homogeneous space, and assume that $N_G(H)/H$ is abelian and $\sigma_{qs}(H) = H$. Then*

- (i) X has a (G, σ_{qs}) -equivariant real structure; and
- (ii) X has a (G, σ) -equivariant real structure if and only if $\Delta_H(\sigma) = 1$.

Proof. Unraveling the definition of Δ_H in §1.3, we verify that the cohomological condition $\Delta_H(\sigma) = 1$ is equivalent to the second condition in Lemma 2.7. Then the result then follows directly from Lemma 2.7. \square

Remark 2.9. In [Bor], Borovoi considers the case where X is a quasi-projective G -variety over an arbitrary field of characteristic zero which admits a (G, σ_{qs}) -equivariant real structure. He obtains a similar cohomological criterion for the existence of a (G, σ) -equivariant real structure on X , but his point of view is quite different from ours as he uses exclusively Galois cohomology while we go through group-theoretical considerations (Lemmas 2.4 and 2.7).

Remark 2.10. If (G, σ) is a simply-connected simple algebraic group with a real group structure, then the Tits class of (G, σ) is often trivial (see tables in Appendix A) in which case by Proposition 2.8 the existence of a (G, σ) -equivariant real structure on G/H reduces to the study of the conjugacy class of H .

The previous results provide conditions for the existence of an equivariant real structure on a homogeneous space, but they tell nothing about the number of equivalence classes of such equivariant real structures. The classical way to determine this number is via Galois cohomology.

Lemma 2.11. (Equivariant real structures and Galois cohomology.)

Let (G, σ) be a complex algebraic group with a real group structure, and let (X, μ_0) be a homogeneous space with a (G, σ) -equivariant real structure. The Galois group $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\text{Aut}^G(X)$ by μ_0 -conjugacy. Then the map

$$H^1(\Gamma, \text{Aut}^G(X)) \rightarrow \{(G, \sigma)\text{-eq. real structures on } X\}/\text{equiv.}, \quad \varphi \mapsto \varphi \circ \mu_0$$

is a bijection that sends the identity element to the equivalence class of μ_0 .

Proof. Note that any antiregular automorphism of X is of the form $\mu = \varphi \circ \mu_0$, where φ is a regular automorphism on X . Now μ defines a (G, σ) -equivariant real structure if and only if $\mu \circ \mu = \text{id}$ and φ is G -equivariant. With the notation of Definition 1.15, this is equivalent to requiring that $\varphi \in Z^1(\Gamma, \text{Aut}^G(X))$. Finally, $\mu = \varphi \circ \mu_0$ is equivalent to $\mu' = \varphi' \circ \mu_0$, with $\varphi, \varphi' \in Z^1(\Gamma, \text{Aut}^G(X))$, if and only if there exists $\psi \in \text{Aut}^G(X)$ such that $\psi \circ \mu' \circ \psi^{-1} = \mu$, that is, $\psi \circ \varphi \circ (\psi^{-1} \circ \mu_0) = \varphi' \circ \mu_0$. This is precisely the equivalence condition defining $H^1(\Gamma, \text{Aut}^G(X))$. \square

The interested reader may also consult [Ser02, §III.1] for more general results on the classification of real structures via Galois cohomology.

If $X = G/H$ is a homogeneous space, then we recall that $\text{Aut}^G(G/H) \simeq N_G(H)/H$ (see e.g. [Tim11, Proposition 1.8]). In particular, if $N_G(H) = H$, then $\text{Aut}^G(X)$ is the trivial group.

Corollary 2.12. *Let (G, σ) be a complex reductive algebraic group with a real group structure, and let $X = G/H$ be a homogeneous space such that $N_G(H) = H$. Then X has a (G, σ) -equivariant real structure μ if and only if $\sigma(H) = cHc^{-1}$ for some $c \in G$. Moreover, if μ exists then it is equivalent to $\mu : gH \mapsto \sigma(g)cH$.*

Proof. If $N_G(H) = H$, then Condition (1) in Lemma 2.4 implies Condition (2), and so G/H has a (G, σ) -equivariant real structure if and only if $\sigma(H)$ is conjugate to H . As $\text{Aut}^G(X) \simeq N_G(H)/H = \{1\}$ the uniqueness part of the statement follows from Lemma 2.11. The last statement is given by Lemma 2.4. \square

Example 2.13. Let (G, σ) be a complex reductive algebraic group with a real group structure, and let $X = G/P$ be a flag variety. Then X has an equivariant real structure μ if and only if $\sigma(P)$ is conjugate to P . Moreover, if μ exists then it is equivalent to $\mu : gP \mapsto \sigma(g)cP$, where $c \in G$ satisfies $\sigma(P) = cPc^{-1}$.

3. EQUIVARIANT REAL STRUCTURES ON HOROSPHERICAL VARIETIES

Because of the particular features of horospherical varieties, the determination of their equivariant real structures can be achieved by the study of their combinatorial data. The main part of this section deals with the question of existence of equivariant real structures on horospherical homogeneous spaces.

Let H be a horospherical subgroup of a reductive algebraic group G , and let $N_G(H)$ be the normalizer of H in G . Then $\mathbb{T} = N_G(H)/H$ is a torus and we will apply the results of the previous sections (where $N_G(H)/H$ is assumed to be abelian) to determine whether there exists an equivariant real structure on G/H . We will explain also how to determine the number of equivalence classes of equivariant real structures on G/H . Then we will describe when the equivariant real structures extend to a given horospherical variety. Finally, we will end this section with the study of equivariant real structures on classical examples of horospherical varieties that arise in algebraic geometry.

3.1. Setting and first definitions. We fix once and for all a triple (G, B, T) , where G is a complex reductive algebraic group, $B \subset G$ is a Borel subgroup and $T \subset B$ is a maximal torus. Let $\mathcal{S} = \mathcal{S}(G, B, T)$ be the corresponding root system, let $\mathbb{X} = \mathbb{X}(T)$ be the character group of T , and let $\mathbb{X}^\vee = \mathbb{X}^\vee(T)$ be the cocharacter group of T . We denote by σ_0 a split real group structure on G such that $\sigma_0(B) = B$ and $\sigma_0(T) = T$.

Definition 3.1. A subgroup H of G is *horospherical* if it contains a maximal unipotent subgroup of G . In this case, the normalizer $P = N_G(H)$ of H in G is a parabolic subgroup of G and the quotient group $\mathbb{T} = P/H \simeq \text{Aut}^G(G/H)$ is a torus (see [Pas08, Proposition 2.2 and Remarque 2.2]). A homogeneous space G/H is *horospherical* if H is a horospherical subgroup of G .

Remark 3.2. Let σ be a real group structure on G . Then σ maps a maximal unipotent subgroup of G to a maximal unipotent subgroup of G and so the set of horospherical subgroups of G is preserved by σ .

Example 3.3. Tori and flag varieties are examples of horospherical homogeneous spaces. Let U be a maximal unipotent subgroup of SL_2 , then SL_2/U is a horospherical homogeneous space isomorphic to the affine plane minus the origin $\mathbb{A}^2 \setminus \{0\}$.

Definition 3.4. A *horospherical G -variety* is a normal G -variety with an open horospherical G -orbit.

Example 3.5. Toric varieties and flag varieties are examples of horospherical varieties. It follows from the combinatorial description of spherical embeddings (see e.g. [Tim11, §15.1]) that the horospherical SL_2 -varieties with open orbit SL_2 -isomorphic to SL_2/U are the following: $\mathbb{A}^2 \setminus \{0\}$, \mathbb{A}^2 , \mathbb{P}^2 , $\mathbb{P}^2 \setminus \{0\}$, $\mathrm{Bl}_0(\mathbb{A}^2)$, and $\mathrm{Bl}_0(\mathbb{P}^2)$.

Other examples of horospherical varieties can be found in [Pas06, Pas15b]. Also, in §3.6 we will consider examples of smooth projective horospherical varieties of Picard rank one.

We now repeat the combinatorial description of the horospherical subgroups stated in the introduction. We recall that there is a 1-to-1 correspondence between the power set of \mathcal{S} and the set of conjugacy classes of parabolic subgroups of G (see [Hum75, §29.3]). For $I \subset \mathcal{S}$, we denote by P_I the *standard parabolic subgroup* generated by B and the $U_{-\alpha}$ with $\alpha \in I$, where $U_{-\alpha}$ is the unipotent subgroup of G associated with the simple root α . Let $I \subset \mathcal{S}$ and let M be a sublattice of \mathbb{X} such that $\langle \chi, \alpha^\vee \rangle = 0$ for all $\alpha \in I$ and all $\chi \in M$, where $\langle \cdot, \cdot \rangle : \mathbb{X} \times \mathbb{X}^\vee \rightarrow \mathbb{Z}$ is the duality bracket, and $\alpha^\vee \in \mathbb{X}^\vee$ is the coroot associated to the root α ; this condition simply means that χ extends to a character of the parabolic subgroup P_I . Then

$$(C) \quad H_{(I,M)} = \bigcap_{\chi \in M} \mathrm{Ker}(\chi)$$

is a horospherical subgroup of G whose normalizer is P_I (see [Pas08, §2] for details).

Definition 3.6. The horospherical subgroup $H_{(I,M)}$ defined by (C) is called *standard horospherical subgroup*. If H is a horospherical subgroup, then by [Pas08, Proposition 2.4] there exists a unique pair (I, M) as above such that H is conjugate to $H_{(I,M)}$; we say that (I, M) is the (*horospherical*) *datum* of H .

Example 3.7. The datum of a parabolic subgroup conjugate to P_I is $(I, \{0\})$. The datum of a maximal unipotent subgroup of G is (\emptyset, \mathbb{X}) .

3.2. Quasi-split case. We denote by σ_{qs} a quasi-split real group structure on G preserving B and T as before.

Lemma 3.8. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$ acting on the set \mathcal{S} as in Definition 1.13. If $I \subset \mathcal{S}$, then $\sigma_{qs}(P_I) = P_{\gamma I}$, where $\gamma I = \{\gamma s \mid s \in I\} \subset \mathcal{S}$.

Proof. First note that $\sigma_{qs}(B) = B$ implies that $\sigma(P_I)$ is a parabolic subgroup of the form P_J for some $J \subset \mathcal{S}$ (see [Hum75, §29.4, Th.]). Since Γ acts on \mathcal{S} by permutation (see Definition 1.13), we have $\gamma\alpha \in \mathcal{S}$ for each $\alpha \in \mathcal{S}$. Also, if $\alpha \in \mathcal{S}$ and $U_{-\alpha}$ is the corresponding unipotent subgroup of G , then $\sigma_{qs}(U_{-\alpha}) = U_{-\gamma\alpha}$ by definition of $U_{-\alpha}$ and of the Γ -action on \mathcal{S} . But $P_I = \langle B, U_{-\alpha} \text{ with } \alpha \in I \rangle$, and so the last sentence implies the result. \square

Proposition 3.9. Let H be a horospherical subgroup of G with datum (I, M) . Then the horospherical datum of $\sigma_{qs}(H)$ is $\gamma(I, M) := (\gamma I, \gamma M)$ with the Γ -action on \mathbb{X} introduced in Definition 1.13. Also, $\sigma_{qs}(H_{(I,M)}) = H_{\gamma(I,M)}$, and so $\sigma_{qs}(H_{(I,M)})$ is conjugate to $H_{(I,M)}$ if and only if $\sigma_{qs}(H_{(I,M)}) = H_{(I,M)}$.

Proof. First of all, by Lemma 3.8, we have $\sigma_{qs}(P_I) = P_{\gamma I}$. Thus, the normalizer of $\sigma_{qs}(H)$ is $\sigma_{qs}(P_I) = P_{\gamma I}$. Also, by definition of the Γ -action on the coroot lattice \mathbb{X}^\vee (see Definition 1.13), we have $\langle \gamma m, \gamma(\alpha^\vee) \rangle = \langle m, \alpha^\vee \rangle = 0$ for all $\alpha \in I$ and $m \in M$. Thus the sublattice M' of \mathbb{X} associated with $\sigma_{qs}(H)$ is ${}^\gamma M$. This proves the first part of the proposition.

Now we have $\sigma_{qs}(P_I) = P_{\gamma I}$ by Lemma 3.8, and

$$\sigma_{qs}(H_{(I,M)}) = \bigcap_{m \in M} \sigma_{qs}(\text{Ker}(m)) = \bigcap_{m \in M} \text{Ker}({}^\gamma m) = H_{\gamma(I,M)}.$$

Therefore, if $\sigma_{qs}(H_{(I,M)}) = H_{\gamma(I,M)}$ is conjugate to $H_{(I,M)}$, they have the same horospherical data, that is, $\gamma(I, M) = (I, M)$, which finishes the proof. \square

Remark 3.10. The previous result can also be obtained as a consequence of [CF15, Theorem 3] which states that for a spherical subgroup $H \subset G$, the subgroups H and $\sigma(H)$ are conjugate in G if and only if the Luna-Vust invariants of G/H are Γ -stable.

Remark 3.11. Let H be a spherical subgroup of G such that $\sigma_{qs}(H)$ is conjugate to H . We do not know whether there always exists a spherical subgroup H' conjugate to H such that $\sigma_{qs}(H') = H'$. It follows from the proof of [ACF14, Theorem 4.14] that this is true when $N_G(N_G(H)) = N_G(H)$, which is the case if H is horospherical.

Corollary 3.12. *Let H be a horospherical subgroup of G with datum (I, M) . Then $X = G/H$ has a (G, σ_{qs}) -equivariant real structure if and only if $\gamma(I, M) = (I, M)$.*

Proof. By remark 2.5, we may assume that $H = H_{(I,M)}$.

If $X = G/H$ has a (G, σ_{qs}) -equivariant real structure, then H and $\sigma_{qs}(H)$ are conjugate by Lemma 2.4, and so $\gamma(I, M) = (I, M)$ by Proposition 3.9.

Conversely, if $\gamma(I, M) = (I, M)$, then $\sigma_{qs}(H) = H$ by Proposition 3.9. Thus we see that the two conditions of Lemma 2.4 are satisfied for $g = 1$, which means that X has a (G, σ_{qs}) -equivariant real structure. \square

Remark 3.13. If $\sigma_{qs} = \sigma_0$ is a split real group structure on G , then $X = G/H$ has always a (G, σ_{qs}) -equivariant real structure since Γ acts trivially on \mathbb{X} .

Example 3.14. If $H = U$ is a maximal unipotent subgroup of G , then its horospherical datum is (\emptyset, \mathbb{X}) which is of course Γ -stable, and so there always exists a (G, σ_{qs}) -equivariant real structure on G/U , namely $\overline{\sigma_{qs}}(gU) = \sigma_{qs}(g)U$.

Example 3.15. Let T be the maximal torus of $G = \text{SL}_4$ formed by diagonal matrices, and let B be the Borel subgroup formed by upper-triangular matrices. For $i \in \{1, 2, 3, 4\}$, we denote $L_i : T \rightarrow \mathbb{G}_m, (t_1, t_2, t_3, t_4) \mapsto t_i$. Then the simple roots of (G, B, T) are $\alpha_1 = L_1 - L_2$, $\alpha_2 = L_2 - L_3$, and $\alpha_3 = L_3 - L_4$ (see [FH91, Chp. 3, §15] for details).

Let $P \subset G$ be the standard parabolic subgroup associated with $I = \{\alpha_2\}$, and let H be the kernel of the character $\chi = L_1 + L_4$ in P . Then H is a horospherical subgroup of G with datum $(I, M) = (\{\alpha_2\}, \mathbb{Z}\langle \chi \rangle)$.

The group G has two inequivalent quasi-split real group structures, namely $\sigma_s(g) = \bar{g}$ (split) and $\sigma_{qs}(g) = \text{inn}_h \circ \sigma_c(g)$ (non split quasi-split), where $\sigma_c(g) =$

${}^t\bar{g}^{-1}$ is the compact real group structure on G and $h = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$. The Γ -

action on $\mathbb{X}(T)$ induced by σ_s is trivial (Remark 3.13), and so $\gamma(I, M) = (I, M)$. On the other hand, the Γ -action induced by σ_{qs} is determined by the relations $\gamma L_1 = -L_4$ and $\gamma L_2 = -L_3$. Thus, we still have $\gamma(I, M) = (I, M)$, but the Γ -action on M is non-trivial since $\gamma\chi = -\chi$. By Corollary 3.12, the homogeneous space $X = G/H$ has a (G, σ_s) - and a (G, σ_{qs}) -equivariant real structure.

3.3. General case. We now consider the case where σ is any real group structure on the complex reductive algebraic group G .

By Theorem 1.20, there exists a quasi-split real structure σ_{qs} (uniquely defined by σ , up to equivalence) and an element $c \in G$ such that σ is equivalent to $\text{inn}_c \circ \sigma_{qs}$. We may replace σ and σ_{qs} in their equivalence classes and assume that $\sigma_{qs}(B) = B$, $\sigma_{qs}(T) = T$, and $\sigma = \text{inn}_c \circ \sigma_{qs}$ to avoid technicalities. Then we still have a well-defined Γ -action on \mathbb{X} (preserving \mathcal{S}) induced by σ_{qs} (see Definition 1.13).

Lemma 3.16. *Let H be a horospherical subgroup with datum (I, M) . If $X = G/H$ has a (G, σ) -equivariant real structure, then $\gamma(I, M) = (I, M)$.*

Proof. By Remark 2.5, we may assume that $H = H_{(I, M)}$ is a standard horospherical subgroup. If X has a (G, σ) -equivariant real structure, then the first condition of Lemma 2.4 yields that $\sigma(H)$ and H are conjugate. Thus $\sigma_{qs}(H) = \text{inn}_{c^{-1}} \circ \sigma(H) = c^{-1}\sigma(H)c$ and H are also conjugate, and so $\gamma(I, M) = (I, M)$ by Proposition 3.9. \square

The previous lemma means that the existence of a (G, σ) -equivariant real structure on G/H implies the existence of a (G, σ_{qs}) -equivariant real structure on G/H . However, we will see that these two conditions are not equivalent.

Proposition 3.17. *Let $\sigma = \text{inn}_c \circ \sigma_{qs}$ be a real group structure on G as above. Let H be a horospherical subgroup of G with datum (I, M) , and assume that $X = G/H$ has a (G, σ_{qs}) -equivariant real structure. Then X has a (G, σ) -equivariant real structure if and only if $\Delta_H(\sigma) = 1$, where Δ_H is the map defined in §1.3.*

Proof. First, by Remark 2.5, we may assume that $H = H_{(I, M)}$ is a standard horospherical subgroup. By assumption, X has a (G, σ_{qs}) -equivariant real structure, and so $\sigma_{qs}(H) = H$ by Proposition 3.9. Now the result follows from Proposition 2.8. \square

We now summarize our main results in the following theorem:

Theorem 3.18. *Let H be a horospherical subgroup of G with datum (I, M) . Let $\sigma = \text{inn}_c \circ \sigma_{qs}$ be a real group structure on G , and let σ_{qs} be the corresponding quasi-split real group structure preserving B and T . Then the following four conditions are equivalent:*

- (i) G/H has a (G, σ_{qs}) -equivariant real structure;
- (ii) $\gamma(I, M) = (I, M)$;
- (iii) H is conjugate to $\sigma(H)$;
- (iv) H is conjugate to $\sigma_{qs}(H)$.

Moreover G/H has a (G, σ) -equivariant real structure if and only if the (equivalent) conditions (i)-(iv) are satisfied and $\Delta_H(\sigma) = 1$, where Δ_H is the map defined in §1.3.

Proof. The equivalence of (i) and (ii) is Corollary 3.12. The equivalence of (iii) and (iv) is straightforward since $\sigma = \text{inn}_c \circ \sigma_{qs}$. The equivalence of (i) and (iv) follows from Proposition 3.9 and Lemma 2.4.

The second part of the theorem is Proposition 3.17. \square

Corollary 3.19. *With the notation of Theorem 3.18, the horospherical homogeneous space G/H has a (G, σ) -equivariant real structure if $\gamma(I, M) = (I, M)$ and one of the following conditions holds:*

- (i) $H^2(\Gamma, Z(G)) = \{1\}$; or
- (ii) $H^2(\Gamma, N_G(H)/H) = \{1\}$.

Proof. Let us note that if (i) or (ii) holds, then the map Δ_H is trivial. By Theorem 3.18, if $\gamma(I, M) = (I, M)$ and Δ_H is trivial, then the horospherical homogeneous space G/H has a (G, σ) -equivariant real structure. \square

Example 3.20. (Equivariant real structures on flag varieties.)

Let $X = G/P$ be a flag variety, and let $I \subset \mathcal{S}$ such that the standard parabolic subgroup P_I is conjugate to P . By Remark 2.5 we may assume that $P = P_I$. We saw in Example 2.13 that X has a (G, σ) -equivariant real structure if and only if $\sigma(P)$ is conjugate to P , which we also recover from Theorem 3.18 since $\Delta_P(\sigma)$ is always trivial. From a combinatorial point of view: $X = G/P_I$ has a (G, σ) -equivariant real structure if and only if $\gamma I = I$, in which case this structure is unique up to equivalence. This is for instance the case when σ is an inner twist of a split real group structure on G .

Example 3.21. Let (G, σ) be a simply-connected simple algebraic group with a real group structure, and let $H = U$ be a maximal unipotent subgroup of G . By Example 3.14, there always exists a (G, σ_{qs}) -equivariant real structure on G/U . Thus, by Theorem 3.18 and Lemma 1.24, there exists a (G, σ) -equivariant real structure on G/U if and only if the Tits class $\delta(\sigma)$ is trivial.

Example 3.22. We resume Example 3.15. The complex algebraic group $G = \text{SL}_4$ has five inequivalent real group structures (see Appendix A): a split one σ_s (with real part $\text{SL}_4(\mathbb{R})$) and an inner twist σ'_s (with real part $\text{SL}_2(\mathbb{H})$), the non split quasi-split one σ_{qs} (with real part $\text{SU}(2, 2)$) and two inner twists σ'_{qs} and σ_c (with real parts $\text{SU}(3, 1)$ and $\text{SU}(4)$). By Theorem 3.18 and Example 3.15, the homogeneous space $X = G/H$ has a (G, σ) -equivariant real structure if and only if $\Delta_H(\sigma)$ is trivial.

Let $\mathbb{T} = P/H \simeq \mathbb{G}_m$. If $\sigma = \sigma_s$ resp. $\sigma = \sigma_{qs}$, then the real group structure on \mathbb{T} induced by σ is equivalent to σ_0 resp. to σ_1 (this follows from a direct computation). Therefore, by Proposition 1.19, the group $H^2(\Gamma, \mathbb{T})$ is trivial when the Γ -action on \mathbb{T} comes from σ_{qs} , and so $\Delta_H(\sigma)$ is trivial for the inner twists of σ_{qs} . It remains to consider the case $\sigma = \sigma'_s$. Note that for this case, there exists $c \in G$ such that $\sigma'_s = \text{inn}_c \circ \sigma_s$, and that $c\sigma_s(c) \in Z(G)$ and is fixed by σ_s . This means that $c\sigma_s(c) = \pm 1 \in H$. Thus $\Delta_H(\sigma'_s)$ is trivial, and so by Theorem 3.18 there exists a (G, σ'_s) -equivariant real structure on X .

3.4. Number of equivalence classes of equivariant real structures. Let $X = G/H$ be a horospherical homogeneous space, and let σ be a real group structure on G . As before, we may assume choose σ such that $\sigma = \text{inn}_c \circ \sigma_{qs}$, for some $c \in G$, where σ_{qs} is a quasi-split real group structure on G stabilizing B and T .

In this section we suppose that there exists a (G, σ) -equivariant real structure μ_0 on X . By Remark 2.5, we may replace H by a standard horospherical subgroup conjugate to H and assume that $\sigma_{qs}(H) = H$. Then σ_{qs} induces a real group structure $\overline{\sigma_{qs}}$ on the torus $\mathbb{T} = N_G(H)/H \simeq \text{Aut}^G(X)$. By Proposition 1.5, the real group structure $\overline{\sigma_{qs}}$ is equivalent to $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$ for some $n_0, n_1, n_2 \in \mathbb{N}$ such that $n_0 + n_1 + 2n_2 = \dim(\mathbb{T})$.

Proposition 3.23. *With the notation and assumptions above, there are exactly 2^{n_1} equivalence classes of (G, σ) -equivariant real structures on $X = G/H$.*

Proof. We treat first the case when $\sigma = \sigma_{qs}$. Note that for this case, there exists a natural equivariant real structure on X , namely $\mu_0 = \overline{\sigma_{qs}}$ the real structure induced by σ_{qs} . Then, by Lemma 2.11, there is a bijection between the set of equivalence classes of (G, σ_{qs}) -equivariant real structures on X and $H^1(\Gamma, \text{Aut}^G(X))$, where Γ acts on $\text{Aut}^G(X)$ by μ_0 -conjugation. Identifying $\text{Aut}^G(X)$ with $\mathbb{T} = N_G(H)/H$, the Γ -action on $\text{Aut}^G(X)$ coincides with the Γ -action on \mathbb{T} induced by σ_{qs} . Thus, the number of equivalence classes of (G, σ) -equivariant real structures on X equals the cardinality of $H^1(\Gamma, \mathbb{T})$, where the action of Γ on \mathbb{T} is given by $\gamma(aH) = \sigma_{qs}(a)H$ for any $a \in N_G(H)$. Since $\overline{\sigma_{qs}}$ is equivalent to $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$ by assumption, the cardinality of $H^1(\Gamma, \mathbb{T})$ is 2^{n_1} by Proposition 1.19.

The calculation of the number of equivalence classes of equivariant (G, σ) -real structures when $\sigma = \text{inn}_c \circ \sigma_{qs}$ is not a quasi-split real group structure is a bit more complicated. First of all, in general, there is no particular privileged choice of the equivariant real structure μ_0 on X to define the Γ -action on $\text{Aut}^G(X)$, as there was in the quasi-split case, namely $\mu_0 = \overline{\sigma_{qs}}$. However we assumed that such a μ_0 exists and this will be sufficient for our purposes. By Lemma 2.7, this means there exists $n \in N_G(H)$ such that $\sigma_{qs}(c)c\sigma_{qs}(n)n \in H$, and then we can take μ_0 defined by $\mu_0(kH) = \sigma(k)cnH$ for all $k \in N_G(H)$. Now we identify $\text{Aut}^G(X)$ with \mathbb{T} and calculate the Γ -action on \mathbb{T} corresponding to the Γ -action by μ_0 -conjugation on $\text{Aut}^G(X)$. Let $\varphi \in \text{Aut}^G(X)$ and $a \in N_G(H)$ such that $\varphi(kH) = kaH$, for all $k \in G$. Then

$$\begin{aligned} \gamma\varphi(kH) &= \mu_0 \circ \varphi \circ \mu_0(kH) = \mu_0 \circ \varphi(\sigma(k)cnH) \\ &= \mu_0(\sigma(k)cnaH) = k\sigma(cna)cnH = kc\sigma_{qs}(cna)nH. \end{aligned}$$

Note that $kc\sigma_{qs}(cna)nH = c\sigma_{qs}(cn)n(n^{-1}\sigma_{qs}(a)n)H$. Since $c\sigma_{qs}(cn)n \in H$, and $(n^{-1}\sigma_{qs}(a)n) \in N_G(H)$, we find

$$\gamma\varphi(kH) = kn^{-1}\sigma_{qs}(a)nH.$$

But \mathbb{T} is abelian, and so $\gamma\varphi(kH) = k\sigma_{qs}(a)H$, and therefore the Γ -action on \mathbb{T} corresponding to the μ_0 -conjugation is given by $\gamma(aH) = \sigma_{qs}(a)H$ as in the quasi-split case that we treated first. Thus the number of equivalence classes of (G, σ) -equivariant real structures on X is again equal to the cardinality of $H^1(\Gamma, \mathbb{T})$, where the Γ -action on \mathbb{T} is the one induced by σ_{qs} . This finishes the proof. \square

Remark 3.24. Suppose that $\text{Out}(G) = \{1\}$. By Theorem 1.20 any real group structure on G is an inner twist of the split one σ_s , and so the induced real group

structure on \mathbb{T} is equivalent to $\sigma_0^{\times n_0}$. Thus Proposition 3.23 implies that when a (G, σ) -equivariant real structure exists on G/H , then it is unique up to equivalence.

Example 3.25. Let $G = T$ be a torus with a real group structure σ equivalent to $\sigma_0^{\times n_0} \times \sigma_1^{\times n_1} \times \sigma_2^{\times n_2}$ for some $n_0, n_1, n_2 \in \mathbb{N}$. Then the number of equivalence classes of (G, σ) -equivariant real structures on $X = T$ is 2^{n_1} . This could also be seen directly from Example 2.6.

Corollary 3.26. *With the same notation and assumptions as in Proposition 3.23, if we moreover assume that $H = U$ is a maximal unipotent subgroup of G , then there is a unique equivalence class of (G, σ) -equivariant real structures on $X = G/U$.*

Proof. If $H = U$, then the inclusion $T \hookrightarrow B$ yields an isomorphism $T \simeq B/U = \mathbb{T}$, and so the induced real group structure $\overline{\sigma_{qs}}$ on \mathbb{T} coincides with the restriction $(\sigma_{qs})|_T$. Since $\sigma_{qs}(B) = B$, this structure must preserve the positive roots of (G, B, T) , which means that $(\sigma_{qs})|_T$ is equivalent to a product $\sigma_0^{n_0} \times \sigma_2^{n_2}$. Therefore $n_1 = 0$ and the result follows from Proposition 3.23. \square

Example 3.27. Let $G = \mathrm{SL}_3$, let $H = U$ be the unipotent radical of a Borel subgroup B , let T be a maximal torus contained in B . In Example 1.12 we saw that G has three inequivalent real group structures, namely σ_s (split), σ_{qs} (non split quasi-split), and σ_c (compact, inner twist of σ_{qs}). Since σ_s and σ_{qs} stabilize B (up to conjugate), they stabilize also U and so there exists an equivariant real structure on $X = G/U$ in these two cases. Moreover, as the Tits class of (G, σ_c) is trivial (see Table 2 in Appendix A), it follows from Theorem 3.18 and Lemma 1.24 that there exists also a (G, σ_c) -equivariant real structure on X . Finally, by Corollary 3.26, each of these three equivariant real structures is unique up to equivalence.

Example 3.28. We resume Example 3.22. We saw that if $\sigma = \sigma_s$ resp. $\sigma = \sigma_{qs}$, then the real group structure on \mathbb{T} induced by σ is equivalent to σ_0 resp. to σ_1 . Thus, by Proposition 3.23, there is one equivalence class of (G, σ) -equivariant real structures on G/H when σ is equivalent to an inner twist of σ_s , and there are two equivalence classes of (G, σ) -equivariant real structures on G/H when σ is equivalent to an inner twist of σ_{qs} .

3.5. Extension of the equivariant real structures. As before we fix a triple (G, B, T) , where G is a complex reductive algebraic group, $B \subset G$ is a Borel subgroup, and $T \subset B$ is a maximal torus. Let σ be a real group structure on G . In this section we determine when a given (G, σ) -equivariant real structure on a horospherical homogeneous space G/H extends to a (G, σ) -equivariant real structure on a horospherical variety whose open orbit is G/H .

For the sake of brevity, we will not recall the *theory of spherical embeddings*, and how to describe such embeddings in terms of the combinatorial gadgets called *colored fan*. Therefore the reader is referred to [Kno91, Tim11, Gan18] for more information on this theory.

For the next theorem, we will consider the Γ -action on the set of colored cones, defined by Hurugen in [Hur11b]. Let σ be an inner twist of a quasi-split real group structure σ_{qs} on G , and consider all the colored fans defining G -equivariant embeddings $G/H \hookrightarrow X$. Then the Γ -action on the character group \mathbb{X} defined by σ_{qs} (see Definition 1.13) induces a Γ -action on this set of colored fans. In particular, if $\sigma_{qs} = \sigma_0$ is split, then this Γ -action is trivial.

Theorem 3.29. ([Hur11b, Theorem 2.23] and [Wed, Theorem 9.1].)

Let μ be a (G, σ) -equivariant real structure on a **spherical** homogeneous space G/H , and let X be a spherical G -variety with open orbit G/H . Then the real structure μ extends on X if and only if the colored fan of the spherical embedding $G/H \hookrightarrow X$ is Γ -invariant.

Remark 3.30. If the equivariant real structure μ on G/H extends to X , then the corresponding real form X/Γ always exists as a real algebraic space but not necessarily as a real variety. Indeed, X need not be a union of Γ -stable quasi-projective G -varieties; see [Hur11b, §2.4] for such an example.

Corollary 3.31. Let μ be a (G, σ) -equivariant real structure on a **horospherical** homogeneous space G/H , and let X be a horospherical G -variety with open orbit G/H . Then the real structure μ extends on X if and only if the colored fan of the embedding $G/H \hookrightarrow X$ is Γ -invariant in which case the corresponding real form X/Γ is a real variety.

Proof. The only thing to check is that X/Γ is a real variety (and not just a real algebraic space). Since $\Gamma = \{1, \gamma\} \simeq \mu_2$, for every colored cone (C, F) of the colored fan associated with the embedding $G/H \hookrightarrow X$, the colored fan consisting of the cones $(\gamma C, \gamma F)_{\gamma \in \Gamma}$ and their faces has only one or two maximal cones. Also, for horospherical embeddings, the relative interiors of two maximal cones do not meet (as the valuation cone is the whole space in this case). Hence, the characterization of quasi-projective spherical varieties via the existence of a strictly convex \mathbb{Q} -valued piecewise linear function (see e.g. [Per14, Corollary 3.2.12]) implies that X is the union of Γ -stable quasi-projective G -varieties, and so the quotient X/Γ is a real variety. \square

Example 3.32. Consider, for example, the horospherical embeddings of SL_2/U given in Example 3.5. There are two inequivalent real group structures on SL_2 : σ_0 which is split, and σ_1 , whose real part is compact. Note that σ_1 is an inner twist of σ_0 . We deduce from Example 3.21 and Remark 3.24 (or Corollary 3.26) that there exists a unique equivalence class of (SL_2, σ_0) -equivariant real structure on SL_2/U , but that there is no (SL_2, σ_1) -equivariant structure on SL_2/U as the Tits class $\delta(\sigma_1)$ is non-trivial. Since σ_0 is split, it induces a trivial Γ -action on the set of colored fans defining a SL_2 -equivariant embedding of $SL_2/U \hookrightarrow X$. Thus, any (SL_2, σ_0) -equivariant real structure on SL_2/U extends to X .

Example 3.33. We resume Example 3.28. Let σ be an inner twist of a non split quasi-split real group structure on $G = SL_4$. We saw in Example 3.15 that the Γ -action on $N = M^\vee = \mathbb{Z} \langle \chi^\vee \rangle \simeq \mathbb{Z}$ induced by σ satisfies $\gamma \chi^\vee = -\chi^\vee$. Let $\mathcal{F} \subset N_{\mathbb{Q}}$ be a colored fan corresponding to a G -equivariant embedding $G/H \hookrightarrow Y$. Then, by Corollary 3.31, a (G, σ) -equivariant real structure on G/H extends to Y if and only if the colored fan \mathcal{F} is symmetric with respect to the origin of $N_{\mathbb{Q}}$. It follows that either $Y = G/H$ (case $\mathcal{F} = \{(\{0\}, \emptyset)\}$) or Y is a \mathbb{P}^1 -bundle over G/P which is the union of two G -orbits of codimension 1, the two G -invariant sections of the structure morphism $Y \rightarrow G/P$, and the open G -orbit (case $\mathcal{F} = \{(\mathbb{Z}_+ \langle \chi^\vee \rangle, \emptyset), (\mathbb{Z}_- \langle \chi^\vee \rangle, \emptyset)\}$).

3.6. Smooth projective horospherical varieties of Picard rank 1. Let G be a complex simply-connected semisimple algebraic group, let $B \subset G$ a Borel subgroup, and let $T \subset B$ a maximal torus as before. In this section we will apply

the results obtained in the previous sections to classify the real structures on the smooth projective horospherical G -varieties of Picard rank 1.

Examples of such varieties are given by the flag varieties $X = G/P$, with P a maximal parabolic subgroup, and the odd symplectic grassmannians; these correspond to the case (3) in Theorem 3.34 and were studied for example in [Mih07, Pec13].

The smooth projective horospherical G -varieties of Picard rank 1 were classified by Pasquier in [Pas09], and since then their geometry has been very much studied (see e.g. [PP10, Hon16, GPPS]). Pasquier proved the following result:

Theorem 3.34. ([Pas09, Theorem 0.1]) *Let X be a smooth projective horospherical G -variety of Picard rank 1. Then either $X = G/P$ is a flag variety (with P a maximal parabolic subgroup) or X has three G -orbits and can be constructed in a uniform way from a triple $(\text{Dyn}(G), \varpi_Y, \varpi_Z)$ belonging to the following list:*

- (1) $(B_n, \varpi_{n-1}, \varpi_n)$ with $n \geq 3$;
- (2) $(B_3, \varpi_1, \varpi_3)$;
- (3) $(C_n, \varpi_m, \varpi_{m-1})$ with $n \geq 2$ and $m \in [2, n]$;
- (4) $(F_4, \varpi_2, \varpi_3)$;
- (5) $(G_2, \varpi_1, \varpi_2)$,

where ϖ_Y, ϖ_Z are fundamental weights such that the two closed orbits of X are G -isomorphic to $G/P(\varpi_Y)$ and $G/P(\varpi_Z)$.

We already looked at equivariant real structures on flag varieties in Examples 2.13 and 3.20. Therefore, we will only consider equivariant real structures in the non-homogeneous cases.

Theorem 3.35. *We keep the notation of Theorem 3.34. Let σ be a real group structure on G , let G_0 be the corresponding real part, and let X be a non-homogeneous smooth projective horospherical G -variety of Picard rank 1 associated with a triple $(\text{Dyn}(G), \varpi_Y, \varpi_Z)$. Then X admits a (G, σ) -equivariant real structure if and only if $(\text{Dyn}(G), G_0, \varpi_Y, \varpi_Z)$ belongs to the following list:*

- (1) $(B_n, G_0, \varpi_{n-1}, \varpi_n)$ with $G_0 = \text{Spin}_{n+4t, n+1-4t}(\mathbb{R})$ and $n \geq 3, t \in \mathbb{Z}$;
- (2) $(B_3, G_0, \varpi_1, \varpi_3)$ with $G_0 = \text{Spin}_7(\mathbb{R})$ or $\text{Spin}_{3,4}(\mathbb{R})$;
- (3) $(C_n, \text{Sp}(2n, \mathbb{R}), \varpi_m, \varpi_{m-1})$ with $n \geq 2$ and $m \in [2, n]$;
- (4) $(F_4, G_0, \varpi_2, \varpi_3)$ with G_0 the real part of one of the three inequivalent real group structures on F_4 ; or
- (5) $(G_2, G_0, \varpi_1, \varpi_2)$ with G_0 the real part of one of the two inequivalent real group structures on G_2 (the split one and the compact one).

Moreover, when such a structure exists, then it is unique up to equivalence.

Proof. In cases (1)-(5) of Theorem 3.34, we observe that $\text{Aut}(\text{Dyn}(G)) = \{1\}$. Thus, by Remark 3.24, if a (G, σ) -equivariant real structure exists on G/H , then it is unique up to equivalence. Also, by Theorem 1.20 (iv) any real group structure σ on G is an inner twist of the split real group structure σ_0 on G . Therefore the induced Γ -action on $\mathbb{X}(T)$ is trivial (Remark 1.10). This has two important consequences: First, by Theorem 3.18, the open orbit $X_0 = G/H$ always admits a (G, σ_0) -equivariant real structure. Second, by Corollary 3.31, any (G, σ) -equivariant real structure on X_0 extends to X .

Moreover, again by Theorem 3.18, the homogeneous space X_0 admits a (G, σ) -equivariant real structure if and only if $\Delta_H(\sigma)$ is trivial. (We recall that the Tits

class $\delta(\sigma)$ of (G, σ) can be found in the tables in Appendix A.) One can check that the cases where $\delta(\sigma)$ is trivial are exactly the cases that appear in the statement of the theorem. Therefore, to finish the proof of this theorem, it suffices to prove that the homomorphism χ_H^* defined in §1.3 is injective in the cases where $\delta(\sigma)$ is non-trivial. This will indeed imply that $\Delta_H(\sigma)$ is non-trivial.

The cases left to consider are those where G is of type B_n , for $n \geq 3$, or of type C_n , for $n \geq 2$. Recall that $\mathbb{T} = N_G(H)/H$ is a quotient of the maximal torus T obtained by composing the following homomorphisms:

$$T \hookrightarrow B \twoheadrightarrow B/U \rightarrow N_G(H)/H = \mathbb{T},$$

where the map $B/U \rightarrow N_G(H)/H$ is the map induced by the inclusion $B \hookrightarrow N_G(H)$. Note that $N_G(H) = BH = TUH = TH$, and so the homomorphism $T \rightarrow \mathbb{T}$ is indeed onto, with kernel $T \cap H$. Thus the induced real structure $\bar{\sigma}_s$ on \mathbb{T} (which is a 1-dimensional torus in cases (1)-(3)) is obtained from $\sigma_s|_T \sim \sigma_0 \times \sigma_0$, and so $\bar{\sigma}_s$ is equivalent to σ_0 . By Proposition 1.19, this means that $H^2(\Gamma, \mathbb{T}) \simeq \mu_2$. A generator of this group is given by the class of the σ_0 -invariant element -1 .

We now consider the group $H^2(\Gamma, Z(G))$. In type B_n and C_n , note that the center of the simply-connected simple group G is $Z(G) \simeq \mu_2$. Since we are considering the cases where the Tits classes are non-trivial, the group $H^2(\Gamma, Z(G))$ is non-trivial, and, since $Z(G) \simeq \mu_2$, it is isomorphic to $Z(G)$ on which Γ acts trivially.

Recall that the homomorphism $\chi_H^* : H^2(\Gamma, Z(G)) \simeq Z(G) \rightarrow H^2(\Gamma, \mathbb{T})$ is induced by the homomorphism $\psi : Z(G) \rightarrow \mathbb{T}$. Since $H^2(\Gamma, \mathbb{T})$ is generated by the class of -1 , either ψ is injective, and then χ_H^* is an isomorphism, or else ψ is trivial, and then χ_H^* is trivial. We will show that ψ is injective, i.e., that $Z(G)$ is not contained in H and this will finish the proof of this theorem.

We are left to prove that $Z(G)$ is not contained in H . For this, use the following construction of H from [GPPS, §1.3]. For each triple $(\text{Dyn}(G), \varpi_1, \varpi_2)$ in Theorem 3.34, consider the projective space $\mathbb{P}(V_1 \oplus V_2)$, where V_1 and V_2 are the irreducible G -modules with highest weights ϖ_1 and ϖ_2 respectively. Let v_i be a highest weight vector of V_i for $i = 1, 2$. Then H is the stabilizer of the line generated by $v_1 \oplus v_2 \in \mathbb{P}(V_1 \oplus V_2)$. With this description, we see that $Z(G)$ is contained in H if and only if $Z(G)$ acts on $V_1 \oplus V_2$ by ± 1 . However, in each case one can check (for instance using the description of the fundamental representations in [FH91]) that $Z(G)$ acts trivially on V_i and by -1 on V_j with $\{i, j\} = \{1, 2\}$. Hence, $Z(G)$ is not contained in H and χ_H^* is an isomorphism. \square

Remark 3.36. Let $(\text{Dyn}(G), \varpi_1, \varpi_2)$ be a triple from Theorem 3.34. Let $\sigma = \sigma_s$ be a split real group structure on G , and let μ_s be a (G, σ_s) -equivariant real structure on X . Then the geometric construction of $X(\mathbb{C})$ from [GPPS, §1.3] yields a construction of the real part of (X, μ_s) provided that one takes \mathbb{R} as a base field in the construction instead of \mathbb{C} .

APPENDIX A. TITS CLASSES FOR THE SIMPLY-CONNECTED SIMPLE ALGEBRAIC GROUPS (BY MIKHAIL BOROVOI)

In this appendix we give the list of all pairs (G, σ) , where G is a complex simply-connected simple algebraic group with a real group structure σ and corresponding real form G_0 , and we give the Tits class in each case (see §1.3 for the definition of the Tits classes). We list all real forms G_0 for a given G (instead of all the possible σ) and write $t(G_0)$ instead of $\delta(\sigma)$ to denote the Tits class.

We now explain how one can compute $t(G_0)$ in Tables 1 and 2 below. Our exposition is based on Skip Garibaldi's answer [Gar18] to the author's MathOverflow question.

Let G be a complex simply-connected simple algebraic group, let $T \subset G$ be a maximal torus, and let $B \subset G$ be a Borel subgroup containing T . For any dominant weight λ of (G, B, T) , let ρ_λ denote the irreducible representation of G with highest weight λ .

Let G_0 be a real form of G . Then the Galois group $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$ acts via the $*$ -action (see [Con14, Remark 7.1.2] for the definition of the $*$ -action) on $\text{Dyn}(G)$ and on the set of dominant weights: $\lambda \mapsto \gamma(\lambda)$. Moreover, Γ naturally acts on the set of isomorphism classes of irreducible complex representations of G_0 : $\rho \mapsto \gamma\rho$. It is known that

$$\gamma(\rho_\lambda) \simeq \rho_{\gamma(\lambda)}.$$

We wish to know whether ρ_λ can be defined over \mathbb{R} . Clearly, if $\gamma(\lambda) \neq \lambda$, then $\gamma(\rho_\lambda) \not\simeq \rho_\lambda$, and hence, ρ_λ cannot be defined over \mathbb{R} . However, even if $\gamma(\lambda) = \lambda$, it might happen that the representation ρ_λ cannot be defined over \mathbb{R} . The obstruction is the Tits algebra of λ .

We assume that $\gamma(\lambda) = \lambda$. Write $Z_0 = Z(G_0)$. The dominant weight λ induces a homomorphism $\lambda|_Z: Z \rightarrow \mathbb{G}_m$, which is defined over \mathbb{R} (because $\gamma(\lambda) = \lambda$). The obtained homomorphism $Z_0 \rightarrow \mathbb{G}_{m, \mathbb{R}}$ induces a homomorphism on second cohomology

$$\lambda_*: H^2(\mathbb{R}, Z_0) \rightarrow H^2(\mathbb{R}, \mathbb{G}_{m, \mathbb{R}}) = \{\pm 1\}.$$

By definition, the *Tits algebra of λ* is $\lambda_*(t(G_0)) \in H^2(\mathbb{R}, \mathbb{G}_{m, \mathbb{R}}) = \{\pm 1\}$. The irreducible complex representation ρ_λ of G_0 can be defined over \mathbb{R} if and only if the corresponding Tits algebra is 1; otherwise, $\rho_\lambda \oplus \gamma\rho_\lambda$ can be defined over \mathbb{R} , but ρ_λ cannot; see Tits [Tit71].

Proposition A.1 ([Gar12], Proposition 7). *The natural map*

$$\prod \lambda_*: H^2(\mathbb{R}, Z_0) \rightarrow \prod H^2(\mathbb{R}, \mathbb{G}_{m, \mathbb{R}})$$

is injective, where the products run over the minuscule weights λ such that $\gamma(\lambda) = \lambda$.

In the following we write λ_k for the fundamental weight such the $\langle \lambda_k, \alpha_i \rangle = \delta_{ki}$, where α_i is the i -th simple root with numeration of Bourbaki, and δ_{ki} is Kronecker's symbol. We write $\rho_k = \rho_{\lambda_k}$.

First we consider the case when $H^2(\mathbb{R}, Z_0) = \{\pm 1\}$. This is the case when G_0 is of one of the types A_{2m-1} , B_n , C_n , D_{2m+1} , E_7 . In the tables of Tits [Tit67], for all real forms G_0 of G and for all fundamental weights λ of G , in particular, for all minuscule weights, it is written whether the complex representation $\gamma(\rho_\lambda)$ of G_0 is equivalent to ρ_λ , and if yes, then whether ρ_λ can be defined over \mathbb{R} . This permits us to determine whether $t(G_0) = 1 \in \{\pm 1\}$. If $t(G_0) \neq 1$, then $t(G_0) = -1$.

Example A.2. Let G_0 be of type ${}^2\mathbf{A}_{2m-1}$, namely, $G_0 = \mathrm{SU}(m+s, m-s)$. Then all fundamental weights are minuscule weights. By [Tit67, page 28] $\gamma(\lambda_k) = \lambda_{2m-k}$, hence, the only minuscule weight λ_k with $\gamma(\lambda_k) = \lambda_k$ is λ_m . Tits writes also that corresponding representation ρ_m can be defined over \mathbb{R} if and only if s is even. Thus the Tits class $t(G_0)$ equals 1 if and only if s is even. Thus $t(\mathrm{SU}(m+s, m-s)) = (-1)^s$.

It remains to treat the case ${}^1\mathbf{D}_{2m}$. Consider the fundamental weights λ_{2m-1} and λ_{2m} , they define a homomorphism

$$\lambda_{2m-1} \times \lambda_{2m} : Z \rightarrow \mathbb{G}_m \times \mathbb{G}_m$$

and an isomorphism

$$\lambda_{2m-1} \times \lambda_{2m} : Z \xrightarrow{\sim} \mu_2 \times \mu_2.$$

We identify Z with $\mu_2 \times \mu_2$ using this isomorphism. Note that the minuscule weights are $\lambda_1, \lambda_{2m-1}, \lambda_{2m}$.

We consider the case $G_0 = \mathrm{Spin}(2m+2s, 2m-2s)$. Then ρ_1 is defined over \mathbb{R} for all s , while, according to Tits [Tit67, p. 39], the half-spin representations ρ_{2m-1} and ρ_{2m} can be defined over \mathbb{R} if and only if s is even. Thus $t(\mathrm{Spin}(2m+2s, 2m-2s)) = ((-1)^s, (-1)^s)$.

We consider the case $G_0 = \mathrm{Spin}^*(4m)$. Then, according to Tits [Tit67, p. 40], the representation ρ_1 cannot be defined over \mathbb{R} . Moreover, *exactly one* of the two half-spin representations ρ_{2m-1} and ρ_{2m} can be defined over \mathbb{R} . Thus, $t(\mathrm{Spin}^*(4m)) = \pm(1, -1)$.

The author of the appendix is grateful to Skip Garibaldi for answering the author's MathOverflow question; see [Gar18].

Notation. In the following tables, the column $\mathrm{Dyn}(G)$ indicates the Dynkin diagram of G , $Z = Z(G)$ indicates the center of G , the real algebraic group G_0 corresponds to (G, σ) , the real algebraic group G_{qs} is a quasi-split inner form of G_0 , the abelian group $H^2(\mathbb{R}, Z_{\mathrm{qs}}) := H^2(\Gamma, Z_{\mathrm{qs}}(\mathbb{C}))$ is the second Galois cohomology group, where $Z_{\mathrm{qs}} = Z(G_{\mathrm{qs}}) = Z(G_0)$, and $t(G_0)$ is the Tits class of G_0 . When the group $H^2(\mathbb{R}, Z_{\mathrm{qs}})$ is trivial (which is the case, for instance, if $\mathrm{Dyn}(G) = \mathbf{A}_{2m}$), the details on all the terms are not written, because the Tits class is clearly trivial in this case.

TABLE 1. Tits classes for the simply-connected exceptional groups

$\mathrm{Dyn}(G)$	Z	G_{qs}	$H^2(\mathbb{R}, Z_{\mathrm{qs}})$	G_0	$t(G_0)$
\mathbf{E}_6	μ_3		1		1
\mathbf{E}_7	μ_2	$\mathbf{E}_{7(7)}$	μ_2	$\begin{cases} \mathbf{E}_{7(7)}, \mathbf{E}_{7(-25)} \\ \mathbf{E}_{7(-133)}, \mathbf{E}_{7(-5)} \end{cases}$	$\begin{matrix} 1 \\ -1 \end{matrix}$
\mathbf{E}_8	1		1		1
\mathbf{F}_4	1		1		1
\mathbf{G}_2	1		1		1

TABLE 2. Tits classes for the simply-connected classical groups

$\text{Dyn}(G)$	G	Z	G_{qs}	$H^2(\mathbb{R}, Z_{\text{qs}})$	G_0	$t(G_0)$
A_{2m}	$m \geq 1$ $\text{SL}(2m+1)$	μ_{2m+1}		1		1
${}^1A_{2m-1}$	$m \geq 1$ $\text{SL}(2m)$	μ_{2m}	$\text{SL}(2m, \mathbb{R})$	μ_2	$\begin{cases} \text{SL}(2m, \mathbb{R}) \\ \text{SL}(m, \mathbb{H}) \end{cases}$	$\begin{matrix} 1 \\ -1 \end{matrix}$
${}^2A_{2m-1}$	$m \geq 2$ $\text{SL}(2m)$	μ_{2m}	$\text{SU}(m, m)$	μ_2	$\text{SU}(m+s, m-s)$	$(-1)^s$
B_n	$n \geq 2$ $\text{Spin}(2n+1)$	μ_2	$\text{Spin}(n, n+1)$	μ_2	$\text{Spin}(n+2s, n+1-2s)$	$(-1)^s$
C_n	$n \geq 3$ $\text{Sp}(2n, \mathbb{C})$	μ_2	$\text{Sp}(2n, \mathbb{R})$	μ_2	$\begin{cases} \text{Sp}(2n, \mathbb{R}) \\ \text{Sp}(s, n-s) \end{cases}$	$\begin{matrix} 1 \\ -1 \end{matrix}$
${}^1D_{2m+1}$	$m \geq 2$ $\text{Spin}(4m+2)$	μ_4	$\text{Spin}(2m+1, 2m+1)$	μ_2	$\text{Spin}(2m+1+2s, 2m+1-2s)$	$(-1)^s$
${}^2D_{2m+1}$	$m \geq 2$ $\text{Spin}(4m+2)$	μ_4	$\text{Spin}(2m+2, 2m)$	μ_2	$\begin{cases} \text{Spin}(2m+2+2s, 2m-2s) \\ \text{Spin}^*(4m+2) \end{cases}$	$\begin{matrix} 1 \\ -1 \end{matrix}$
${}^1D_{2m}$	$m \geq 2$ $\text{Spin}(4m)$	$\mu_2 \times \mu_2$	$\text{Spin}(2m, 2m)$	$\mu_2 \times \mu_2$	$\begin{cases} \text{Spin}(2m+2s, 2m-2s) \\ \text{Spin}^*(4m) \end{cases}$	$\begin{matrix} ((-1)^s, (-1)^s) \\ \pm(-1, 1) \end{matrix}$
${}^2D_{2m}$	$m \geq 2$ $\text{Spin}(4m)$	$\mu_2 \times \mu_2$	$\text{Spin}(2m+1, 2m-1)$	1	$\text{Spin}(2m+1+2s, 2m-1-2s)$	1

REFERENCES

- [ACF14] Dmitri Akhiezer and Stéphanie Cupit-Foutou. On the canonical real structure on wonderful varieties. *J. Reine Angew. Math.*, 693:231–244, 2014.
- [Akh15] Dmitri Akhiezer. Satake diagrams and real structures on spherical varieties. *Internat. J. Math.*, 26(12):1550103, 13, 2015.
- [BM13] Victor Batyrev and Anne Moreau. The arc space of horospherical varieties and motivic integration. *Compos. Math.*, 149(8):1327–1352, 2013.
- [Ben16] Mohamed Benzerga. Real structures on rational surfaces (PhD’s thesis). <https://tel.archives-ouvertes.fr/tel-01471071>, 2016.
- [BG] Mikhail Borovoi, with an appendix by Giuliano Gagliardi. Equivariant models of spherical varieties. *arXiv:1710.02471*.
- [Bor] Mikhail Borovoi. Existence of equivariant models of G-varieties. *arXiv:1804.08475*.
- [Con14] Brian Conrad. Reductive group schemes. In *Autour des schémas en groupes. Vol. I*, volume 42/43 of *Panor. Synthèses*, pages 93–444. Soc. Math. France, Paris, 2014.
- [CF15] Stéphanie Cupit-Foutou. Anti-holomorphic involutions and spherical subgroups of reductive groups. *Transform. Groups*, 20(4):969–984, 2015.
- [FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.
- [Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
- [Gan18] Jacopo Gandini. Embeddings of spherical homogeneous spaces. *J. Acta. Math. Sin.-English Ser. (2018) 34: 299*.
- [Gar12] Skip Garibaldi. Outer automorphisms of algebraic groups and determining groups by their maximal tori. *Michigan Math. J.*, 61(2):227–237, 2012.
- [Gar18] Skip Garibaldi (<https://mathoverflow.net/users/6486/skip>). The Tits classes of simply connected simple real groups. MathOverflow. <https://mathoverflow.net/q/303176> (version: 2018-06-19).
- [GPPS] Richard Gonzales, Clélia Pech, Nicolas Perrin, and Alexander Samokhin. Geometry of horospherical varieties of Picard rank one. *arXiv:1803.05063*.
- [Hon16] Jaehyun Hong. Smooth horospherical varieties of Picard number one as linear sections of rational homogeneous varieties. *J. Korean Math. Soc.*, 53(2):433–446, 2016.
- [Hum75] James E. Humphreys *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.
- [Hur11a] Mathieu Huruguen. Compactification of spherical homogeneous spaces over an arbitrary field (PhD’s thesis). <https://tel.archives-ouvertes.fr/tel-00716402>, 2011.
- [Hur11b] Mathieu Huruguen. Toric varieties and spherical embeddings over an arbitrary field. *J. Algebra*, 342:212–234, 2011.
- [Kna02] Anthony W. Knap. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.
- [Kno91] Friedrich Knop. The Luna-Vust theory of spherical embeddings. In *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*, pages 225–249. Manoj Prakashan, Madras, 1991.
- [KMRT98] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The Book of Involutions*. American Mathematical Society Colloquium Publications, 44. American Mathematical Society, Providence, RI, 1998.
- [LPR] Kevin Langlois, Clélia Pech, and Michel Raibaut. Stringy invariants for horospherical varieties of complexity one. *arXiv:1511.03852, to appear in Algebraic Geometry*.
- [Man17] Frédéric Mangolte. *Variétés algébriques réelles*, volume 24 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2017.
- [Mih07] Ion Alexandru Mihai. Odd symplectic flag manifolds. *Transform. Groups*, 12(3):573–599, 2007.
- [OV90] A. L. Onishchik and È. B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

- [Pas06] Boris Pasquier. Horospherical Fano varieties (PhD's thesis). <https://tel.archives-ouvertes.fr/tel-00111912>, 2006.
- [Pas08] Boris Pasquier. Variétés horosphériques de Fano. *Bull. Soc. Math. France*, 136(2):195–225, 2008.
- [Pas09] Boris Pasquier. On some smooth projective two-orbit varieties with Picard number 1. *Math. Ann.*, 344(4):963–987, 2009.
- [Pas10] Boris Pasquier. The pseudo-index of horospherical Fano varieties. *Internat. J. Math.*, 21(9):1147–1156, 2010.
- [PP10] Boris Pasquier and Nicolas Perrin. Local rigidity of quasi-regular varieties. *Math. Z.*, 265(3):589–600, 2010.
- [Pas15a] Boris Pasquier. An approach of the minimal model program for horospherical varieties via moment polytopes. *J. Reine Angew. Math.*, 708:173–212, 2015.
- [Pas15b] Boris Pasquier. Birational geometry of horospherical varieties (HDR). Available on the author's webpage, 2015.
- [Pas18] Boris Pasquier. The log minimal model program for horospherical varieties via moment polytopes. *Acta Math. Sin. (Engl. Ser.)*, 34(3):542–562, 2018.
- [Pau81] Franz Pauer. Normale Einbettungen von G/U . *Math. Ann.*, 257(3):371–396, 1981.
- [Pec13] Clélia Pech. Quantum cohomology of the odd symplectic Grassmannian of lines. *J. Algebra*, 375:188–215, 2013.
- [Per14] Nicolas Perrin. On the geometry of spherical varieties. *Transform. Groups*, 19(1):171–223, 2014.
- [Ser02] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.
- [Tim11] Dmitry A. Timashev. *Homogeneous spaces and equivariant embeddings*, volume 138 of *Encyclopaedia of Mathematical Sciences*. Springer, Heidelberg, 2011. Invariant Theory and Algebraic Transformation Groups, 8.
- [Tit66] Jacques Tits. *Classification of algebraic semisimple groups*. Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math. 9, Boulder, Colo., 1965) pp. 33-62 Amer. Math. Soc., Providence, R.I., 1966.
- [Tit67] Jacques Tits. *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*. Springer-Verlag, Berlin-New York, 1967.
- [Tit71] Jacques Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. *J. Reine Angew. Math.*, 247:196–220, 1971.
- [Vos98] V. E. Voskresenskiĭ. *Algebraic groups and their birational invariants*, volume 179 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1998. Translated from the Russian manuscript by Boris Kunyavski [Boris È. Kunyavskiĭ].
- [Wed] Torsten Wedhorn. Spherical spaces. *arXiv:1512.01972v1*, to appear in *Annales de l'Institut Fourier*.

UNIV. BOURGOGNE FRANCHE-COMTÉ, INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR5584,
 9 AVENUE ALAIN SAVARY, BP 47870 - 21078 DIJON CEDEX, FRANCE
Email address: lucy.moser-jauslin@u-bourgogne.fr

UNIV. BOURGOGNE FRANCHE-COMTÉ, INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UMR5584,
 9 AVENUE ALAIN SAVARY, BP 47870 - 21078 DIJON CEDEX, FRANCE
Email address: ronan.terpereau@u-bourgogne.fr